Lectures on

Higher Topos Theory

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Contents

Abstract

This course surveys the theory of ∞-topoi. After introducing the basic notions (descent, univalence, modalities, ...), we discuss general constructions (limits and colimits, congruences, parametrization, Grothendieck topologies) and a range of examples, including Goodwillie calculus, finitary functors, geometric structures and schemes in Lurie's framework, and analytic stacks in the sense of Clausen–Scholze.

Preface

These are lecture notes based on the course *Higher Topos Theory* taught by Marc Hoyois at the University of Regensburg in the winter term 2025/2026, written by Bastiaan Cnossen. The presentation is at times different from the original lectures. The notations and terminology follow the preferred conventions of the note-taker, which usually match those of the lecturer. The following are exceptions:

- We say 'animae' for the plural of 'anima'.
- The presheaf category of a category C is denoted $\mathrm{PSh}(C)$ rather than $P(C)$.
- Given topoi S and T, we refer to a left exact colimit-preserving functor $f^*\colon S \to T$ as a *morphism of logoi* rather than a *geometric morphism*. We refer to the right adjoint $f_*\colon T \to S$ as a *morphism of topoi*. We write Logos and Topos for the resulting (non-full) subcategories of Cat, so that $\mathrm{Logos} \simeq \mathrm{Topos}^{\mathrm{op}}$. This terminology is due to Anel and Joyal [AJ21].

Some added material not treated during the course:

- Grothendieck sites (Section 6.1);
- Groupoid actions, classifying stacks and principal bundles (Section 6.2);
- Shape theory (Section 6.3);
- Univalent families and Uemura's proof of closure of $\mathrm{Topos}^{\text{ét}}$ under colimits (Section 4.5.1);
- Hypercovers (Section 6.4).

1 Introduction

Topos theory originated in Grothendieck's pursuit of the Weil conjectures, where he introduced topoi to define *étale cohomology*. Since cohomological constructions are inherently derived, this means that higher-categorical phenomena already appear at the origin of the subject.

Higher topos theory provides a richer framework than classical topos theory. The perspective of these notes is that higher topos theory is the primary object of study, with classical topos theory appearing as its 0-truncated fragment. We therefore use "topos" to mean higher topos and "classical topos" for the classical notion.

As a first approximation, we have the following analogy:

$$\text{Topos theory} \quad \sim \quad \text{Classical topos theory} \ + \ \text{animae}.$$

This is only a loose analogy, since topos theory exhibits various phenomena absent in the classical theory. Two fundamental examples are *Postnikov towers* and *Goodwillie towers*. Given a topos T, these form sequences of subcategories:

$$T_{\leq 0} \subseteq T_{\leq 1} \subseteq T_{\leq 2} \subseteq \cdots \subseteq T_{\leq \infty} = T^{\mathrm{hyp}} \subseteq T^{(1)} \subseteq T^{(2)} \subseteq \cdots \subseteq T^{(\infty)} \subseteq T.$$

These towers are invisible from the perspective of the classical topos $T_{\leq 0}$.

Descent

The organizing idea in topos theory is *descent*: reconstructing global objects from local data together with gluing information.

Consider first the familiar setting of vector bundles on a manifold M. Given an open cover $\{U_i\}$ of M, a vector bundle on M is completely determined by its restrictions to each open set U_i, together with transition functions on the overlaps $U_i \cap U_j$ satisfying the usual cocycle condition on triple overlaps. In categorical terms, the category $\mathrm{Vect}(M)$ of vector bundles on M satisfies descent for open covers.

This same pattern appears in algebraic geometry. A scheme is built by gluing affine pieces $\mathrm{Spec}(R)$ along Zariski open subschemes. For an affine scheme $\mathrm{Spec}(R)$, a quasi-coherent sheaf is simply an R-module. For a general scheme X obtained by gluing affines, the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X can be defined via descent: we specify compatible R-modules for all affine maps $\mathrm{Spec}(R) \to X$, where these modules must satisfy gluing conditions on overlaps. The assignment $X \mapsto \mathrm{QCoh}(X)$ thus satisfies descent for affine covers, by construction.

These examples motivate the classical notion of a *Grothendieck topos*. One starts with a small category C of "local models" (such as open subsets of a topological space, or affine schemes), equips it with a notion of "coverings" generalizing open covers, and forms the category of sheaves on C with respect to these coverings. This sheaf category, which is defined as a left exact localization of the presheaf category of C, may be thought of as obtained from C by universally adding all colimits while demanding that colimits along coverings must be preserved. Via descent, every contravariant assignment $C^{\mathrm{op}} \to \mathrm{Cat}$ of categories compatible with coverings universally extends to the entire sheaf category.

But the presentation in terms of a category C with coverings is not intrinsic to the resulting topos: different choices of $(C, \text{coverings})$ can give rise to equivalent sheaf categories. The invariant content is the descent property itself, and this can be recognized directly on the category. Informally, it says that for any covering $\{X_i\}$ of an object X, a map into X is determined by its base changes to the objects X_i, together with coherent gluing data on overlaps. This leads to the definition of a topos as a presentable category satisfying descent for all colimits.

Topoi as spaces

Even though a topos is defined as a category satisfying descent, we often think of it as a geometric object in its own right, or simply as a kind of generalized space. This viewpoint is justified already by the basic example coming from topology.

Indeed, every topological space X gives rise to a topos $\mathrm{Shv}(X)$, its category of sheaves of animae. This assignment determines a functor $\mathrm{Top} \to \mathrm{Topos}$, which is fully faithful when restricted to sober spaces (in particular, all spaces typically considered in topology and geometry). Thus we may "do topology" by studying topoi rather than topological spaces. In fact, the terminology "topos" itself was chosen by the Grothendieck school to reflect the view that topology is fundamentally the study of topoi [McL90].

Classical geometric structures generalize naturally to this setting. For instance, the notion of a *locally ringed space*, a topological space equipped with a structure sheaf of rings, extends to that of a *locally ringed topos*. One can develop a theory of schemes as certain locally ringed topoi, recovering classical algebraic geometry while allowing for higher categorical phenomena.

Taking the geometric viewpoint seriously leads to a natural convention regarding morphisms. Every continuous map $f\colon X \to Y$ of topological spaces induces a pullback functor $f^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$. This functor is left exact and preserves all colimits; in other words, it preserves the descent structure on the sheaf categories. Given topoi S and T, a natural class of functors $f^*\colon S \to T$ between them is the class of left exact colimit-preserving functors. Following the geometric viewpoint, we treat such a functor as a morphism of topoi from T to S, rather than the other way around.

Content

In the remainder of these notes, we develop the general theory of topoi systematically. The organization is as follows.

Chapter 2: Foundations. We make precise the concept of descent in a presentable category and analyze the resulting definition of a topos. We study effective epimorphisms, including their relation to groupoid objects and the epi–mono factorization. We also give alternative characterizations of topoi, showing in particular that any topos can be written as a left exact localization of a presheaf category; this lets us reduce many general statements to claims about the category An of animae.

Chapter 3: Truncation and connectivity. We develop the theory of n-truncated and n-connected morphisms in a topos, showing that they form orthogonal factorization systems. This leads to the definition of homotopy group objects $\pi_n(X)$ and the study of Postnikov towers. We also discuss hypercompletion and gerbes.

Chapter 4: The categories of topoi and logoi. We study the category of topoi from both the geometric perspective (Topos) and the algebraic perspective (Logos). We introduce presentations of topoi in terms of generators and relations and use them to compute limits and colimits. We also study étale morphisms, which play an analogous role to local homeomorphisms of topological spaces.

Chapter 5: Modalities. We study modalities and their relation to epimorphisms, acyclic maps, and congruences, and we prove the Blakers–Massey theorem. We analyze open and closed immersions and develop structure theory for topoi via triple factorizations. We also study products of modalities and the corresponding topos-theoretic interpretation of Goodwillie calculus.

Chapter 6: Topics in topos theory. We discuss several further foundational topics: sheaf topoi arising from Grothendieck sites, groupoid actions and principal bundles, essential morphisms and shape theory, univalent families and colimits of topoi along étale morphisms, hypercovers, Postnikov-completeness and boundedness, dimension theory, coherence, pretopoi, compactly assembled categories and exponentiability, and parametrized objects.

Chapter 7: Geometry. The final chapter is an outlook on geometric applications of the general theory developed earlier in the course. We show how classical scheme theory fits into the topos framework and how this perspective extends to Lurie's framework of geometries and fractured topoi. We then indicate how six-functor formalisms give rise to fractured topoi. We conclude with two examples: analytic stacks in the sense of Clausen–Scholze, and Scholze–Stefanich's framework of Gestalten. Treatments are brief and most proofs are omitted.

Appendices. Appendix A collects background on factorization systems and saturated classes used throughout the text. Appendix B records basic facts about limits and colimits in categories of sections of cartesian and cocartesian fibrations.

2 Foundations

In this chapter, we define topoi as presentable categories satisfying descent (Section 2.1) and study effective epimorphisms, including their relation to groupoid objects and the epi–mono factorization (Section 2.2). We then give alternative characterizations of topoi in Section 2.3; in particular, every topos is a left exact localization of a presheaf category, which reduces many arguments to the case of An.

2.1 Topoi and descent

The definition of topoi relies on the concept of *descent*.

Definition 2.1. Let I be a small category, and let T be a category with pullbacks and I-indexed colimits. We say that T *satisfies descent for I-indexed colimits*, or that *I-indexed colimits are van Kampen*, if the functor

$$T^{\mathrm{op}} \to \mathrm{Cat}, \qquad X \mapsto T_{/X}$$

preserves I-indexed limits.

Definition 2.2. A *topos* is a presentable category satisfying descent for all colimits. We will alternatively refer to a topos as a *logos*.

Let us unwind this definition. Given a small category I and an I-indexed diagram $X_\bullet : I \to T$, set $X := \mathrm{colim}_i X_i$. The descent condition demands that the canonical functor $T_{/X} \to \lim_i T_{/X_i}$, obtained from the base change functors along the maps $X_i \to X$, is an equivalence.

To understand this condition better, we provide an alternative description of this limit in terms of cartesian natural transformations.

Definition 2.3. Given two functors $F, G : I \to T$, a natural transformation $\alpha : F \to G$ is called *cartesian* if for every morphism $i \to j$ in I the naturality square

$$\begin{tikzcd} F(i) \arrow[r, "\alpha(i)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G(i) \arrow[d] \\ F(j) \arrow[r, "\alpha(j)"] & G(j) \end{tikzcd}$$

is a pullback square. We denote by $\mathrm{Fun}^{\mathrm{cart}}(I, T) \subseteq \mathrm{Fun}(I, T)$ the wide subcategory spanned by the cartesian natural transformations.

Given a diagram $X_\bullet \colon I \to T$, evaluation at $i \in I$ determines a functor $\mathrm{ev}_i \colon \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to T_{/X_i}$. Furthermore, by definition of cartesianness these evaluation functors are compatible with base change, resulting in a comparison functor $\mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet} \to \lim_{i\in I^{\mathrm{op}}} T_{/X_i}$.

Lemma 2.4. *Let C be a category with pullbacks and let $X_\bullet \colon I \to C$ be a functor. Then the functor $\mathrm{Fun}^{\mathrm{cart}}(I,C)_{/X_\bullet} \to \lim_{i\in I^{\mathrm{op}}} C_{/X_i}$ is an equivalence.*

More informally, objects of the limit are given by I-indexed diagrams $Y_\bullet$ in C equipped with a cartesian natural transformation $Y_\bullet \to X_\bullet$.

Proof. Recall that a limit of a diagram $I^{\mathrm{op}} \to \mathrm{Cat}$ of categories may be computed as the category of cartesian sections of the cartesian unstraightening of the diagram. By definition, the functor $C_{/-} \colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is the cartesian straightening of the target map $t \colon \mathrm{Ar}(C) \to C$, so the unstraightening of $C_{/-}$ is t. It follows that the unstraightening of the functor $i \mapsto C_{/X_i}$ is the base change of t along $X_\bullet$:

$$\begin{tikzcd} C/X_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}(C) \arrow[d, "t"] \\ I \arrow[r, "X_\bullet"] & C. \end{tikzcd}$$

The category of sections $I \to C/X_\bullet$ of this map is equivalent to the category of maps $I \to \mathrm{Ar}(C)$ whose target projection is $X_\bullet \colon I \to C$, which in turn is equivalent to the slice-category $\mathrm{Fun}(I,C)_{/X_\bullet}$. It thus remains to show that a section $\sigma \colon I \to C/X_\bullet$ is cartesian precisely if the corresponding natural transformation $Y_\bullet \to X_\bullet$ is a cartesian natural transformation. But this is an immediate consequence of the fact that the t-cartesian morphisms in $\mathrm{Ar}(C)$ are precisely the pullback squares in C. $\square$

With this alternative description of $\lim_i T_{/X_i}$ at hand, we can immediately deduce that the functor $T_{/X} \to \lim_i T_{/X_i}$ admits a left adjoint:

Lemma 2.5. *Let T be a category with pullbacks and I-indexed colimits, and let $X = \mathrm{colim}_i X_i$ be a colimit in T. Then the functor $T_{/X} \to \lim_i T_{/X_i}$ admits a left adjoint*

$$\mathrm{colim} \colon \lim_i T_{/X_i} \to T_{/X}$$

sending a cartesian transformation $Y_\bullet \to X_\bullet$ to its colimit $\mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$.

Proof. The colimit functor $\mathrm{colim} \colon \mathrm{Fun}(I,T) \to T$ admits a right adjoint $\mathrm{const} \colon T \to \mathrm{Fun}(I,T)$ sending an object Y to the constant functor on Y. By slicing this adjunction over an arbitrary diagram $X_\bullet \in \mathrm{Fun}(I,T)$, with colimit denoted X, we obtain another adjunction

$$\mathrm{colim} \colon \mathrm{Fun}(I,T)_{/X_\bullet} \rightleftarrows T_{/X},$$

see [Lur09, Proposition 5.2.5.1]. The right adjoint in this adjunction is given by the composite $T_{/X} \xrightarrow{\mathrm{const}} \mathrm{Fun}(I,T)_{/\mathrm{const}\,X} \to \mathrm{Fun}(I,T)_{/X_\bullet}$, with the second map given by base change along the colimit cocone $X_\bullet \to \mathrm{const}_X$. We now make the following two observations:

- The category $\mathrm{Fun}(I,T)_{/X_\bullet}$ contains the limit $\lim_{i\in I^{\mathrm{op}}} T_{/X_i} \simeq \mathrm{Fun}^{\mathrm{cart}}(I,T)_{/X_\bullet}$ as a full subcategory: given transformations $Y_\bullet \to Z_\bullet \to X_\bullet$, if both $Z_\bullet \to X_\bullet$ and $Y_\bullet \to X_\bullet$ are cartesian transformations, then so is $Y_\bullet \to Z_\bullet$ by the pasting law for pullback squares.

- Since any transformation $\mathrm{const}\, Y \to \mathrm{const}\, X$ between constant diagrams is cartesian, and cartesian transformations are closed under base change, the right adjoint $T_{/X} \to \mathrm{Fun}(I, T)_{/X_\bullet}$ lands in this full subcategory.

It follows that the adjunction restricts to an adjunction $\lim_i T_{/X_i} \rightleftarrows T_{/X}$, as desired. □

We can now characterize when the functor $T_{/X} \to \lim_i T_{/X_i}$ is an equivalence. By Lemma 2.5, this happens precisely when the unit and counit of the adjunction are natural isomorphisms. Explicitly, this means:

(1) **Counit:** For any map $Y \to X = \mathrm{colim}_i X_i$ in T, the canonical map $\mathrm{colim}_i (X_i \times_X Y) \to Y$ is an isomorphism.

(2) **Unit:** Given a cartesian transformation $Y_\bullet \to X_\bullet$ of functors $I \to T$, each of the maps $Y_i \to (\mathrm{colim}_i Y_i) \times_{\mathrm{colim}_i X_i} X_i$ is an isomorphism.

Since these conditions will frequently show up in the remainder of these notes, it is useful to give them names:

Definition 2.6 (Effective colimits)**.** Let T be a category with pullbacks and I-indexed colimits. We say that *I-indexed colimits in T are effective* if, for every cartesian transformation $Y_\bullet \to X_\bullet$ of functors $I \to T$, the extended natural transformation $\overline{Y}_\bullet \to \overline{X}_\bullet$ of colimit cocones $I^\triangleright \to T$ is again cartesian, i.e. for every $j \in I$ the commutative square

$$\begin{tikzcd} Y_j \arrow[r] \arrow[d] & \mathrm{colim}_i Y_i \arrow[d] \\ X_j \arrow[r] & \mathrm{colim}_i X_i \end{tikzcd}$$

is a pullback square.

Definition 2.7 (Universal colimits)**.** Let T be a category with pullbacks and I-indexed colimits. We say that *I-indexed colimits in T are universal* if, for every morphism $f\colon A \to B$, the pullback functor $f^*\colon T_{/B} \to T_{/A}$ preserves I-indexed colimits.

Lemma 2.8. *For a category T with pullbacks and I-indexed colimits, the following conditions are equivalent:*

(1) *I-indexed colimits are universal in T;*

(2) *For every diagram $X_\bullet\colon I \to T$, the counit of the adjunction $T_{/X} \leftrightarrows \lim_i T_{/X_i}$ is an isomorphism;*

(3) *For every diagram $X_\bullet\colon I \to T$, the functor $T_{/X} \to \lim_i T_{/X_i}$ is fully faithful.*

Proof. The equivalence between (2) and (3) is standard. It is clear that (1) implies (2) by taking $A = Y$ and $B = X = \mathrm{colim}_i X_i$. The implication (2) $\implies$ (1) follows from the observation that for a diagram $X_\bullet$ in $T_{/B}$ the composition map $T_{/\mathrm{colim}_i X_i} \to T_{/B}$ preserves and creates colimits. □

With this terminology, we may summarize the above discussion as follows:

Corollary 2.9. *Let T be a category with pullbacks and I-indexed colimits. Then T satisfies descent for I-indexed colimits if and only if I-indexed colimits are both effective and universal.* □

In particular, we get:

Corollary 2.10. *A category T is a topos if and only if it is presentable and I-indexed colimits are both effective and universal for every small category I.* □

2.1.1 Examples of descent

Let us illustrate the descent condition by giving some examples.

Example 2.11 (Initial object)**.** Let T be a category with an initial object. Then the initial object $\emptyset$ is van Kampen if and only if it is *strictly initial*, meaning that any morphism $X \to \emptyset$ is an isomorphism.

Indeed, by definition $\emptyset$ is van Kampen if and only if the functor $T_{/\emptyset} \to *$ is an equivalence. This functor admits a fully faithful left adjoint $* \to T_{/\emptyset}$ hitting the identity $\mathrm{id}_\emptyset$. For the functor to be an equivalence, the counit $\emptyset \to X$ must be an isomorphism for every $X \in T_{/\emptyset}$, which is precisely the universality condition.

Example 2.12 (Coproducts)**.** For a category T with arbitrary coproducts, we say that *coproducts are disjoint* if for all $X, Y \in T$ the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \sqcup Y \end{tikzcd}$$

is a pullback square. We claim that arbitrary coproducts in T are van Kampen if and only if they are universal and T has disjoint coproducts.

By definition, coproducts are van Kampen if and only if they are universal and for every collection of maps $\{Y_i \to X_i\}_{i \in I}$ the commutative square

$$\begin{tikzcd} Y_j \arrow[r] \arrow[d] & Y := \coprod_{i \in I} Y_i \arrow[d] \\ X_j \arrow[r] & X := \coprod_{i \in I} X_i \end{tikzcd}$$

is a pullback square for every $j \in I$. To see that this reduces to disjointness, consider $X' := \coprod_{i \in I \setminus \{j\}} X_i$ and $Y' := \coprod_{i \in I \setminus \{j\}} Y_i$. We may write $X = X_j \sqcup X'$ and $Y = Y_j \sqcup Y'$. Invoking universality of coproducts once more shows that the condition reduces to disjointness of binary coproducts.

Example 2.13 (Group actions)**.** Let T be a topos, and let G be a group object in T (i.e. a groupoid object $G_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to T$ satisfying $G_0 = *$). Let $\mathbf{B}G := \mathrm{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} G_n$. Then there is an equivalence

$$T_{/\mathbf{B}G} \quad \simeq \quad \lim_{[n] \in \mathbf{\Delta}} T_{/G^n} \quad =: \quad \mathrm{Mod}_G(T).$$

In other words, a *module over G* is a simplicial object $X_\bullet$ with a cartesian transformation $X_\bullet \to G_\bullet$, which we can depict (suppressing degeneracies) as a simplicial diagram of face maps:

$$\begin{tikzcd} \cdots \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d] & X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d] & X_0 \arrow[d] \\ \cdots \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & G \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & * \end{tikzcd}$$

with each square cartesian. In particular, this implies $X_1 \simeq G \times X_0$. The two maps $G \times X_0 \simeq X_1 \to X_0$ can then be interpreted as the projection map and an *action map*. The rest of the simplicial diagram $X_\bullet$ records the higher coherences of this G-action on X_0.

Example 2.14. Let T be a topos. Given an anima $A \in \mathrm{An}$, we obtain an equivalence

$$\mathrm{Fun}(A, T) \quad = \quad \lim_A T_{/*} \quad \simeq \quad T_{/\operatorname{colim}_A *}.$$

We refer to this equivalence as the *Grothendieck construction*.

Exercise 2.15. Figure out which colimits are van Kampen in the following categories:

- The category Set of sets, the category $\mathrm{An}_{\leq n}$ of n-truncated animae;
- A stable cocomplete category;
- The opposite category $\mathrm{An}^{\mathrm{op}}$.

2.1.2 Descent via pullback squares

The descent condition can be reformulated in terms of colimits in the category $\mathrm{Ar}^{\mathrm{pb}}(T)$ of pullback squares in T. This is defined as the wide subcategory of $\mathrm{Ar}(T)$ whose morphisms are the pullback squares in T.

Proposition 2.16 (Rezk). *A presentable category T is a topos if and only if $\mathrm{Ar}^{\mathrm{pb}}(T)$ admits colimits and the inclusion $\mathrm{Ar}^{\mathrm{pb}}(T) \hookrightarrow \mathrm{Ar}(T)$ preserves colimits.*

Proof. First assume that T is a topos. We claim that the subcategory $\mathrm{Ar}^{\mathrm{pb}}(T) \hookrightarrow \mathrm{Ar}(T)$ is closed under colimits. To see this, note that a functor $I \to \mathrm{Ar}^{\mathrm{pb}}(T)$ corresponds under currying to a cartesian natural transformation $Y_\bullet \to X_\bullet$. Passing to colimits in T extends this to a natural transformation of colimit cocones $\overline{Y}_\bullet \to \overline{X}_\bullet$. By effectivity of I-indexed colimits, this remains a cartesian natural transformation, hence corresponds to a cocone $I^{\triangleright} \to \mathrm{Ar}^{\mathrm{pb}}(T)$.

We must show this is a colimit cocone. In other words, we must show that a commutative square in T of the form

$$\begin{tikzcd} \operatorname{colim}_i Y_i \arrow[r] \arrow[d] & A \arrow[d] \\ \operatorname{colim}_i X_i \arrow[r] & B \end{tikzcd}$$

is a pullback square if and only if each of the composite squares

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{colim}_i Y_i \arrow[r] \arrow[d] & A \arrow[d] \\ X_i \arrow[r] & \operatorname{colim}_i X_i \arrow[r] & B \end{tikzcd}$$

is a pullback square. This is an immediate consequence of universality of colimits.

For the converse, assume that $\mathrm{Ar}^{\mathrm{pb}}(T)$ admits colimits and the inclusion preserves them. The fact that colimit cocones in $\mathrm{Ar}^{\mathrm{pb}}(T)$ agree with colimit cocones in $\mathrm{Ar}(T)$ immediately implies effectivity of colimits. Universality of colimits follows from observing that the commutative square

$$\begin{tikzcd} \operatorname{colim}_i X_i \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{colim}_i X_i \arrow[r, equal] & X \end{tikzcd}$$

is a pullback square. □

Corollary 2.17. *Let T be a topos and let $L\colon T \to T'$ be a left exact functor localization. Then also T' is a topos.*

Proof. Let $R\colon T' \hookrightarrow T$ denote the fully faithful right adjoint to L. Then this adjunction induces an adjunction

$$L_*\colon \mathrm{Ar}(T) \rightleftarrows \mathrm{Ar}(T')\colon R_*$$

on arrow categories. Moreover, since both L and R preserve pullback squares, this adjunction restricts to an adjunction

$$\mathrm{Ar}^{\mathrm{pb}}(T) \rightleftarrows \mathrm{Ar}^{\mathrm{pb}}(T').$$

The claim now follows from Proposition 2.16, as $\mathrm{Ar}^{\mathrm{pb}}(T')$ inherits its colimits from $\mathrm{Ar}^{\mathrm{pb}}(T)$ and these are compatible with the once in $\mathrm{Ar}(T')$. □

2.2 Effective epimorphisms

We will now introduce an important class of morphisms $X \twoheadrightarrow Y$ in a topos, known as the *effective epimorphisms*. There are various ways of thinking about such morphisms:

(1) They are higher categorical analogues of the effective epimorphisms from 1-category theory, i.e. those morphisms that exhibit Y as the quotient of X at the equivalence relation $X \times_Y X \subseteq X \times X$;

(2) They are those morphisms for which the pullback functor $T_{/Y} \to T_{/X}$ is conservative, allowing us to check various conditions of maps in $T_{/X}$ after pulling back to Y;

(3) They are the topos-theoretic analogues of the *covers* in a Grothendieck site.

Following Lurie, we take perspective (1) as our definition. To make this precise, we first recall the notion of groupoid objects.

2.2.1 Groupoid objects

Recall that a *(classical) groupoid* is a classical category $\mathcal{G}$ in which all morphisms are invertible. Concretely, it consists of a set $\mathcal{G}_0$ of objects, a set $\mathcal{G}_1$ of morphisms, source/target maps $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$, unit and inverse maps $e\colon \mathcal{G}_0 \to \mathcal{G}_1$ and $i\colon \mathcal{G}_1 \to \mathcal{G}_1$, and a composition map $m\colon \mathcal{G}_1 \times_{s,\mathcal{G}_0,t} \mathcal{G}_1 \to \mathcal{G}_1$, satisfying the expected relations.

The higher categorical analogue is the following:

Definition 2.18. Let C be a category. A simplicial object $\mathcal{G}\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is called a *groupoid object* if for every $[n] \in \mathbf{\Delta}$ and every (not necessarily order-preserving) partition

$$[n] \simeq \{i_0, \ldots, i_k\} \sqcup_{\{i_k\}} \{i_k, \ldots, i_n\},$$

the induced diagram

$$\begin{tikzcd} \mathcal{G}_n \arrow[r, "(i_0,\ldots,i_k)^*"] \arrow[d, "(i_k,\ldots,i_n)^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_k \arrow[d, "i_k^*"] \\ \mathcal{G}_{n-k} \arrow[r, "i_k^*"] & \mathcal{G}_0 \end{tikzcd}$$

is a pullback square in C. We let $\mathrm{Grpd}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ denote the full subcategory spanned by the groupoid objects in C.

Exercise 2.19. Show that a groupoid object in Set is nothing but a classical groupoid.

For the purposes of topos theory, we may think of groupoid objects as higher categorical analogues of equivalence relations. Given a groupoid $\mathcal{G}$ in the category of sets, the image of the map $(s, t)\colon \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ is an equivalence relation on $\mathcal{G}_0$. Indeed, reflexivity follows from the identity maps, transitivity follows from composition of morphisms, and symmetry follows by inverting morphisms. Conversely, equivalence relations $R \subseteq X \times X$ on a set X correspond bijectively to groupoids in sets $\mathcal{G}$ such that $\mathcal{G}_0 = X$ and the map $(s, t)\colon \mathcal{G}_1 \hookrightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is injective. Under this correspondence, the colimit $|\mathcal{G}|$ of the groupoid corresponds to the quotient X/R.

Given a groupoid object $\mathcal{G}_\bullet$ in C with $X = \mathcal{G}_0$, we similarly think of the colimit $|\mathcal{G}_\bullet|$ as a "quotient of X by the relations contained in $\mathcal{G}$". The main difference from the classical situation is that $\mathcal{G}$ may now contain various non-equivalent relations between two points in X that contribute non-trivially to the quotient object.

Lemma 2.20 ([Lur09, Proposition 6.1.2.11])**.** *Let C be a category and consider an augmented simplicial object $U^+_\bullet\colon \mathbf{\Delta}_+^{\mathrm{op}} \to C$. The following conditions are equivalent:*

(1) *The diagram $U^+_\bullet$ is right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}} \subseteq \mathbf{\Delta}_+^{\mathrm{op}}$;*

(2) *The simplicial object $U_\bullet := U^+_\bullet|_{\mathbf{\Delta}}$ is a groupoid object and the square*

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$

is a pullback square.

Proof. This is left to the reader. □

The definitions of universal/effective colimits specialize to groupoids as follows:

Definition 2.21. Let C be a category with pullbacks and geometric realizations.

(1) A geometric realization $X = |X_\bullet| := \operatorname{colim}_n X_n$ is called a *groupoid colimit* if the simplicial object $X_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is a groupoid.

(2) We say *groupoid colimits are effective in C* if for every cartesian transformation $Y_\bullet \to X_\bullet$ of groupoid objects, the commutative square

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {|Y_\bullet|} \arrow[d] \\ X_0 \arrow[r] & {|X_\bullet|} \end{tikzcd}$$

is a pullback square.

(3) We say that *groupoid colimits are universal* if for every morphism $f \colon A \to B$ in C, the pullback functor $f^* \colon C_{/B} \to C_{/A}$ preserves groupoid colimits.

Lemma 2.22. *Let C be a category with pullbacks and geometric realizations. Then C satisfies descent for groupoid colimits if and only if groupoid colimits are universal and effective.*

Proof. This follows as in Corollary 2.9. □

Corollary 2.23. *Groupoid colimits are effective in a topos.* □

2.2.2 Effective epimorphisms and Čech nerves

In 1-category theory, a morphism $f \colon U \to X$ is called an *effective epimorphism* if it exhibits its target as the quotient of its source at the equivalence relation $U \times_X U$ determined by f. Equivalently, X is the coequalizer of the two projection maps $U \times_X U \rightrightarrows U$.

The definition in the higher categorical setting is similar. We replace the equivalence relation $U \times_X U$ by a certain groupoid object $\check{C}_\bullet(f)$ called the *Čech nerve of f*. In degrees ≤ 1, this groupoid is given by the two projection maps $U \times_X U \rightrightarrows U$.

Construction 2.24 (Čech nerve)**.** Let C be a category with pullbacks. For a morphism $f \colon U \to X$ in C, we define its *Čech nerve* $\check{C}_\bullet(f) \in \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ as the image of $f \in \operatorname{Ar}(C)$ under the following functor:

$$\operatorname{Fun}([1], C) \cong \operatorname{Fun}((\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}, C) \xrightarrow{j_*} \operatorname{Fun}(\mathbf{\Delta}_+^{\mathrm{op}}, C) \xrightarrow{i^*} \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

Here $\mathbf{\Delta}_+$ is the augmented simplex category, $i \colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+$ is the canonical inclusion, $j \colon \mathbf{\Delta}_+^{\leq 0} \hookrightarrow \mathbf{\Delta}_+$ is the inclusion of the objects $[-1]$ and $[0]$, and $[1] \cong (\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$ is the canonical isomorphism sending 0 to $[0]$ and 1 to $[-1]$. The pointwise formula for the right Kan extension j_* shows that the Čech nerve is given by

$$\check{C}_n(f) \;\cong\; U^{\times_X^{n+1}} \;=\; \underbrace{U \times_X U \times_X \cdots \times_X U}_{n+1 \text{ times}}.$$

This computation also shows that the right Kan extension functor j_* exists.

Notation 2.25. Given a morphism $f\colon U \to X$, we will write $\check{C}^+_\bullet(f)$ for the augmented simplicial diagram $j_*(f)\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$, and refer to this as the *augmented Čech nerve of f*. Under the canonical equivalence between ${\mathbf{\Delta}_+}^{\mathrm{op}}$ and $(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}$, we may think of $\check{C}^+_\bullet(f)$ as encoding a cocone on $\check{C}_\bullet(f)$ with cone point X.

Lemma 2.26. *Let C be a category with pullbacks. Then the Čech nerve $\check{C}_\bullet(f)$ of any morphism f is a groupoid object.*

Proof. This is immediate from Lemma 2.20, since the augmented Čech nerve $\check{C}^+_\bullet(f)$ of f is by definition right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}} \subseteq \mathbf{\Delta}_+^{\mathrm{op}}$. □

Definition 2.27. Let C be a category with pullbacks. A morphism $f\colon U \to X$ is called an *effective epimorphism* if the cocone $\check{C}^+_\bullet(f)$ is a colimit diagram, exhibiting X as the geometric realization $|\check{C}_\bullet(f)|$ of its Čech nerve. We denote by

$$\mathrm{EffEpi}(C) \subseteq \mathrm{Ar}(C)$$

the full subcategory spanned by the effective epimorphisms.

Warning 2.28. This terminology is standard but rather unfortunate: not every effective epimorphism is an epimorphism in the categorical sense, see Example 5.7 below for a counterexample. The terminology is a historical accident. In a 1-category, effective epimorphisms *are* always epimorphisms, but this fails in higher categories.

Lemma 2.29. *Let C be a category with pullbacks such that groupoid colimits are universal in C (e.g. C is a topos).*

(1) *A morphism $f\colon U \to X$ in C is an effective epimorphism if and only if the functor $f^*\colon C_{/X} \to C_{/U}$ is conservative.*

(2) *Effective epimorphisms are closed under base change.*

Proof. (1) Assume first that f is an effective epimorphism. By Lemma 2.26, $X \simeq \mathrm{colim}_n\, \check{C}_n(f)$ is a groupoid colimit. By our assumption on C, the functor $C_{/X} \hookrightarrow \lim_n C_{/\check{C}_n(f)}$ is fully faithful, hence conservative. In particular, the functors $C_{/X} \to C_{/\check{C}_n(f)}$ for $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ are jointly conservative. Since each of these functors factors through $C_{/U}$, we conclude that f^* itself must be conservative.

Conversely, assume that f^* is conservative. We must show that the map $\mathrm{colim}_n\, \check{C}_n(f) \to X$ is an isomorphism in C. Since f^* is conservative and preserves groupoid colimits, it suffices to show that $\mathrm{colim}_n\, \check{C}_n(f) \times_X U \to U$ is an isomorphism. This follows from the fact that the simplicial diagram $[n] \mapsto U^{\times_{/X}^{(n+1)}} \times_X U \cong U^{\times_{/X}^{(n+2)}}$ admits an extra degeneracy.

(2) Consider a pullback square

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

such that f is an effective epimorphism. Since the functor $g^*\colon C_{/X} \to C_{/X'}$ preserves pullbacks, it sends $\check{C}_\bullet(f)$ to $\check{C}_\bullet(f')$. By assumption on C, it also preserves groupoid colimits, hence we see that the map $\mathrm{colim}_n\, \check{C}_n(f') \simeq (\mathrm{colim}_n\, \check{C}_n(f)) \times_X X' \to X \times_X X' = X'$ is an equivalence, implying that also f' is an effective epimorphism. □

Corollary 2.30. *Let C be a category with pullbacks such that groupoid colimits are universal in C (e.g. C is a topos). Consider a commutative diagram in C as follows:*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d, "f''"'] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y'' \arrow[r, two heads] & Y' \arrow[r] & Y. \end{tikzcd}$$

Assume that the map $Y'' \to Y'$ is an effective epimorphism. If both the left-hand square and the outher square are pullback squares, then so is the right-hand square.

Proof. We need to show that the map $X' \to X \times_Y Y'$ is an isomorphism. By the previous lemma, this may be tested after pullback along the effective epimorphism $Y'' \twoheadrightarrow Y'$. The claim now follows by applying the 2-out-of-3 property to the following commutative diagram:

$$\begin{tikzcd} X'' \arrow[rr, "\simeq"] \arrow[d, "\simeq"'] & & X' \times_{Y'} Y'' \arrow[d] \\ X \times_Y Y'' \arrow[r, "\simeq"] & (X \times_Y Y') \times_{Y'} Y''. \end{tikzcd}$$ □

We now show that the category of effective epimorphisms in a topos is equivalent to the category of groupoids, via passage to the Čech nerve. We start with the following simple observation:

Lemma 2.31. *Let C be a category with pullbacks and geometric realizations. Then the functor $\check{C}_\bullet \colon \operatorname{Ar}(C) \to \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$ admits a left adjoint*

$$\operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \xrightarrow{i_!} \operatorname{Fun}(\mathbf{\Delta}_+^{\mathrm{op}}, C) \xrightarrow{j^*} \operatorname{Fun}((\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}, C) \cong \operatorname{Fun}([1], C),$$

sending a simplicial object $X_\bullet$ to the map $X_0 \to \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n$.

Proof. We have $\check{C}_\bullet(-) = i^* j_*$. The functor i^* has a left adjoint given by left Kan extension along $i \colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$, and the functor j_* has a left adjoint given by the restriction functor j^*. □

We may use this adjunction to reformulate the condition of effectivity of groupoids from Definition 2.21:

Lemma 2.32. *Let C be a category with pullbacks and geometric realizations, and assume that groupoid colimits are universal. Then the following three conditions are equivalent:*

(1) *The category C satisfies descent for groupoid colimits;*

(2) *Groupoid colimits are effective in C;*

(3) *For every groupoid object $\mathcal{G}$ the unit $\mathcal{G} \to \check{C}_\bullet(\mathcal{G}_0 \to |\mathcal{G}|)$ of the adjunction from Lemma 2.31 is an isomorphism.*

Proof. The equivalence between (1) and (2) is Lemma 2.22.

To show that (2) implies (3), consider a groupoid object $\mathcal{G}$. By setting $\mathcal{G}_{-1}^+ := |\mathcal{G}| = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_n$, we may extend $\mathcal{G}$ to an augmented simplicial object $\mathcal{G}_\bullet^+$. (More precisely, this is the left Kan extension of $\mathcal{G}$ along $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_+^{\mathrm{op}}$.) We will show that $\mathcal{G}_\bullet^+$ is right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$. Observe that the first face maps define a transformation $\mathcal{G}_{\bullet+1}^+ \to \mathcal{G}_\bullet^+$ of functors $\mathbf{\Delta}_+^{\mathrm{op}} \to T$. The fact that $\mathcal{G}$ is a groupoid object implies that the restriction of this transformation to $\mathbf{\Delta}^{\mathrm{op}}$ is cartesian. By effectivity of groupoid colimits, this implies that the square

$$\begin{tikzcd}
\mathcal{G}_{n+1} \arrow[r] \arrow[d] & \operatorname{colim}_{[n]} \mathcal{G}_{n+1} \arrow[d] \\
\mathcal{G}_n \arrow[r] & \mathcal{G}_{-1}^+ = \operatorname{colim}_{[n]} \mathcal{G}_n
\end{tikzcd}$$

is a pullback square. The top right corner $\operatorname{colim}_{[n]} \mathcal{G}_{n+1}$ is equivalent to $\mathcal{G}_0$ because the simplicial object $\mathcal{G}_{\bullet+1}^+$ admits extra degeneracies. Taking $n = 0$, we see that the square

$$\begin{tikzcd}
\mathcal{G}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G}_0 \arrow[d] \\
\mathcal{G}_0 \arrow[r] & \mathcal{G}_{-1}^+
\end{tikzcd}$$

is a pullback square. It then follows from Lemma 2.20 that $\mathcal{G}_\bullet^+$ is right Kan extended from $(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}}$ and thus agrees with the augmented Čech nerve of the morphism $\mathcal{G}_0 \to |\mathcal{G}|$.

To show that (3) implies (2), we first observe that assumption (3) immediately implies that the map $f\colon \mathcal{G}_0 \to |\mathcal{G}|$ is an effective epimorphism for every groupoid object $\mathcal{G}$. Indeed, the isomorphism $\mathcal{G} \xrightarrow{\sim} \check{C}_\bullet(f)$ of groupoid objects induces the desired isomorphism $|\mathcal{G}| \xrightarrow{\sim} |\check{C}_\bullet(f)|$.

Now consider a cartesian transformation $X_\bullet \to Y_\bullet$ of groupoid objects in C. We need to show that the right square in the following commutative diagram is a pullback square:

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\
Y_0 \arrow[r, two heads] & {|Y_\bullet|} \arrow[r] & {|X_\bullet|}.
\end{tikzcd}$$

The left-hand square is a pullback square by assumption (3), and since we just argued that $Y_0 \twoheadrightarrow |Y_\bullet|$ is an effective epimorphism, it suffices by Corollary 2.30 to show that the outer rectangle is a pullback square. This outer rectangle may be decomposed in terms of the following two squares:

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d] \\
Y_0 \arrow[r] & X_0 \arrow[r] & {|X_\bullet|}.
\end{tikzcd}$$

Here the left square is a pullback square by the cartesianness assumption, while the right square is cartesian by assumption (3). This finishes the proof. □

Lemma 2.33. *Let C be a category satisfying descent for groupoid colimits (e.g. a topos). Then the Čech nerve functor restricts to an equivalence*

$$\check{C}_\bullet\colon \operatorname{EffEpi}(C) \xrightarrow{\sim} \operatorname{Grpd}(C)$$

The inverse sends $\mathcal{G}$ to the map $\mathcal{G}_0 \to |\mathcal{G}|$.

Proof. Consider the adjunction $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \rightleftarrows \mathrm{Ar}(C)$ from Lemma 2.31. The unit transformation is an isomorphism by Lemma 2.32. In particular, the colimit functor $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \to \mathrm{Ar}(C)$ is fully faithful.

Its essential image consists precisely of those morphisms $f\colon U \to X$ for which the counit of the adjunction is an isomorphism. This counit takes the form of a commutative square

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d] & U \arrow[d, "f"] \\ \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(f) \arrow[r] & X, \end{tikzcd}$$

and by definition this is an isomorphism if and only if f is an effective epimorphism in C. □

Definition 2.34. Let T be a topos. A *pointed connected object* is an object X equipped with an effective epimorphism $* \twoheadrightarrow X$. We denote by $T_*^{\geq 1} \subseteq T_*$ the subcategory of pointed connected objects.

Proposition 2.35 (Delooping principle)**.** *Let T be a topos. Then there is an equivalence of categories*

$$\mathbf{B}\colon \mathrm{Grp}(T) \quad \rightleftarrows \quad T_*^{\geq 1} : \mathbf{\Omega}.$$

Proof. The equivalence $\mathrm{Grpd}(T) \simeq \mathrm{EffEpi}(T)$ from Lemma 2.33 sits in a commutative triangle

$$\begin{tikzcd} \mathrm{Grpd}(T) \arrow[rr, "\simeq"] \arrow[dr, "G\mapsto G_0"'] & & \mathrm{EffEpi}(T) \arrow[dl, "(U\to X)\mapsto U"] \\ & T. & \end{tikzcd}$$

Taking fibers over $* \in T$ gives the claim. □

2.2.3 The epi-mono factorization system

In a topos, every morphism can be factored into an effective epimorphism followed by a monomorphism. This factorization is in fact unique. We formulate this precisely using *factorization systems*, which are recalled in Chapter A.

Proposition 2.36. *Let C be a category, and assume that groupoids are effective and universal. Then C admits a factorization system (E, M) with E given by the effective epimorphisms and M given by the monomorphisms in C.*

Proof. Existence of factorizations: Consider an arbitrary morphism $f\colon U \to X$. Define $V := \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(f)$. This gives a factorization of f as

$$U \xrightarrow{e} V \xrightarrow{m} X.$$

The map $e\colon U \to V$ is an effective epimorphism by Lemma 2.33, since effectivity of groupoids implies $\check{C}(e) = \check{C}(f)$.

To see that $m\colon V \to X$ is a monomorphism, it suffices to show that the first projection $p_1\colon V\times_X V \to V$ is an isomorphism (since the diagonal $V \to V \times_X V$ is a section of p_1, each being an iso implies the other). Since $e: U \to V$ is an effective epimorphism, it suffices by Lemma 2.29(1) to show that the pullback $V \times_X U \to U$ of p_1 along e is an isomorphism. By universality of groupoid colimits,

$$V \times_X U \;\simeq\; \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_{n+1}(f).$$

The augmented simplicial object $[n] \mapsto \check{C}_{n+1}(f)$ over U admits extra degeneracies via the diagonal $U \to U \times_X U = \check{C}_1(f)$, hence $V \times_X U \to U$ is an isomorphism.

Orthogonality: Given any solid diagram in C of the form

$$\begin{tikzcd} U \arrow[r] \arrow[d, "f"'] & A \arrow[d, hook, "i"] \\ X \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

where f is an effective epimorphism and $i: A \hookrightarrow B$ is a monomorphism, the anima of dashed fillers is contractible.

Since $A \hookrightarrow B$ is a monomorphism, it suffices to show that there exists a map $X \to A$ making the bottom triangle commute. The top triangle will then automatically commute as well. Working in the slice $C_{/B}$, the object X is a colimit of the Čech nerve $\check{C}_\bullet(f)$. Since each object $\check{C}_n(f) = U^{\times_X^{n+1}}$ lies in the subcategory $C_{/A} \subseteq C_{/B}$, so must this colimit. □

2.2.4 Properties of effective epimorphisms

We now prove some basic properties of effective epimorphisms. Throughout this subsection, we fix a category C in which groupoids are effective and universal.

Lemma 2.37. *Effective epimorphisms are closed under composition.*

Proof. The effective epimorphisms are the left class of a factorization system, which is closed under composition by Proposition A.3. □

Lemma 2.38. *Any morphism which has a section is an effective epimorphism.*

Proof. Assume $f: X \to Y$ admits a section $s: Y \to X$. Then the augmented Čech nerve $\check{C}^+_\bullet(f): \mathbf{\Delta}_+^{\mathrm{op}} \to C$ admits an extra degeneracy, hence is a colimit diagram. □

Corollary 2.39. *Any morphism which is both a monomorphism and an effective epimorphism is an isomorphism. In particular, every monomorphism with a section is an isomorphism.*

Proof. For the first claim, recall that if a category C is equipped with a factorization system (L, R), then any morphism $f: X \to Y$ in both L and R is an isomorphism (see Lemma A.4). The second claim follows immediately from the first combined with Lemma 2.38. □

Lemma 2.40. *Given a commutative triangle*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "gf"] & & Z, \end{tikzcd}$$

if gf is an effective epimorphism then so is g.

Proof. We may factor the map g as

$$Y \xhookrightarrow{i_Y} Y \sqcup_X Z \xrightarrow{\langle g, \mathrm{id}_Z \rangle} Z,$$

where the first map is the inclusion of the Y-component and the second map is induced by the maps $g\colon Y \to Z$ and $\mathrm{id}_Z\colon Z \to Z$. The first map is an effective epimorphism, since it is a cobase change of the effective epimorphism $gf\colon X \to Z$. The second map is an effective epimorphism by Lemma 2.38 since the canonical map $Z \to Y \sqcup_X Z$ is a section. □

2.3 Characterizations of topoi

We now provide various alternative characterizations of topoi. Throughout this section, we fix a presentable category T.

Definition 2.41. An *object classifier* is an object $U \in \widehat{T} := \mathrm{Fun}^{\lim}(T^{\mathrm{op}}, \widehat{\mathrm{An}})$ such that $\mathrm{Hom}(X, U) \simeq (T_{/X})^{\simeq}$. Note that T admits an object classifier if and only if the assignment $X \mapsto (T_{/X})^{\simeq}$ preserves limits.

Theorem 2.42. *Let T be a presentable category. Then the following are equivalent:*

(1) *The category T is a topos.*

(2) *Colimits are universal and there exists an object classifier.*

(3) *The Giraud axioms are satisfied:*

- *Colimits are universal (Definition 2.7);*
- *Coproducts are disjoint (Example 2.12);*
- *Groupoids are effective (Definition 2.21, Lemma 2.32);*

(4) *There exists a small category C and a left exact localization* $\mathrm{PSh}(C) \to T$.

Proof. (1) $\implies$ (2): Clear, since $(-)^{\simeq}\colon \mathrm{Cat} \to \mathrm{An}$ preserves limits.

(2) $\implies$ (1): Assume (2) holds. We must show that the functor $T_{/X} \to \lim_i T_{/X_i}$ is an equivalence for any diagram. Full faithfulness holds by universality of colimits (Lemma 2.8). Essential surjectivity follows from the existence of an object classifier.

(1) $\implies$ (3): We verify the three conditions:

- Universality of colimits holds by definition of a topos.
- Coproducts are disjoint by Example 2.12.
- All groupoids are effective by Corollary 2.23.

(4) $\implies$ (1): Since An is a topos, it follows that $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, T)$ is a topos, and hence so is any left exact localization of $\mathrm{PSh}(C)$ by Corollary 2.17.

(3) $\implies$ (4): We use the proposition below. By the theory of presentable categories, we can write $T = \mathrm{Ind}_\kappa(C)$ for some small category C *with finite limits* and some regular cardinal κ.

Since the inclusion $C \hookrightarrow \mathrm{Ind}_\kappa(C) = T$ is left exact, the proposition guarantees that the extension $\mathrm{PSh}(C) \to \mathrm{Ind}_\kappa(C)$ is a left exact localization, giving (4). □

Proposition 2.43. *Let C be a category which has finite limits, and let T be a presentable category satisfying the Giraud axioms. Let $f: C \to T$ be a left exact functor. Then the unique colimit-preserving extension $F\colon \mathrm{PSh}(C) \to T$ of f is left exact.*

Proof. We say that a presheaf $X \in \mathrm{PSh}(C)$ is *good* if the functor F preserves every pullback square of the form

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

By universality of colimits, X is good if and only if F preserves all such pullback squares where Y and Z are representable. In particular, since f preserves pullbacks, every $X \in C$ is good. Since coproducts are van Kampen, coproducts of good objects are good.

It now suffices to show that if $X_0 \twoheadrightarrow X$ is an effective epimorphism with X_0 good, then X is also good. To see this, we first reduce to a pullback square where Y and Z are representables. Since effective epimorphisms in a presheaf topos are precisely the maps pointwise surjective on π_0, the maps $Y \to X$ and $Z \to X$ factor through X_0. The pullback square thus factors as follows:

$$\begin{tikzcd} Y \times_X Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times_X Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d] \\ Y \times_X X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times_X X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \arrow[d] \\ Y \arrow[r] & X_0 \arrow[r] & X \end{tikzcd}$$

Since X_0 is good, F preserves the three pullback squares on the top and left. It remains to show that F preserves the bottom right pullback square, which is now independent of Y and Z.

Let $X_\bullet \to X$ be the Čech nerve of the map $X_0 \to X$. We must show that F preserves this Čech nerve. Since F preserves colimits, $F(X_\bullet) \to F(X)$ is a colimit diagram. Since X_0 is good, $F(X_\bullet)$ is a groupoid. By effectivity of groupoids, $F(X_\bullet)$ is the Čech nerve of the map $F(X_0) \to F(X)$, as desired. □

Remark 2.44. Marc mentioned he did not yet manage to find a direct proof that (3) implies (1) that does not go through the presentation of a topos as a left exact localization of a presheaf topos.

2.3.1 Exactness in topoi

Characterization (4) immediately implies the following exactness properties in topoi:

Proposition 2.45. *The following statements hold true in a topos:*

(1) *Filtered colimits commute with finite limits.*

(2) *Sifted colimits commute with finite products.*

(3) *Anima-indexed colimits commute with weakly contractible limits. For example,*

$$(X \times_Z Y)/G \cong X/G \times_{Z/G} Y/G.$$

(4) *Given a diagram of simplicial objects*

$$\begin{tikzcd} & Y_\bullet \arrow[d] \\ X_\bullet \arrow[r] & Z_\bullet \end{tikzcd}$$

such that $\tau_0 Z_\bullet$ is constant, we get an isomorphism

$$\mathrm{colim}_n(X_n \times_{Z_n} Y_n) \xrightarrow{\sim} (\mathrm{colim}_n X_n) \times_{\mathrm{colim}_n Z_n} (\mathrm{colim}_n Y_n).$$

Proof. (1) Writing T as a left exact localization of a presheaf topos, this follows from the analogous statement in An, where it is a classical fact. (A model-independent argument for this may be found in [SW25].)

(2) Given two diagrams $X_\bullet, Y_\bullet \colon I \to T$, with I sifted, we need to show that the canonical map

$$\mathrm{colim}_{i\in I}(X_i \times Y_i) \to (\mathrm{colim}_{i\in I} X_i) \times (\mathrm{colim}_{i\in I} Y_i)$$

induced by the projections is an isomorphism. We may factor this map as a composite

$$\mathrm{colim}_{i\in I}(X_i \times Y_i) \to \mathrm{colim}_{i\in I}\,\mathrm{colim}_{j\in I}(X_i \times Y_j) \to (\mathrm{colim}_{i\in I} X_i) \times (\mathrm{colim}_{j\in I} Y_j),$$

where the first map is induced by the diagonal functor $I \to I \times I$. Since I is sifted, this map is final by definition, so the first map is an isomorphism. Moreover, it follows from descent that the functor $- \times - \colon T \times T \to T$ preserves colimits in both variables separately, so that also the second map is an isomorphism.

(3) Given $A \in \mathrm{An}$, we get an equivalence $\mathrm{Fun}(A, T) \cong T_{/\,\mathrm{colim}_A *}$ by descent. The colimit functor $\mathrm{Fun}(A, T) \to T$ then corresponds to the forgetful functor $T_{/\,\mathrm{colim}_A *} \to T$, which preserves weakly contractible limits.

(4) Since $Z_0 \to \tau_0 Z_0$ is an effective epimorphism, the pullback functor $T_{/\,\tau_0 Z} \to T_{/Z}$ is conservative by Lemma 2.29. So without loss of generality, we may assume that $Z_\bullet \colon \Delta^{\mathrm{op}} \to T_*^{\geq 1}$. Using the delooping result $T_*^{\geq 1} \simeq \mathrm{Grp}(T)$, we see that $Z = \mathbf{B}G_\bullet$ for some simplicial group object $G_\bullet$. We may then write $X = X'/G$ and $Y = Y'/G$, so that $X \times_{\mathbf{B}G} Y = (X' \times Y')/G$. This now involves only finite products and simplicial colimits, both of which commute with simplicial colimits. □

2.3.2 The subobject classifier

Characterization (2) implies the existence of *classifying objects* for every sufficiently small local class of morphisms.

Definition 2.46. Let Σ be a class of morphisms in a topos T. We say that Σ is a *local class* if the following conditions are satisfied:

(1) It is closed under base change;

(2) It is closed under small coproducts;

(3) Given a pullback square in T of the form

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ X' \arrow[r, "g", two heads] & X \end{tikzcd}$$

in which $f' \in \Sigma$ and g is an effective epimorphism, we also have $f \in \Sigma$.

Proposition 2.47. *Let Σ be a class of morphisms in a topos T which is stable under base change. The following conditions are equivalent:*

(1) *The class Σ is a local class;*

(2) *The functor $T^{\mathrm{op}} \to \widehat{\mathrm{Cat}}$ given by sending $X \in T$ to the full subcategory of $T_{/X}$ spanned by the morphisms in Σ preserves limits;*

(3) *The functor $T^{\mathrm{op}} \to \widehat{\mathrm{An}}$ given by sending $X \in T$ to the full subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ preserves limits;*

(4) *The full subcategory of $\mathrm{Ar}^{\mathrm{pb}}(T)$ spanned by the morphisms in Σ is closed under colimits.*

(5) *The class Σ is closed under coproducts, and for every commutative cube in T of the form*

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "a"] & & B \arrow[dr] \arrow[dd, "b" near start] & \\ & C \arrow[rr] \arrow[dd, "c" near start] & & D \arrow[dd, "d"] \\ A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\ & C' \arrow[rr] & & D', \end{tikzcd}$$

if the top and bottom squares are pushout squares, the left and back squares are pullback squares, and $a, b, c \in \Sigma$, then also $d \in \Sigma$.

Proof. Since fully faithful functors and monomorphisms of animae are closed under limits, conditions (2) and (3) are both equivalent to the following: for every diagram $X_\bullet$ in T with colimit X, a morphism $f\colon Y \to X$ lies in Σ whenever its base changes $f_i\colon Y_i := Y \times_X X_i \to X_i$ lie in Σ.

(1) $\implies$ (2+3): Assume $f_i \in \Sigma$ for all $i \in I$. Then $\bigsqcup_i f_i\colon \bigsqcup_i Y_i \to \bigsqcup_i X_i$ also lies in Σ. Since this map is a base change of f along the effective epimorphism $\bigsqcup_i X_i \twoheadrightarrow X$, we have $f \in \Sigma$.

(2+3) $\implies$ (1): Closure under small coproducts follows by taking $X = \bigsqcup_i X_i$. Detection after base change along an effective epimorphism follows by writing $X \simeq \mathrm{colim}_{[n]\in\boldsymbol{\Delta}^{\mathrm{op}}} X'^{\times_X^{n+1}}$.

Thus (1), (2) and (3) are equivalent.

The equivalence between (2+3) and (4) follows from a close inspection of the proof of Proposition 2.16: both conditions are saying that for a cartesian natural transformation $Y_\bullet \to X_\bullet$ such that each map $Y_i \to X_i$ is in Σ, also the map $\mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$ is in Σ.

Finally, (4) is equivalent to asking that this subcategory of $\mathrm{Ar}^{\mathrm{pb}}(T)$ is closed under small coproducts and pushouts, which amounts to condition (5). ◻

Definition 2.48. Let Σ be a local class in T. We say that a morphism $f_\Sigma \colon Y_\Sigma \to X_\Sigma$ *classifies* Σ if for every object $X \in T$ the map

$$\mathrm{Hom}_T(X, X_\Sigma) \to (T_{/X})^{\simeq}, \qquad g \mapsto g^*(f_\Sigma)$$

is a monomorphism of animae whose image is given by the full subanima spanned by the morphisms in Σ. In this situation, we also say that the object X_Σ *classifies* Σ.

By the Yoneda lemma, a classifying object is unique if it exists.

Corollary 2.49. *Let Σ be a local class in T. Then there exists a classifying object for Σ if and only if for every object $X \in T$ the subcategory of $T_{/X}$ spanned by the morphisms in Σ is small.*

Proof. If a classifying object exists, then for every object $X \in T$ the subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ is equivalent to the small anima $\mathrm{Hom}_T(X, X_\Sigma)$, hence is itself small.

Conversely, if for every object $X \in T$ the subanima of $(T_{/X})^{\simeq}$ spanned by the morphisms in Σ is small, then these subanimae determine a functor $T^{\mathrm{op}} \to \mathrm{An}$. By Proposition 2.47, this functor preserves limits. The adjoint functor theorem then implies it is represented by some object X_Σ of T. ◻

In a topos T, the subcategory $(T_{/X})_{\leq -1}$ of monomorphisms $Y \hookrightarrow X$ is small, giving us the following definition:

Definition 2.50. For a topos T, we denote by Ω the classifying object for the class of monomorphisms, and call it the *subobject classifier*.

Lemma 2.51. *The subobject classifier is 0-truncated. That is, for every object $X \in T$, the anima $\mathrm{Hom}_T(X, \Omega)$ is 0-truncated.*

Proof. By definition, $\mathrm{Hom}_T(X, \Omega)$ is the subanima of $(T_{/X})^{\simeq}$ spanned by the monomorphisms $U \hookrightarrow X$. For a monomorphism, any automorphism $U \xrightarrow{\sim} U$ in $T_{/X}$ must be the identity map. It follows that $(T_{/X})^{\simeq}$ has no non-trivial self-identifications, i.e. it is 0-truncated. ◻

Lemma 2.52. *Let $Y \to \Omega$ be the universal monomorphism. Then Y is the terminal object.*

Proof. Write $p \colon Y \to \Omega$ for the universal monomorphism, and let $t \colon * \to \Omega$ be the map classifying the identity $\mathrm{id}_* \colon * \to *$, exhibited by a pullback square of the form

$$\begin{tikzcd} * \arrow[r, "s"] \arrow[d, "\mathrm{id}_*"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ * \arrow[r, "t"'] & \Omega. \end{tikzcd}$$

We claim s is an inverse to the canonical map $q \colon Y \to *$, showing Y is terminal. Since clearly $q \circ s \cong \mathrm{id}_*$, it remains to show that $s \circ q \cong \mathrm{id}_Y$.

To this end, observe that we have two pullback diagrams of the following form:

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "\mathrm{id}_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[r, "s"] \arrow[d, "\mathrm{id}_*"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ Y \arrow[r, "q"] & * \arrow[r, "t"] & \Omega \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Y \arrow[r, "\mathrm{id}_Y"] \arrow[d, "\mathrm{id}_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "p"] \\ Y \arrow[r, "p"] & \Omega. \end{tikzcd}$$

In particular, both bottom maps $Y \to \Omega$ are classifying maps for id_Y, and it follows that $p \cong t \circ q$. We then get the chain of identifications

$$p \circ (s \circ q) = (p \circ s) \circ q = t \circ q = p = p \circ \mathrm{id}_Y,$$

so $s \circ q$ defines an endomorphism of p in the slice category $T_{/\Omega}$. Since p is a monomorphism, the anima of endomorphisms of p in $T_{/\Omega}$ is (-1)-truncated and nonempty (it contains id_Y), hence contractible. Therefore $s \circ q \cong \mathrm{id}_Y$, as desired. $\square$

3 Truncation and connectivity

The epi-mono factorization system from Proposition 2.36 can be generalized. For every $n \geq -2$, we obtain notions of *n-truncated* and *n-connected* morphisms, which also form a factorization system.

We recall truncated morphisms in Section 3.1, where we also establish the key fact that effective epimorphisms can be checked on 0-truncations. We introduce connectivity in Section 3.2 and establish the claimed factorization system. In Section 3.3, we introduce the *homotopy group objects* $\pi_n(X) \in T_{/X}$ for an object $X \in T$, and use these to prove the important result that a morphism in T is n-connected if and only if it is an effective epimorphism and its diagonal is $(n-1)$-connected. In Section 3.4, we define the *hypercompletion* of a topos via a notion of ∞-connected morphisms. We end in Section 3.5 with a discussion of *gerbes*: objects that are both n-truncated and $(n-1)$-connected for some n.

3.1 Truncation

We start with a recollection of truncated objects in a category.

Definition 3.1. Let C be a category with pullbacks. A morphism $f\colon X \to Y$ in C is called (-2)-*truncated* if it is an isomorphism. For $n \geq -1$, we say f is *n-truncated* if $\Delta_f\colon X \to X \times_Y X$ is $(n-1)$-truncated. We denote by $C_{\leq n} \subseteq C$ the full subcategory of n-truncated objects.

If the inclusion $C_{\leq n} \hookrightarrow C$ admits a left adjoint, we denote this adjoint by

$$\tau_n\colon C \to C_{\leq n},$$

and call it the *n-truncation functor*.

Remark 3.2. Being (-1)-truncated is the same as being a monomorphism.

Remark 3.3. The truncation functor $\tau_n\colon C \to C_{\leq n}$ always exists when C is presentable, hence in particular when T is a topos.

To see this, recall that C admits a tensoring and a cotensoring by the category An. There are unique functors

$$- \otimes -\colon \mathrm{An} \times C \to C \qquad \text{and} \qquad (-)^{(-)}\colon \mathrm{An}^{\mathrm{op}} \times C \to C$$

satisfying $* \otimes X \cong X \cong X^*$ for all $X \in C$ and preserving colimits (resp. limits) in the first variable. By an easy induction argument, an object $X \in C$ is n-truncated if and only if the map $X \to X^{S^{n+1}}$ induced by $S^{n+1} \to *$ is an isomorphism.

Since C is presentable, there is a small collection of objects $\{Y\}$ that detect equivalences. Since $\mathrm{Hom}_C(Y, X^{S^{n+1}}) \simeq \mathrm{Hom}_C(Y \otimes S^{n+1}, X)$, the n-truncated objects are precisely the local objects with respect to the small collection of morphisms $\{S^{n+1} \otimes Y \to * \otimes Y = Y\}$. In particular, the n-truncated objects form a reflective subcategory of C, admitting a reflector $\tau_n \colon C \to C_{\leq n}$.

Exercise 3.4. Show that a stable category does not admit any non-zero n-truncated objects.

We now prove some basic properties of truncated objects.

Lemma 3.5. *The n-truncated morphisms in C are closed under base change.*

Proof. Immediate from the definitions, details left to the reader. □

Lemma 3.6. *A morphism $f \colon X \to Y$ is n-truncated if and only if the induced map of animae $f_* \colon \mathrm{Hom}_C(Z, X) \to \mathrm{Hom}_C(Z, Y)$ is n-truncated for all $Z \in C$.*

Proof. Immediate from Yoneda lemma and the fact that $\mathrm{Hom}_C(Z, -)$ preserves pullbacks. □

Lemma 3.7. *Consider morphisms $f \colon X \to Y$ and $g \colon Y \to Z$.*

(1) *If g is n-truncated, then f is n-truncated if and only if gf is n-truncated.*

(2) *If gf is n-truncated and g is $(n+1)$-truncated, then f is n-truncated.*

Proof. (1) We prove the claim by induction on n. The case $n = -2$ is clear by 2-out-of-3.

For $n \geq -1$, consider the following commutative triangle:

$$\begin{tikzcd} & X \times_Y X \arrow[dr] & \\ X \arrow[ur, "\Delta_f"] \arrow[rr, "\Delta_{gf}"] & & X \times_Z X. \end{tikzcd}$$

The right diagonal map is a base change of $\Delta_g \colon Y \to Y \times_Z Y$, hence is $(n-1)$-truncated. By the induction hypothesis, Δ_f is $(n-1)$-truncated if and only if Δ_{gf} is $(n-1)$-truncated. This completes the proof.

(2) We may factor f as the composite

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The first map is a base change of the n-truncated map $\Delta_g \colon Y \to Y \times_Z Y$. The second map is a base change of the n-truncated map $gf \colon X \to Z$. By (1) and Lemma 3.5, f is also n-truncated. □

Lemma 3.8. *Consider a functor $F \colon C \to D$.*

(1) *If F preserves pullbacks, then it preserves n-truncated objects for all $n \geq -2$.*

(2) *If F furthermore admits a right adjoint, and C and D admit n-truncation functors, then F commutes with n-truncation: for any $X \in C$ the canonical map*

$$\tau_n(F(X)) \to F(\tau_n X)$$

is an isomorphism.

Proof. Part (1) is clear, since n-truncatedness is formulated in terms of pullbacks.

For part (2), the claim is equivalent to the claim that the right adjoint $G\colon D \to C$ of F preserves n-truncated objects. This is an instance of (1). $\square$

Corollary 3.9. *For a morphism $f\colon X \to Y$ in a topos T, the pullback functor $f^*\colon T_{/Y} \to T_{/X}$ preserves n-truncated objects and commutes with n-truncation: for a map $Z \to Y$ we have $\tau_n(f^*Z/X) \cong f^*\tau_n(Z/Y)$.*

Proof. Lemma 3.8 applies: the functor f^* preserves limits (as it has a left adjoint) and preserves colimits (by universality of colimits), so that it admits a right adjoint $f_*\colon T_{/X} \to T_{/Y}$ by the adjoint functor theorem. $\square$

Proposition 3.10 ([ABFJ20, Proposition 2.2.6])**.** *Consider a pushout square*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in which the map $A \hookrightarrow B$ is a monomorphism. Then also $C \to D$ is a monomorphism and the square is a pullback square.

Proof. Consider the cube

$$\begin{tikzcd} A \arrow[rr, equal] \arrow[dr] \arrow[dd, equal] & & A \arrow[dr] \arrow[dd] & \\ & C \arrow[rr, equal] \arrow[dd, equal] & & C \arrow[dd] \\ A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\ & C \arrow[rr] & & D. \end{tikzcd}$$

The top and bottom faces are pushouts, and the left and back faces are pullbacks. By descent (Mather's first cube lemma), the front and right faces are also pullback squares. This says precisely that the starting square is a pullback square and that the map $C \to D$ is a monomorphism. $\square$

Lemma 3.11. *Let $U \hookrightarrow X$ be a monomorphism. Then also $\tau_0 U \hookrightarrow \tau_0 X$ is a monomorphism, and the square*

$$\begin{tikzcd} U \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \tau_0 U \arrow[d, hook] \\ X \arrow[r, two heads] & \tau_0 X \end{tikzcd}$$

is a pullback square.

Proof. This statement immediately reduces to the case $T = \mathrm{An}$. In this case, the monomorphism takes the form of an inclusion $U \hookrightarrow U \sqcup V$, for which the claim is clear. $\square$

We now prove a crucial fact: effective epimorphisms in a topos can be tested on 0-truncations.

Lemma 3.12 (Key lemma)**.** *Let T be a topos.*

(1) *The map $X \to \tau_0 X$ is an effective epimorphism for every $X \in T$.*

(2) *A morphism $f\colon Y \to X$ is an effective epimorphism if and only if the map $\tau_0 Y \to \tau_0 X$ is an effective epimorphism.*

Remark 3.13. There is a circularity in the proof of this lemma in HTT: the proof appearing in Chapter 6 uses results from Chapter 7, which in turn rely on this result from Chapter 6.

While this circularity can be fixed, Lurie also forgets to prove that if X is 0-connected then $X \to X \times X$ is an effective epimorphism.

Proof. (1) Factor the map $X \to \tau_0 X$ into an effective epimorphism followed by a monomorphism:

$$X \twoheadrightarrow U \hookrightarrow \tau_0 X.$$

We must show that the second map is an isomorphism. Since it is a monomorphism ((-1)-truncated) and $\tau_0 X$ is 0-truncated, Lemma 3.7 implies that U is also 0-truncated. By definition, $\tau_0 X$ is the initial 0-truncated object equipped with a map from X. Therefore the map $U \hookrightarrow \tau_0 X$ admits a section $\tau_0 X \to U$. In particular, it is an isomorphism by Corollary 2.39.

(2) The "only if" direction is clear from the commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, two heads] & X \arrow[d, two heads] \\ \tau_0 Y \arrow[r, "\tau_0 f"] & \tau_0 X. \end{tikzcd}$$

Indeed, by part (1) the map $X \to \tau_0 X$ is an effective epimorphism. If f is also an effective epimorphism, then so is the composite $Y \to \tau_0 X$ by Lemma 2.37. Then $\tau_0 Y \to \tau_0 X$ is also an effective epimorphism by Lemma 2.40.

Conversely, assume that $\tau_0 f$ is an effective epimorphism. Consider the epi–mono factorization $Y \twoheadrightarrow U \hookrightarrow X$ of f and the induced diagram

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d] & U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \tau_0 Y \arrow[r] & \tau_0 U \arrow[r] & \tau_0 X \end{tikzcd}$$

By Lemma 3.11, the map $\tau_0 U \to \tau_0 X$ is again a monomorphism, and the right-hand square is a pullback square. Since $\tau_0 Y \to \tau_0 X$ is an effective epimorphism by assumption, Corollary 2.39 implies that $\tau_0 U \to \tau_0 X$ is even an isomorphism. It then follows that $U \to X$ is also an isomorphism, as desired. □

Corollary 3.14. *A map $* \to X$ in a topos T is an effective epimorphism if and only if $\tau_0 X$ is the terminal object of T.* □

3.2 Connectivity

There is a dual notion to truncatedness, called *connectivity*:

Definition 3.15. Let T be a topos, and let $n \geq -2$.

- We say that $X \in T$ is *n-connected* if $\tau_n X = *$.
- We say $f\colon X \to Y$ is *n-connected* if it is n-connected in $T_{/Y}$.

We denote by $T^{\geq n+1}$ the full subcategory of T spanned by the n-connected objects.

Warning 3.16. In the classical algebraic topology literature, there is an unfortunate shift in terminology, where n-connected morphisms are referred to as *$(n-1)$-connected* for historical reasons. The 'correct' definition from a topos-theoretic perspective is the one given in Definition 3.15. We should treat slice topoi just as topoi themselves, and constructions done in slice topoi should simply be special cases of the constructions for general topoi.

Note that every object is (-2)-connected. Observe that f is n-connected if and only if for every solid commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & U \arrow[d, "g"] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y \end{tikzcd}$$

where g is n-truncated, the anima of diagonal fillers is contractible.

Proposition 3.17. *Let T be a topos.*

(1) *The pair* (n-connected, n-truncated) *forms a factorization system on T.*

(2) *The n-connected maps are stable under base change.*

(3) *A map $f\colon X \to Y$ is n-connected if and only if the functor $f^*\colon (T_{/Y})_{\leq n} \to (T_{/X})_{\leq n}$ is fully faithful.*

Proof. (1) We first show that any map $f\colon X \to Y$ factors as an n-connected map followed by an n-truncated map. Let $X_n := \tau_n(X/Y)$ be the n-truncation of $X \in T_{/Y}$. The map $X_n \to Y$ is n-truncated by definition.

To show that $X \to X_n$ is n-connected, consider any n-truncated morphism $U \to X_n$. We must show that the map

$$\mathrm{Hom}_{/X_n}(X_n, U) \to \mathrm{Hom}_{/X_n}(X, U)$$

is an equivalence. Since $T_{/X_n} \simeq (T_{/Y})_{/X_n}$, this map is induced on vertical fibers in the following diagram of Hom animae in $T_{/Y}$:

$$\begin{tikzcd} \mathrm{Hom}_{/Y}(X_n, U) \arrow[r] \arrow[d] & \mathrm{Hom}_{/Y}(X, U) \arrow[d] \\ \mathrm{Hom}_{/Y}(X_n, X_n) \arrow[r] & \mathrm{Hom}_{/Y}(X, X_n). \end{tikzcd}$$

Since U and X_n are n-truncated objects in $T_{/Y}$, the universal property of $X_n = \tau_n(X/Y) \in T_{/Y}$ guarantees that both horizontal maps are equivalences. Thus the induced map on fibers is also an equivalence.

Next, we show that any n-connected map $f\colon X \to Y$ is left orthogonal to any n-truncated map $g\colon W \to Z$. We must show that the anima of diagonal fillers in any commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

is contractible. Since n-truncated maps are closed under base change, we may replace g by the projection $W \times_Z Y \to Y$. This reduces the problem to finding diagonal fillers in the commutative square:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \times_Z Y \arrow[r] \arrow[d, "\mathrm{pr}"] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "g"] \\ Y \arrow[r, equal] \arrow[ur, dashed] & Y \arrow[r] & Z. \end{tikzcd}$$

The anima of such diagonal fillers is contractible by n-connectedness of f.

(2) Consider a pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

By Corollary 3.9, we have $\tau_n(X'/Y') \cong g^*\tau_n(X/Y)$. If f is n-connected, then $\tau_n(X/Y) = Y$, so $\tau_n(X'/Y') = Y'$. Hence f' is also n-connected.

(3) We need to show that f is n-connected if and only if for each two n-truncated maps $Z \to Y$ and $Z' \to Y$, the induced map

$$f^*\colon \operatorname{Hom}_{/Y}(Z, Z') \to \operatorname{Hom}_{/X}(X \times_Y Z, X \times_Y Z')$$

is an equivalence. By the universal property of $X \times_Y Z'$, the latter is equivalent to the condition that composition with the projection $X \times_Y Z \to Z$ induces an equivalence

$$\operatorname{Hom}_{/Y}(Z, Z') \xrightarrow{\sim} \operatorname{Hom}_{/Y}(X \times_Y Z, Z').$$

In other words, we have to show that f is n-connected if and only if the map $X \times_Y Z \to Z$ is left orthogonal to any n-truncated map $Z' \to Y$. This is immediate from parts (1) and (2). □

Corollary 3.18. *A morphism is (-1)-connected if and only if it is an effective epimorphism.*

Proof. Both the (-1)-connected maps as well as the effective epimorphisms are characterized as the morphisms that are left orthogonal to the monomorphisms/(-1)-truncated maps. □

Corollary 3.19. *Every pointed object $X \in T_*$ is (-1)-connected.*

Proof. A basepoint $x\colon * \to X$ determines a section of the map $X \to *$, which by Lemma 2.38 implies that $X \to *$ is an effective epimorphism, hence (-1)-connected. □

3.3 Homotopy group objects

Our goal in this section is to prove the following inductive characterization of n-connected morphisms, which resembles the definition of n-truncated morphisms:

Theorem 3.20. *Let $n \geq 0$. Then a morphism $f\colon X \to Y$ is n-connected if and only if it is an effective epimorphism and $\Delta_f\colon X \to X \times_Y X$ is $(n-1)$-connected.*

This theorem is rather non-trivial, and its proof will require us to introduce the notion of homotopy group objects.

Definition 3.21. Let $X \in T$ and let $n \geq 0$. Define the *n-th homotopy object* $\pi_n(X)$ as the following static (0-truncated) object of $T_{/X}$:

$$\pi_n(X) := \tau_0\big(X^{S^n} \to X\big) \quad \in \quad (T_{/X})_{\leq 0} \quad \subseteq \quad T_{/X}.$$

Here X^{S^n} denotes the cotensoring of X by S^n, cf. Remark 3.3. Since S^n is an E_n-cogroup in An and the map $* \to S^n$ is a map of E_n-cogroups, $\pi_n(X)$ is an E_n-group. In particular:

- $\pi_0(X)$ is a pointed object,
- $\pi_1(X)$ is a group,
- $\pi_n(X)$ is an abelian group for $n \geq 2$.

For this reason, we will often somewhat abusively speak of *homotopy group objects*, even though $\pi_0(X)$ is only a pointed object.

Example 3.22. Since $X^{S^0} = X \times X$, we see that $\pi_0(X) \cong \tau_0 X \times X \in T_{/X}$. It becomes a pointed object via the canonical section $X \xrightarrow{\Delta} X \times X \to \tau_0 X \times X$.

Lemma 3.23. *Let $F\colon T \to T'$ be a functor preserving colimits and finite limits. Then F preserves homotopy groups: for all $X \in T$ we have a natural isomorphism*

$$F(\pi_n(X)) \cong \pi_n(F(X)) \quad \in \quad T_{/F(X)}.$$

Proof. Since $F\colon T_{/X} \to T'_{/F(X)}$ preserves finite limits and X^{S^n} is a finite limit, we see that $F(X^{S^n}) \cong F(X)^{S^n}$. Furthermore, F commutes with τ_0 by Lemma 3.8. □

Lemma 3.24. *The functor $\pi_n\colon T \to T$ preserves finite products.*

Proof. Reduce to An. □

Notation 3.25. Given $n \geq 0$ and a morphism $f\colon X \to Y$, write

$$\pi_n(f) \in (T_{/Y})_{/f} \cong T_{/X}$$

for the n-th homotopy object of f regarded as an object of $T_{/Y}$. *(Note: We do <u>not</u> mean the induced map $\pi_n(X) \to \pi_n(Y)$.)*

Remark 3.26. Note that we have

$$\pi_n(f) \cong \pi_{n-1}(\Delta_f)$$

for $n \geq 1$.

Exercise 3.27. Show that every subgroup of $\pi_1(X)$ is normal.

Proposition 3.28. *Given a morphism $f\colon X \to Y$, there is a long exact sequence in $(T_{/X})_{\leq 0}$ of the form*

$$\cdots \to \pi_n(f) \to \pi_n(X) \to f^*\pi_n(Y) \to \pi_{n-1}(f) \to \ldots.$$

Proof. When $T = \mathrm{An}$, this is a standard fact from classical homotopy theory. The claim then immediately follows for any presheaf topos $\mathrm{PSh}(C) = \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$, where everything is computed pointwise. By Theorem 2.42, any topos T may be written as a left exact localization of a presheaf topos $\mathrm{PSh}(C)$, and the relevant homotopy groups are computed by first forming them in $\mathrm{PSh}(C)$ and then applying the localization functor. □

Proposition 3.29. *Let $n \geq -1$.*

(1) *Assume that f is n-truncated. Then $\pi_k(f) = 0$ for $k > n$. The converse holds if f is m-truncated for some m.*

(2) *A morphism f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) = *$ for all $k \leq n$.*

Proof. We prove both statements by induction on n.

(1) If f is (-1)-truncated, then Δ_f is an isomorphism. In particular, the homotopy groups of Δ_f are trivial. Since $\pi_n(\Delta_f) = \pi_{n+1}(f)$, all higher homotopy groups are trivial.

It remains to show that $\pi_0(f) = 0$. Without loss of generality, take $f\colon X \to *$. Since X is (-1)-truncated, it is in particular 0-truncated, so $\pi_0(X) = (X \times X \to X)$. The map $\Delta_X\colon X \to X \times X$ is an isomorphism, so $X \times X \to X$ is also an isomorphism. Thus $\pi_0(X)$ is trivial.

For the induction step, assume f is n-truncated. Then Δ_f is $(n-1)$-truncated, giving $\pi_k(f) = \pi_{k-1}(\Delta_f) = 0$ for all $k > n$.

For the converse, assume f is m-truncated for some m and $\pi_n(f) = *$. Then $\pi_{n-1}(\Delta_f) = *$. By the induction hypothesis, Δ_f is $(n-2)$-truncated, meaning that f is $(n-1)$-truncated. For the base case $n = 0$, if X is 0-truncated and $\pi_0(X) = * = (X \times X \to X)$, then $\Delta_X\colon X \to X \times X$ is an isomorphism.

(2) Assume that X is n-connected, i.e. $\tau_n X = *$. Then X is in particular k-connected for all $k \leq n$. We have already seen that X is an effective epimorphism. It remains to show the homotopy groups vanish. This reduces to An as follows. We may write T as a reflective localization $L\colon \mathrm{PSh}(C) \to T$ of a presheaf topos. Then $L(\tau_n^{\mathrm{pre}} X) = *$. This gives a functor $\mathrm{PSh}(C)_{/\tau_n^{\mathrm{pre}} X} \to T$, and the source is still a presheaf category.

For the converse, the case $n = -1$ was treated before. For $n \geq 0$, consider the map $t\colon X \to \tau_n X$. This is n-connected. By the long exact sequence,

$$* = \pi_k(X) \xrightarrow{\cong} t^*\pi_k(\tau_n X)$$

for all $k \leq n$. Since t is an effective epimorphism, $\pi_k(\tau_n X) = *$ for all $k \leq n$. By part (1), we also have $\tau_0 X \cong *$, since $X \to *$ is an effective epimorphism (Corollary 3.14). □

We are now ready to prove our characterization of n-connected maps:

Proof of Theorem 3.20. We must show that f is n-connected if and only if it is an effective epimorphism and Δ_f is $(n-1)$-connected.

By the proposition, f is n-connected if and only if it is an effective epimorphism and $\pi_k(f) = *$ for $k \le n$. Similarly, Δ_f is $(n-1)$-connected if and only if Δ_f is an effective epimorphism and $\pi_k(f) = *$ for $k \le n$.

It remains to show that if f is 0-connected, then Δ_f is an effective epimorphism. By the Key Lemma (Lemma 3.12), it suffices to show that the map $\tau_0 X \to \tau_0 X \times \tau_0 X$ is an effective epimorphism. But $\tau_0 X = *$ by Corollary 3.14. □

This theorem has various useful consequences.

Corollary 3.30. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. If gf is n-connected and g is $(n+1)$-connected, then f is n-connected.*

Proof. Dual to the proof of Lemma 3.7(2), we may factor f as $X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y$, where the first map is a base change of Δ_g, hence n-connected by the theorem, and the second map is a base change of gf. □

Corollary 3.31. *Consider a 0-connected pointed object $X \in T_*^{\ge 0}$. Then X is n-connected if and only if ΩX is $(n-1)$-connected. Similarly, X is n-truncated if and only if ΩX is $(n-1)$-truncated.*

Proof. By the theorem, X is n-connected if and only if $\Delta\colon X \to X \times X$ is $(n-1)$-connected. Given a base point $x\colon * \to X$, it follows from 0-connectedness and the previous lemma that x is (-1)-connected. Hence $(x,x)\colon * \to X \times X$ is an effective epimorphism. This implies that Δ is $(n-1)$-connected if and only if its base change $\Omega X \to *$ is $(n-1)$-connected.

The proof for truncatedness is analogous. □

Corollary 3.32. *Let $f\colon X \to Y$ be n-connected. Then the functor $f^*\colon (T_{/Y})_{\le n-1} \to (T_{/X})_{\le n-1}$ is an equivalence.*

Proof. Full faithfulness of f^* was proved in Proposition 3.17.

For essential surjectivity, we may replace T by $T_{/Y}$ and assume $Y = *$. Given an $(n-1)$-truncated morphism $U \to X$, we must show that it is in the image of the functor $(-)\times X\colon T_{\le n-1} \to (T_{/X})_{\le n-1}$. We claim that the map $U \to \tau_{n-1}U \times X$ is an isomorphism. By the factorization system from Proposition 3.17, it suffices to show it is both $(n-1)$-truncated and $(n-1)$-connected.

- The map is $(n-1)$-truncated since it lives over X, and both maps to X are $(n-1)$-truncated. The claim follows by left cancellation.
- To see it is $(n-1)$-connected, note that it factors as

  $$U \longrightarrow U \times X \longrightarrow \tau_{n-1}U \times X.$$

  The first map is $(n-1)$-connected, as it is a base change of the diagonal $X \to X \times X$, which is $(n-1)$-connected by the theorem. The second map is $(n-1)$-connected since it is a base change of the map $U \to \tau_{n-1}U$. □

3.4 Hypercompletion

The notion of n-connectedness has an analogue for $n = \infty$.

Definition 3.33. A morphism $f \colon X \to Y$ in T is called ∞-*connected* if it is n-connected for every n. It is called ∞-*truncated* (or *hypercomplete*) if it is right orthogonal to the class of ∞-connected maps.

Definition 3.34. Let $T_{\leq\infty}$ denote the full subcategory of ∞-truncated objects in T. An alternative notation for this subcategory is T^{hyp}. We say that T is *hypercomplete* if every object is ∞-truncated.

Remark 3.35. The term "hypercomplete" is historical but slightly misleading: hypercompleteness is a separation condition rather than a completeness condition. "Postnikov separated" might have been a better term. In [ABFJ25], the authors use the term "hyperreduced" instead.

Proposition 3.36. *Let T be a topos.*

(1) *The ∞-connected maps form a strongly saturated class stable under base change and diagonals.*

(2) *The inclusion $T_{\leq\infty} \hookrightarrow T$ admits a left adjoint $\tau_\infty \colon T \to T_{\leq\infty}$.*

(3) *The pair* (∞-connected, ∞-truncated) *is a factorization system on T.*

Proof. (1) The n-connected maps form a saturated class (i.e. closed under pushouts and closed under colimits in $\mathrm{Ar}(T)$), by the factorization system from Proposition 3.17. In particular, the ∞-connected maps form a saturated class.

For the 2-out-of-3 property, it remains to check left cancellation, which follows from Corollary 3.30. This also implies closure under diagonals. Closure under base change follows from part (2) of Proposition 3.17.

(2) The ∞-truncated objects are precisely the local objects with respect to this class. The general theory of local objects in presentable categories then tells us that the left adjoint τ_∞ exists. We refer to [Lur09, Proposition 5.5.4.15] for details.

For (3), it remains to show that any morphism $f \colon X \to Y$ can be factored into an ∞-connected morphism followed by an ∞-truncated one. The same argument as in part (1) of Proposition 3.17 shows that the map $X \to \tau_\infty(X/Y)$ provides such a factorization. □

Corollary 3.37. *The left adjoint $\tau_\infty \colon T \to T_{\leq\infty}$ is left exact. In particular, $T_{\leq\infty}$ is again a topos.*

Proof. For a finite diagram $\{X_i\}_{i\in I}$ in T, consider the map $\lim_i X_i \to \lim_i \tau_\infty X_i$. Since ∞-connected maps are closed under base change, they are closed under finite limits. Hence this map is ∞-connected. Since ∞-truncated objects in T are closed under limits, the target is ∞-truncated. It follows that the map induces an equivalence $\tau_\infty(\lim_i X_i) \xrightarrow{\sim} \lim_i \tau_\infty X_i$. □

3.5 Gerbes

Consider an object $X \in T$. We have the Postnikov tower

$$X \to \cdots \to \tau_n X \to \tau_{n-1} X \to \cdots \to \tau_0 X \to \tau_{-1} X \to \tau_{-2} X = *.$$

The object $\tau_n X \in T_{/\, \tau_{n-1} X}$ is n-truncated by Lemma 3.7 and $(n-1)$-connected by Proposition 3.17. Maps with this property are known as *gerbes*:

Definition 3.38. Given $n \geq 0$, we say an object $X \in T$ is an *n-gerbe* if it is n-truncated and $(n-1)$-connected. We define an *Eilenberg–MacLane object of degree n* to be a pointed n-gerbe. We denote by

$$\mathrm{Gerb}_n(T) \subseteq T \qquad \text{and} \qquad \mathrm{EM}_n(T) \subseteq T_*$$

the resulting full subcategories. More generally, we refer to a morphism $X \to Y$ as an *n-gerbe* if it is an n-gerbe in the slice topos $T_{/Y}$.

Example 3.39. Let K be an $(n+1)$-gerbe in T, and consider two maps $a, b\colon * \to K$. Define P as the following pullback:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "b"] \\ * \arrow[r, "a"] & K. \end{tikzcd}$$

Then P is an n-gerbe. Indeed, the map $b\colon * \to K$ is an n-gerbe by the cancellation properties of truncated and connected maps from Lemma 3.7 and Corollary 3.30, hence the base change $P \to *$ is also an n-gerbe.

The Eilenberg–MacLane objects can in fact be completely classified in terms of abelian group objects in T, just as for the Eilenberg–MacLane spaces in classical algebraic topology. To do this, we need an 'iterated' version of the Delooping principle from Proposition 2.35.

Definition 3.40. Given a category C with finite products, we inductively define for every $n \geq 0$ the category $E_n\mathrm{Grp}(C)$ of *E_n-groups in C* by setting

$$E_0\mathrm{Grp}(C) := C_* \qquad \text{and} \qquad E_{n+1}\mathrm{Grp}(C) := \mathrm{Grp}(E_n\mathrm{Grp}(C)).$$

Proposition 3.41 (Iterated delooping principle)**.** *For every $n \geq 0$, there is an equivalence*

$$\mathbf{B}^n\colon E_n\mathrm{Grp}(T) \rightleftarrows T_*^{\geq n} :\mathbf{\Omega}^n.$$

Proof. First observe that there is an equivalence $T_*^{\geq 0} \simeq T_*$. That is, every pointed object in T is automatically (-1)-connected. Indeed, any map $x\colon * \to X$ is a section of $X \to *$, so the latter is an effective epimorphism by Lemma 2.38.

Consider now the 'delooping equivalence'

$$\mathbf{B}\colon \mathrm{Grp}(T) \rightleftarrows T_*^{\geq 1} :\mathbf{\Omega}$$

from Proposition 2.35. By passing to E_{n-1}-groups, this results in an equivalence $E_n\mathrm{Grp}(T) \simeq E_{n-1}\mathrm{Grp}(T_*^{\geq 1})$. Since the equivalence is implemented by taking loops, Corollary 3.31 implies that it restricts to an equivalence

$$\mathbf{B}\colon E_n\mathrm{Grp}(T^{\geq k-1}) \rightleftarrows E_{n-1}\mathrm{Grp}(T_*^{\geq k}) :\mathbf{\Omega}$$

for every $k \geq 1$. Since $\mathrm{Grp}(C_*) \simeq \mathrm{Grp}(C)$ for any C, we may now paste together all these equivalences to obtain the result:

$$\mathbf{B}^n \colon E_n\mathrm{Grp}(T) \simeq E_{n-1}\mathrm{Grp}(T_*^{\geq 1}) \simeq \cdots \simeq E_1\mathrm{Grp}(T_*^{\geq n-1}) \simeq E_0\mathrm{Grp}(T_*^{\geq n}) = T_*^{\geq n} \colon \mathbf{\Omega}^n. \qquad \square$$

Corollary 3.42. *For $n \geq 0$ there are equivalences*

$$\mathrm{EM}_0(T) \simeq (T_{\leq 0})_*, \qquad \mathrm{EM}_1(T) \simeq \mathrm{Grp}(T_{\leq 0}), \qquad \textit{and} \qquad \mathrm{EM}_n(T) \simeq \mathrm{Ab}(T_{\leq 0}) \quad \text{for } n \geq 2.$$

Proof. By Corollary 3.31, the equivalence $\mathbf{\Omega}^n \colon T_*^{\geq n} \xrightarrow{\sim} E_n\mathrm{Grp}(T)$ from Proposition 3.41 restricts to an equivalence

$$\mathbf{\Omega}^n \colon \mathrm{EM}_n(T) \xrightarrow{\sim} E_n\mathrm{Grp}(T_{\leq 0})$$

for all n, with inverse given by $\mathbf{B}^n$. But since $T_{\leq 0}$ is a 1-category, we have $E_n\mathrm{Grp}(T_{\leq 0}) \simeq \mathrm{Ab}(T_{\leq 0})$ for all $n \geq 2$. $\square$

The n-gerbes play an important role in *obstruction theory*. Given objects $Y, X \in T$ and a map $Y \to \tau_0 X$, we could ask whether we can lift this map to X. Often, we can proceed inductively along the Postnikov tower of X. Assuming we have already found a lift $Y \to \tau_{n-1}X$, we may ask whether this further lifts to $X \to \tau_n X$:

$$\begin{tikzcd} & \tau_n X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "?"] & \tau_{n-1}X. \end{tikzcd}$$

In the remainder of this section we will explain how to reformulate this question in terms of the vanishing of a certain class in the cohomology of Y.

Lemma 3.43. *Let $n \geq 2$, and let X be an n-gerbe. Then there exists a unique $A \in \mathrm{Ab}(T_{\leq 0})$ such that $\pi_n X \cong X \times A$.*

Proof. Since X is $(n-1)$-connected, the functor $X \times - \colon T \to T_{/X}$ induces an equivalence $T_{\leq n-2} \xrightarrow{\sim} (T_{/X})_{\leq n-2}$ on $(n-2)$-truncated objects, by Corollary 3.32. In particular, the abelian group object $\pi_n(X) \in \mathrm{Ab}((T_{/X})_{\leq 0})$ corresponds to an object $A \in \mathrm{Ab}(T_{\leq 0})$ under this equivalence. $\square$

Remark 3.44. If $n = 1$, then $A \in \mathrm{Grp}(T_{\leq 0})$ is unique if it exists, but it might not exist.

Definition 3.45. Let X be an n-gerbe and consider $A \in \mathrm{Ab}(T_{\leq 0})$. We say that X is *banded by A* if it comes equipped with an isomorphism $\pi_n X \cong X \times A$ in $\mathrm{Ab}(T_{/X})$. We denote by

$$\mathrm{Gerb}_n^A(T)$$

the category of pairs consisting of an n-gerbe X together with an isomorphism $\pi_n X \cong X \times A$.

Lemma 3.46. *The trivial n-gerbe $\mathbf{B}^n A$ is bounded by A. Moreover, if X is any Eilenberg–MacLane object bounded by A, then $X \simeq \mathbf{B}^n A$.*

Proof. For the first claim, consider the two canonical maps $\mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$ and $\Omega^n \mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$. Using that $\mathbf{B}^n A \in \mathrm{CGrp}(T)$, this induces a map

$$\mathbf{B}^n A \times A \simeq \mathbf{B}^n A \times \Omega^n \mathbf{B}^n A \to (\mathbf{B}^n A)^{S^n}$$

in $T_{/\mathbf{B}^n A}$. We need to show that this induces an isomorphism on 0-truncations. Since the map $* \twoheadrightarrow \mathbf{B}^n A$ is an effective epimorphism, this can be checked after pullback along this map. But there it is clear as the map becomes the identity on $\Omega^n \mathbf{B}^n A$.

For the second claim, let X be an Eilenberg-MacLane object of degree n, so that $X \simeq \mathbf{B}^n A'$ for some $A' \in \mathrm{Ab}(T_{\leq 0})$. By the first part, we have $\pi_n(X) \cong X \times A'$ in $\mathrm{Ab}(T_{/X})$. If X were also banded by A, this would imply $A \cong A'$, hence $X \cong \mathbf{B}^n A' \cong \mathbf{B}^n A$. □

Theorem 3.47. *Let $A \in \mathrm{Ab}(T_{\leq 0})$ and $n \geq 1$. Then for any object $X \in T$ there is an equivalence*

$$\mathrm{Hom}_T(X, \mathbf{B}^{n+1} A) \quad \simeq \quad \mathrm{Gerb}_n^A(T_{/X}), \qquad\qquad f \mapsto \mathrm{fib}_0(f).$$

Proof. First, note that the functor is well-defined. By Example 3.39, the map $0\colon * \to \mathbf{B}^{n+1} A$ is an n-gerbe. The claim is that this map is in fact the *universal* n-gerbe.

To see this, consider an n-gerbe $\widetilde{X} \to X$ banded by A. There is a *unique* cartesian square of the form

$$\begin{tikzcd} \widetilde{X} \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ X \arrow[r, dashed] & \mathbf{B}^{n+1} A. \end{tikzcd}$$

More formally, we claim that $\mathrm{Hom}_{\mathrm{Ar}^{\mathrm{pb}}(T)}(\widetilde{X} \to X,\ * \to \mathbf{B}^{n+1} A)$ is contractible. We proceed in two steps.

Step 1: We first prove the claim in the case where $\widetilde{X} \to X$ admits a section $s\colon X \to \widetilde{X}$. By regarding $\widetilde{X}$ as an n-gerbe in $T_{/X}$, this section makes it into an Eilenberg-MacLane object, hence it is of the form $X \times \mathbf{B}^n A$ by Lemma 3.46. Let us now rewrite the data contained in such a pullback square:

- It consists of a map $X \to \mathbf{B}^{n+1} A$ together with an identification of $\widetilde{X} = X \times \mathbf{B}^n A$ with the pullback $P := X \times_{\mathbf{B}^{n+1} A} *$.
- An identification $X \times \mathbf{B}^n A \cong P$ is the same as providing a section of $P \to *$, which in turn is equivalent to providing a lift $X \to *$ of the map $X \to \mathbf{B}^{n+1} A$.
- But the anima of pairs of a map $X \to \mathbf{B}^n A$ together with a lift to a map $X \to *$ is just the same as the anima of maps $X \to *$, which is clearly contractible.

This yields the claim.

Step 2: We now prove the claim in general. Consider an effective epimorphism $f\colon U \twoheadrightarrow X$ such that the pullback map $\widetilde{X} \times_X U \to U$ admits a section; for example, we may take $U = \widetilde{X}$ and take the section to the the diagonal $\widetilde{X} \to \widetilde{X} \times_X \widetilde{X}$. Let $\check{C}(f)$ denote the Čech nerve of f, and write $U_n := \check{C}(f)_n$ and $\widetilde{U}_n := \widetilde{X} \times_X \check{C}(f)_n$ for short. Then each map $\widetilde{U}_n \to U_n$ is an n-gerbe banded by A admitting a section, hence by Step 1 we know that

$$\mathrm{Hom}_{\mathrm{Ar}^{\mathrm{pb}}(T)}(\widetilde{U}_n \to U_n,\ * \to \mathbf{B}^{n+1} A) \ \simeq\ *$$

for all n. The claim now follows, since by descent the map $\widetilde{X} \to X$ is a colimit in $\mathrm{Ar}^{\mathrm{pb}}(T)$ of the maps $\widetilde{U}_n \to U_n$. □

Remark 3.48. Under the equivalence $T_{/\mathbf{B}G} \simeq \mathrm{Mod}_G(T)$, passing to fibers over T (with respect to the forgetful functor $T_{/\mathbf{B}G} \to T$ and the quotient functor $\mathrm{Mod}_G(T) \to T$) yields

$$\mathrm{Hom}_*(*, \mathbf{B}G) \;\cong\; \{G\text{-torsors}\}.$$

Let us now return to the question of obstruction theory. Given an object X, the map $\tau_n X \to \tau_{n-1} X$ is an n-gerbe, banded by some $A \in \mathrm{Ab}(T_{/\tau_{n-1}X})$. We may thus write it as a pullback

$$\begin{tikzcd} \tau_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ \tau_{n-1} X \arrow[r] & \mathbf{B}^{n+1} A. \end{tikzcd}$$

In particular, finding a lift $Y \to \tau_n X$ of a map $Y \to \tau_{n-1} X$ is equivalent to lifting the composite $Y \to \mathbf{B}^{n+1} A$ to $Y \to *$. We wish to think of this as the question of whether a certain cohomology class vanishes:

Definition 3.49. Let A be an abelian group object in $T_{\leq 0}$, and let $Y \in T$. We define the *n-th cohomology of Y with coefficients in A* as

$$H^n(Y, A) := \pi_0 \,\mathrm{Hom}_T(Y, \mathbf{B}^n A) \in \mathrm{Ab}.$$

4 The categories of topoi and logoi

In this chapter, we investigate the category of topoi. There are two natural choices for a 'category of topoi', each with its own merits: one could either work with the left-exact colimit-preserving functors $f^*\colon S \to T$, or one could work with their right adjoints $f_*\colon T \to S$. We denote the resulting categories by Logos and Topos, respectively; this terminology is explained in Section 4.1.

In Section 4.2, we introduce the notion of a *presentation* of a topos (or logos) in terms of generators and relations. We use this in Section 4.3 to compute limits and colimits of topoi. In Section 4.4, we introduce the notion of an *étale morphism* of topoi, which plays an analogous role to that of a local homeomorphism of topological spaces. The resulting wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ turns out to be closed under colimits; we prove this later in Section 4.5, after developing univalent families.

4.1 Topoi and logoi

As mentioned before, there are two natural perspectives on the direction of morphisms between topoi: the *algebraic* and the *geometric* perspective. While the data in such a morphism is the same in both cases, the fact that the maps point in opposite directions can lead to confusing terminology, especially when discussing constructions with universal properties like limits and colimits of topoi.

In [AJ21], Anel and Joyal proposed to emphasize the duality between these perspectives by strictly reserving the term 'topos' for the geometric perspective and introducing the new word 'logos' for the algebraic perspective. Just as affine schemes are dual to commutative rings and compact Hausdorff spaces are dual to commutative C^*-algebras, they argue that topoi should be dual to logoi.

As we will see, certain constructions (like presentations of topoi) are more naturally explained using the algebraic perspective than the geometric one. It is convenient to have language available that directly reflects this perspective.

Definition 4.1. Given two topoi S and T, a *morphism of topoi* (or *geometric morphism*) from S to T is a functor $\varphi_*\colon S \to T$ which admits a left exact colimit-preserving left adjoint $\varphi^*\colon T \to S$. We denote by

$$\mathrm{Topos} \quad \subseteq \quad \mathrm{Cat}$$

the subcategory spanned by the topoi and the morphisms of topoi.

Definition 4.2. A category T is called a *logos* if it is a topos, i.e. if it is presentable and satisfies descent for all colimits. Given two logoi S and T, a *morphism of logoi* (sometimes called *algebraic* *morphism*[1]) from S to T is a left exact colimit-preserving functor $\varphi^*\colon S \longrightarrow T$. We denote by

$$\mathrm{Logos} \quad \subseteq \quad \mathrm{Cat}$$

the subcategory spanned by the logoi and the morphisms of logoi.

Remark 4.3. By Lemma 3.8, every morphism of logoi preserves n-truncated objects and commutes with n-truncations for all n.

Remark 4.4. By the adjoint functor theorem, every morphism of logoi $\varphi^*\colon S \to T$ admits a right adjoint $\varphi_*\colon T \to S$, which is then a morphism of topoi. This assignment can be made functorial: there is an equivalence of categories

$$\mathrm{Topos} \quad \xrightarrow{\sim} \quad \mathrm{Logos}^{\mathrm{op}}, \qquad T \mapsto T, \qquad \varphi_* \mapsto \varphi^*.$$

More precisely, this equivalence arises as a restriction of the equivalence $\mathrm{Pr}^{\mathrm{R}} \xrightarrow{\sim} (\mathrm{Pr}^{\mathrm{L}})^{\mathrm{op}}$ from [Lur09, Corollary 5.5.3.4].

The 'logos'-terminology is not yet very common. Marc did not use this terminology in his lectures, instead defining a 'geometric morphism' from S to T to be a left exact colimit-preserving functor $\varphi^*\colon T \to S$. However, I have found it clarifying to use two separate phrases for the two different perspectives on topoi.

As we will see repeatedly throughout this course, there are certain aspects of topos theory in which topoi behave like 'geometric' objects, taking the role of generalized spaces. In other aspects, they have a more 'algebraic' flavor, where they behave analogously to commutative rings. For example, the following notions are most usefully seen through the geometric viewpoint:

- In Definition 4.31, we will define the notion of an 'étale map' of topoi, which mimics the notion of a local homeomorphism between topological spaces;
- In Definition 5.40, we define 'open immersions' of topoi, mimicking an open embedding of topological spaces.

On the other hand, the following definitions are best seen through an algebraic lens:

- In Section 4.2, we will construct presentations of logoi in terms of 'generators and relations', like for ring theory;
- In Chapter 5, we will study *congruences*: those classes of morphisms in a logos that arise as 'kernels' of morphisms of logoi. For any congruence K in a logos T, we obtain the *quotient logos* $T/K := T[K^{-1}]$.

We now upgrade Logos and Topos to 2-categories.

[1] We warn the reader that this terminology clashes with the notion of 'algebraic morphism' used by Lurie [Lur09, Definition 6.3.6.1].

Construction 4.5. The category Cat is cartesian closed (the functor $C \times - : \mathrm{Cat} \to \mathrm{Cat}$ admits a right adjoint $\mathrm{Fun}(C, -) : \mathrm{Cat} \to \mathrm{Cat}$ for all C) and thus enriched over itself. This gives rise to a 2-category $\mathbb{C}\mathrm{at}$, whose objects are categories, whose morphisms are functors, and whose 2-morphisms are natural transformations between functors: $\mathrm{Hom}_{\mathbb{C}\mathrm{at}}(C, D) = \mathrm{Fun}(C, D)$.

This in particular allows us to upgrade the category Logos to a 2-category, by regarding it as a locally full subcategory of $\mathbb{C}\mathrm{at}$:

$$\mathbb{L}\mathrm{ogos} \quad \subseteq \quad \mathbb{C}\mathrm{at}.$$

The Hom-categories $\mathbb{L}\mathrm{ogos}$ are given by the full subcategories

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T) \quad \subseteq \quad \mathrm{Fun}(S, T)$$

spanned by the morphisms of logoi. We may similarly upgrade Topos to a 2-category, by setting

$$\mathbb{T}\mathrm{opos} \quad := \quad \mathbb{L}\mathrm{ogos}^{\mathrm{op}} \quad \subseteq \quad \mathbb{C}\mathrm{at}^{\mathrm{op}}.$$

In other words, we define the Hom-categories of $\mathbb{T}\mathrm{opos}$ to be $\mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(T, S) := \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T)$.

Remark 4.6. The subcategory $\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(S, T)$ of $\mathrm{Fun}(S, T)$ is denoted by $\mathrm{Fun}^*(S, T)$ by Lurie. Marc in his lectures denoted $\mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(T, S)$ by $\mathrm{Geom}(T, S)$.

A point in a topological space X is the same as a continuous map $* \to X$ from the one-point space, i.e. the terminal object in the category of topological spaces. Given that the terminal object of Topos is An (see Example 4.12 below), we get the following analogous notion for topoi:

Definition 4.7. A *point* of a topos T is a geometric morphism $\mathrm{An} \to T$. We write

$$\mathrm{Pt}(T) \quad := \quad \mathrm{Fun}_{\mathbb{T}\mathrm{opos}}(\mathrm{An}, T)$$

for the category of points.

Example 4.8. Let $T = \mathrm{PSh}(C)[\Sigma^{-1}]$ be a left exact localization of a presheaf category. Then the inclusion

$$\mathrm{Pt}(T) \quad \simeq \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T, \mathrm{An}) \quad \hookrightarrow \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C), \mathrm{An}) \quad \simeq \quad \mathrm{Pro}(C)^{\mathrm{op}}$$

identifies $\mathrm{Pt}(T)$ with the Σ-*local pro-objects in* C, i.e. those formal systems $(X_i)_{i \in I}$ for which the functor

$$\mathrm{PSh}(C) \to \mathrm{An}, \qquad F \longmapsto \mathrm{colim}_i\, F(X_i)$$

inverts all morphisms in Σ.

4.2 Presentations of logoi

Algebraic objects can often be presented in terms of 'generators and relations'; think of presentations of groups or rings. In the case of commutative rings, one first forms the free commutative ring $\mathbb{Z}[S]$ on a given set of generators S (the polynomial ring), and then forms the quotient of $\mathbb{Z}[S]$ by an ideal generated by a given set of relations. There is an analogous notion of presentations for logoi:

- In Section 4.2.1 we construct the *free logos* $\mathrm{An}[C]$ on a given category of generators C;

- In Section 4.2.2, we define the *quotient logos* T/K of a logos T by a congruence K.
- In Section 4.2.3, we introduce *logos presentations* (C, Σ), consisting of a small category C and a small collection of morphisms Σ in $\mathrm{An}[C]$, and define the logos presented by (C, Σ) as $\langle C \mid \Sigma \rangle := \mathrm{An}[C]/\Sigma^c$, where Σ^c is the congruence generated by Σ.

4.2.1 Free logoi

We start by constructing the free logos on a small category.

Construction 4.9. Let C be a small category. We define the *free logos on* C as

$$\mathrm{An}[C] \quad := \quad \mathrm{PSh}(C^{\mathrm{lex}}),$$

where $C^{\mathrm{lex}} \subseteq \mathrm{PSh}(C^{\mathrm{op}})^{\mathrm{op}}$ is defined as the smallest subcategory containing C which is closed under finite limits. The category $\mathrm{An}[C]$ comes equipped with a canonical inclusion

$$C \hookrightarrow C^{\mathrm{lex}} \hookrightarrow \mathrm{PSh}(C^{\mathrm{lex}}) = \mathrm{An}[C].$$

Proposition 4.10. *For any small category C, the inclusion $C \hookrightarrow \mathrm{An}[C]$ exhibits $\mathrm{An}[C]$ as the free logos generated by C: for any other logos T, restriction along the inclusion induces an equivalence of categories*

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], T) \quad \xrightarrow{\sim} \quad \mathrm{Fun}(C, T).$$

Proof. Recall from [Lur09, Proposition 5.3.6.2] that for a small category C, the subcategory $C^{\mathrm{rex}} \subseteq \mathrm{PSh}(C)$ generated by the representables under finite colimits is the free finite cocompletion of C. It follows that $C^{\mathrm{lex}} = ((C^{\mathrm{op}})^{\mathrm{rex}})^{\mathrm{op}}$ is the free finite limit completion: for any other category D with finite limits, restriction along $C \hookrightarrow C^{\mathrm{lex}}$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{lex}}, D) \quad \xrightarrow{\sim} \quad \mathrm{Fun}(C, D).$$

By the universal property of the presheaf category, we also see that restriction along $C^{\mathrm{lex}} \hookrightarrow \mathrm{PSh}(C^{\mathrm{lex}})$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C^{\mathrm{lex}}), T) \quad \xrightarrow{\sim} \quad \mathrm{Fun}(C^{\mathrm{lex}}, T).$$

By Proposition 2.43, this functor restricts to the desired equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], T) \quad = \quad \mathrm{Fun}^{\mathrm{colim,lex}}(\mathrm{PSh}(C^{\mathrm{lex}}), T) \quad \xrightarrow{\sim} \quad \mathrm{Fun}^{\mathrm{lex}}(C^{\mathrm{lex}}, T). \qquad \square$$

Remark 4.11. We may think of this result as a categorification of the isomorphism $\mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[S], R) \cong \mathrm{Hom}_{\mathrm{Set}}(S, R)$ for every set S and every commutative ring R.

Example 4.12. For $C = \emptyset$, we get $\emptyset^{\mathrm{lex}} = *$ and so $\mathrm{An}[\emptyset] = \mathrm{PSh}(*) = \mathrm{An}$. In particular, An is the *initial logos*:

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}, T) \quad \simeq \quad \mathrm{Fun}(\emptyset, T) \quad \simeq \quad *.$$

By dualizing, it follows that $\mathrm{An} \in \mathrm{Topos}$ is the *terminal topos*.

Example 4.13. When $C = * = \{X\}$ is the terminal category, we get $C^{\mathrm{lex}} = (\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}$, the opposite of the category of finite animae. In particular, the *free logos on one generator* is the logos

$$\mathrm{An}[X] \quad := \quad \mathrm{An}[\{X\}] \quad = \quad \mathrm{PSh}((\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}}) \quad = \quad \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An}).$$

4.2.2 Congruences and quotients of logoi

In classical algebra, we can form the quotient of a ring R by an ideal I. As we discuss in this section, we may analogously form the quotient of a logos T by a class of morphisms K. This produces a new logos T/K in which the morphisms of K have been inverted. We begin by defining the precise conditions on K that ensure such a quotient behaves well.

Definition 4.14. Let T be a logos. A collection of morphisms K in T is called a *congruence of small generation* if it satisfies the following conditions:

(1) K is strongly saturated of small generation (see Section A.3).

(2) K is closed under base change.

Recall from Proposition A.15 that condition (1) ensures that the localization $T \to T[K^{-1}]$ exists and is an accessible Bousfield localization (i.e., the target is presentable and the localization functor admits a fully faithful right adjoint). Condition (2) ensures that the localization is compatible with the logos structure, i.e., that it is left exact:

Lemma 4.15 (cf. [Lur09, Proposition 6.2.1.1])**.** *Let $L\colon C \to D$ be a Bousfield localization, with fully faithful right adjoint $R\colon D \hookrightarrow C$. Assume that C admits finite limits. Then the following conditions are equivalent:*

(1) *The localization functor L is left exact.*

(2) *The class of morphisms inverted by L is closed under finite limits in* $\mathrm{Ar}(C)$.

(3) *The class of morphisms inverted by L is closed under base change.*

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are clear. For (3) $\implies$ (1), let K be the class of morphisms inverted by L, and assume K is closed under base change. Observe that an object of C lies in the essential image of R if and only if it is local with respect to K. Since the terminal object 1 of C is clearly K-local, it follows that $L(1) \xrightarrow{\sim} 1$, and so L preserves the terminal object. It remains to show that L also preserves pullbacks, so consider two morphisms $X \to Y \leftarrow Z$. Since K-local objects are closed under limits in C, it follows that $RLX \times_{RLY} RLZ$ is K-local; in particular, it defines the pullback in D. It now remains to show that the canonical map $X \times_Y Z \to RLX \times_{RLY} RLZ$ lies in K. We may factor this map as a composite

$$X \times_Y Z \to X \times_{RLY} Z \to RLX \times_{RLY} Z \to RLX \times_{RLY} RLZ.$$

The second and third maps are base changes of morphisms in K, hence lie in K again by assumption. The first map is a base change of the diagonal of $Y \to RLY$, which is therefore a section of the projection $\mathrm{pr}_1\colon Y \times_{RLY} Y \to Y$. Since K is closed under 2-out-of-3, it suffices to show that pr_1 is in K, which is true as it is a base change of $Y \to RLY$. □

We now combine these results with the properties of saturated classes discussed in the appendix to obtain the following characterization. Recall from Definition A.13 that the *kernel* of a cocontinuous functor φ between presentable categories is defined as the class of morphisms inverted by φ.

Corollary 4.16. *Let T be a logos and K a class of morphisms in T. The following are equivalent:*

(1) *K is a congruence of small generation;*

(2) *K is strongly saturated of small generation and is closed under finite limits in* $\mathrm{Ar}(T)$*;*

(3) *The localization functor $L\colon T \to T[K^{-1}]$ is left exact and admits a fully faithful right adjoint (i.e., it is a Bousfield localization);*

(4) *K is the kernel of some morphism of logoi $\varphi\colon T \to S$.*

Proof. The equivalence (1) $\iff$ (3) follows immediately from Definition 4.14 and Lemma 4.15, combined with Proposition A.15.

The implication (3) $\implies$ (4) follows since K is the kernel of $L\colon T \to T[K^{-1}]$, see Proposition A.15. For (4) $\implies$ (2), if K is the kernel of a morphism of logoi φ, then K is closed under finite limits because φ is left exact. Furthermore, since φ is a morphism of logoi, it preserves colimits, and thus its kernel is strongly saturated of small generation by [Lur09, Proposition 5.5.4.16] (see Proposition A.14).

Finally, for (2) $\implies$ (1), it suffices to note that a class closed under finite limits is in particular closed under base change. □

We can now define the quotient of a logos.

Definition 4.17. Let K be a congruence of small generation in a logos T. We define the *quotient of T by K* as the localization

$$T/K \quad := \quad T[K^{-1}].$$

By Corollary 4.16, T/K is a logos and the localization map $T \to T/K$ is a morphism of logoi.

This construction satisfies the expected universal property:

Proposition 4.18. *For every logos S, restriction along $T \to T/K$ induces a fully faithful functor*

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T/K, S) \quad \hookrightarrow \quad \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(T, S).$$

Its essential image consists of those morphisms of logoi $\varphi\colon T \to S$ that invert K (i.e., $K \subseteq \ker(\varphi)$).

Proof. By definition of localization, restriction along $T \to T/K$ induces a fully faithful functor

$$\mathrm{Fun}(T/K, S) \hookrightarrow \mathrm{Fun}(T, S),$$

with essential image those functors $T \to S$ that invert K. It remains to show that a functor $T/K \to S$ preserves colimits and finite limits if and only if the composite $T \to T/K \to S$ does.

The 'only if' direction is clear. For the 'if' direction, note that every diagram in T/K may be regarded as a diagram in T via the inclusion. Since $T \to T/K$ preserves colimits and finite limits, colimits and finite limits in T/K may be computed by forming them first in T and then localizing. Thus if $T \to S$ is a logos morphism, then so is $T/K \to S$. □

Definition 4.19. A morphism of logoi $\varphi\colon T \to S$ is called a *quotient map* if it is a Bousfield localization, i.e., if its right adjoint is fully faithful.

Note that any quotient map $\varphi\colon T \to S$ induces an isomorphism of logoi $T/\ker(\varphi) \xrightarrow{\sim} S$.

Lemma 4.20. *Any quotient map of logoi is an epimorphism in the category of logoi.*

Proof. Let $\varphi^*\colon T \to S$ be a quotient map. Then its right adjoint φ_* is fully faithful, hence the counit map $\varphi^*\varphi_* \to \mathrm{id}$ is an isomorphism. Note that for every third logos T', precomposition by φ^* and φ_* induces another adjunction

$$\mathrm{Fun}(S, T') \rightleftarrows \mathrm{Fun}(T, T').$$

whose unit and counit are induced by the unit and counit of the adjunction $\varphi^* \dashv \varphi_*$. It follows that the restriction functor $\mathrm{Fun}(S, T') \to \mathrm{Fun}(T, T')$ is fully faithful as well, and hence so is

$$(\varphi^*)^*\colon \mathrm{Hom}_{\mathrm{Logos}}(S, T') = \mathrm{Fun}^{\mathrm{lex,colim}}(S, T')^{\simeq} \hookrightarrow \mathrm{Fun}^{\mathrm{lex,colim}}(T, T')^{\simeq} = \mathrm{Hom}_{\mathrm{Logos}}(T, T').$$

This shows that φ^* is an epimorphism. □

Remark 4.21. The previous lemma implies that any quotient map of logoi is a *monomorphism* when regarded as a morphism in Topos. This is analogous to the fact that an epimorphism $R \to S$ of commutative rings induces a monomorphism $\mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ of affine schemes. Recall that there are two common constructions for commutative rings that give rise to epimorphisms:

(1) Given a commutative ring R and an ideal I, we may form the quotient ring $R \to R/I$ by killing off all elements in I;

(2) Given a commutative ring R and a set S of elements of R, we may form the localization $R \to R[S^{-1}]$ by formally inverting all elements in S.

The quotient map $R \to R/I$ corresponds to a *closed immersion* of affine schemes, while the localization map $R \to R[S^{-1}]$ corresponds to an *open immersion* of affine schemes. We will discuss the analogous notions of open/closed immersions of topoi in Section 5.4 below.

We conclude with two fundamental examples of congruences.

Example 4.22. Let Σ be a small set of morphisms in a logos T. The *congruence generated by* Σ, denoted Σ^c, is defined as the intersection of all strongly saturated classes containing Σ that are closed under base change. It is a non-trivial fact that Σ^c is of small generation (and thus a congruence in the sense of Definition 4.14). This is proved by Lurie [Lur09, Proposition 6.2.1.2]. We will give an independent proof below, see Remark 5.38.

Example 4.23. Let $\varphi\colon S \to T$ be a morphism of logoi. Given a congruence of small generation K in T, we define its *preimage* as

$$\varphi^{-1}(K) \quad := \quad \{g \in \mathrm{Ar}(S) \mid \varphi(g) \in K\}.$$

This is again a congruence of small generation. Indeed, consider the composite morphism of logoi

$$S \xrightarrow{\varphi} T \xrightarrow{L} T/K.$$

As the kernel of L is K, the kernel of this composite is precisely $\varphi^{-1}(K)$. By Corollary 4.16, the kernel of any morphism of logoi is a congruence of small generation, which proves the claim.

4.2.3 Presentations of logoi

We now combine free logoi with quotients of logoi to arrive at the following definition:

Definition 4.24. A *logos presentation* is a pair (C, Σ) consisting of a small category C together with a small class of morphisms Σ in $\mathrm{An}[C]$. We define the *logos presented by* (C, Σ) as the quotient

$$\langle C \mid \Sigma \rangle \quad := \quad \mathrm{An}[C] \,/\, \Sigma^c$$

of the free logos on C by congruence Σ^c from Example 4.22. In particular, $\langle C \mid \Sigma \rangle$ is a logos and the quotient map $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ is a logos morphism.

We wish to think of C as the 'generators' of $\langle C \mid \Sigma \rangle$, and of Σ as the 'relations'.

More explicitly, $\langle C \mid \Sigma \rangle$ may be identified with the full subcategory of $\mathrm{PSh}(C^{\mathrm{lex}})$ spanned by the Σ^c-local presheaves.

Lemma 4.25. *Let* (C, Σ) *be a logos presentation. Then the quotient functor* $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ *is universal among logos morphisms* $\mathrm{An}[C] \to T$ *that invert* Σ.

Proof. By Proposition 4.18, $\mathrm{An}[C] \to \langle C \mid \Sigma \rangle$ is universal among logos morphisms $\varphi\colon \mathrm{An}[C] \to T$ satisfying $\Sigma^c \subseteq \ker(\varphi)$. But since $\ker(\varphi)$ is a congruence, this is equivalent to the condition that $\Sigma \subseteq \ker(\varphi)$, i.e. that φ inverts Σ. □

We now show that every logos admits a presentation.

Proposition 4.26. *Let* T *be a logos. Then there exists a logos presentation* (C, Σ) *and an equivalence of logoi*

$$\langle C \mid \Sigma \rangle \xrightarrow{\sim} T.$$

Proof. We saw in (the proof of) Theorem 2.42 that we may write T as an accessible left exact localization of some presheaf category $\mathrm{PSh}(C)$, where C is a small category with finite limits. By accessibility, the class of morphisms inverted by the localization functor $\mathrm{PSh}(C) \to T$ is the saturation of some small class of morphisms Σ' in $\mathrm{PSh}(C)$.

Let $F\colon C^{\mathrm{lex}} \to C$ be the unique left exact functor extending the identity on C. Using the universal property of C^{lex} we get an isomorphism $\mathrm{Hom}_C(X, FY) \cong \mathrm{Hom}_{C^{\mathrm{lex}}}(X, Y)$ for all $X \in C$ and $Y \in C^{\mathrm{lex}}$, showing that F is right adjoint to the inclusion $C \hookrightarrow C^{\mathrm{lex}}$. In particular, the left Kan extension functor $F_!\colon \mathrm{PSh}(C^{\mathrm{lex}}) \to \mathrm{PSh}(C)$ admits a fully faithful left adjoint, given by left Kan extension along the inclusion. It follows that also the restriction functor F^* is fully faithful, and in particular $F_!$ is a left exact localization. We now consider the composite

$$\mathrm{An}[C] = \mathrm{PSh}(C^{\mathrm{lex}}) \xrightarrow{F_!} \mathrm{PSh}(C) \to T,$$

which is again a left exact localization. Using the inclusion $\mathrm{PSh}(C) \hookrightarrow \mathrm{PSh}(C^{\mathrm{lex}}) = \mathrm{An}[C]$, we may regard Σ' as a collection of morphisms in $\mathrm{An}[C]$. Define the class Σ as the union of Σ' with the counit morphisms $FX \to X$ for all $X \in C$. One may then check that T is precisely the localization of $\mathrm{An}[C]$ at the closure Σ^c, giving the desired presentation of T. □

Example 4.27. The following are some examples of presentations of logoi:

(1) As discussed, the initial logos (no generators and no relations) is An:

$$\langle \emptyset \mid \emptyset \rangle \quad = \quad \mathrm{An}.$$

(2) The free logos on a single generator X is $\mathrm{An}[X]$:

$$\langle \{X\} \mid \emptyset \rangle \quad = \quad \mathrm{An}[X] \quad = \quad \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An}).$$

(3) More generally, the logos presented by $(C, \emptyset)$ is simply the free logos generated by C:

$$\langle C \mid \emptyset \rangle \quad = \quad \mathrm{An}[C] \quad = \quad \mathrm{PSh}(C^{\mathrm{lex}}).$$

(4) The free logos generated by some initial object X is simply An:

$$\langle \{X\} \mid \emptyset \to X \rangle \quad \simeq \quad \mathrm{An}.$$

Indeed, logos morphisms $F\colon \langle \{X\} \mid \emptyset \to X \rangle \to T$ correspond to logos morphisms $\mathrm{An}[X] = \langle \{X\} \mid \emptyset \rangle \to T$ sending the generator X to the initial object of T, but by definition of $\mathrm{An}[X]$ there is a unique such morphism.

(5) The free logos in which the initial object is terminal is the one-object logos:

$$\langle \{X\} \mid \emptyset \to * \rangle \quad \simeq \quad *.$$

Indeed, any logos in which the initial and terminal objects agree is trivial: for all Y we have $\emptyset \simeq \emptyset \times Y \simeq * \times Y \simeq Y$.

(6) The free logos generated by an n-truncated object is

$$\langle \{X\} \mid X \to X^{S^{n+1}} \rangle \quad \simeq \quad \mathrm{Fun}((\mathrm{An}_{\leq n})^{\mathrm{fin}}, \mathrm{An}),$$

where $(\mathrm{An}_{\leq n})^{\mathrm{fin}} \subseteq \mathrm{An}_{\leq n}$ is the subcategory generated under finite colimits by the point. Note that the left-hand side is the logos classifying the assignment $T \mapsto T_{\leq n}$ of n-truncated objects. To see that the right-hand side has the same universal property, observe that a left exact functor $(\mathrm{An}^{\mathrm{fin}})^{\mathrm{op}} \to T$ factors through $T_{\leq n}$ if and only if it factors through the truncation functor $\tau_{\leq n}\colon \mathrm{An}^{\mathrm{fin}} \to (\mathrm{An}_{\leq n})^{\mathrm{fin}}$.

(7) The free logos generated by an n-connected object X is

$$\langle \{X\} \mid \tau_n(X) \to * \rangle.$$

It arises as a localization of $\mathrm{An}[X] = \langle * \mid \emptyset \rangle$.

(8) The *pointed object classifier* is the logos

$$\langle [1] \mid \mathrm{source} = * \rangle \quad \simeq \quad \mathrm{PSh}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}).$$

Note that a logos morphism $\langle [1] \mid \mathrm{source} = * \rangle \to T$ is just a morphism in T whose source is terminal, i.e. a pointed object. But the right-hand side also classifies pointed objects: we have

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}), T) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, T) \simeq \mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, T_*) \xrightarrow{\sim} T_*.$$

(9) Given a Lawvere theory L, there is the *L-algebra classifier* $\langle L \mid \text{finite products} \rangle$: for every logos T we have

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\langle L \mid \text{finite products} \rangle, T) \quad \simeq \quad \mathrm{Alg}_L(T) \quad := \quad \mathrm{Fun}^{\times}(L, T).$$

This logos may be explicitly described as

$$\langle L \mid \text{finite products} \rangle \quad \simeq \quad \mathrm{PSh}((\mathrm{PSh}_\Sigma(L)^{\mathrm{fin}})^{\mathrm{op}}).$$

4.3 Limits and colimits of topoi

We now compute limits and colimits of topoi.

Theorem 4.28. *The categories* Topos *and* Logos *admit small limits and colimits. Furthermore, the following properties are satisfied:*

(1) *The inclusion* $\mathrm{Logos} \hookrightarrow \mathrm{Cat}$, $\varphi \mapsto \varphi^*$, *preserves limits.*

(2) *The inclusion* $\mathrm{Topos} \hookrightarrow \mathrm{Cat}$, $\varphi \mapsto \varphi_*$, *preserves filtered limits.*

(3) *The inclusion* $\mathrm{Logos} \hookrightarrow \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{colim}})$ *preserves finite coproducts. In particular:*

- An *is the terminal topos.*
- *The product* $T \times S$ *of two topoi is computed as the Lurie tensor product* $T \otimes S$ *of cocomplete categories.*

Remark 4.29. This theorem does not give an explicit general description of all limits in Topos (e.g. pullbacks or cosimplicial limits); these are in general more subtle to describe.

Proof. Colimits: We start by computing colimits in Topos, or equivalently limits in Logos. Given a diagram $I \to \mathrm{Logos}, i \mapsto T_i$ of logoi, define $T := \lim_i T_i$, which is again a presentable category. Given an indexing diagram K, a functor $K^{\triangleright} \to T$ is a colimit diagram if and only if it becomes a colimit diagram in each T_i. The analogous statement holds for finite limit diagrams. In particular, colimits and finite limits in T are computed pointwise in the categories T_i. Note that we have

$$\mathrm{Ar}(T) \xrightarrow{\sim} \lim_i \mathrm{Ar}(T_i) \qquad \text{and} \qquad \mathrm{Ar}^{\mathrm{pb}}(T) \xrightarrow{\sim} \lim_i \mathrm{Ar}^{\mathrm{pb}}(T_i).$$

It then follows immediately from the characterization of logoi given in Proposition 2.16 that T is again a logos. Furthermore, we see that a functor $S \to T$ is a logos morphism if and only if each composite $S \to T_i$ is a logos morphism. It follows that the limit in Cat is in fact a limit in Logos, as desired.

Limits Next, we construct limits in Topos, or equivalently colimits in Logos. Given a diagram $I \to \mathrm{Logos}, i \mapsto T_i$, choose a regular cardinal κ such that each T_i is κ-accessible and each transition functor is κ-accessible. Let $C_i := (T_i)^{\kappa}$ be the full subcategory of κ-compact objects, so that $T_i \simeq \mathrm{Ind}_{\kappa}(C_i)$. Let Σ_i be the collection of morphisms inverted by the composite functor

$$\mathrm{PSh}(C_i^{\mathrm{lex}}) \to \mathrm{PSh}(C_i) \to \mathrm{Ind}_{\kappa}(C_i) = T_i.$$

Then, as shown in Proposition 4.26, this induces an equivalence $\langle C_i \mid \Sigma_i \rangle \xrightarrow{\sim} T_i$ for all $i \in I$. In other words: we have found a functorial presentation for the logoi T_i.

Consider now the small category $C := \mathrm{colim}_i C_i$, and let Σ be the union of the images of the morphisms Σ_i inside $\mathrm{An}[C] = \mathrm{PSh}(C^{\mathrm{lex}})$. We now consider the logos

$$T := \langle C \mid \Sigma \rangle.$$

The functors $C_i \to C$ induce functorial maps $T_i = \langle C_i \mid \Sigma_i \rangle \to \langle C \mid \Sigma \rangle = T$, producing a cocone $I^{\triangleright} \to \mathrm{Logos}$. First note that for any topos S we have

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C], S) \quad \simeq \quad \mathrm{Fun}(C, S) \quad \simeq \quad \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}(C_i, S) \quad \simeq \quad \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{An}[C_i], S),$$

showing that the free logos on C is the colimit in Logos of the free logoi on the C_i. In light of Lemma 4.25, it thus remains to show that a logos morphism $\mathrm{An}[C] \to S$ inverts the morphisms in Σ if and only if each restriction $\mathrm{An}[C_i] \to \mathrm{An}[C] \to S$ inverts the morphisms in Σ_i. But this holds by construction of Σ.

Filtered limits: We now sketch how to compute filtered limits in Topos. Given a functor $I \to \mathrm{Topos}$, consider the associated cartesian fibration $p\colon E \to I^{\mathrm{op}}$. One can show that the category of sections $\Gamma(p)$ admits colimits and finite limits, which are computed fiberwise: see Chapter B for details. In particular, $\Gamma(p)$ is a topos, and each of the evaluation functors $\mathrm{ev}_i\colon \Gamma(p) \to E_i$ is a morphism of topoi, see [Lur09, Lemma 6.3.3.2]. Assuming that I is filtered, we now claim that the subcategory $\Gamma_{\mathrm{cart}}(p)$ of cartesian sections is a left exact localization of $\Gamma(p)$. To see this, consider an arbitrary section $s \in \Gamma(p)$. We need to turn s into a cartesian section. Fixing $i \in I$, we get for every morphism $j \to i$ a functor $(f_{ji})_*\colon E_j \to E_i$, and hence an object $(f_{ji})_*(s(j)) \in E_i$. These objects are functorial in $j \in I_{/i}$: given morphisms $k \to j \to i$ in I, the map $s(j) \to s(k)$ in E corresponds to a map $s(j) \to (f_{kj})_* s(k)$, which thus induces a map $(f_{ji})_*(s(j)) \to (f_{ji})_*(f_{kj})_*(s(k)) \simeq (f_{ki})_*(s(k))$. With more care, this defines a diagram $I_{/i}^{\mathrm{op}} \to E_i$, and we define

$$s^{(1)}(i) := \mathrm{colim}_{j\to i}(f_{ji})_*(s(j)).$$

The canonical maps $s(i) \to (f_{ji})_*(s(j))$ produce a map $s \to s^{(1)}$, and one can show that for any cartesian section $t \in \Gamma_{\mathrm{cart}}(p)$, restriction along this map induces an equivalence $\mathrm{Hom}(s^{(1)}, t) \xrightarrow{\sim} \mathrm{Hom}(s, t)$. For an ordinal α, we define $s^{(\alpha)}$ the α-iteration of this construction. This still comes equipped with a map $s \to s^{(\alpha)}$ and we still have $\mathrm{Hom}(s^{(\alpha)}, t) \xrightarrow{\sim} \mathrm{Hom}(s, t)$ for all $t \in \Gamma_{\mathrm{cart}}(p)$. Now let κ be a cardinal such that each $(f_{ji})_*$ preserves κ-filtered colimits for all $j \to i$. Then one can show that $s^{(\kappa)}$ is a cartesian section, and it follows that the assignment $s \mapsto s^{(\kappa)}$ determines a left adjoint to the inclusion $\Gamma_{\mathrm{cart}}(p) \hookrightarrow \Gamma(p)$. It follows in particular that the category $\Gamma_{\mathrm{cart}}(p) \simeq \lim_i T_i$ is a topos.

To finish the proof, it remains to show that a functor $f_*\colon S \to \Gamma_{\mathrm{cart}}(p)$ is a morphism of topoi if and only if each of the maps $S \to \Gamma_{\mathrm{cart}}(p) \xrightarrow{\mathrm{ev}_i} T_i$ is a morphism of topoi. This is [Lur09, Proposition 6.3.3.7].

Terminal object: We argued in Example 4.12 that An is the terminal topos.

Products: To compute the product of T and S in Topos, write $T = \mathrm{PSh}(C)[\Sigma^{-1}]$ and $S = \mathrm{PSh}(D)[\Gamma^{-1}]$ with $C, D \in \mathrm{Cat}^{\mathrm{lex}}$. We claim that a coproduct of C and D in $\mathrm{Cat}^{\mathrm{lex}}$ is given by the product category $C \times D$. We have left exact inclusions $C \simeq C \times \{*\} \hookrightarrow C \times D$ and $D \simeq \{*\} \times D \hookrightarrow C \times D$, and restriction along them produces a functor

$$\mathrm{Fun}^{\mathrm{lex}}(C \times D, E) \quad \to \quad \mathrm{Fun}^{\mathrm{lex}}(C, E) \times \mathrm{Fun}^{\mathrm{lex}}(D, E).$$

We claim that this functor is an equivalence, with inverse sending a pair (f, g) to the functor $h\colon C \times D \to E$ defined by

$$h(x, y) \simeq f(x) \times g(y).$$

Indeed it is easy to see that $h(x, *) = f(x)$ and $h(*, y) = g(y)$, and conversely we always have $h(x, y) = h(x, *) \times h(*, y)$ for any h since we have the relation $(x, y) = (x, *) \times (*, y)$ in $C \times D$. Using Proposition 2.43, we conclude that for every topos T' we have an equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C \times D), T') \xrightarrow{\sim} \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(C), T') \times \mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(D), T'),$$

and hence the logos $\mathrm{PSh}(C \times D) \simeq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$ is a coproduct of $\mathrm{PSh}(C)$ and $\mathrm{PSh}(D)$ in Logos, i.e. it is a product of topoi.

Now consider the following commutative diagram of tensor products in Pr^{L}:

$$\begin{tikzcd} \mathrm{PSh}(C) \otimes \mathrm{PSh}(D) \arrow[r] \arrow[d] & \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)[\Lambda^{-1}] \arrow[d] \\ \mathrm{PSh}(C)[\Sigma^{-1}] \otimes \mathrm{PSh}(D) \arrow[r] & \mathrm{PSh}(C)[\Sigma^{-1}] \otimes \mathrm{PSh}(D)[\Lambda^{-1}] = T \otimes S \end{tikzcd}$$

Using the universal properties of localizations and tensor products, we observe that this is a pushout square in Pr^{L}. Passing to right adjoints then produces a pullback square in Pr^{R}, and we see that we may identify $T \otimes S$ with the intersection

$$(T \otimes \mathrm{PSh}(D)) \cap (\mathrm{PSh}(C) \otimes S)$$

as a full subcategory of $\mathrm{PSh}(C) \otimes \mathrm{PSh}(D) = \mathrm{PSh}(C \times D)$.

To conclude, we use that for every small category E and presentable category U there is a natural equivalence

$$\mathrm{PSh}(E) \otimes U \simeq \mathrm{Fun}(E^{\mathrm{op}}, U).$$

Hence if $L \colon U' \to U$ is a left exact localization with fully faithful right adjoint R, then the induced functor

$$\mathrm{PSh}(E) \otimes U' \to \mathrm{PSh}(E) \otimes U$$

identifies with postcomposition by L on $\mathrm{Fun}(E^{\mathrm{op}}, -)$. Note that this is again a left exact localization: it is left exact since finite limits in functor categories are computed pointwise, and its right adjoint is pointwise postcomposition by R, hence fully faithful. Applying this to $\mathrm{PSh}(D) \to S$ (with $E = C$) shows that $\mathrm{PSh}(C) \otimes S \subseteq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$ is a left exact localization; by symmetry, the same holds for $T \otimes \mathrm{PSh}(D) \subseteq \mathrm{PSh}(C) \otimes \mathrm{PSh}(D)$. Intersections of left exact localizations remain left exact localizations (via transfinite iteration of the localization functors).

Thus the intersection

$$(T \otimes \mathrm{PSh}(D)) \cap (\mathrm{PSh}(C) \otimes S)$$

is the left exact localization of $\mathrm{PSh}(C \times D)$ at the union of the two induced localizing classes (equivalently, at the strongly saturated class they generate). In particular, $T \otimes S$ is a topos. By comparing universal properties, we now conclude that $T \otimes S$ is the coproduct of T and S in Logos, i.e. the product of T and S in Topos. □

Exercise 4.30. The category Topos is tensored and cotensored over Cat: for a topoi S, T and a category C, there exist topoi S^C and $C \otimes T$ such that

$$\mathrm{Geom}(T, S^C) \simeq \mathrm{Fun}(C, \mathrm{Geom}(T, S)) \simeq \mathrm{Geom}(C \otimes T, S).$$

Show that $C \otimes T \simeq \mathrm{Fun}(C, T)$, and describe S^C using a presentation of S.

4.4 Étale morphisms

Recall that a continuous map $f \colon X \to Y$ of topological spaces is called *étale* if it is a local homeomorphism, in the sense that around every point $x \in X$ there exists an open neighborhood U

of X such that the composite $U \hookrightarrow X \xrightarrow{f} Y$ is an open embedding. We now investigate an analogous notion of étale morphisms in the setting of topos theory.

Definition 4.31. A morphism of logoi $\varphi^*\colon T \to S$ is called *étale* if there exists an object $X \in T$ and an isomorphism of logoi $S \xrightarrow{\sim} T_{/X}$ such that the composite

$$T \xrightarrow{\varphi^*} S \xrightarrow{\sim} T_{/X}$$

is the functor $U \mapsto U \times X$. A morphism $\varphi_*\colon S \to T$ of topoi is étale if the corresponding morphism $\varphi^*\colon T \to S$ is étale. We denote by

$$\mathrm{Logos}^{\text{ét}} \subseteq \mathrm{Logos} \qquad \text{and} \qquad \mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$$

the wide subcategories spanned by étale morphisms.

Remark 4.32. As we will see below in Example 6.21, every continuous map of topological spaces $f\colon X \to Y$ gives rise to a morphism of topoi $f_*\colon \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ between their *sheaf categories*. One can show that f is an étale morphism of topological spaces (that is, a local homeomorphism) if and only if f_* is an étale morphism of topoi, justifying the terminology.

Proposition 4.33. *A morphism of logoi $\varphi^*\colon T \to T'$ is étale if and only if:*

(1) *The functor φ^* admits a left adjoint $\varphi_!$;*

(2) *The functor $\varphi_!$ is conservative;*

(3) *The functor $\varphi_!$ satisfies the projection formula:*

$$\varphi_!\big(\varphi^*X \times_{\varphi^*Y} Z\big) \;\simeq\; X \times_Y \varphi_!(Z)$$

for all $X, Y \in T$ and $Z \in T'$.

Proof. If φ^* is étale, choose $X \in T$ and an identification $T' \simeq T_{/X}$ under which φ^* is the functor $U \mapsto U \times X$. Then $\varphi_!$ exists, is conservative, and satisfies the projection formula by standard properties of slice topoi. Conversely, given such a left adjoint $\varphi_!$, consider the induced functor

$$\varphi_!\colon T' \longrightarrow T_{/\varphi_!(*)}, \qquad X \mapsto (\varphi_!(X) \to \varphi_!(*)).$$

This functor admits a right adjoint

$$(Y \to \varphi_!(*)) \longmapsto \varphi^*(Y) \times_{\varphi^*\varphi_!(*)} *.$$

One checks that the unit and counit are isomorphisms (using conservativity and the projection formula), so φ^* identifies T' with the slice topos $T_{/\varphi_!(*)}$. $\square$

Proposition 4.34. *Let T be a topos.*

(1) *There is an equivalence of 2-categories*

$$T \;\simeq\; \mathrm{Topos}^{\text{ét}}_{/T}, \qquad X \longmapsto T_{/X}.$$

In particular, $\mathrm{Topos}^{\text{ét}}_{/T}$ *is a 1-category.*

(2) *For every morphism of topoi $\varphi\colon S \to T$ and every $X \in T$, the square*

$$\begin{tikzcd} S_{/\varphi^* X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T_{/X} \arrow[d] \\ S \arrow[r, "\varphi"] & T \end{tikzcd}$$

is a pullback in Topos. *In particular, the category* $\mathrm{Topos}^{\text{ét}}$ *admits pullbacks, and the inclusion* $\mathrm{Topos}^{\text{ét}} \hookrightarrow \mathrm{Topos}$ *preserves pullbacks.*

Proof. Given a morphism of topoi $\varphi\colon S \to T$ and an object $X \in T$, we claim there is a natural equivalence

$$\mathrm{Geom}_{/T}(S, T_{/X}) \;\simeq\; \mathrm{Hom}_S(*, \varphi^* X).$$

This immediately gives (2), and it gives (1) by taking $S = T_{/Y}$ and noticing that $\mathrm{Hom}_{T_{/Y}}(Y, X \times Y) \simeq \mathrm{Hom}_T(Y, X)$.

To prove the claim, we will start by producing a natural equivalence

$$\mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S) \;\simeq\; \mathrm{Hom}_S(*, \varphi^* X).$$

- Given $F\colon T_{/X} \to S$ in $\mathrm{Fun}^{\mathrm{lex}}_{T/}$, consider the morphism
$$f_F := F(\Delta)\colon F(X) \longrightarrow F(X \times X)$$
induced by the diagonal $\Delta\colon X \to X \times X$, viewed as a morphism in $T_{/X}$. Since (X, id_X) is terminal in $T_{/X}$, we have $F(X) \simeq *$. Moreover, because F is a functor under T, we identify $F(X \times X)$ with $\varphi^* X$. So f_F is a point in $\mathrm{Hom}_S(*, \varphi^* X)$.

- Conversely, given a morphism $f\colon * \to \varphi^* X$, we define $F\colon T_{/X} \to S$ by sending $U \to X$ to the pullback
$$\begin{tikzcd} F_f(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \varphi^*(U) \arrow[d] \\ * \arrow[r, "f"] & \varphi^*(X). \end{tikzcd}$$
It is clear that F_f preserves the terminal object and pullbacks, and so it is left exact. Moreover, when $U = X \times Y$ for some $Y \in T$, we get $\varphi^*(U) \simeq \varphi^*(X) \times \varphi^*(Y)$ and thus $F_f(U) = \varphi^*(Y)$, showing that F_f is a functor under T.

Now, applying F_f to the diagonal map $\Delta\colon X \to X \times X$ reproduces the original map f. Conversely, given any left exact functor $F\colon T_{/X} \to S$ under T and any $U \in T_{/X}$, we may always write U as the pullback in $T_{/X}$ of the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & U \times X \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X, \end{tikzcd}$$

and thus we get a pullback square

$$\begin{tikzcd} F(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F(U \times X) \simeq \varphi^*(U) \arrow[d] \\ F(X) \arrow[r, "F(\Delta)"] & F(X \times X) \simeq \varphi^*(X), \end{tikzcd}$$

showing that F is naturally isomorphic to F_{f_F}. This finishes the construction of the equivalence $\mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S) \simeq \mathrm{Hom}_S(*, \varphi^* X)$. It also follows that every left exact functor $T_{/X} \to S$ under T is of the form F_f for some f, and using descent in S it follows that F preserves colimits. In other words, the forgetful functor

$$\mathrm{Geom}_{/T}(S, T_{/X}) \to \mathrm{Fun}^{\mathrm{lex}}_{T/}(T_{/X}, S)$$

is an equivalence. This finishes the proof. $\square$

By part (2) of the proposition, étale morphisms are closed under base change in Topos. In particular, given a morphism of topoi $\varphi\colon S \to T$, pullback along φ defines a functor $\varphi^*\colon \mathrm{Topos}^{\text{ét}}_{/T} \to \mathrm{Topos}^{\text{ét}}_{/S}$.

Corollary 4.35. *The functor* $\mathrm{Topos}^{\mathrm{op}} \to \widehat{\mathrm{Cat}}, T \mapsto \mathrm{Topos}^{\text{ét}}_{/T}$ *is equivalent to the composite* $\mathrm{Topos}^{\mathrm{op}} \simeq \mathrm{Logos} \hookrightarrow \mathrm{Cat} \hookrightarrow \widehat{\mathrm{Cat}}$.

Proof sketch. The functor $\mathrm{Topos}^{\text{ét}}_{/T} \to T$ sending $(\varphi\colon T' \to T)$ to $\varphi_!(*) \in T$ is compatible with base change: it follows from (2) of Proposition 4.34 that for a pullback square

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "\varphi'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & T' \arrow[d, "\varphi"] \\ S \arrow[r, "\psi"] & T, \end{tikzcd}$$

we have $\psi^* \varphi_!(*) \simeq \varphi'_!(*)$. $\square$

Corollary 4.36. *Consider morphisms of topoi* $T \xrightarrow{\varphi} T' \xrightarrow{\psi} S$. *If* ψ *and* $\psi\varphi$ *are étale morphisms, then so is* φ.

Proof. Without loss of generality, $T = S_{/X}$ and $T' = S_{/Y}$. By the proposition, the map φ is then induced by some morphism $f\colon X \to Y$ in S. We may identify this morphism as an object $f \in S_{/Y} = T'$, and one may deduce that $T \simeq T'_{/f}$. $\square$

4.5 Colimits of topoi along étale morphisms

Recall from Definition 4.31 the wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ spanned by the étale morphisms of topoi. The goal in this section is the following somewhat surprising theorem:

Theorem 4.37. *The category* $\mathrm{Topos}^{\text{ét}}$ *admits colimits, and these colimits are preserved by the inclusion* $\mathrm{Topos}^{\text{ét}} \hookrightarrow \mathrm{Topos}$.

This is an interesting statement: it tells us that $\mathrm{Topos}^{\text{ét}}$ behaves very much like a topos itself. Indeed, recall from Theorem 4.28 that colimits in Topos are computed as limits in $\widehat{\mathrm{Cat}}^{\mathrm{op}}$. Thus the previous statement is equivalent to the claim that the composite $(\mathrm{Topos}^{\text{ét}})^{\mathrm{op}} \hookrightarrow \mathrm{Topos}^{\mathrm{op}} \hookrightarrow \widehat{\mathrm{Cat}}$ preserves limits. By Corollary 4.35, this is equivalent to the condition that the functor

$$(\mathrm{Topos}^{\text{ét}})^{\mathrm{op}} \to \widehat{\mathrm{Cat}}, \quad T \mapsto (\mathrm{Topos}^{\text{ét}})_{/T}$$

preserves limits, which is precisely descent for limits in $\mathrm{Topos}^{\text{ét}}$! Of course $\mathrm{Topos}^{\text{ét}}$ is not actually a topos, since it is too large to be presentable. But we may think of it as a "topos up to size issues".

To understand what is going on, it is useful to make a comparison to the formalism of geometric structures on topoi introduced by [Lur18, Part VI], which we will discuss in more detail in Sections 7.2 and 7.3 below. In various geometric situations, every geometric object X naturally comes with two topoi, a "small topos" and a "large topos". For example:

- For every topological space X, the "small topos" associated to X is the category $\mathrm{Shv}(X)$ of sheaves on X. The "large topos" is obtained by equipping the category Top of topological spaces with the structure of a site, forming the sheaf category Shv(Top) and passing to the slice $\mathrm{Shv}(\mathrm{Top})_{/X}$.
- Similarly, for a scheme X there is the "small topos" $\mathrm{Shv}(X_{\text{ét}})$, given by sheaves on the étale site of X. The "large topos" in this case could for example be defined as the slice topos $\mathrm{Shv}(\mathrm{Aff})_{/X}$ where we equip the category $\mathrm{Aff} := \mathrm{CRing}^{\mathrm{op}}$ of affine schemes with the Zariski topology, and take the "functor of points" perspective in which the scheme X is regarded as an object in $\mathrm{Shv}(\mathrm{Aff})$.

What these examples have in common is that the small topos arises from the large topos by restricting to a suitable notion of "étale maps" over X. For topological spaces, an étale map is defined as a local homeomorphism, while for schemes it is the usual notion of étale morphism. These maps give rise to wide subcategories $\mathrm{Shv}(\mathrm{Top})^{\text{ét}}$ and $\mathrm{Shv}(\mathrm{Aff})^{\text{ét}}$ consisting of the 'representable étale maps', and there are equivalences

$$\mathrm{Shv}(\mathrm{Top})^{\text{ét}}_{/X} \simeq \mathrm{Shv}(X)$$

and

$$\mathrm{Shv}(\mathrm{Aff})^{\text{ét}}_{/X} \simeq \mathrm{Shv}(X_{\text{ét}}).$$

The situation for topoi looks surprisingly similar. If we think of the topos T as a "geometric object", then we may think of the category Topos as a "category of geometric objects". While Topos is not itself a topos, we may embed it into a bigger category Shv(Topos) which is a topos ("up to size issues"). The category Shv(Topos) inherits a notion of 'étale morphism' from Topos, giving rise to a wide subcategory $\mathrm{Shv}(\mathrm{Topos})^{\text{ét}} \subseteq \mathrm{Shv}(\mathrm{Topos})$. The slice over a topos T is given by

$$\mathrm{Shv}(\mathrm{Topos})^{\text{ét}}_{/T} \simeq \mathrm{Topos}^{\text{ét}}_{/T} \simeq T,$$

and in particular each such slice is a topos. We think of $\mathrm{Shv}(\mathrm{Topos})_{/T}$ as the "big topos" associated to T, while T itself is the "small topos" associated to T.

Lurie's proof of Theorem 4.37 is quite long and technical. We will instead give a conceptual proof due to Uemura [Uem25]; we thank David Wärn for brining Uemara's article to our attention.

4.5.1 Univalent families

In this section, we discuss a construction of Uemura [Uem25] that will be used in the proof of Theorem 4.37. The key point is that every family in a topos admits a universal approximation by a *univalent* family. This provides a convenient replacement for the object-classifier technology that one might otherwise use. All material in this section is from [Uem25].

Definition 4.38. Let T be a topos. We write

$$\mathrm{Fam}(T) := \mathrm{Ar}^{\mathrm{pb}}(T)$$

for the wide subcategory of $\mathrm{Ar}(T)$ whose morphisms are pullback squares. We call an object $u\colon E \to B$ of $\mathrm{Fam}(T)$ a *family in T*, and we call its codomain B the *base* of the family.

A family $u \in \mathrm{Fam}(T)$ is *univalent* if it is a (-1)-truncated object of the category $\mathrm{Fam}(T)$. We write

$$\mathrm{Fam}^{\mathrm{univ}}(T) \subseteq \mathrm{Fam}(T)$$

for the full subcategory spanned by the univalent families.

Informally, a morphism $u\colon E \to B$ is univalent if it is a *property* for an arbitrary morphism v in T to be a pullback of u, i.e. if the anima $\mathrm{Hom}_{\mathrm{Fam}(T)}(v, u)$ is (-1)-truncated. Once we know that $\mathrm{Fam}(T)$ admits binary products, this is equivalently the condition that the diagonal

$$u \longrightarrow u \times u$$

is an isomorphism in $\mathrm{Fam}(T)$.

Lemma 4.39. *Let $u\colon E \to B$ be a family in a topos T. The functor*

$$\mathrm{Fam}(T)_{/u} \longrightarrow T_{/B}$$

sending a pullback square $v \to u$ to the induced morphism from the base of v to B is an equivalence.

Proof. The inverse sends a morphism $f\colon X \to B$ to the pullback family f^*u. Since morphisms in $\mathrm{Fam}(T)$ are by definition pullback squares, these two constructions are inverse to each other. □

Lemma 4.40. *The wide subcategory* $\mathrm{Fam}(T) \subseteq \mathrm{Ar}(T)$ *is closed under small colimits and pullbacks.*

Proof. The assertion about colimits is precisely the topos direction of Proposition 2.16, after identifying $\mathrm{Fam}(T)$ with $\mathrm{Ar}^{\mathrm{pb}}(T)$.

For pullbacks, note that $\mathrm{Fam}(T)$ admits pullbacks if and only if for each $u \in \mathrm{Fam}(T)$ the slice category $\mathrm{Fam}(T)_{/u}$ admits binary products, which is immediate from Lemma 4.39. □

Proposition 4.41. *The category* $\mathrm{Fam}(T)$ *has binary products, and these products preserve small colimits in each variable.*

Proof. Let $u_1\colon E_1 \to B_1$ and $u_2\colon E_2 \to B_2$ be families. For an object $X \to B_1 \times B_2$, write $f_i\colon X \to B_i$ for the two projections. Consider the presheaf on $T_{/B_1\times B_2}$ which sends X to

$$\mathrm{Iso}_{T_{/X}}\big(f_1^*u_1, f_2^*u_2\big)\,,$$

the anima of isomorphisms in $T_{/X}$ between the pullbacks of u_1 and u_2 to X. This presheaf preserves limits: if $X \simeq \mathrm{colim}_i X_i$ in $T_{/B_1\times B_2}$, then descent identifies $T_{/X}$ with $\lim_i T_{/X_i}$, and under this identification the two pulled-back families are the compatible systems of their pullbacks to the X_i. Thus isomorphisms between them are computed as the limit of the isomorphism animae over the X_i. By the adjoint functor theorem, the presheaf is therefore representable by some object $P \to B_1 \times B_2$.

Pulling back either u_1 or u_2 along the universal isomorphism gives a family over P, and this family is the product $u_1 \times u_2$ in $\mathrm{Fam}(T)$.

It remains to prove preservation of colimits in each variable. Fix a family $u_1 \colon E_1 \to B_1$. To show that $u_1 \times - \colon \mathrm{Fam}(T) \to \mathrm{Fam}(T)$ preserves colimits, we may equivalently show that for any family $v \colon E' \to B'$ the functor

$$\mathrm{Fam}(T)_{/v} \longrightarrow \mathrm{Fam}(T)_{/u_1 \times v}$$

induced by $u_1 \times -$ preserves colimits, since slices detect colimits. Let P' denote the base of the product $u_1 \times v$. We claim that under the equivalences $\mathrm{Fam}(T)_{/v} \simeq T_{/B'}$ and $\mathrm{Fam}(T)_{/u_1 \times v} \simeq T_{/P'}$ from Lemma 4.39, the functor identifies with pullback along the projection map $P' \to B'$. Since this functor preserves colimits by descent in T, this will finish the proof. To show the claim, we must show that for every $f \colon u_2 \to v$ in $\mathrm{Fam}(T)$ with base $f_B \colon B_2 \to B'$, the square of bases

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \\ B_1 \times B_2 \arrow[r, "\mathrm{id} \times f_B"] & B_1 \times B' \end{tikzcd}$$

is a pullback square in T. We compare the presheaves represented by the two objects over $B_1 \times B_2$. Let $X \to B_1 \times B_2$ be an object, and write $g_1 \colon X \to B_1$ and $g_2 \colon X \to B_2$ for the two projections. The object P represents the presheaf

$$X \longmapsto \mathrm{Iso}_{T_{/X}}\left(g_1^* u_1, g_2^* u_2\right).$$

On the other hand, the pullback $(B_1 \times B_2) \times_{B_1 \times B'} P'$ represents the presheaf

$$X \longmapsto \mathrm{Iso}_{T_{/X}}\left(g_1^* u_1, (f_B g_2)^* v\right),$$

because P' represents the analogous equivalence presheaf for u_1 and v over $B_1 \times B'$. Since $f \colon u_2 \to v$ is a morphism in $\mathrm{Fam}(T)$, the family u_2 is the pullback $f_B^* v$, and hence $g_2^* u_2 \simeq (f_B g_2)^* v$. These two presheaves are therefore naturally equivalent, proving that the square is a pullback. ◻

Corollary 4.42. *Let $u_\bullet \colon [\omega] \to \mathrm{Fam}(T)$ be a sequence of univalent families. Then $\mathrm{colim}_n\, u_n$ is univalent.*

Proof. By Proposition 4.41, we have

$$(\mathrm{colim}_n\, u_n) \times (\mathrm{colim}_m\, u_m) \simeq \mathrm{colim}_{(n,m) \in [\omega] \times [\omega]} (u_n \times u_m). \tag{4.1}$$

The diagonal functor $[\omega] \to [\omega] \times [\omega]$ is final, so this is further equivalent to $\mathrm{colim}_n (u_n \times u_n)$. The diagonal of $\mathrm{colim}_n\, u_n$ is therefore the colimit of the diagonals $u_n \to u_n \times u_n$, which are isomorphisms by univalence. ◻

Proposition 4.43. *The inclusion*

$$\mathrm{Fam}^{\mathrm{univ}}(T) \hookrightarrow \mathrm{Fam}(T)$$

admits a left adjoint. We call it the univalent completion functor.

Proof. Let u be a family. We construct a sequence $u_0 \to u_1 \to u_2 \to \cdots$ in $\mathrm{Fam}(T)$ by setting $u_0 = u$ and defining u_{n+1} by the pushout square

$$\begin{tikzcd} u_n \times u_n \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & u_n \arrow[d] \\ u_n \arrow[r] & u_{n+1}. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Let $u_\infty := \mathrm{colim}_n\, u_n$. We claim that u_∞ is univalent. Indeed, the pushout square gives a canonical map $q_n\colon u_n \times u_n \to u_{n+1}$ for which both composites $u_n \rightrightarrows u_n \times u_n \to u_{n+1}$ agree with the structure map $u_n \to u_{n+1}$. In other words, q_n is a diagonal filler for

$$\begin{tikzcd} u_n \arrow[r] \arrow[d] & u_{n+1} \arrow[d] \\ u_n \times u_n \arrow[r] & u_{n+1} \times u_{n+1}. \end{tikzcd}$$

Moreover, the composites

$$u_n \longrightarrow u_n \times u_n \xrightarrow{q_n} u_{n+1} \qquad \text{and} \qquad u_n \times u_n \xrightarrow{q_n} u_{n+1} \longrightarrow u_{n+1} \times u_{n+1}$$

are the structure maps of the shifted sequence $(u_{n+1})_n$ and of the sequence $(u_n \times u_n)_n$, respectively. Since the shift $[\omega]_{\geq 1} \hookrightarrow [\omega]$ is final, and since (4.1) holds without assuming the u_n are univalent, passing to colimits identifies the resulting map with the diagonal

$$u_\infty \longrightarrow u_\infty \times u_\infty$$

and the maps q_n induce an inverse. Hence this diagonal is an isomorphism.

Now let v be a univalent family. We show that restriction along $u \to u_\infty$ induces an equivalence

$$\mathrm{Hom}_{\mathrm{Fam}(T)}(u_\infty, v) \longrightarrow \mathrm{Hom}_{\mathrm{Fam}(T)}(u, v).$$

Given a map $u_n \to v$, the two induced maps $u_n \times u_n \rightrightarrows v$ agree because v is (-1)-truncated in $\mathrm{Fam}(T)$. Hence the map $u_n \to v$ extends uniquely across the pushout defining u_{n+1}. Iterating and passing to the limit gives the desired isomorphism. This proves the adjunction. □

Remark 4.44. This construction is the join construction for propositional truncation, transplanted from homotopy type theory to the category $\mathrm{Fam}(T)$; compare [Uem25].

From this point on, we formulate the proof in the logos direction. Thus an étale morphism of logoi is, after choosing an object X, the canonical functor $T \to T_{/X}$ sending U to $U \times X$, dual to the corresponding geometric morphism of topoi.

4.5.2 Dependent products

If T is a logos and $p\colon X \to Y$ is a morphism in T, then the pullback functor

$$p^*\colon T_{/Y} \longrightarrow T_{/X}$$

preserves colimits by descent, hence admits a right adjoint Π_p, called *dependent product along p*.

Definition 4.45. Let $\varphi^*\colon T \to S$ be a morphism of logoi. We say that φ^* *preserves dependent products* if, for every morphism $p\colon X \to Y$ in T, the canonical comparison

$$\varphi^*(\Pi_p Z) \longrightarrow \Pi_{\varphi^*(p)}\varphi^*(Z)$$

is an isomorphism for every $Z \in T_{/X}$.

Definition 4.46. Let $\varphi^*\colon T \to S$ be a morphism of logoi and suppose that φ^* admits a left adjoint $\varphi_\sharp\colon S \to T$. We say that $\varphi_\sharp$ is *T-indexed* if, for every morphism $p\colon X' \to X$ in T and every pullback square in S of the form

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi^*(X') \arrow[d, "\varphi^*(p)"] \\ Y \arrow[r] & \varphi^*(X), \end{tikzcd}$$

the transposed square

$$\begin{tikzcd} \varphi_\sharp(Y') \arrow[r] \arrow[d, "\varphi_\sharp(q)"'] & X' \arrow[d, "p"] \\ \varphi_\sharp(Y) \arrow[r] & X \end{tikzcd}$$

is a pullback in T.

Lemma 4.47. *Let $\varphi^*\colon T \to S$ be a morphism of logoi. Then φ^* preserves dependent products if and only if it admits a T-indexed left adjoint.*

Proof. First suppose that φ^* admits a left adjoint $\varphi_\sharp\colon S \to T$. For every object $X \in T$, the induced functor

$$\varphi^*_X\colon T_{/X} \longrightarrow S_{/\varphi^*(X)}$$

has a left adjoint

$$\varphi_{\sharp,X}\colon S_{/\varphi^*(X)} \longrightarrow T_{/X},$$

which sends a map $Y \to \varphi^*(X)$ to its transpose $\varphi_\sharp(Y) \to X$.

Now let $p\colon X' \to X$ be a morphism in T. Since φ^* is left exact, we have a commutative square of pullback functors

$$\begin{tikzcd} T_{/X} \arrow[r, "\varphi^*_X"] \arrow[d, "p^*"'] & S_{/\varphi^*(X)} \arrow[d, "\varphi^*(p)^*"] \\ T_{/X'} \arrow[r, "\varphi^*_{X'}"] & S_{/\varphi^*(X')}. \end{tikzcd}$$

The assertion that φ^* preserves dependent products along p is precisely that the corresponding right Beck–Chevalley comparison

$$\varphi^*_X \Pi_p \longrightarrow \Pi_{\varphi^*(p)}\varphi^*_{X'}$$

is an isomorphism. Taking mates under the adjunctions $\varphi_{\sharp,X} \dashv \varphi^*_X$, $\varphi_{\sharp,X'} \dashv \varphi^*_{X'}$, $p^* \dashv \Pi_p$, and $\varphi^*(p)^* \dashv \Pi_{\varphi^*(p)}$, this is equivalent to the left Beck–Chevalley comparison

$$\varphi_{\sharp,X'}\varphi^*(p)^* \longrightarrow p^*\varphi_{\sharp,X}$$

being an isomorphism. Evaluating this comparison on a map $Y \to \varphi^*(X)$ gives the canonical map from the transpose of the pullback square

$$\begin{tikzcd}
Y \times_{\varphi^*(X)} \varphi^*(X') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi^*(X') \arrow[d, "\varphi^*(p)"] \\
Y \arrow[r] & \varphi^*(X)
\end{tikzcd}$$

to the pullback of $\varphi_\sharp(Y) \to X$ along p. Thus the left Beck–Chevalley comparison is an isomorphism exactly when every such transposed square is a pullback. This is precisely the condition that $\varphi_\sharp$ is T-indexed.

It remains to note that preservation of dependent products implies that φ^* has a left adjoint. Let $\{X_i\}_{i\in I}$ be a small family of objects in T. The product $\prod_i X_i$ can be constructed as the dependent product of the object

$$\coprod_{i\in I} X_i \longrightarrow \coprod_{i\in I} *$$

along the fold map $\coprod_{i\in I} * \to *$. Since φ^* preserves colimits, the coproducts in this construction are carried to the corresponding coproducts in S, and by assumption φ^* preserves the dependent product. Hence φ^* preserves small products. Together with left exactness, this implies that φ^* preserves all small limits. Since φ^* is an accessible functor between presentable categories, the adjoint functor theorem gives a left adjoint to φ^*. The first part of the proof then shows that this left adjoint is T-indexed. $\square$

Lemma 4.48. *Let $\varphi^* \colon T \to S$ be a morphism of logoi which preserves dependent products. Then φ^* sends univalent families in T to univalent families in S.*

Proof. Let $\varphi_\sharp$ be the T-indexed left adjoint of φ^* from Lemma 4.47. The adjunction gives a natural isomorphism on arrow categories

$$\mathrm{Hom}_{\mathrm{Ar}(S)}(v, \varphi^*(u)) \simeq \mathrm{Hom}_{\mathrm{Ar}(T)}(\varphi_\sharp(v), u).$$

Under this isomorphism, the T-indexed condition says that any morphism $v \to \varphi^*(u)$ whose square in S is a pullback is sent to a morphism $\varphi_\sharp(v) \to u$ whose square in T is a pullback. Thus we get a map

$$\mathrm{Hom}_{\mathrm{Fam}(S)}(v, \varphi^*(u)) \longrightarrow \mathrm{Hom}_{\mathrm{Fam}(T)}(\varphi_\sharp(v), u).$$

This map is a monomorphism of animae: both sides are subanimae of the corresponding arrow-category mapping animae, and the ambient map is an isomorphism. If u is univalent, then $\mathrm{Hom}_{\mathrm{Fam}(T)}(\varphi_\sharp(v), u)$ is (-1)-truncated; hence its subanima $\mathrm{Hom}_{\mathrm{Fam}(S)}(v, \varphi^*(u))$ is also (-1)-truncated. Thus $\varphi^*(u)$ is univalent. $\square$

4.5.3 Characterization of étale morphisms

We now use the notions of univalent families and dependent products to provide alternative characterizations of étale morphisms.

Definition 4.49. Let $\varphi^* \colon T \to S$ be a morphism of logoi.

(1) We say that φ^* *provides enough families* if, for every family v in S, there exists a family u in T and a morphism $v \to \varphi^*(u)$ in $\mathrm{Fam}(S)$.

(2) If φ^* preserves dependent products, we say that φ^* *provides enough univalent families* if the same condition holds after restricting to univalent families.

(3) We say that φ^* is *object-generating* if the closure of the image of φ^* under colimits and finite limits is all of S.

Lemma 4.50. *Let $\varphi^* \colon T \to S$ be a morphism of logoi with a T-indexed left adjoint $\varphi_\sharp$. Then φ^* factors as*

$$T \xrightarrow{(-)\times\varphi_\sharp(*)} T_{/\varphi_\sharp(*)} \xrightarrow{\psi^*} S,$$

where ψ^* is fully faithful.

Proof. Let $\eta\colon \mathrm{id}_S \to \varphi^*\varphi_\sharp$ be the unit of the adjunction $\varphi_\sharp \dashv \varphi^*$. Define

$$\psi^* \colon T_{/\varphi_\sharp(*)} \longrightarrow S$$

as the composite

$$T_{/\varphi_\sharp(*)} \xrightarrow{\varphi^*} S_{/\varphi^*\varphi_\sharp(*)} \xrightarrow{\eta_*^*} S,$$

where $\eta_*\colon * \to \varphi^*\varphi_\sharp(*)$ is the unit at the terminal object of S and the second functor is pullback along η_*. For $X \in T$, the object $\psi^*(X \times \varphi_\sharp(*))$ is the pullback of $\varphi^*(X) \times \varphi^*\varphi_\sharp(*) \to \varphi^*\varphi_\sharp(*)$ along η_*, hence is equivalent to $\varphi^*(X)$.

The left adjoint of ψ^* is $\varphi_\sharp$ followed by the evident map to $\varphi_\sharp(*)$. The counit is identified, by the T-indexed condition, with the transpose of a pullback square, and is therefore an isomorphism. Hence ψ^* is fully faithful. □

Proposition 4.51 (Uemura)**.** *Let $\varphi^* \colon T \to S$ be a morphism of logoi which preserves dependent products. Then the following conditions are equivalent:*

(1) *φ^* is an étale morphism.*

(2) *The unit $\mathrm{id}_S \to \varphi^*\varphi_\sharp$ of the indexed adjunction is cartesian.*

(3) *φ^* provides enough families.*

(4) *φ^* provides enough univalent families.*

(5) *φ^* is object-generating.*

If these conditions hold, then φ^* is equivalent, under T, to the canonical étale morphism $T \to T_{/\varphi_\sharp(*)}$, $Y \mapsto Y \times \varphi_\sharp(*)$.

Proof. Assume first that φ^* is an étale morphism. Choose $X \in T$ and identify S with $T_{/X}$ so that φ^* is the functor $Y \mapsto Y \times X$. Then the left adjoint is the forgetful functor $T_{/X} \to T$, and the unit is cartesian because a morphism in a slice is recovered by pulling back its total space along the corresponding map to X. This gives (1) $\Rightarrow$ (2).

If the unit is cartesian and $v\colon Y' \to Y$ is a family in S, then the naturality square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d, "v"'] \arrow[dr, phantom, "\lrcorner", very near start] & \varphi^*\varphi_\sharp(Y') \arrow[d, "\varphi^*\varphi_\sharp(v)"] \\ Y \arrow[r] & \varphi^*\varphi_\sharp(Y) \end{tikzcd}$$

exhibits v as a pullback of the family $\varphi_\sharp(v)$ after applying φ^*. Thus (2) implies (3).

Condition (3) implies (5): applying (3) to the family $Y \to *$ gives a family $u\colon E \to B$ in T and a pullback square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \varphi^*(E) \arrow[d, "\varphi^*(u)"] \\ * \arrow[r] & \varphi^*(B). \end{tikzcd}$$

Since $* \simeq \varphi^*(*)$, the object Y is a finite limit of objects in the image of φ^*. Hence φ^* is object-generating.

Now assume (5). By Lemma 4.50, the morphism φ^* factors as an étale morphism followed by a fully faithful morphism ψ^*. Since φ^* is object-generating, so is ψ^*. A fully faithful morphism of logoi which is object-generating is an equivalence, because its essential image is already closed under colimits and finite limits. Hence φ^* is an étale morphism. This proves (5) $\Rightarrow$ (1).

It remains to compare (3) and (4). The implication (4) $\Rightarrow$ (3) follows by applying univalent completion from Proposition 4.43 to the target family. Conversely, if (3) holds and v is univalent in S, choose a family u in T and a map $v \to \varphi^*(u)$. Let u^{univ} be the univalent completion of u. By Lemma 4.48, $\varphi^*(u^{\mathrm{univ}})$ is univalent, and the composite

$$v \longrightarrow \varphi^*(u) \longrightarrow \varphi^*(u^{\mathrm{univ}})$$

shows that (4) holds. $\square$

Proposition 4.52. *For every logos C, the functor*

$$C^{\mathrm{op}} \longrightarrow \mathrm{Logos}^{\text{ét}}_{C/}, \qquad X \longmapsto (C \to C_{/X})$$

is an equivalence. Under this equivalence, the inclusion $\mathrm{Logos}^{\text{ét}}_{C/} \hookrightarrow \mathrm{Logos}_{C/}$ *preserves limits.*

Proof. Essential surjectivity and full faithfulness are the slice classification of étale morphisms from Proposition 4.34, translated to logoi. Explicitly, an étale morphism $\varphi^*\colon C \to S$ is equivalent to $C \to C_{/\varphi_\sharp(*)}$ by Proposition 4.51.

For preservation of limits, let $X_\bullet\colon I \to C$ be a diagram with colimit X. Descent gives

$$C_{/X} \simeq \lim_i C_{/X_i}$$

in $\mathrm{Logos}_{C/}$. Passing to C^{op}, this says exactly that $X \mapsto C_{/X}$ preserves the relevant limits. $\square$

4.5.4 Proof of the main theorem

We are now in a position to prove Theorem 4.37, i.e. the fact that $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$ is closed under colimits, or equivalently that $\mathrm{Logos}^{\text{ét}} \subseteq \mathrm{Logos}$ is closed under limits.

Lemma 4.53. *Let $\{C_i\}_{i\in I}$ be a small family of logoi. Then every projection*

$$\prod_{j\in I} C_j \longrightarrow C_i$$

is an étale morphism.

Proof. Let $D = \prod_{j\in I} C_j$. Define an object $x \in D$ by taking $x_i = *$ and $x_j = \emptyset$ for $j \neq i$. Then

$$D_{/x} \simeq \prod_{j\in I} (C_j)_{/x_j} \simeq C_i,$$

since the slice over the initial object is terminal. Under this equivalence, the canonical étale functor $D \to D_{/x}$ is the projection to C_i. □

Lemma 4.54. *Let $C_\bullet : I \to \mathrm{Logos}^{\text{ét}}$ be a small diagram, and let*

$$C_{-\infty} := \lim_{i\in I} C_i$$

be its limit in Logos*. Then every projection $\pi_i : C_{-\infty} \to C_i$ is an étale morphism.*

Proof. Let $D := \prod_{i\in I} C_i$. By Lemma 4.53, it suffices to prove that the forgetful morphism

$$\pi : C_{-\infty} \longrightarrow D$$

is an étale morphism. By Proposition 4.51, it is enough to show that π preserves dependent products and provides enough univalent families.

Dependent products in $C_{-\infty}$ are computed pointwise, and each transition functor in the diagram preserves dependent products because it is an étale morphism. Thus π preserves dependent products.

Let $u = (u_i)_{i\in I}$ be a univalent family in D, i.e. a tuple of univalent families $u_i \in \mathrm{Fam}^{\mathrm{univ}}(C_i)$ without imposing the transition equivalences. Since limits of logoi are computed in Cat, we have $\mathrm{Fam}(C_{-\infty}) \simeq \lim_i \mathrm{Fam}(C_i)$. Hence $\mathrm{Fam}^{\mathrm{univ}}(C_{-\infty})$ is the full subposet of $\mathrm{Fam}^{\mathrm{univ}}(D)$ spanned by the tuples equipped with equivalences $s^*(u_i) \simeq u_j$ for all arrows $s : i \to j$ in I; the higher coherences are automatic because univalent families form a poset. We will enlarge u to such a compatible tuple.

For a morphism $s : i \to j$ in I, write $s^* : C_i \to C_j$ for the corresponding étale morphism of logoi, and write $s_! : C_j \to C_i$ for its left adjoint. Since s^* is an étale morphism, both s^* and $s_!$ send families to families, and they preserve colimits in the categories of families.

Define an endofunctor Φ of $\mathrm{Fam}(D)$ by

$$\Phi(u)_i := u_i \sqcup \coprod_{(s:\, j\to i)} s^*(u_j) \sqcup \coprod_{(s:\, i\to j)} s_!(u_j).$$

Let G be the endofunctor of $\mathrm{Fam}^{\mathrm{univ}}(D)$ obtained by applying univalent completion componentwise to Φ. There is a natural map $u \to G(u)$. Moreover, for every $s : i \to j$ there are natural maps

$$s^*(u_i) \longrightarrow G(u)_j \qquad \text{and} \qquad u_j \longrightarrow s^* G(u)_i,$$

the second obtained by adjunction from $s_!(u_j) \to G(u)_i$.

Now form the chain

$$u \longrightarrow G(u) \longrightarrow G^2(u) \longrightarrow \cdots$$

and set $u^\infty := \operatorname{colim}_n G^n(u)$. The functor G preserves sequential colimits: this follows from the construction of Φ, the fact that the functors s^* and $s_!$ preserve colimits of families, and the fact that univalent completion is a left adjoint by Proposition 4.43. Hence, by the usual Adámek fixed-point argument [Ada74], the map $G(u^\infty) \to u^\infty$ is an equivalence.

For every arrow $s\colon i \to j$, the displayed maps for G pass to the colimit and give morphisms

$$s^*(u_i^\infty) \longrightarrow u_j^\infty \qquad \text{and} \qquad u_j^\infty \longrightarrow s^*(u_i^\infty)$$

between univalent families. Since univalent families form a poset inside $\mathrm{Fam}(C_j)$, these two maps exhibit $s^*(u_i^\infty)$ and u_j^∞ as equivalent. By the observation above, u^∞ is therefore a univalent family in the limit logos $C_{-\infty}$, and the map $u \to u^\infty$ shows that π provides enough univalent families. □

Proposition 4.55. *Let C be a category with small colimits, and let $D \subseteq C$ be a wide subcategory. Suppose that:*

(1) *For every object $X \in C$, the wide subcategory $D_{/X} \subseteq C_{/X}$ is closed under small colimits.*

(2) *For every small diagram $X_\bullet\colon I \to D$, if $X = \operatorname{colim}_i X_i$ is computed in C, then all structure maps $X_i \to X$ lie in D.*

Then D is closed under small colimits in C.

Proof. Let $X_\bullet\colon I \to D$ be a diagram and let $X = \operatorname{colim}_i X_i$ in C. We need to show that X has the universal property of a colimit in D. For any object Y, a morphism $f\colon X \to Y$ belongs to D if and only if all composites $X_i \to X \to Y$ belong to D: the forward direction follows from (2), and the reverse direction follows from (1), since f is the colimit in $C_{/Y}$ of the diagram of the composites $X_i \to Y$. This identifies the mapping anima from X to Y in D with the limit of the mapping animae from the X_i to Y in D, as desired. □

We finally get to the main result.

Proof of Theorem 4.37. We apply Proposition 4.55 to the wide subcategory $\mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Topos}$.

First fix a topos S. We claim that $(\mathrm{Topos}^{\text{ét}})_{/S} \subseteq \mathrm{Topos}_{/S}$ is closed under colimits. Under the equivalence

$$S \simeq \mathrm{Topos}^{\text{ét}}_{/S}, \qquad X \longmapsto S_{/X}$$

from Proposition 4.34, a diagram of étale morphisms over S corresponds to a diagram $X_\bullet$ in S. If $X = \operatorname{colim}_i X_i$, descent in S gives

$$S_{/X} \simeq \lim_i S_{/X_i},$$

which says that the corresponding colimit in $\mathrm{Topos}_{/S}$ is again étale over S.

It remains to verify the structure-map condition. Let $T_\bullet\colon I \to \mathrm{Topos}^{\text{ét}}$ be a diagram, and let $T = \operatorname{colim}_i T_i$ be its colimit in Topos. Passing to logoi, this is the limit of the opposite diagram in $\mathrm{Logos}^{\text{ét}}$. By Lemma 4.54, each projection $T \to T_i$ on the logos side is an étale morphism. Equivalently, each structure morphism $T_i \to T$ on the topos side is étale.

The criterion therefore shows that $\mathrm{Topos}^{\text{ét}}$ is closed under colimits in Topos, and that the colimit computed in Topos has the universal property in $\mathrm{Topos}^{\text{ét}}$. Hence the inclusion preserves these colimits. $\square$

5 Modalities

In the previous chapters, we have seen that for every $-2 \leq n \leq \infty$ there exists a factorization system on a topos T consisting of the n-connected and n-truncated morphisms. Since each slice $T_{/X}$ is again a topos, each of the slices also obtain such factorization systems. Moreover, they behave well with respect to base change: given a morphism $f: X \to Y$ in T, the pullback functor $f^*: T_{/Y} \to T_{/X}$ preserves finite limits and colimits, hence preserves the unique decompositions into n-connected and n-truncated morphisms for all n.

In a series of articles [ABFJ18; ABFJ20; ABFJ22; ABFJ24; ABFJ25], Anel, Biedermann, Finster and Joyal investigated classes of factorization systems on topoi that have analogous compatibilities with respect to slice categories, called modalities:

Definition 5.1. A *modality in T* is a factorization system (L, R) in T such that L is stable under base change.

Terminology 5.2. Recall from Proposition A.3 that the morphisms in R are precisely those that are right orthogonal to the morphisms in L. In particular, a modality is completely determined by the class L. In what follows, we may sometimes refer to a class of morphisms L as a *modality* if the pair $(L, L^{\perp})$ is a modality.

Lemma 5.3. *In a modality (L, R) on T, the classes L and R are both local classes, in the sense of Definition 2.46.*

Proof. Let $f: X \to Z$ be a morphism in T, let $Z' \to Z$ be an effective epimorphism, and assume that the base change $f': X \times_Z Z' \to Z'$ is in L. We wish to show that f lies in L. To this end, consider the (L, R)-factorization $X \xrightarrow{l} Y \xrightarrow{r} Z$ of f. Since L and R are closed under base change, the maps $X \times_Z Z' \to Y \times_Z Z' \to Z \times_Z Z' = Z'$ provide the unique factorization of f'. Since f' lies in L we get that the map $Y \times_Z Z' \to Z'$ is an isomorphism. Since the pullback functor $T_{/Z} \to T_{/Z'}$ is conservative, this implies that already $Y \to Z$ is an isomorphism, hence $f \in L$.

If $f' \in R$, we instead deduce that $X \times_Z Z' \to Y \times_Z Z'$ and hence $X \to Y$ is an isomorphism, so that $f \in R$. $\square$

This chapter discusses several aspects of the theory of modalities, following Anel, Biedermann, Finster and Joyal. The main topics are:

(1) The fact that the epimorphisms in a topos form a modality, and an identification of the epimorphisms in An in terms of acyclic maps, see Section 5.1;

(2) A generalization of the classical Blakers–Massey theorem from algebraic topology which can be formulated for an arbitrary modality; see Section 5.2;

(3) The relationship between saturated classes, modalities and congruences. Importantly, we will discuss the fact that the congruence Σ^c generated by a small class of maps may be identified with the modality $(\Sigma^\Delta)^m$ generated by the collection Σ^Δ of all iterated diagonals of morphisms in Σ, see Section 5.3. These results provide a simple proof of the fact that the inclusion $\mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C)$ of sheaves on a Grothendieck site into all presheaves admits a left exact left adjoint, see Section 6.1 for details;

(4) A simple construction of the closed complement of an open immersion $\mathcal{U} \hookrightarrow \mathcal{X}$ of topoi, see Section 5.4;

(5) A 'structure theorem' for topoi, where any morphism of logoi admits a unique triple factorization into a 'monogenic quotient' followed by an 'epigenic quotient', followed by a conservative morphism, see Section 5.5;

(6) Products of modalities and congruences, see Section 5.6;

(7) The resulting topos-theoretic interpretation of Goodwillie calculus, see Section 5.7.

5.1 Epimorphisms and acyclic maps

In this section, we discuss in detail the example of a modality given by the *epimorphisms* in a topos. For An, this recovers the acyclic maps: those that induce isomorphisms on homology with local coefficients.

Definition 5.4 (Cotruncation)**.** We say that a map $f\colon X \to Y$ is *n-cotruncated* if it is n-truncated in T^{op}. Inductively, this equivalently means that the (-2)-cotruncated maps are precisely the isomorphisms, and that a map f is $(n+1)$-cotruncated if and only if its codiagonal $\nabla_f\colon Y \sqcup_X Y \to Y$ is n-cotruncated.

Lemma 5.5. *Let L be the collection of n-cotruncated morphisms in T. Then the pair $(L, L^\perp)$ forms a modality on T.*

Proof. We start by showing that $(L, L^\perp)$ is a factorization system on T. In light of Proposition A.11, it suffices to show that the n-cotruncated morphisms form a saturated class of small generation. The fact that they form a saturated class is clear, since colimits commute with pushouts. For small generation, first observe that for a fixed object $X \in T$, the n-cotruncated morphisms form an accessible subcategory of $T_{/X}$: it is given by the following pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T_{/X} \arrow[d, "-\otimes S^{n+1}"] \\ * \arrow[r, "\mathrm{id}_X"] & T_{/X}, \end{tikzcd}$$

and since both the bottom functor and right vertical functor are accessible, the pullback is too by [Lur09, Proposition 5.4.7.3]. As a consequence, there exists a small collection of n-cotruncated morphisms to X that generate everything under sufficiently filtered colimits. Now, if we let X range over a small collection of generators of T, the resulting small collection of n-cotruncated maps forms a generating set for L.

To show that the n-cotruncated maps are stable under base change, consider a pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

and assume that f is n-cotruncated. We will show that also f' is n-cotruncated. By induction, it will suffice to show that the following square is a pullback square:

$$\begin{tikzcd} Y' \sqcup_{X'} Y' \arrow[r] \arrow[d, "\nabla_{f'}"'] & Y \sqcup_X Y \arrow[d, "\nabla_f"] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

But this is clear from universality of colimits. $\square$

Definition 5.6. A morphism $f\colon X \to Y$ in T is called an *epimorphism* if it is (-1)-cotruncated, i.e. if the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a pushout square. We denote by $(-)^+\colon T \to T$ the left adjoint to the inclusion of the (-1)-cotruncated objects, so that we obtain a factorization

$$X \to X^+ \to *$$

into an epimorphism followed by a map which is right orthogonal to the epimorphisms.

As already mentioned in Warning 2.28, the terminology regarding 'epimorphism' and 'effective epimorphism' is a rather unfortunate historical accident: effective epimorphisms are not generally epimorphisms:

Example 5.7. For $T = \mathrm{An}$, the map $S^0 \to *$ is an effective epimorphism (as it is a surjection on path components), but it is not an epimorphism: the pushout of $S^0 \to *$ along itself is S^1, which is not isomorphic to $*$.

The situation is made even more confusing by the fact that the converse *does* hold:

Proposition 5.8. *Every epimorphism in T is 0-connected, hence in particular an effective epimorphism.*

Proof. We may assume that $Y = *$ by passing to the slice topos: the forgetful functor $T_{/Y} \to T$ preserves all colimits, hence preserves and detects epimorphisms. So we need to show: if $X \to *$ is an epimorphism, then $\tau_0 X$ is terminal.

Since the suspension functor $\Sigma^\infty\colon T \to \mathrm{Sp}(T)$ preserves colimits, it follows that the commutative square

$$\begin{tikzcd}
\Sigma^\infty(X) \arrow[r] \arrow[d] & \Sigma^\infty(*) \arrow[d, equal] \\
\Sigma^\infty(*) \arrow[r, equal] & \Sigma^\infty(*)
\end{tikzcd}$$

is a pushout square, hence a pullback square by stability. It follows that the map $\Sigma^\infty(X) \to \Sigma^\infty(*)$ is an isomorphism. The suspension functor factors as

$$T \xrightarrow{F^{\mathrm{CGrp}}} \mathrm{CGrp}(T) \xhookrightarrow{\mathbb{B}^\infty} \mathrm{Sp}(T),$$

where the first functor is the free commutative group functor and the second functor is the classifying spectrum functor which identifies commutative groups with connective spectra. By conservativity of $\mathbb{B}^\infty$ we conclude that $F^{\mathrm{CGrp}}(X) \xrightarrow{\sim} F^{\mathrm{CGrp}}(*)$. Now, since the truncation functor $\tau_0\colon T \to T_{\leq 0}$ commutes with finite products, it induces a functor $\tau_0\colon \mathrm{CGrp}(T) \to \mathrm{CGrp}(T_{\leq 0})$. Since also the inclusion $T_{\leq 0} \hookrightarrow T$ preserves finite products, it follows that τ_0 also commutes with the free commutative group construction, giving a commutative diagram as follows:

$$\begin{tikzcd}
T \arrow[r, "F^{\mathrm{CGrp}}"] \arrow[d, "\tau_0"'] & \mathrm{CGrp}(T) \arrow[d, "\tau_0"] \\
T_{\leq 0} \arrow[r, "F^{\mathrm{CGrp}}"] & \mathrm{CGrp}(T_{\leq 0}).
\end{tikzcd}$$

In particular, we conclude that $F^{\mathrm{CGrp}}(\tau_0 X) \xrightarrow{\sim} F^{\mathrm{CGrp}}(*)$. To deduce that $\tau_0 X = *$, note that for any 0-truncated object X' the unit map $X' \to F^{\mathrm{CGrp}}(X')$ is a monomorphism. Indeed, this is clear for $T = \mathrm{An}$ (where $F^{\mathrm{CGrp}}(X') = \mathbb{Z}[X']$ is the free abelian group generated by the set X'), thus also for any presheaf topos, and then finally for any left exact localization $f^*\colon \mathrm{PSh}(C) \to T$ of a presheaf topos as f^* preserves monomorphisms and commutes with the adjunction $T \rightleftarrows \mathrm{CGrp}(T)$. From the commutative diagram

$$\begin{tikzcd}
\tau_0 X \arrow[r, hook] \arrow[d] & F^{\mathrm{CGrp}}(\tau_0 X) \arrow[d, "\sim"] \\
* \arrow[r, hook] & F^{\mathrm{CGrp}}(*),
\end{tikzcd}$$

it follows that $\tau_0 X \to *$ is a monomorphism in $T_{\leq 0}$. Since $\tau_0(-)$ preserves colimits, it is also an epimorphism. But then we are done, as in a classical topos, every monomorphism which is also an epimorphism is an isomorphism, see Lemma 5.9. $\square$

Lemma 5.9. *Let T be a topos, and let $f\colon X \to Y$ be a morphism in $T_{\leq 0}$ which is both an epimorphism and a monomorphism. Then f is an isomorphism.*

Proof. Since f is a monomorphism, it is contained in $(T_{/Y})_{\leq -1}$. The fact that it is an epimorphism in $T_{\leq 0}$ means that the map $\tau_0(Y \sqcup_X Y) \to Y$ induced by the codiagonal is an isomorphism. This 0-truncation is a priori computed in T, but since Y is 0-truncated it may as well be computed in $T_{/Y}$, so that the map $f\colon X \to Y$ is also an epimorphism in $(T_{/Y})_{\leq 0}$. This reduces the claim to $Y = *$.

In this case, the map $f: X \to *$ to the terminal object is a monomorphism. By definition of the subobject classifier Ω from Definition 2.50, this means that there is a pullback square in T of the following form:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, hook] \\ * \arrow[r, "g"] & \Omega. \end{tikzcd}$$

Recall from Lemma 2.51 that Ω is 0-truncated, hence this is a pullback square in $T_{\leq 0}$. Since f is assumed to be an epimorphism in $T_{\leq 0}$, it follows that the bottom map g must agree with the universal monomorphism $* \hookrightarrow \Omega$. But then X is the pullback of the map $* \hookrightarrow \Omega$ along itself, which implies that $X \xrightarrow{\sim} *$. This finishes the proof. $\square$

We now identify the epimorphisms in the topos of animae.

Proposition 5.10. *Consider the topos $T = \mathrm{An}$.*

(1) *A map of animae $f: X \to Y$ is an epimorphism if and only if f is* acyclic*, meaning that for any local system A of abelian groups on Y, the induced map*

$$H^*(Y, A) \xrightarrow{\cong} H^*(X, f^*A)$$

is an isomorphism.

(2) *A map of animae $f: X \to Y$ is in the right orthogonal class to the epimorphisms if and only if the group $\pi_1(f) \in \mathrm{Grp}((T_{/Y})_{\leq 0})$ is* hypoabelian.

(3) *For any anima X, the unique factorization $X \to *$ as*

$$X \xrightarrow{\text{epi}} X^+ \xrightarrow{\pi_1 \text{ hypoabelian}} *$$

is Quillen's +-construction.

In (2), we say that a group G is *perfect* if it is left orthogonal to abelian groups (i.e. any map $G \to A$ into an abelian group is trivial, or equivalently the abelianization G^{ab} of G is trivial), and we say that a group A is *hypoabelian* if it is right orthogonal to perfect groups (i.e. any map $G \to A$ from a perfect group is trivial, or equivalently A does not admit any perfect subgroups.)

Proof sketch. (1) The 'only if' direction follows from the Mayer-Vietoris sequence associated to the pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & Y. \end{tikzcd}$$

For the 'if'-direction, consider $C := Y \sqcup_X Y$. It follows from the Mayer-Vietoris sequence that the map $i_1: Y \to C$ is acyclic. It follows from the Seifert-van-Kampen theorem that the square

$$\begin{tikzcd} \pi_1(X, x) \arrow[r, "f"] \arrow[d, "f"] & \pi_1(Y, f(x)) \arrow[d] \\ \pi_1(Y, f(x)) \arrow[r] & \pi_1(C, i_1(f(x))) \end{tikzcd}$$

is a pushout square in the category of groups for all $x \in X$. But since $\pi_1(X,x) \to \pi_1(Y, f(x))$ is surjective, it is an epimorphism in the category of groups, hence the map $(i_1)_*\colon \pi_1(Y,y) \to \pi_1(C, i_1(y))$ is an isomorphism. It then follows from Hurewicz that $i_1\colon Y \to C$ is an isomorphism in An, hence so is $\nabla_f\colon C \to Y$.

(2) See [Hoy19a, Corollary 10].

(3) By definition, Quillen's +-construction is the reflection of X into the subcategory of animae with hypoabelian fundamental group. By (2), this subcategory precisely consists of the objects Y for which $Y \to *$ is right orthogonal to the epimorphisms, and the claim follows. □

5.2 The Blakers–Massey theorem

Recall from classical algebraic topology the important *Blakers–Massey theorem*:

Theorem 5.11 (Classical Blakers–Massey theorem)**.** *Let X be a topological space and let $A, B \subseteq X$ be two subspaces whose interiors cover X. Assume that A and B are connected and that their intersection $C := A \cap B$ is (connected and) simply connected. If the pair (A, C) is $(m-1)$-connected and the pair (B, C) is $(n-1)$-connected, $m, n \geq 3$, then the triple $(X; A, B)$ is $(m+n-2)$-connected.*

In homotopy-theoretic language, we may think of X, A, B and C as animae, where $X = A \sqcup_C B$ is the (homotopy) pushout of two maps $f\colon C \to A$ and $g\colon C \to B$. The assumptions then require f to be n-connected and g to be m-connected (recall from Warning 3.16 the shift in terminology) and the conclusion translates to the condition that the map $C \to A \times_X B$ is $(m+n)$-connected.

It turns out that this statement holds in an arbitrary topos, and in fact admits a purely topos-theoretic proof. It will be a special case of a more general version of the Blakers–Massey theorem, due to Anel, Biedermann, Finster, and Joyal [ABFJ20], which can be formulated for an arbitrary modality L.

Notation 5.12 (Day convolution on slices)**.** Let $\square_X$ denote Day convolution in $\mathrm{Ar}(T_{/X}) = \mathrm{Fun}([1], T_{/X})$, with respect to the minimum operation on $[1]$. Given maps $u\colon A \to B$ and $v\colon C \to D$ in $T_{/X}$, the map $u \square_X v$ is the dashed arrow in the pushout square:

$$\begin{tikzcd}
A \times_X C \arrow[r, "A\times v"] \arrow[d, "u\times C"'] & A \times_X D \arrow[d] \arrow[ddr, bend left, "u\times D"] & \\
B \times_X C \arrow[r] \arrow[drr, bend right, "B\times v"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "u\square_X v"] & \\
& & B \times_X D.
\end{tikzcd}$$

This operation is also known as the *pushout product of u and v over X*.

Theorem 5.13 (Generalized Blakers–Massey, Anel, Biedermann, Finster, and Joyal [ABFJ20])**.** *Let (L, R) be a modality in T, and consider a pushout square*

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\
X \arrow[r] & W. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

If $\Delta_f \square_Z \Delta_g$ lies in L, then also the gap map $Z \to X \times_W Y$ lies in L.

In the statement we regard Δ_f and Δ_g as maps in $T_{/Z}$ via the projections $p_1 \colon Z \times_X Z \to Z$ and $p_1 \colon Z \times_Y Z \to Z$.

We prove this theorem below. We first deduce the ordinary Blakers–Massey theorem.

Exercise 5.14. Let u and v be maps in $T_{/Z}$. If u is m-connected and v is n-connected, show that $u \square_Z v$ is $(m+n+2)$-connected.

Corollary 5.15 (Blakers–Massey)**.** *Assume f is m-connected and g is n-connected. Then the map $Z \to X \times_W Y$ is $(m+n)$-connected.*

Proof. The map Δ_f is $(m-1)$-connected while Δ_g is $(n-1)$-connected. By the exercise this means that $\Delta_f \square_Z \Delta_g$ is $(m-1+(n-1)+2)$-connected, and thus the theorem applies. □

Remark 5.16. While the proof of the Blakers–Massey theorem is quite involved, the *dual Blakers–Massey theorem* is easy to prove. It states that for a pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z, \end{tikzcd}$$

if $f \square_Z g \in L$, then the cogap map $(X \sqcup_W Y \to Z)$ lies in L. To see this, we note that the cogap map $X \sqcup_W Y \to Z$ may be written as the base change along $\Delta_Z \colon Z \to Z \times Z$ of $f \square_Z g$, so the claim follows from stability of L under base change.

5.2.1 Modal descent

We first give a proof of Theorem 5.13 using modal descent, which is the proof given by Anel, Biedermann, Finster, and Joyal [ABFJ20]. We start by introducing some terminology.

Definition 5.17. A commutative square in T is called *L-cartesian* if its gap map lies in L.

Proposition 5.18 (Properties of L-cartesian squares)**.** *Consider two composable squares*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "B"] & X'' \arrow[d] \\ Y \arrow[r, "f"'] & Y' \arrow[r] & Y''. \end{tikzcd}$$

(1) *If A and B are L-cartesian, then their composite $A+B$ is L-cartesian.*

(2) *If A and $A+B$ are L-cartesian and f is an effective epimorphism, then B is L-cartesian.*

(3) *If B is cartesian and $A+B$ is L-cartesian, then A is L-cartesian.*

Proof. The gap map $X \to Y \times_{Y''} X''$ of $A+B$ may be factored as

$$X \to Y \times_{Y'} X' \to Y \times_{Y''} X'',$$

where the first map is the gap map of A and the second map is the base change along $Y \to Y'$ of the gap map of B. This immediately implies (1). For (2), it follows from right cancellation that the base change along $Y \to Y'$ of the gap map of B lies in L, hence so does the gap map itself by Lemma 5.3. Part (3) is clear: if B is cartesian, then the gap map of A agrees with that of $A + B$. $\square$

Lemma 5.19. *Consider a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\in L"'] & X' \arrow[d, "\in L"] \\ Z \arrow[r] \arrow[d, "\in R"'] & Z' \arrow[d, "\in R"] \\ Y \arrow[r] & Y', \end{tikzcd}$$

where the four maps lie in L and R as indicated. If the outer square is L-cartesian, then the bottom square is cartesian.

Proof. By left cancellation, the gap map $Z \to Y \times_{Y'} Z'$ of the bottom square lies in R. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\in L"] \arrow[d, "\in L"'] & Y \times_{Y'} X' \arrow[d, "\in L"] \\ Z \arrow[r] & Y \times_{Y'} Z'. \end{tikzcd}$$

Then the three maps lie in L as indicated, hence by right cancellation so does the map $Z \to Y \times_{Y'} Z'$. Since (L, R) is a factorization system, it follows that this map is an isomorphism, as desired. $\square$

Proposition 5.20 (Modal descent)**.** *Let $X_\bullet, Y_\bullet \in \mathrm{Fun}(I, T)$ be two diagrams with colimits X and Y. If we are given an L-cartesian transformation $X_\bullet \to Y_\bullet$, then the square*

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d] & X \arrow[d] \\ Y_\bullet \arrow[r] & Y \end{tikzcd}$$

is L-cartesian. Equivalently: the subcategory $\mathrm{Ar}^{L\text{-}\mathrm{cart}}(T) \subseteq \mathrm{Ar}(T)$ is closed under colimits.

Proof. Let $P_\bullet$ be the pullback of the square. Since the map $X_\bullet \to P_\bullet$ is again L-cartesian, while $P_\bullet \to Y_\bullet$ is even cartesian, we may replace $Y_\bullet$ by $P_\bullet$ and assume that $X \xrightarrow{\cong} Y$.

Consider now the factorization $X_\bullet \xrightarrow{\in L} Y_\bullet \xrightarrow{\in R} Z_\bullet$, and consider the resulting commutative diagram:

$$\begin{tikzcd} X_\bullet \arrow[r] \arrow[d, "\in L"'] & X \arrow[d] \\ Z_\bullet \arrow[r] \arrow[d, "\in R"'] & Z \arrow[d] \\ Y_\bullet \arrow[r] & Y. \end{tikzcd}$$

By Lemma 5.19, the transformation $Z_\bullet \to Y_\bullet$ is cartesian, hence by descent the bottom square is a pullback square. Since morphisms in L are closed under colimits in $\mathrm{Ar}(T)$, the map $X \to Z$ lies in L. Since $X \xrightarrow{\cong} Y$ is an isomorphism, also $Z \to Y$ lies in L, hence so do all maps $Z_i \to Y_i$. It follows that $Z_\bullet \to Y_\bullet$ is an isomorphism, hence so is $Z \to Y$, and consequently also $X \to Z$. But then the gap maps $X_i \to Y_i \times_Y X$ are precisely the maps $X_i \to Z_i$, which lie in L by construction. □

Proof of the Generalized Blakers–Massey theorem. Consider a pushout square

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & W, \end{tikzcd}$$

where $\Delta_f \square_Z \Delta_g \in L$. We must show that also $Z \to X \times_W Y \in L$.

We first reduce to the case where f is an effective epimorphism. Indeed, letting X' be the image of f, we get a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, two heads, "f'"'] & Y \arrow[d, two heads] \\ X' \arrow[r] \arrow[d, hook] \arrow[ur, phantom, "\ulcorner"] & W' \arrow[d, hook] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & W, \end{tikzcd}$$

where the two displayed squares are still pushouts. By Proposition 3.10, the lower square is a pullback square. We then observe that the diagonal Δ_f is isomorphic to the diagonal $\Delta_{f'}$, and that the gap map $Z \to X \times_W Y$ agrees with the gap map $Z \to X' \times_{W'} Y$ of the top square. It thus suffices to prove the statement for the top square, or equivalently in the case where f is an effective epimorphism.

Unwinding definitions, we see that the map

$$\Delta_f \square_Z \Delta_g \colon \big(Z \times_X Z\big) \sqcup_Z \big(Z \times_Y Z\big) \longrightarrow \big(Z \times_X Z\big) \times_Z \big(Z \times_Y Z\big)$$

is given on the first component by (p_1, p_2, p_1, p_1) and on the second by (p_1, p_1, p_1, p_2). In particular, this map is the gap map of the top face of the following commutative cube:

$$\begin{tikzcd} (Z \times_X Z) \sqcup_Z (Z \times_Y Z) \arrow[rr, "(p_1,p_2)+(p_1,p_1)"] \arrow[dr, "(p_1,p_1)+(p_1,p_2)"] \arrow[dd, "p_1+p_2"'] & & Z \times_X Z \arrow[dr, "p_1"] \arrow[dd, "gp_2"' near start] & \\ & Z \times_Y Z \arrow[rr, "p_1" near start] \arrow[dd, "fp_2" near start] & & Z \arrow[dd] \\ Z \arrow[rr, "g" near end] \arrow[dr, "f"] & & Y \arrow[dr] & \\ & X \arrow[rr] & & W. \end{tikzcd}$$

We now make the following series of claims:

- The top face is a pushout. More generally, for objects A, B in some pointed category C (here $C = T_{Z//Z}$) the lower-right corner of

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r] \arrow[d] & A \vee B \arrow[r] \arrow[d] & B \arrow[d] \\
* \arrow[r] & A \arrow[r] & *
\end{tikzcd}$$

  is a pushout, by the pasting property of pushout squares.

- The left and back faces are L-cartesian. By symmetry it suffices to do this for the left face. Writing $\sigma\colon Z \times_Y Z \xrightarrow{\cong} Z \times_Y Z$ for the swap map, this face decomposes as

$$\begin{tikzcd}
(Z \times_X Z) \sqcup_Z (Z \times_Y Z) \arrow[r, "(\Delta_f \square_Z \Delta_g)\circ(\mathrm{id} \sqcup \sigma)"] \arrow[d, "p_1 + p_2"] & (Z \times_X Z) \times_Z (Z \times_Y Z) \arrow[r, "(p_1, p_4)"] \arrow[d, "p_1"] & Z \times_Y Z \arrow[d, "f p_2"] \\
Z \arrow[r, equal] & Z \arrow[r, "f"] & X.
\end{tikzcd}$$

  The left square is L-cartesian by hypothesis. The right square is a pullback square. It follows that the outer rectangle is L-cartesian.

- By modal descent applied when I is the span diagram $\ulcorner$, the front and right faces are L-cartesian.

- Finally, we may decompose the right face of the diagram as the composite of two squares:

$$\begin{tikzcd}
Z \times_X Z \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d] \\
Z \arrow[r, two heads, "f"] & X \arrow[r] & W.
\end{tikzcd}$$

  Here the left square is cartesian, f was assumed to be an effective epimorphism, and the outer rectangle is L-cartesian by the previous point. It follows from Proposition 5.18 that the right square is L-cartesian.

This gives the claim. □

5.2.2 Wärn's alternative proof

We now explain a second proof of Theorem 5.13, following the description of path spaces of pushouts due to Wärn [Wär25]. The main point of this approach is that the pullback $B \times_D C$ of a pushout

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\
B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & D
\end{tikzcd}$$

can be built as a sequential colimit of objects of zigzags. The first non-trivial step in this filtration is the relative pushout product $\Delta_f \square_A \Delta_g$. So if we can show that all later maps in the filtration lie in the modality L, then so does the gap map $A \to B \times_D C$.

Construction 5.21 (Making one leg cartesian)**.** Fix a span $B \leftarrow A \rightarrow C$. A *span over* this span is a diagram of the form

$$\begin{tikzcd} Q \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] & R \arrow[d] \\ B & A \arrow[l] \arrow[r] & C. \end{tikzcd}$$

Given such a span, we define two new spans over $B \leftarrow A \rightarrow C$ as follows.

(1) The span obtained by *making the left leg cartesian* is

$$Q^\ell := Q, \qquad P^\ell := Q \times_B A, \qquad R^\ell := P^\ell \sqcup_P R.$$

(2) The span obtained by *making the right leg cartesian* is

$$R^r := R, \qquad P^r := R \times_C A, \qquad Q^r := P^r \sqcup_P Q.$$

Lemma 5.22. *In both cases, the evident map from the original span to the new span induces an equivalence on pushouts.*

Proof. For the left construction, the new pushout is

$$Q^\ell \sqcup_{P^\ell} R^\ell \simeq Q \sqcup_{P^\ell} (P^\ell \sqcup_P R) \simeq Q \sqcup_P R,$$

by pasting of pushout squares. The right construction is symmetric. $\square$

Construction 5.23 (Zigzag construction, Wärn [Wär25])**.** Starting with a span $Q_0 \leftarrow P_0 \rightarrow R_0$ over $B \leftarrow A \rightarrow C$, define a sequence of spans

$$Q_n \leftarrow P_n \rightarrow R_n$$

by alternately making the left and right legs cartesian: for odd n we apply the left construction to the previous span, and for even $n \geq 2$ we apply the right construction. Let

$$Q_\infty := \operatorname{colim}_n Q_n, \qquad P_\infty := \operatorname{colim}_n P_n, \qquad R_\infty := \operatorname{colim}_n R_n.$$

The zigzag construction computes pullbacks of pushouts in the following precise sense.

Theorem 5.24 (Wärn [Wär25, Theorem 3.6])**.** *Let*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout square in T. Let $Q_0 \leftarrow P_0 \rightarrow R_0$ be a span over $B \leftarrow A \rightarrow C$, and write

$$S := Q_0 \sqcup_{P_0} R_0.$$

Then the colimit span $Q_\infty \leftarrow P_\infty \rightarrow R_\infty$ from Construction 5.23 has pushout S, and the canonical maps

$$Q_\infty \rightarrow S \times_D B, \qquad P_\infty \rightarrow S \times_D A, \qquad R_\infty \rightarrow S \times_D C$$

are isomorphisms.

Proof. By Construction 5.21, each step in the zigzag construction preserves the pushout of the span. Since colimits commute with colimits, the pushout of $Q_\infty \leftarrow P_\infty \rightarrow R_\infty$ is again S.

For odd n, the map $P_n \rightarrow Q_n$ is cartesian over $A \rightarrow B$ by construction. Since the odd natural numbers are cofinal in $\mathbb{N}$, and since sequential colimits in a topos are universal, it follows that $P_\infty \rightarrow Q_\infty$ is cartesian over $A \rightarrow B$. Similarly, using the even stages, $P_\infty \rightarrow R_\infty$ is cartesian over $A \rightarrow C$.

Now compare the pushout square $Q_\infty \sqcup_{P_\infty} R_\infty \simeq S$ with the original pushout square $B \sqcup_A C \simeq D$:

$$
\begin{tikzcd}
P_\infty \arrow[rr] \arrow[dr] \arrow[dd] & & R_\infty \arrow[dr] \arrow[dd] & \\
& Q_\infty \arrow[rr] \arrow[dd] & & S \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & C \arrow[dr] & \\
& B \arrow[rr] & & D
\end{tikzcd}
$$

The left and back faces of the cube are cartesian by the previous paragraph, while the top and bottom faces are pushouts. Descent for pushouts therefore implies that the front and right faces are cartesian. This gives $Q_\infty \xrightarrow{\sim} S \times_D B$ and $R_\infty \xrightarrow{\sim} S \times_D C$, and then also $P_\infty \xrightarrow{\sim} S \times_D A$. $\square$

We wish to apply this result to the Blakers–Massey problem, where we start with $B \leftarrow \emptyset \rightarrow \emptyset$. The following is the starting observation:

Lemma 5.25. *For the zigzag construction associated to the initial span $B \leftarrow \emptyset \rightarrow \emptyset$ over $B \xleftarrow{f} A \xrightarrow{g} C$, the map $P_2 \rightarrow P_3$ is a pushout of $\Delta_f \square_A \Delta_g$.*

Proof. The first step gives

$$Q_1 \simeq B, \qquad P_1 \simeq A, \qquad R_1 \simeq A.$$

The second step makes the right leg cartesian, hence gives

$$R_2 \simeq A, \qquad P_2 \simeq A \times_C A, \qquad Q_2 \simeq (A \times_C A) \sqcup_A B,$$

where the map $A \rightarrow A \times_C A$ is Δ_g and $A \rightarrow B$ is f. The third step makes the left leg cartesian, so

$$P_3 \simeq Q_2 \times_B A \simeq ((A \times_C A) \times_B A) \sqcup_{A \times_B A} A,$$

where we used universality of pushouts. Observe that we may rewrite $(A \times_C A) \times_B A$ as $(A \times_C A) \times_A (A \times_B A)$. Consider now the following commutative diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & (A \times_C A) \arrow[d] & \\
(A \times_B A) \arrow[r] \arrow[d] & (A \times_C A) \sqcup_A (A \times_B A) \arrow[r, "\Delta_g \square_A \Delta_f"] \arrow[d] & (A \times_C A) \times_A (A \times_B A) \arrow[d] \\
A \arrow[r] & (A \times_C A) \arrow[r] & P_3
\end{tikzcd}
$$

We just argued that the bottom rectangle is a pushout square. Since the left upper square and left rectangle are also pushout squares, it follows from the pasting law of pushout squares that also the right bottom square is a pushout square. This shows that the map $P_2 = A \times_C A \to P_3$ is a pushout of $\Delta_f \square_A \Delta_g$, as desired. $\square$

The next observation is that passing to the next stage of the zigzag construction preserves any existing connectivity estimates of the current stage.

Lemma 5.26. *Let L be a modality, and consider the zigzag construction associated to a span $Q_0 \leftarrow P_0 \to R_0$ over $B \leftarrow A \to C$. If $P_{n-1} \to P_n$ lies in L for some $n \geq 2$, then also $P_n \to P_{n+1}$ lies in L.*

Proof. Suppose first that P_n is obtained from P_{n-1} by making the left leg cartesian, so that the next step makes the right leg cartesian. Since $n \geq 2$, the preceding step made the right leg cartesian, so

$$P_{n-1} \xrightarrow{\sim} R_{n-1} \times_C A.$$

Unwinding the definitions of these two consecutive steps gives

$$R_n \xrightarrow{\sim} P_n \sqcup_{P_{n-1}} R_{n-1} \qquad \text{and} \qquad P_{n+1} \xrightarrow{\sim} R_n \times_C A.$$

Since pushouts are universal, it follows that

$$P_{n+1} \xrightarrow{\sim} (P_n \times_C A) \sqcup_{P_{n-1} \times_C A} (R_{n-1} \times_C A) \xrightarrow{\sim} (P_n \times_C A) \sqcup_{P_{n-1} \times_C A} P_{n-1}.$$

Under this identification, the composite $P_{n-1} \to P_n \to P_{n+1}$ is the canonical map from the second summand. It is therefore the pushout of

$$P_{n-1} \times_C A \longrightarrow P_n \times_C A$$

along $P_{n-1} \times_C A \to P_{n-1}$. The displayed map is a base change of $P_{n-1} \to P_n$, hence lies in L. It follows that the composite $P_{n-1} \to P_{n+1}$ lies in L. Since $P_{n-1} \to P_n$ lies in L by assumption, right cancellation for the class L now implies that $P_n \to P_{n+1}$ lies in L.

The case where P_n is obtained by making the right leg cartesian is symmetric, with B in place of C. $\square$

Combining the previous two lemmas, we obtain a neat simple proof of the generalized Blakers–Massey theorem.

Wärn's zigzag proof of Theorem 5.13. Consider a pushout square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\ B \arrow[r] \arrow[ur, phantom, "\ulcorner" near end] & D \end{tikzcd}$$

and let L be a modality such that $\Delta_f \square_A \Delta_g$ lies in L. We must show that the gap map $A \to B \times_D C$ lies in L.

Apply the zigzag construction to the span $B \leftarrow \emptyset \rightarrow \emptyset$. The first step produces the span

$$B \leftarrow A \xrightarrow{=} A,$$

so that $R_1 \simeq A$. Note that $S = B \sqcup_{\emptyset} \emptyset = B$, so by Theorem 5.24 have $R_\infty \xrightarrow{\sim} B \times_D C$. The gap map $A \to B \times_D C$ identifies with

$$R_1 \longrightarrow R_\infty.$$

Since morphisms in L are closed under transfinite composites, it thus remains to show that each map $R_m \to R_{m+1}$ is in L for $m \geq 1$. For m odd, these maps are isomorphisms. For m even, these maps are pushouts of $P_m \to P_{m+1}$, so it remains to show that all these maps are in L. By Lemma 5.26, it suffices to check this for $m = 2$. But by Lemma 5.25, the map $P_2 \to P_3$ is a pushout of $\Delta_f \square_A \Delta_g$, hence lies in L by assumption. $\square$

5.3 Congruences, modalities, and saturated classes

In previous sections, we have introduced the following four types of classes of morphisms Σ in a topos T:

- *Saturated classes:* Those classes that contain all isomorphisms, are closed under composition, and are closed under colimits in $\mathrm{Ar}(T)$ (see Definition A.8). Recall that a saturated class of *small generation* is the left class of a factorization system on T (Proposition A.11).
- *Modalities:* Factorization systems (L, R) such that L is closed under base change. Every saturated class of small generation closed under base change defines a modality.
- *Strongly saturated classes:* Saturated classes satisfying the 2-out-of-3 property. Recall that strongly saturated classes of small generation are precisely the kernels of accessible localizations (Proposition A.15).
- *Congruences:* Strongly saturated classes that are additionally closed under base change. By Corollary 4.16, the congruences of small generation are precisely the kernels of morphisms of logoi.

In this section, we will study some elementary relations between these types of classes, due to [ABFJ25]. The following are the main takeaways:

- We show in Proposition 5.29 that the modality Σ^m generated by a set of morphisms Σ agrees with the saturated class $(\Sigma^{bc})^s$ generated by the class Σ^{bc} of base changes of morphisms in Σ;
- We show in Theorem 5.36 that the congruence Σ^c generated by Σ agrees with the modality $(\Sigma^\Delta)^m$ generated by the set Σ^Δ of iterated diagonals of morphisms in Σ;
- We show in Proposition 5.32 that a class of morphisms K is a congruence precisely when it is a modality and is closed under diagonals.

These results will be our main tool for checking that localizations of topoi are again topoi. As an application, they give a short proof that the category of sheaves on a Grothendieck site is a topos; we spell this out in Section 6.1. This application seems to be a new observation by Marc Hoyois which, to the note-taker's knowledge, does not appear in the literature.

The rest of the section is devoted to proving these results.

Notation 5.27. We denote by

$$\mathrm{Sat}(T), \qquad \mathrm{SSat}(T), \qquad \mathrm{Mdl}(T), \qquad \text{and} \qquad \mathrm{Cong}(T)$$

the partially ordered sets of saturated classes, strongly saturated classes, modalities, and congruences in T, with partial order given by inclusion. There are obvious inclusions $\mathrm{Cong}(T) \subseteq \mathrm{SSat}(T) \subseteq \mathrm{Sat}(T)$ and $\mathrm{Cong}(T) \subseteq \mathrm{Mdl}(T) \subseteq \mathrm{Sat}(T)$.

Notation 5.28. Given a class of morphisms $\Sigma \subseteq \mathrm{Ar}(T)$, we define the following closures:

- Σ^s: the *saturation* (closure under composition, identities, and colimits).
- Σ^{ss}: the *strong saturation* (closure under composition, colimits, and 2-out-of-3).
- Σ^m: the smallest saturated class closed under base change containing Σ.
- Σ^c: the smallest strongly saturated class closed under base change containing Σ.

Recall from Example 4.22 the non-obvious fact that Σ^c is in fact of small generation, hence a congruence, whenever Σ is small. It directly follows that Σ^c is in fact the *smallest congruence containing* Σ.

We now show that also the modality Σ^m is of small generation, so that it is the smallest modality containing Σ.

Proposition 5.29 ([ABFJ22, Lemma 3.2.13, Proposition 3.2.18, Corollary 3.2.19])**.** *Let T be a topos, let Σ be a small class of morphisms, and let $\mathcal{G}$ be a set of generators of T. Let Σ^{bc} denote the set base changes of morphisms in Σ to an object in $\mathcal{G}$. Then we have*

$$\Sigma^m \quad = \quad (\Sigma^{bc})^s.$$

In particular, Σ^m is a modality of small generation.

Proof. Write $\Sigma' := (\Sigma^{bc})^s$, the saturation of Σ^{bc}. It is clear that $\Sigma' \subseteq \Sigma^m$, so it remains to show that also $\Sigma^m \subseteq \Sigma'$. To this end, consider the following class of morphisms

$$\Sigma'' := \{\, u \in \Sigma' \mid \text{every base change of } u \text{ lies in } \Sigma' \,\}.$$

We have $\Sigma'' \subseteq \Sigma'$, so it remains to show that in fact $\Sigma^m \subseteq \Sigma''$. We will do this by showing that Σ'' is a saturated class closed under base change which contains Σ.

It is clear that Σ'' is closed under base change. We claim that it is also saturated. It is clear that Σ'' is closed under composition and contains all isomorphisms, so it remains to show that it is closed under colimits. Consider a diagram of maps $X_\bullet \to Y_\bullet$ in Σ'', and let $Z \to \mathrm{colim}_i Y_i$ be a map. We need to show that the base change $Z \times_{\mathrm{colim}_i Y_i} \mathrm{colim}_i X_i \to Z$ lies in Σ'. By descent, we may write Z as $\mathrm{colim}_i Z_i$, and hence this base change map may be written as the colimit of the maps $Z_i \times_{Y_i} X_i \to Z_i$. Since each of these base changes lies in Σ' by definition of Σ'', the claim follows from the fact that Σ' is closed under colimits.

It remains to show that Σ'' contains Σ. In other words, given a morphism $u\colon A \to B$ in Σ and an arbitrary map $Z \to B$, we need to show that the base change $Z \times_B A \to Z$ lies in Σ'. To this end, let $\mathcal{E} \subseteq T_{/B}$ be the full subcategory spanned by those $Z \to B$ for which this condition holds. By definition of Σ', we have $\Sigma^{bc} \subseteq \Sigma'$, and in particular $\mathcal{E}$ contains all maps $Z \to B$ with $Z \in \mathcal{G}$. Since Σ' is closed under colimits, it follows from descent that $\mathcal{E} \subseteq T_{/B}$ is closed under colimits. But since $\mathcal{G}$ is a set of generators, it follows that $\mathcal{E} = T_{/B}$, showing the claim. $\square$

Remark 5.30. It follows that the assignments $\Sigma \mapsto \Sigma^m$ and $\Sigma \mapsto \Sigma^c$ define left adjoints to the inclusions

$$\mathrm{Mdl}(T) \hookrightarrow \mathrm{Sat}(T) \qquad \text{and} \qquad \mathrm{Cong}(T) \hookrightarrow \mathrm{Sat}(T).$$

The previous proposition gives a simple description of the modality generated by Σ. We would like to have a similarly simple description of the congruence Σ^c. Since congruences are defined via the 2-out-of-3 property, the definition of Σ^c a priori requires the closure under 2-out-of-3, which is an operation over which we have little explicit control. To deal with this, we will now show that congruences admit several useful characterizations involving limits and diagonals that avoid mentioning the 2-out-of-3 property.

Lemma 5.31. *Let $\mathcal{E}$ be a category with finite limits and let K be a class of morphisms in $\mathcal{E}$ closed under finite limits. Then K satisfies the left cancellation property. Dually, a class closed under finite colimits satisfies the right cancellation property.*

Proof. It suffices to prove the first claim, as the second one is dual. Given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, the following commutative square in $\mathrm{Ar}(T)$ exhibits f as a limit in $\mathrm{Ar}(T)$ of the morphisms id_Y, gf and g:

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "f"] \arrow[dd, "f"'] & & X \arrow[dr, "f"] \arrow[dd, "gf"' near start] & \\
& Y \arrow[rr, equal] \arrow[dd, "\mathrm{id}_Y" near start] & & Y \arrow[dd, "g"] \\
Y \arrow[rr, "g" near start] \arrow[dr, equal] & & Z \arrow[dr, equal] & \\
& Y \arrow[rr, "g"'] & & Z.
\end{tikzcd}$$

Since K contains identities and is closed under finite limits, if $g, gf \in K$, then $f \in K$. $\square$

Proposition 5.32. *The following conditions are equivalent for a class of morphisms K:*

(1) *K is a congruence;*

(2) *K is a modality and satisfies 2-out-of-3;*

(3) *K is strongly saturated and closed under finite limits;*

(4) *K is saturated and closed under finite limits;*

(5) *K is a modality and is closed under diagonals: if $f\colon X \to Y$ is in K, then so is $\Delta_f\colon X \to X \times_Y X$.*

Proof. The equivalence between (1) and (2) holds by unwinding definitions. The equivalence between (1) and (3) follows just as in the proof of Lemma 4.15. Clearly (3) implies (4). Conversely, if (4) is satisfied, then it follows from Lemma 5.31 that K satisfies left cancellation. Since it is saturated, it always satisfies closure under composition and right cancellation, so that it in fact satisfies 2-out-of-3, showing (3).

It is clear that (2) implies (5), since Δ_f is a section of the projection pr_1, which is a base change of f. Conversely, assume (5). Since K is a modality, it is the left class of a factorization system, so it is closed under composition and has right cancellation. For 2-out-of-3, it remains to check left cancellation. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ with $g, gf \in K$, we factor f as

$$X \xrightarrow{(\mathrm{id}, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The second map is a base change of $gf \in K$, hence in K. The first is a base change of Δ_g, which is in K by assumption. Thus $f \in K$. □

In the remainder of this section, we will show that for a class of morphisms Σ, the class Σ^c may be obtained by first closing up Σ under diagonals and then taking the modality generated by the resulting class. The main technical tool for this is the following *décalage* construction, which provides a means to extract the largest congruence contained in a modality.

Definition 5.33 ([ABFJ24, Section 2.2.7])**.** Let L be a modality in T. We define the *décalage* of L by

$$D(L) := \{\, u \in L \mid \Delta_u \in L \,\} \subseteq L.$$

We inductively define $D^n(L)$ by $D^0(L) := L$ and $D^{n+1}(L) := D(D^n(L))$. Finally, we set $D^\infty(L) := \cap_n D^n(L)$.

Proposition 5.34. *If L is a modality of small generation, then $D(L)$ is again a modality of small generation.*

Proof. Stability under base change is clear. Next, we show that $D(L)$ is saturated. It is clear that it contains all isomorphisms. For composition, let $A \xrightarrow{f} B \xrightarrow{g} C$ be in $D(L)$. Then Δ_{gf} factors as

$$A \xrightarrow{\Delta_f} A \times_B A \longrightarrow A \times_C A,$$

where the second map is a base change of Δ_g. Since $\Delta_f, \Delta_g \in L$ and L is a modality, $\Delta_{gf} \in L$.

For colimits, let $f_\bullet\colon Y_\bullet \to X_\bullet$ be a diagram in $\mathrm{Ar}(T)$ where each map lies in $D(L)$. We know the colimit map $f\colon Y \to X$ lies in L, and we must show that also its diagonal $\Delta_f\colon Y \to Y \times_X Y$ lies in L. By universality of colimits, we may write $Y \times_X Y$ as the colimit of the base changes $Y_i \times_X Y$, and the diagonal Δ_f is the colimit of the maps $Y_i \to Y_i \times_X Y$. Since L is a modality, it will suffice to show that this map is contained in L for every $i \in I$. Note that this map factors as $Y_i \to Y_i \times_{X_i} Y_i \to Y_i \times_{X_i} Y$.

The first is in L by assumption, as $f_i \in D(L)$. The second map is a base change of $X_i \to X_i \times_X Y$, and so it remains to show that this map lies in L, or equivalently that the commutative square

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\ X_i \arrow[r] & X \end{tikzcd}$$

is L-cartesian. By modal descent, Proposition 5.20, it suffices to show that $Y_\bullet \to X_\bullet$ is L-cartesian, i.e. that for a morphism $i \to j$ in I the commutative square

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d, "f_i"'] & Y_j \arrow[d, "f_j"] \\ X_i \arrow[r] & X_j \end{tikzcd}$$

is L-cartesian. In other words, we must show that the gap map $Y_i \to X_i \times_{X_j} Y_j$ lies in L. We may factor this map as

$$Y_i \to Y_i \times_{X_j} Y_j \to X_i \times_{X_j} Y_j.$$

The first map is a base change of $\Delta_{f_j} \colon Y_j \to Y_j \times_{X_j} Y_j$ and the second map is a base change of $f_i \colon Y_i \to X_i$, so the claim follows from the assumptions on f_i and f_j. □

Corollary 5.35. *For a modality L in T, the class $D^\infty(L)$ is a congruence. Moreover, it is the largest congruence contained in L: the assignment $L \mapsto D^\infty(L)$ defines a right adjoint* $\mathrm{Mdl}(T) \to \mathrm{Cong}(T)$ *to the inclusion* $\mathrm{Cong}(T) \hookrightarrow \mathrm{Mdl}(T)$.

Proof. By definition, $D^\infty(L) = \bigcap_{n \geq 0} D^n(L)$. Using Proposition 5.34 inductively, each $D^n(L)$ is a modality, and intersections of modalities are modalities, so $D^\infty(L)$ is a modality. Moreover, it is closed under diagonals, so is a congruence by Proposition 5.32. □

We now state and prove the main result of this section. Given a class of morphisms Σ, write Σ^Δ for the closure of Σ under diagonals $f \mapsto \Delta_f$. In other words, Σ^Δ consists of all morphisms in Σ and their iterated diagonals.

Theorem 5.36 (Anel–Biedermann–Finster–Joyal, [ABFJ22, Proposition 4.2.12])**.** *Let T be a topos and let $\Sigma \subseteq \mathrm{Ar}(T)$ be a class of maps. Then the congruence Σ^c is given by*

$$\Sigma^c = (\Sigma^\Delta)^m,$$

the modality generated by the iterated diagonals of Σ.

Proof of Theorem 5.36. Set $K := D^\infty((\Sigma^\Delta)^m)$, which by the previous corollary is the largest congruence contained in $(\Sigma^\Delta)^m$. Note that it contains Σ: any iterated diagonal of a morphism in Σ by definition lies in Σ^Δ and thus also in $(\Sigma^\Delta)^m$. So K is a congruence containing Σ, which implies $\Sigma^c \subseteq K$. Since we also have the inclusions $K \subseteq (\Sigma^\Delta)^m \subseteq \Sigma^c$, this finishes the proof. □

Warning 5.37. The formula $\Sigma^m = (\Sigma^{bc})^s$ for the modality generated by Σ does *not* generalize: we generally have $\Sigma^c \neq (\Sigma^{bc})^{ss}$. Indeed, the right-hand side generally has no reason to be closed under base change.

Remark 5.38. If Σ is just a *set* of morphisms, then Σ^Δ is still just a set, and it follows from Proposition 5.29 that the congruence $\Sigma^c = (\Sigma^\Delta)^m$ is of small generation.

Corollary 5.39. *If $\Sigma \subseteq \mathrm{Ar}(T)$ consists of monomorphisms, then $\Sigma^c = \Sigma^m$.*

Proof. If f is a monomorphism, Δ_f is an isomorphism, hence automatically contained in Σ^m. It follows that $\Sigma^c = (\Sigma^\Delta)^m = \Sigma^m$. □

5.4 Open and closed immersions

The notion of an open embedding $U \hookrightarrow X$ of topological spaces admits an analogue in the setting of topoi:

Definition 5.40. A morphism $\mathcal{U} \to \mathcal{X}$ is an *open immersion* if it is both étale and a monomorphism. A morphism in Logos is an *open localization* if its corresponding morphism in Topos is an open immersion.

Lemma 5.41. *Let $j_*\colon \mathcal{X}_{/U} \to \mathcal{X}$ be an étale morphism of topoi, $U \in \mathcal{X}$. Then j_* is an open immersion if and only if U is (-1)-truncated.*

Proof. We saw in Proposition 4.34 that the base change of j_* along itself is given by the functor $(\mathcal{X}_{/U})_{/U\times U} \to \mathcal{X}_{/U}$. This is an equivalence if and only if the object $(U \times U) \in \mathcal{X}_{/U}$ is terminal, i.e. if the projection map $\mathrm{pr}_1 : U \times U \to U$ is an isomorphism. This in turn is equivalent to the diagonal $\Delta\colon U \to U \times U$ being an isomorphism, which by definition means that U is (-1)-truncated. □

If $U \hookrightarrow X$ is an open embedding of topological spaces, then by definition its complement $Z := X \setminus U$ is a closed subspace. Its toposic analogue is as follows:

Definition 5.42. Let $j_*\colon \mathcal{U} \hookrightarrow \mathcal{X}$ be an open immersion of topoi, and let $j^*\colon \mathcal{X} \to \mathcal{U}$ be its left adjoint. We define the *closed subtopos complementary to $\mathcal{U}$* as

$$\mathcal{X}/\mathcal{U} \quad := \quad \{\, X \in \mathcal{X} \mid j^*(X) = * \,\} \quad \subseteq \quad \mathcal{X}.$$

More generally, we say that a morphism of topoi $i\colon \mathcal{Z} \to \mathcal{X}$ is a *closed immersion* if there exists a (-1)-truncated object $U \in \mathcal{X}$ such that $i_*\colon \mathcal{Z} \to \mathcal{X}$ induces an equivalence $\mathcal{Z} \xrightarrow{\sim} \mathcal{X}/\mathcal{U}$.

Lemma 5.43. *Given an open immersion $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$, the closed complement $\mathcal{Z} := \mathcal{X}/\mathcal{U}$ is a topos, and the inclusion $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$ is a morphism of topoi.*

Proof. By Lemma 5.41, we may identify $\mathcal{U}$ with the slice $\mathcal{X}_{/U}$ for some (-1)-truncated object U of $\mathcal{X}$. Given $X \in \mathcal{X}$, we then have $X \in \mathcal{Z}$ if and only if the projection map $X \times U \to U$ is an isomorphism. Equivalently, given an object $Y \in \mathcal{X}$, every morphism $Y \to U$ admits a unique lift $Y \to U \times X$. We conclude that the subcategory $\mathcal{Z} = \mathcal{X}/\mathcal{U}$ consists precisely of those objects that are local with respect to the following class of morphisms:

$$S_U := \{\, \emptyset \to V \mid \text{there exists a map } V \to U \,\}.$$

It in fact suffices to let the V range over a small set of generators of $\mathcal{X}$, so by Proposition A.15 the inclusion $\mathcal{Z} \hookrightarrow \mathcal{X}$ admits a left adjoint $\mathcal{X} \to \mathcal{Z}$. It remains to show that this left adjoint preserves finite limits, or equivalently that the saturation $(S_U)^s$ is a congruence. Since S_U is closed under base change, we have $(S_U)^s = (S_U)^m$ by Theorem 5.36. Moreover, since S_U consists of monomorphisms, we see that $(S_U)^m = (S_U)^c$ by Corollary 5.39. We deduce that $(S_U)^s$ is a congruence, hence the localization $\mathcal{X} \to \mathcal{Z}$ is left exact, exhibiting $\mathcal{Z}$ as a topos and the inclusion $\mathcal{Z} \hookrightarrow \mathcal{X}$ as a morphism of topoi. □

If $j\colon \mathcal{U} \hookrightarrow \mathcal{X}$ is an open immersion with closed complement $i\colon \mathcal{Z} \hookrightarrow \mathcal{X}$, we may recover the topos $\mathcal{X}$ from the topoi $\mathcal{U}$ and $\mathcal{Z}$ together with the composite $i^* j_*\colon \mathcal{U} \to \mathcal{Z}$.

Construction 5.44. Let $\mathcal{U}$ and $\mathcal{Z}$ be categories with finite limits. Given a functor $F\colon \mathcal{U} \to \mathcal{Z}$, we define the category $\mathcal{X}$ as the comma category of F:

$$\mathcal{X} \quad := \quad \mathcal{U} \times_{\mathcal{Z}} \mathrm{Ar}(\mathcal{Z}).$$

In other words, the objects of $\mathcal{X}$ are triples (U, Z, φ) where $U \in \mathcal{U}$, $Z \in \mathcal{Z}$ and $\varphi\colon Z \to F(U)$. Note that the two forgetful functors $j^*\colon \mathcal{X} \to \mathcal{U}$ and $i^*\colon \mathcal{X} \to \mathcal{Z}$ admit canonical right adjoints

$$j_*\colon \mathcal{U} \hookrightarrow \mathcal{X}, U \mapsto (U, F(U), \mathrm{id}_{F(U)}), \qquad \text{and} \qquad i_*\colon \mathcal{Z} \hookrightarrow \mathcal{X}, Z \mapsto (*, Z, *).$$

Lemma 5.45. *Consider topoi $\mathcal{U}$ and $\mathcal{Z}$ and an accessible left exact functor $F\colon \mathcal{U} \to \mathcal{Z}$.*

(1) *The associated recollement $\mathcal{X}$ is a topos,*

(2) *The functor $j_*\colon \mathcal{U} \to \mathcal{X}$ is an open immersion of topoi,*

(3) *The functor $i_*\colon \mathcal{Z} \to \mathcal{X}$ is a closed immersion of topoi.*

Proof. (1) The fact that $\mathcal{X}$ admits colimits and finite limits is a consequence of the results of Chapter B. (Note though that this uses the version stated in Remark B.4.) The forgetful functor $(j^*, i^*)\colon \mathcal{X} \to \mathcal{U} \times \mathcal{Z}$ is cocontinuous, left exact, and conservative, and it follows that $\mathcal{X}$ is a topos. Parts (2) and (3) are left to the reader. □

5.5 Structure theory for topoi

In this section, we discuss the 'quotient triple factorization' of any morphism of logoi $\varphi\colon T \to S$ into a 'monogenic quotient', an 'epigenic quotient' and a conservative morphism:

$$T \xrightarrow{\varphi^{\mathrm{mono}}} T/K^{\mathrm{mono}} \xrightarrow{\varphi^{\mathrm{epi}}} T/K \xrightarrow{\varphi^{\mathrm{cons}}} S.$$

The morphism φ^{cons} in fact factors further as $T/K \to \langle \varphi(T) \rangle \hookrightarrow S$, where the second morphism is fully faithful.

5.5.1 Monogenic and epigenic modalities

Definition 5.46. For a modality L, we define

$$L^{\mathrm{epi}} := L \cap \mathrm{EffEpi}, \qquad \text{and} \qquad L^{\mathrm{mono}} := (L \cap \mathrm{Mono})^m = (L \cap \mathrm{Mono})^c,$$

where the last equality holds by Corollary 5.39. We say L is *monogenic* if $L = L^{\mathrm{mono}}$ (i.e. generated by its monomorphisms), and *epigenic* if $L = L^{\mathrm{epi}}$ (i.e. it is contained in the effective epimorphisms).

Remark 5.47. Monogenic/epigenic congruences are also called *topological/cotopological* in the literature. We will avoid that terminology, as it is less descriptive.

While it is clear that L^{epi} and L^{mono} are saturated classes closed under base change, it is not immediately clear they are actually modalities. For this, we consider the following lemma:

Lemma 5.48. *Let L be a modality, and let $f\colon A \to B$ be a morphism with epi–mono factorization*

$$A \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \xhookrightarrow{\mathrm{im}(f)} B.$$

Then f is contained in L if and only if both $\mathrm{im}(f)$ *and* $\mathrm{coim}(f)$ *are contained in L.*

Proof. The 'only if'-direction is clear. For the other direction, assume $f \in L$. We first show that $\mathrm{coim}(f) \in L$. Since the class L is local and $\mathrm{coim}(f)$ is an effective epimorphism, we may check this after base change along itself, i.e. it suffices to show that the projection $A \times_{\mathrm{Im}(f)} A \to A$ lies in L. As $\mathrm{Im}(f) \hookrightarrow B$ is a monomorphism, the map $A \times_{\mathrm{Im}(f)} A \to A \times_B A$ is an isomorphism, so we may equivalently show that the projection $A \times_B A \to A$ lies in L. This holds, since this projection is a base change of f. It then follows from right cancellation that also $\mathrm{im}(f) \in L$, finishing the proof. $\square$

Corollary 5.49. *Given a modality L, also L^{epi} is a modality.*

Proof. Let $R := L^{\perp}$ and define $R^{\mathrm{epi}} := (L^{\mathrm{epi}})^{\perp}$. We need to show that the pair $(L^{\mathrm{epi}}, R^{\mathrm{epi}})$ is a factorization system. Orthogonality holds by construction, so it remains to check that any morphism $f\colon A \to B$ admits an $(L^{\mathrm{epi}}, R^{\mathrm{epi}})$-factorization. Since $L^{\mathrm{epi}} \subset L$, we get $R \subseteq R^{\mathrm{epi}}$. Similarly, from $L^{\mathrm{epi}} \subseteq \mathrm{EffEpi}$ it follows that $\mathrm{Mono} \subseteq R^{\mathrm{epi}}$. Consider the (L, R)-factorization $A \xrightarrow{l} C \xrightarrow{r} B$ of f, and consider the epi-mono factorization $A \overset{\mathrm{coim}(l)}{\twoheadrightarrow} \mathrm{Im}(l) \xhookrightarrow{\mathrm{im}(l)} C$ of l. By Lemma 5.48 we have that $\mathrm{coim}(l) \in L \cap \mathrm{EffEpi} = L^{\mathrm{epi}}$, while $r \circ \mathrm{im}(l) \in R \circ \mathrm{Mono} \subseteq R^{\mathrm{epi}}$. $\square$

The question of whether L^{mono} is also a modality in general seems more subtle. We will show that this is at least the case when L is of small generation. We introduce some notation.

Notation 5.50. Given a morphism $f\colon A \to B$ in T, we denote by $A \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \xhookrightarrow{\mathrm{im}(f)} B$ its epi-mono factorization. If Σ is a class of morphisms in T, we define

$$\mathrm{im}(\Sigma) := \{\mathrm{im}(f) \mid f \in \Sigma\} \qquad \text{and} \qquad \mathrm{coim}(\Sigma) := \{\mathrm{coim}(f) \mid f \in \Sigma\}.$$

Remark 5.51. If L is a modality, it follows from Lemma 5.48 that $\mathrm{im}(L) = L \cap \mathrm{Mono}$ and $\mathrm{coim}(L) = L \cap \mathrm{EffEpi}$.

Lemma 5.52 ([ABFJ24, Proposition 4.1.14])**.** *For a class of morphisms Σ in T, we have*

$$(\Sigma^m)^{\mathrm{mono}} = (\mathrm{im}(\Sigma))^m \qquad \textit{and} \qquad (\Sigma^c)^{\mathrm{mono}} = (\mathrm{im}(\Sigma^{\Delta}))^m = (\mathrm{im}(\Sigma^{\Delta}))^c.$$

In particular, if L is a modality of small generation, then so is L^{mono}. If L is a congruence of small generation, then so is L^{mono}.

Proof. The second relation follows from the first, since $\Sigma^c = (\Sigma^\Delta)^m$ by Theorem 5.36. The last two claims immediately follow by writing $L = \Sigma^m$ and noting that $\mathrm{im}(\Sigma)$ and $\mathrm{im}(\Sigma^\Delta)$ are small. We need to show the first relation.

Let $L := \Sigma^m$. By Lemma 5.48, we have $\mathrm{im}(\Sigma) \subseteq L^{\mathrm{mono}}$. Note that L^{mono} is a saturated class closed under base change, so that in fact $\mathrm{im}(\Sigma)^m \subseteq L^{\mathrm{mono}}$. For the converse, consider the class L' of morphisms $f : A \to B$ such that $\mathrm{im}(f) \in \mathrm{im}(\Sigma)^m$. We are done if we can show that $L \subseteq L'$, since then $L \cap \mathrm{Mono} = \mathrm{im}(L) \subseteq \mathrm{im}(\Sigma)^m$ and thus also $L^{\mathrm{mono}} = (L \cap \mathrm{Mono})^m \subseteq \mathrm{im}(\Sigma)^m$. By construction we have $\Sigma \subseteq L'$, so it remains to show that L' is a saturated class closed under base change.

To this end, note that $\mathrm{im}(\Sigma)^m = \mathrm{im}(\Sigma)^c$ by Corollary 5.39, so in particular $\mathrm{im}(\Sigma)^m$ is a congruence of small generation. Consider now the quotient map $T \to T/\mathrm{im}(\Sigma)^m$. This localization preserves epi-mono factorizations, and it follows that $f \in L'$ if and only if the morphism f is sent to an effective epimorphism under this localization functor. Since the class of effective epimorphisms in $T/\mathrm{im}(\Sigma)^c$ is a saturated class closed under base change, the same follows for L', finishing the proof. □

Corollary 5.53. *Let Σ be a small class of monomorphisms in T. Then the congruence Σ^c generated by Σ is monogenic.*

Proof. Since $\mathrm{im}(\Sigma^\Delta) = \mathrm{im}(\Sigma) = \Sigma$, this is immediate from the lemma. □

Remark 5.54. By definition, a monogenic congruence K is completely determined by the class of monomorphisms $K \cap \mathrm{Mono}$. The classes of monomorphisms in T arising this way are precisely the *Grothendieck topologies on T*, see Proposition 6.8.

Corollary 5.55. *For every modality L, we have*

$$L \;=\; L^{\mathrm{epi}} \vee L^{\mathrm{mono}},$$

i.e. L is the smallest modality containing both L^{epi} and L^{mono}.

Proof. It is clear that L contains both L^{epi} and L^{mono}. Conversely, if L' is a modality containing both L^{epi} and L^{mono}, then it follows from the previous lemma that it also contains L. □

Lemma 5.56. *If L is a monogenic modality, then L is a congruence. In particular, L^{mono} is a congruence for every modality L of small generation.*[1]

Proof. By assumption we have $L = L^{\mathrm{mono}} = (L \cap \mathrm{Mono})^m$. Since $L \cap \mathrm{Mono}$ consists of monomorphisms, it follows from Corollary 5.39 that $(L \cap \mathrm{Mono})^m$ agrees with $(L \cap \mathrm{Mono})^c$, hence is a congruence. □

Lemma 5.57. *A congruence K is epigenic if and only if it is contained in the class of ∞-connected maps.*

Proof. If K is contained in Conn_∞, then it is in particular contained in EffEpi, so that K is epigenic. Conversely, assume that $K \subseteq \mathrm{EffEpi}$. For any morphism f in K, the iterated diagonals Δ^n_f of f also lie in K, hence are effective epimorphisms by assumption. By Theorem 3.20 it follows inductively that f is n-connected for all $n \geq 0$, hence ∞-connected. □

[1] It is shown in [ABFJ24, Proposition 4.1.4] that the smallness condition is automatic for monogenic modalities.

5.5.2 The quotient triple factorization

We now discuss the promised triple factorization of morphisms of logoi. The terminology and perspective is due to [ABFJ25].

Definition 5.58. Let $\varphi\colon T \to S$ be a morphism of logoi.

(1) We call φ a *monogenic quotient* if it is a quotient map (Definition 4.19) whose kernel is a monogenic congruence.

(2) We call φ an *epigenic quotient* if it is a quotient map whose kernel is an epigenic congruence.

(3) We call φ *conservative* if it only inverts isomorphisms.

(4) We call φ *weakly conservative* if it only inverts effective epimorphisms.

Notation 5.59. Let $\varphi\colon T \to S$ be a morphism of logoi with kernel $K := \ker(\varphi)$. Then we have an inclusion $K^{\mathrm{mono}} \subseteq K$, giving rise to three morphisms of logoi as follows:

$$T \xrightarrow{\varphi^{\mathrm{mono}}} T/K^{\mathrm{mono}} \xrightarrow{\varphi^{\mathrm{epi}}} T/K \xrightarrow{\varphi^{\mathrm{cons}}} S.$$

We refer to this as the *quotient triple factorization* of φ. We further write

$$\varphi^{\mathrm{quot}} := \varphi^{\mathrm{epi}} \circ \varphi^{\mathrm{mono}} \qquad \text{and} \qquad \varphi^{\mathrm{wcons}} := \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{epi}}.$$

Proposition 5.60. *Let $\varphi\colon T \to S$ be a morphism of logoi.*

(1) *The morphisms φ^{mono}, φ^{epi}, φ^{cons}, φ^{quot} and φ^{wcons} are, respectively, a monogenic quotient, an epigenic quotient, conservative, a quotient map, and weakly conservative;*

(2) *The decomposition $\varphi = \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{quot}}$ is the unique decomposition of φ into a quotient map followed by a conservative map;*

(3) *The decomposition $\varphi = \varphi^{\mathrm{wcons}} \circ \varphi^{\mathrm{mono}}$ is the unique decomposition of φ into a monogenic quotient followed by a weakly conservative map;*

(4) *The decomposition $\varphi = \varphi^{\mathrm{cons}} \circ \varphi^{\mathrm{epi}} \circ \varphi^{\mathrm{mono}}$ is the unique decomposition of φ into a monogenic quotient followed by an epigenic quotient followed by a conservative map.*

Proof. (1) It is clear that φ^{mono} is a monogenic quotient, that φ^{quot} is a quotient map, and that φ^{cons} is conservative. Granting that φ^{epi} is an epigenic quotient, it then follows immediately that φ^{wcons} is weakly conservative.

To see that φ^{epi} is an epigenic quotient, we must show that every morphism inverted by φ^{epi} is an effective epimorphism. Let $f\colon A \to B$ be a morphism in T/K^{mono} whose image under φ^{epi} is an isomorphism. Using the fully faithful right adjoint $T/K \hookrightarrow T$, we may see this as a morphism in T, which lies in the kernel K. Consider its epi-mono factorization $A \twoheadrightarrow \mathrm{Im}(f) \hookrightarrow B$ in T. Then the map $\mathrm{Im}(f) \hookrightarrow B$ lies in K^{mono}, hence gets inverted in T/K^{mono}. It follows that f is isomorphic in T/K^{mono} to the effective epimorphism $A \twoheadrightarrow \mathrm{Im}(f)$, as desired.

(2) Given a decomposition $T \xrightarrow{\psi} T' \to S$ into a quotient map and a conservative map, we see that $\ker(\psi) = \ker(\varphi) = K$. Since ψ is a quotient map, it follows that $T' \cong T/K$.

(3) Given a decomposition $T \xrightarrow{\psi} T' \to S$ into a monogenic quotient map and a weakly conservative map, the fact the second map only inverts effective epimorphisms implies that $\ker(\psi) \cap \mathrm{Mono} = \ker(\varphi) \cap \mathrm{Mono} = K$. Since ψ is a monogenic quotient, we have $\ker(\psi) = \ker(\psi)^{\mathrm{mono}} = K^{\mathrm{mono}}$, and thus $T' \cong T/K^{\mathrm{mono}}$.

(4) This follows immediately by first factoring φ as in (2) and then factoring the resulting quotient map $T \to T/K$ as in (3). □

Remark 5.61. In [ABFJ25, Remark 2.1.34], the authors make the following analogy with ring theory. Given a morphism $\varphi\colon A \to B$ of commutative rings, we may consider its kernel $I := \ker(\varphi)$, and obtain an image-factorization $T \twoheadrightarrow T/\ker(\varphi) \hookrightarrow S$. We may loosely think of the injectivity of the second map as analogous to the conservativity of the map $\varphi^{\mathrm{cons}}\colon T/K \to S$. Let $S \subseteq A$ be the multiplicative subset given by all elements $a \in A$ that are inverted in $T/\ker(phi)$. Then the quotient map $A \to A/I$ factors as

$$A \to A[S^{-1}] \to A/I.$$

The authors propose that this is analogous to the factorization of $T \to T/K$ via T/K^{mono}.

5.5.3 Hypercomplete congruences

We now discuss the hypercompletion of a congruence.

Definition 5.62. Given a congruence K of small generation, we define its *hypercompletion* as the kernel of the composite:

$$K^{\mathrm{hyp}} \quad := \quad \ker\big(T \to T/K \to (T/K)^{\mathrm{hyp}}\big),$$

where $(-)^{\mathrm{hyp}}$ denotes the hypercompletion of a topos (localization at ∞-connected maps). Note that we always have $K \subseteq K^{\mathrm{hyp}}$. We say a congruence K is *hypercomplete* if $K = K^{\mathrm{hyp}}$.

Lemma 5.63. *Hypercompletion does not affect the monogenic part of a congruence: for any congruence K we have*

$$K^{\mathrm{hyp}} \cap \mathrm{Mono} \;=\; K \cap \mathrm{Mono}\,.$$

Furthermore, the hypercompletion only depends on the monogenic part of a congruence: we have

$$(K^{\mathrm{mono}})^{\mathrm{hyp}} = K^{\mathrm{hyp}}.$$

Proof. For the first part, consider the sequence of localizations $T \to T/K \to (T/K)^{\mathrm{hyp}}$. Let f be a monomorphism in T contained in K^{hyp}; we need to show that f is in fact contained in K. Since $T \to T/K$ is left exact, the image of f in T/K is a monomorphism that becomes an equivalence in the hypercompletion $(T/K)^{\mathrm{hyp}}$. Thus, in T/K, the map f is ∞-connected. But a map that is both a monomorphism and ∞-connected is an equivalence. Therefore, f is already inverted in T/K, so $f \in K$.

For the second part, we always have $(K^{\mathrm{mono}})^{\mathrm{hyp}} \subseteq K^{\mathrm{hyp}}$. For the converse, we may consider the monogenic-epigenic factorization of the quotient $T \to T/K$:

$$T \to T/K^{\mathrm{mono}} \to T/K.$$

Since the second map inverts only ∞-connected morphisms, it induces an equivalence $(T/K^{\mathrm{mono}})^{\mathrm{hyp}} \xrightarrow{\sim} (T/K)^{\mathrm{hyp}}$ on hypercompletions, and it follows that $(K^{\mathrm{mono}})^{\mathrm{hyp}} = K^{\mathrm{hyp}}$ as desired. □

Note that it follows from the lemma that the assignment $K \mapsto K^{\mathrm{hyp}}$ defines an equivalence of posets

$$\mathrm{MonoCong}(T) \quad \xrightarrow{\sim} \quad \mathrm{HypCong}(T),$$

with inverse given by $K \mapsto K^{\mathrm{mono}}$. All in all, we get the following diagram of inclusions of posets and their adjoints:

$$\begin{tikzcd}
\mathrm{MonoCong}(T) \arrow[dd, "\simeq"'] \arrow[dr, hook, shift left] & & \\
& \mathrm{Cong}(T) \arrow[ul, shift left, "K \mapsto K^{\mathrm{mono}}"] \arrow[dl, shift left, "K \mapsto K^{\mathrm{hyp}}"] \arrow[r, shift right, "K \mapsto K^{\mathrm{epi}}"'] & \mathrm{EpiCong}(T) \arrow[l, hook', shift right] \\
\mathrm{HypCong}(T) \arrow[ur, hook, shift left] & &
\end{tikzcd}$$

5.5.4 Images

In Remark 5.61, we made the analogy between the factorization $T \to T/K \to S$ of a logos morphism $\varphi\colon T \to S$ and the image factorization $A \twoheadrightarrow A/I \hookrightarrow B$ of a morphism $A \to B$ of commutative rings. One may object that the analogy is not completely fitting, since the morphism of logoi $T/K \hookrightarrow S$ is merely conservative, and not necessarily a monomorphism. We now see that this map can be factored further via a subtopos $\langle\varphi(T)\rangle \subseteq S$.

Construction 5.64. Let $\varphi\colon T \to S$ be a morphism of logoi. We define its *image* $\langle\varphi(T)\rangle$ as the smallest full subcategory of S that contains the objects $\varphi(X)$ for all $X \in T$ and is closed under colimits and finite limits.

Proposition 5.65. *Let $\varphi\colon T \to S$ be a morphism of logoi. Then $\langle\varphi(T)\rangle$ is a logos, and the inclusion $\langle\varphi(T)\rangle \hookrightarrow S$ is a morphism of logoi. In particular, φ admits a factorization in* Logos *as*

$$\mathcal{X} \to \langle\varphi(T)\rangle \hookrightarrow \mathcal{Y},$$

where the second map is a monomorphism in Logos*.*

Proof. To show that $\langle\varphi(T)\rangle$ is a logos, it suffices to show that it is presentable and that the Giraud axioms are satisfied, see Theorem 2.42. For the presentability, we will refer to Lurie's proof, see [Lur09, Proposition 6.3.6.2]. For the Giraud axioms, we note that they are all formulated purely in terms of colimits and finite limits, hence these properties are inherited from the logos S.

The inclusion $\langle\varphi(T)\rangle \hookrightarrow S$ preserves colimits and finite limits by construction, hence is a morphism of logoi. It is also clearly a monomorphism, finishing the proof. $\square$

Remark 5.66. Let $\varphi\colon T \to S$ be a morphism of logoi with kernel $K = \ker(\varphi)$. There is a canonical factorization of φ as

$$T \xrightarrow{(1)} T/K^{\mathrm{mono}} \xrightarrow{(2)} T/K \xrightarrow{(3)} \langle\varphi(T)\rangle \xhookrightarrow{(4)} S.$$

This suggests the following classes:

- (1) monogenic localizations;

- (2) epigenic localizations;
- (3) the intersection of (1+2+3) and (3+4) (no explicit characterization);
- (4) those with φ fully faithful ("of trivial shape" / "cell-like");
- (1+2) left exact localizations;
- (1+2+3) the "algebraic" morphisms (in Lurie's terminology); this class is closed under filtered colimits in Topos and contains all étale morphisms;
- (2+3+4) the "surjective" morphisms: φ preserves and detects effective epimorphisms (equivalently, ∞-connected maps);
- (3+4) those with φ conservative;
- (2+3) the intersection of (1+2+3) and (2+3+4) (no explicit characterization).

In classical topos theory, a morphism is usually called "surjective" if it is in the class (3+4). This is because in classical topos theory the class (2) does not appear.

5.6 Products of modalities

Given two ideals I and J in a commutative ring R, we may form their *product* $IJ \subseteq R$, consisting of all elements of the form ij for $i \in I$ and $j \in J$. There is an analogous construction for logoi: given two congruences K and L in T, there exists a *product* KL. Somewhat surprisingly, it turns out that this only depends on the underlying modalities.

Definition 5.67. For $u\colon A \to B$ and $v\colon C \to D$, the *pushout product*

$$u \mathbin{\square} v\colon (B \times C) \sqcup_{A\times C} (A \times D) \longrightarrow B \times D$$

is the cogap map of the commutative square

$$\begin{tikzcd} A \times C \arrow[r, "A\times v"] \arrow[d, "u\times C"'] & A \times D \arrow[d, "u\times D"] \\ B \times C \arrow[r, "B\times v"] & B \times D. \end{tikzcd}$$

We already encountered this operation before in Notation 5.12.

Definition 5.68. For modalities K and L, define their *product* as the class

$$KL \quad := \quad (K \mathbin{\square} L)^m,$$

where $K \mathbin{\square} L$ is the class of morphisms of the form $u \mathbin{\square} v$ for $u \in K$ and $v \in L$.

Lemma 5.69. *We have an inclusion $KL \subseteq K \cap L$.*

Proof. By symmetry, it suffices to show $KL \subseteq K$. If $u \in K$ and $v \in L$, then also the maps $u \times C$ and $u \times D$ are in K, hence so is the cobase change $A \times D \to (B \times C) \sqcup_{A\times C} (A \times D)$. By right cancellation, it follows that the cogap map $u \mathbin{\square} v$ is in K as well. $\square$

While KL is by definition saturated and closed under base change, it is not clear it is always a modality, i.e. that the pair $(KL, (KL)^{\perp})$ defines a factorization system on T. The following lemma shows that this is fine for modalities of small generation:

Lemma 5.70. *Given two classes of maps Σ, Σ' in T, we have $\Sigma^m \Sigma'^m = (\Sigma \mathbin{\square} \Sigma')^m$. In particular, if K and L are modalities of small generation, then so is KL.*

Proof. The inclusion $(\Sigma \mathbin{\square} \Sigma')^m \subseteq \Sigma^m \Sigma'^m$ is clear. For the converse, it suffices to show that $(\Sigma \mathbin{\square} \Sigma')^m$ contains $\Sigma^m \mathbin{\square} \Sigma'^m$. Let us first show that it contains $\Sigma \mathbin{\square} \Sigma'^m$. For a fixed morphism $u \in \Sigma$, consider the collection of morphisms $v \in \mathrm{Ar}(T)$ such that $u \mathbin{\square} v \in (\Sigma \mathbin{\square} \Sigma')^m$. This collection clearly contains Σ', hence to show it contains all of Σ'^m it remains to show that it is a saturated class closed under base change. Closure under base change is clear: if v' is a base change of v, then $u \mathbin{\square} v'$ is a base change of $u \mathbin{\square} v$. The fact that it contains all isomorphisms and that it is closed under compositions is also straightforward. Finally, closure under colimits follows from the observation that the functor $u \mathbin{\square} - \colon \mathrm{Ar}(T) \to \mathrm{Ar}(T)$ preserves colimits.

We may now repeat the argument to show that for fixed $v \in \Sigma'^m$ the collection of $u \in \mathrm{Ar}(T)$ such that $u \mathbin{\square} v \in (\Sigma \mathbin{\square} \Sigma')^m$ contains Σ and is a saturated class closed under base change, hence contains all of Σ^m. This finishes the proof. □

Remark 5.71. The definition of KL in fact makes sense whenever K and L are arbitrary classes of morphisms. In particular, it defines a product on the poset $\mathrm{Acyc}(T)$ of *acyclic classes*,[2] i.e. those saturated classes that are closed under base change. In fact, this turns $\mathrm{Acyc}(T)$ into a commutative algebra object in the category $\mathrm{Pos}^{\mathrm{cocompl}}$ of cocomplete posets; the unit is the acyclic class All of all morphisms.

Corollary 5.72. *Let K and L be monogenic modalities of small generation. Then so is KL.*

Proof. By Lemma 5.52, we may write $K = \Sigma^m$ and $L = \Sigma'^m$, where Σ and Σ' consist of monomorphisms. Then $\Sigma \mathbin{\square} \Sigma'$ also consists of monomorphisms, and the result follows from Lemma 5.70. □

Corollary 5.73. *Let K be a monogenic modality. Then $K^2 = K$.*

Proof. The inclusion $K^2 \subseteq K$ is immediate from Lemma 5.69. For the converse, we may assume that $K = \Sigma^m$ for a class of monomorphisms Σ, and it will suffice to show that $u \in K^2$ for every morphism $u \colon A \hookrightarrow B$ in Σ. By construction, the morphism $u \mathbin{\square} u \colon (B \times A) \sqcup_{A \times A} (A \times B) \to B \times B$ is in K^2, hence so is its base change along $\Delta \colon B \to B \times B$, which by descent is given by the map $A \sqcup_{A \times_B A} A \to B$. But since u is a monomorphism, we get $A \cong A \times_B A$, so that this map becomes $u \colon A \to B$. □

We now discuss various examples of products of modalities.

Example 5.74. Let $L = \mathrm{Conn}_n$ be the n-connected maps. As a modality this is generated by $S^{n+1} \to *$. Note that the pushout product of $S^{n+1} \to *$ with $S^{m+1} \to *$ is $S^{n+m+3} \to *$, so that by Lemma 5.70 we get

$$(\mathrm{Conn}_n)(\mathrm{Conn}_m) = \mathrm{Conn}_{n+m+2}\,.$$

As a special case, for $\mathrm{EffEpi} = (S^0 \to *)^m$ we get $\mathrm{Conn}_n = \mathrm{EffEpi}^{n+2}$.

[2] This terminology is due to [ABFJ22], in analogy to the acyclic maps in algebraic topology from Section 5.1.

More generally, note that for any morphism $u\colon X \to Y$, the pushout product $(S^0 \to *) \mathbin{\square} u$ is the codiagonal $\nabla_u\colon Y \sqcup_X Y \to Y$. It follows that for every modality L we have

$$\mathrm{EffEpi} \cdot L \;=\; \nabla(L)^m.$$

Example 5.75. For any topos T, one has $\mathrm{Epi}^2 \subseteq \mathrm{Conn}_\infty$. Indeed, recall from Proposition 5.8 that every epimorphism is 0-connected, so that

$$\mathrm{Epi}^2 \subseteq \mathrm{Conn}_0^2 = \mathrm{Conn}_2 \subseteq \mathrm{Conn}_1\,.$$

We claim that we also have an inclusion $\mathrm{Epi} \cap \mathrm{Conn}_1 \subseteq \mathrm{Conn}_\infty$. Indeed, if $f\colon X \to Y$ is an epimorphism, then we get a pushout square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] \arrow[ur, phantom, "\ulcorner", very near end] & Y. \end{tikzcd}$$

If $f \in \mathrm{Conn}_n$ for some n, then $\Delta_f \in \mathrm{Conn}_{n-1}$, hence $\Delta_f \mathbin{\square}_X \Delta_f \in \mathrm{Conn}_{n-1} \mathbin{\square} \mathrm{Conn}_{n-1} = \mathrm{Conn}_{2n}$. Since the gap map of this square is f, it then follows from Blakers–Massey that $f \in \mathrm{Conn}_{2n}$. Assuming $f \in \mathrm{Conn}_1$, it then follows inductively that $f \in \mathrm{Conn}_{2^k}$ for all k, so that $f \in \mathrm{Conn}_\infty$.

Construction 5.76. Given modalities K and L, we define

$$K \backslash L := \{\, f \mid K \mathbin{\square} f \subseteq L \,\}.$$

It is not difficult to see that this is a saturated class closed under base change. If M is a third modality, then we have

$$KM \subseteq L \iff M \subseteq K \backslash L,$$

so that $K \setminus L$ functions as an internal hom in $\mathrm{Acyc}(T)$.

Example 5.77. Since $S^0 \mathbin{\square} u = \nabla(u)$, we see that $u \in \mathrm{EffEpi} \setminus \mathrm{Iso}$ if and only if u is an epimorphism. More generally, we inductively get

$$\mathrm{Cotr}_{n+1} = \mathrm{EffEpi} \setminus \mathrm{Cotr}_n\,.$$

5.6.1 Products of congruences

We now see that if two modalities happen to be congruences, then their product is again a congruence. The following is the key input:

Lemma 5.78 (Key lemma)**.** *Let L be a modality.*

(1) $\Delta\nabla(L) \subseteq L$.

(2) $\Delta^{-1}(L) \cap \mathrm{EffEpi} \subseteq \mathrm{EffEpi} \cdot L$.

Proof. (1) For $u\colon A \to B$ in L, the morphism $\Delta\nabla(u)$ takes the form

$$\Delta\nabla(u)\colon B \sqcup_A B \longrightarrow (B \sqcup_A B) \times_B (B \sqcup_A B).$$

The two summand inclusions $i_k\colon B \to B \sqcup_A B$ induce two maps $i_k \times \mathrm{id}\colon B \times_B (B \sqcup_A B) \to (B \sqcup_A B) \times_B (B \sqcup_A B)$ which are jointly an effective epimorphism. It thus suffices to show that left vertical map in the following pullback square is in L:

$$\begin{tikzcd} B \arrow[r, "i_k"] \arrow[d, "i_k"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \sqcup_A B \arrow[d, "\Delta\nabla(u)"] \\ B \sqcup_A B \arrow[r, "i_k \times \mathrm{id}"] & (B \sqcup_A B) \times_B (B \sqcup_A B). \end{tikzcd}$$

This holds since i_k is a cobase change of u.

(2) Let $u\colon A \twoheadrightarrow B$ be an effective epimorphism with $\Delta_u \in L$. We need to show that $u \in \mathrm{EffEpi} \cdot L$. Since u is effective epi, it suffices to show the projection $A \times_B A \to A$ lies in $\mathrm{EffEpi} \cdot L$. Consider the pushout square

$$\begin{tikzcd} (A \times_B A) \sqcup_A (A \times_B A) \arrow[r] \arrow[d] & A \times_B A \arrow[d] \\ A \times_B A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A. \end{tikzcd}$$

The top horizontal map is $\nabla\Delta_u = (S^0 \to *) \,\square\, \Delta_u$, which lies in $\mathrm{EffEpi} \,\square\, L \subseteq \mathrm{EffEpi} \cdot L$. It follows that the bottom map is in $\mathrm{EffEpi} \cdot L$ as claimed. $\square$

Corollary 5.79. *For any modality L, we have*

$$D(L) \cap \mathrm{EffEpi} \;=\; \mathrm{EffEpi} \cdot L. \qquad \square$$

In particular, if K is a congruence we get

$$K^{\mathrm{epi}} = K \cap \mathrm{EffEpi} = \mathrm{EffEpi} \cdot K.$$

Proof. Since $D(L) = L \cap \Delta^{-1}(L)$, the inclusion "$\subseteq$" holds by (2) of the previous lemma. For "$\supseteq$", we recall that $\mathrm{EffEpi} \cdot L = \nabla(L)^m$, so it remains to show that $\nabla(L) \subseteq \Delta^{-1}(L)$. This is precisely part (1) of the previous lemma.

The last statement holds because $D(K) = K$ for any congruence K. $\square$

We now prove that the product of two congruences is again a congruence.

Theorem 5.80. *Let $K, L \in \mathrm{Cong}(T)$ be congruences. Then their product KL is also a congruence.*

Proof. By (5) of Proposition 5.32, it suffices to show that it is closed under diagonals, i.e., $KL \subseteq D(KL)$. Recall that any modality M can be decomposed as $M = M^{\mathrm{mono}} \vee M^{\mathrm{epi}}$. Applying this to K and L, we can distribute the product:

$$KL = (K^{\mathrm{epi}} \vee K^{\mathrm{mono}})(L^{\mathrm{epi}} \vee L^{\mathrm{mono}}) = K^{\mathrm{epi}}L^{\mathrm{epi}} \vee K^{\mathrm{epi}}L^{\mathrm{mono}} \vee K^{\mathrm{mono}}L^{\mathrm{epi}} \vee K^{\mathrm{mono}}L^{\mathrm{mono}}.$$

Using that $K^{\mathrm{epi}} = \mathrm{EffEpi} \cdot K$ and $L^{\mathrm{epi}} = \mathrm{EffEpi} \cdot L$, we see that the first three terms are contained in $\mathrm{EffEpi} \cdot K \cdot L$, which by Corollary 5.79 is contained in $D(KL)$. It thus remains to show that $K^{\mathrm{mono}}L^{\mathrm{mono}} \subseteq D(KL)$. Note that $K^{\mathrm{mono}}L^{\mathrm{mono}} \subseteq (KL)^{\mathrm{mono}}$. By Lemma 5.56, the modality $(KL)^{\mathrm{mono}}$ is a congruence, hence in particular closed under diagonals. So $(KL)^{\mathrm{mono}} \subseteq D(KL)$. All in all, we see that $KL \subseteq D(KL)$, finishing the proof. $\square$

5.7 The Goodwillie tower of a topos

The ideas presented here are due to Anel, Biedermann, Finster, and Joyal.

Construction 5.81. Let $K \in \mathrm{Cong}(T)$ be a congruence. Since congruences form a subalgebra of $\mathrm{Mdl}(T)$, we can form powers of K, giving congruences K^n for all $n \geq 0$ sitting in a chain

$$\cdots \subseteq K^3 \subseteq K^2 \subseteq K \subseteq \mathrm{All}\,.$$

This gives rise to the *K-adic filtration* of T, defined by the sequence of quotient logoi:

$$T \to \cdots \to T/K^3 \to T/K^2 \to T/K \to T/\mathrm{All} \simeq *.$$

For every object $X \in T$, we define the *associated graded* category $(K^n/K^{n+1})_X \subseteq T_{/X}$ as the full subcategory spanned by those maps $Y \to X$ which are in K^n and are K^{n+1}-local (meaning the map is inverted in T/K^{n+1}). Note that for the terminal object $X = 1$ we may write this as the following pullback:

$$\begin{tikzcd} (K^n/K^{n+1})_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T/K^{n+1} \arrow[d] \\ * \arrow[r, "1"] & T_{/K^n}. \end{tikzcd}$$

A similar expression may be obtained for other X by first replacing T by $T_{/X}$.

Proposition 5.82 ([ABFJ25, Theorem 4.2.11])**.** *Let $n \geq 1$ and consider $X \in T$. In the category $(K^n/K^{n+1})_X$, pullback squares agree with pushout squares.*

Proof idea. This follows from an application of the Generalized Blakers–Massey theorem and its dual. We refer to [ABFJ25] for details. □

We can apply the K-adic filtration to the specific case where K is the congruence of ∞-connected maps.

Definition 5.83. Let T be a topos. We define the *n-th Goodwillie approximation* of T as

$$T^{(n)} := T/(\mathrm{Conn}_\infty)^{n+1}.$$

This gives rise to a tower of logoi

$$T \to \cdots \to T^{(2)} \to T^{(1)} \to T^{(0)} \to T^{(-1)} = *.$$

By passing to right adjoints (geometric morphisms), this may alternatively be regarded as a filtration of T by subtopoi:

$$* = T^{(-1)} \hookrightarrow T^{(0)} \hookrightarrow T^{(1)} \hookrightarrow \cdots \hookrightarrow T.$$

We show that the topos-theoretic filtration recovers classical Goodwillie calculus when applied to functor categories.

Construction 5.84. Let C be a small category with finite colimits and a terminal object, and let T be a topos. Since C is in particular filtered, the colimit functor

$$\operatorname{colim}\colon \operatorname{Fun}(C,T) \to T$$

preserves colimits and finite limits, hence is a morphism of logoi. Observe that it is a quotient map, with fully faithful right adjoint given by the constant diagram functor $X \mapsto \mathrm{const}_X$; it is fully faithful as C is weakly contractible, e.g. due to the existence of an initial object. Note that the constant functors const_X are precisely those functors $F \in \operatorname{Fun}(C,T)$ that are local with respect to the maps $y(f)\colon y(Y) \to y(X)$ for every morphism $f\colon X \to Y$, where $y\colon C^{\mathrm{op}} \hookrightarrow \operatorname{Fun}(C,\mathrm{An}) \to \operatorname{Fun}(C,T)$ is the T-analogue of the Yoneda embedding for C^{op}. In particular, the kernel of colim is the strongly saturated class generated by $y(\mathrm{Ar}(C^{\mathrm{op}}))$.

We denote the monogenic-epigenic factorization of colim as follows:

$$\operatorname{Fun}(C,T) \to \operatorname{Shv}(C^{\mathrm{op}};T) \to T.$$

The middle term corresponds to the localization with respect to the monogenic part of the kernel. This defines a Grothendieck topology on C^{op}. In this specific case, *every* map in C^{op} (regarded via Yoneda in $\operatorname{Fun}(C,T)$) generates a covering sieve, because colim sends every map in C to an isomorphism.

We claim that the tower associated to the logos $\operatorname{Shv}(C^{\mathrm{op}};T)$ is precisely the tower of excisive functors. Recall the definition:

Definition 5.85. A functor $F\colon C \to T$ is called *excisive* if it sends pushout squares in C to pullback squares in T.

For $n \geq 0$, we say F is *n-excisive* if it sends strongly cocartesian $(n+1)$-cubes in C to cartesian $(n+1)$-cubes in T. Recall that an $(n+1)$-cube $X\colon [1]^{n+1} \to C$ is called *strongly cocartesian* if it is left Kan extended from its restriction to the $n+1$ edges $\{0\}^k \times [1] \times \{0\}^{n-k}$, and it is *cartesian* if it is a limit cone.

We denote by $\mathrm{Exc}^n(C,T) \subseteq \operatorname{Fun}(C,T)$ the full subcategory of n-excisive functors.

Remark 5.86. For $n = 0$, we see that a 0-excisive functor F is one that sends any map $X \to Y$ in C to a diagram $F(X) \to F(Y)$ exhibiting $F(X)$ as the limit of $\{F(Y)\}$. This is precisely saying that F is a constant functor.

Theorem 5.87. *Let K be the kernel of the colimit functor* $\operatorname{colim}\colon \operatorname{Fun}(C,T) \to T$*. Then there is an equivalence*

$$\operatorname{Fun}(C,T)/K^{n+1} \simeq \mathrm{Exc}^n(C,T).$$

In particular, if T is hypercomplete, we have an equivalence $\operatorname{Shv}(C^{\mathrm{op}};T)^{(n)} \simeq \mathrm{Exc}^n(C,T)$.

Proof. We may immediately reduce to the case $T = \mathrm{An}$. In particular, we may assume T is hypercomplete.

As observed before, K is the strong saturation of the class $\Sigma = y(\mathrm{Ar}(C^{\mathrm{op}}))$, i.e., the image of all maps in C^{op} under the Yoneda embedding $y\colon C^{\mathrm{op}} \hookrightarrow \operatorname{Fun}(C,\mathrm{An})$. Since C has finite colimits, C^{op} has finite limits. It follows that Σ is closed under diagonals and base change to representables. It follows from Proposition 5.29 and Theorem 5.36 that $K = \Sigma^c$.

We start with the case $n = 1$. We have $K^2 = (\Sigma \mathbin{\square} \Sigma)^m$. For maps $u\colon A \to B$ and $v\colon C \to D$ in C^{op}, the pushout product $u \mathbin{\square} v$ corresponds to the gap map in the following commutative square in C^{op}, seen as a square in C:

$$\begin{tikzcd}
A \sqcup B \arrow[r] \arrow[d] & B \sqcup C \arrow[d] \\
A \sqcup D \arrow[r] & B \sqcup D.
\end{tikzcd}$$

Note that this is a cocartesian square in C. Moreover, one can show that every cocartesian square in C is a cobase change of such a square. We deduce that a functor $F\colon C \to \mathrm{An}$ is local with respect to $(\Sigma \mathbin{\square} \Sigma)^m$ if and only if it sends all cocartesian squares to cartesian squares, i.e., if it is excisive.

Generalizing this argument, the strongly cocartesian $(n+1)$-cubes are precisely the cobase changes of iterated pushout products $u_1 \mathbin{\square} \dots \mathbin{\square} u_{n+1}$, and we conclude that the local objects with respect to the congruence K^{n+1} are precisely the n-excisive functors. $\square$

Remark 5.88. Let D be a differentiable category, meaning D has finite limits and sequential colimits which commute with each other. Then we can apply the theorem to the subtopos

$$T := \mathrm{PSh}_{\mathrm{seq}}(D) \hookrightarrow \mathrm{PSh}(D),$$

defined as the full subcategory spanned by objects of the form $\mathrm{colim}_n\, y(X_n)$ for sequences $X_0 \to X_1 \to X_2 \to \dots$ in D. This inclusion admits a left exact left adjoint

$$\mathrm{PSh}(D) \to \mathrm{PSh}_{\mathrm{seq}}(D)$$

given by localizing at those maps $\mathrm{colim}_n\, y(X_n) \to \mathrm{colim}_n\, y(Y_n)$ such that $\mathrm{colim}_n X_n \xrightarrow{\sim} \mathrm{colim}_n Y_n$.

Lemma 5.89. *Let $C = \mathrm{An}^{\mathrm{fin}}$ be the category of finite animae. Then the inclusion $i\colon \mathrm{An}^{\mathrm{fin}} \hookrightarrow \mathrm{An}$ defines an ∞-connected object in the topos $\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})$. Moreover, it exhibits $\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})$ as the classifying topos for ∞-connected objects: for every topos T, evaluation at i induces an equivalence of categories*

$$\mathrm{Geom}(T, \mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}})) \xrightarrow{\sim} T^{\geq\infty}, \qquad \varphi \mapsto \varphi^*(i).$$

Proof. Every object $X \in T$ determines a morphism of logoi $\mathrm{PSh}(\mathrm{An}^{\mathrm{fin,op}}) \to T$ given on generators by $A \mapsto X^A$. Observe that X is ∞-connected if and only if for every morphism $A \to B$ in $\mathrm{An}^{\mathrm{fin}}$, the induced map $X^B \to X^A$ is an effective epimorphism. Since the maps $A \to B$ generate the kernel K of $\mathrm{colim}\colon \mathrm{Fun}(C, \mathrm{An}) \to \mathrm{An}$, this is in turn equivalent to sending all maps in K to effective epimorphisms. Since this is in turn equivalent to sending all maps in K^{mono} to isomorphisms, i.e. to factoring through the localization $\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}}) = \mathrm{Fun}(\mathrm{An}^{\mathrm{fin}}, \mathrm{An})/K^{\mathrm{mono}}$. All in all, this shows that the equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{PSh}(\mathrm{An}^{\mathrm{fin,op}}), T) \xrightarrow{\sim} T, \qquad \varphi \mapsto \varphi(i)$$

restricts to the desired equivalence

$$\mathrm{Fun}_{\mathbb{L}\mathrm{ogos}}(\mathrm{Shv}(\mathrm{An}^{\mathrm{fin,op}}), T) \xrightarrow{\sim} T^{\geq\infty}.$$

$\square$

Remark 5.90. In the same way, one can show that $\mathrm{Shv}(\mathrm{An}_*^{\mathrm{fin,op}})$ classifies *pointed* ∞-connected objects.

This classification allows us to identify the layers of the Goodwillie tower with known categories of spectra.

Lemma 5.91. *The category* $\mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, T)$ *is equivalent to the category of "T-parametrized spectra":*

$$\{(X, E) \mid X \in T, E \in \mathrm{Sp}(T_{/X})\}.$$

Lemma 5.92. *The category* $\mathrm{Exc}^1(\mathrm{An}^{\mathrm{fin}}, T)$ *is equivalent to the category of "T-parametrized spectral torsors":*

$$\{(X, E, s) \mid X \in T, E \in \mathrm{Sp}(T_{/X}), s \in \Gamma(X, E)\}.$$

Let $P_n\colon \mathrm{Fun}(C, T) \to \mathrm{Exc}^n(C, T)$ be the left adjoint to the inclusion (the n-excision approximation). By Theorem 5.87, this agrees with the localization functor $\mathrm{Fun}(C, T) \to \mathrm{Fun}(C, T)/K^{n+1}$ for $K := \ker(\mathrm{colim}\colon \mathrm{Fun}(C, T) \to T)$, and so the P_n-equivalences are the K^{n+1}-local morphisms. This gives the following immediate corollary:

Corollary 5.93. *We have* $(P_n\text{-equiv})(P_m\text{-equiv}) = (P_{n+m+1}\text{-equiv})$. □

Blakers–Massey then gives:

Corollary 5.94. *Consider a pushout square in* $\mathrm{Fun}(C, T)$ *of the form*

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "g"'] & G \arrow[d] \\ H \arrow[r] & K, \end{tikzcd}$$

where f in a P_n-equivalence and g is a P_m-equivalence. Then the gap map $F \to G \times_K H$ is a P_{n+m+1}-equivalence. □

Definition 5.95. A functor $F\colon C \to T$ is called *n-reduced* if $P_n(F) = *$.

Corollary 5.96 (Goodwillie's structure theorem)**.** *Consider a functor $F\colon C \to T$, and define G via the following pushout square:*

$$\begin{tikzcd} P_nF \arrow[r] \arrow[d] & P_{n-1}F \arrow[d] \\ P_0F \arrow[r] & G. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Then G is an n-reduced object over P_0F, and this square is P_n-cartesian. □

Corollary 5.97. *The functor P_nG is a delooping of the fiber* $\mathrm{fib}_{/P_0F}(P_nF \to P_{n-1}F)$. □

Corollary 5.98. *The category of n-homogeneous functors is stable.* □

6 Topics in topos theory

This chapter collects several further topics in topos theory:

- In Section 6.1, we define Grothendieck sites (C, τ) and their associated sheaf topoi $\mathrm{Shv}_\tau(C)$;
- In Section 6.2, we introduce actions by a groupoid object $\mathcal{G}$ and principal $\mathcal{G}$-bundles, and show that both are classified in a suitable sense by the classifying stack $\mathbf{B}\mathcal{G} := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_n$;
- Section 6.3 discusses essential morphisms and shapes of topoi;
- In Section 6.4, we define hypercovers and show that a topos is hypercomplete if and only if every hypercover is effective;
- The topic of Section 6.5 is Postnikov complete and bounded topoi;
- In Section 6.6 we study dimension theory of topoi, and show that every topos of locally bounded dimension is hypercomplete, and even Postnikov complete if there is a global bound on the dimension;
- Section 6.7 defines the notion of *coherent topoi*;
- Section 6.8 introduces pretopoi;
- Section 6.9 characterizes those topoi that are exponentiable in Topos: they are precisely the compactly assembled topoi.
- Section 6.10 contains a brief digression into the question of when the category $\int_{\mathrm{An}} C$ of An-parametrized objects in a category C is a topos; we will show that this is the case precisely if C is a *locus*, meaning that C is an accessible category with pullbacks and van Kampen weakly contractible colimits.

6.1 Sheaf topoi

In this section, we define the category of τ-sheaves associated to a collection of covering sieves τ and show that it is a topos. This is one of the fundamental ways of constructing examples of topoi.

Definition 6.1. Let C be a category. A *sieve* on an object $X \in C$ is a subfunctor $U \hookrightarrow y(X)$ of the representable presheaf $y(X) = \mathrm{Hom}_C(-, X)$. Equivalently, it is a collection of morphisms with codomain X that is closed under precomposition.

Theorem 6.2 (Lurie)**.** *Let C be a small category, and let τ be a collection of sieves on C stable under pullbacks. Let*

$$\mathrm{Shv}_\tau(C) \quad \subseteq \quad \mathrm{PSh}(C)$$

be the full subcategory spanned by the τ-local objects. Then $\mathrm{Shv}_\tau(C)$ *is a left exact localization of* $\mathrm{PSh}(C)$*. In particular,* $\mathrm{Shv}_\tau(C)$ *is a topos.*

The standard proof of this result (as found for example in [Lur09]) involves a detailed analysis of the "plus-construction", which is fairly intricate. However, using the machinery of modalities and congruences established in Section 5.3, we can give a much simpler and more conceptual proof. We will freely use the notations introduced in that section.

Proof. We know from Proposition A.15 that the inclusion admits an accessible left adjoint $L\colon \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$. We need to show that L is left exact. By Corollary 4.16, it suffices to show that the class of morphisms K inverted by L is a congruence.

The class K is precisely the strong saturation τ^{ss} of τ. We claim that it agrees with τ^c. Since τ consists of monomorphisms, it follows from Corollary 5.39 that $\tau^c = \tau^m$. Furthermore, since $\tau^{bc} = \tau$ by assumption on τ, we get from Proposition 5.29 that $\tau^m = (\tau^{bc})^s = \tau^s$. All in all, we see that $\tau^{ss} \subseteq \tau^c = \tau^m = \tau^s \subseteq \tau^{ss}$, and thus all four classes agree. In particular τ^{ss} is a congruence, finishing the proof. □

6.1.1 Grothendieck sites

While not necessary for the definition of $\mathrm{Shv}_\tau(C)$, the collection τ in the previous theorem is usually taken to be a *Grothendieck topology*. We can make sense of the notion of Grothendieck topologies in an arbitrary topos:

Definition 6.3 ([ABFJ24, Definition 3.1.2])**.** A *Grothendieck topology* on a topos T is a class τ of monomorphisms in T, called the *covering monomorphisms*, such that:

(a) The saturated class τ^s is of small generation.

(b) The class τ is a local class, in the sense of Definition 2.46;

(c) Covering monomorphisms are closed under composition;

(d) Given monomorphisms $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow Z$ in T, if $g \circ f$ is a covering monomorphism, then so is g.

We refer to a τ-local object as a τ-*sheaf* and denote the full subcategory of τ-sheaves by

$$\mathrm{Shv}_\tau(T) \quad \subseteq \quad T.$$

The argument from Theorem 6.2 goes through, showing that the localization functor $L_\tau\colon T \to \mathrm{Shv}_\tau(T)$ is left exact.

Remark 6.4. Let (T, τ) and (S, τ') be topoi equipped with Grothendieck topologies. Observe that a morphism of topoi $\varphi_*\colon T \to S$ restricts to a functor $\varphi^{\mathrm{Shv}}\colon \mathrm{Shv}_\tau(T) \to \mathrm{Shv}_{\tau'}(S)$ if and only if its associated logos morphism $\varphi^*\colon S \to T$ preserves covering monomorphisms. In this case, the functor φ^{Shv} is automatically a morphism of topoi. By passing to left adjoints, we see that φ^* commutes with sheafification:

$$\begin{tikzcd} S \arrow[r, "\varphi^*"] \arrow[d, "L_{\tau'}"'] & T \arrow[d, "L_\tau"] \\ \mathrm{Shv}_{\tau'}(S) \arrow[r, "(\varphi^{\mathrm{Shv}})^*"] & \mathrm{Shv}_\tau(T). \end{tikzcd}$$

When specializing Definition 6.3 to presheaf topoi, we recover the usual notion of a Grothendieck site.

Notation 6.5. Let $f\colon X \to Y$ be a morphism in some category C. For every sieve $U \hookrightarrow y(Y)$, we denote by $f^*U \hookrightarrow y(X)$ the sieve obtained by pullback along $y(f)\colon y(X) \to y(Y)$. Note that it consists of those morphisms $g\colon Z \to X$ such that the composite $f \circ g$ is in the sieve U.

Definition 6.6 (Grothendieck topology, [Lur09, Definition 6.2.2.1])**.** A *Grothendieck topology* on a category C consists of a specification, for each object X of C, of a collection of sieves on C which we will refer to as *covering sieves*. The collections of covering sieves are required to satisfy the following properties:

(1) For every object X of C, the identity $y(X) \to y(X)$ is a covering sieve;

(2) For every morphism $f\colon X \to Y$ in C and every covering sieve $U \hookrightarrow y(Y)$ on Y, the pullback sieve $f^*U \hookrightarrow y(X)$ is a covering sieve on X;

(3) Let X be an object of C, $U \hookrightarrow y(X)$ a covering sieve on X, and $V \hookrightarrow y(X)$ an arbitrary sieve on X. Suppose that, for every morphism $f\colon Y \to X$ belonging to U, the pullback sieve f^*V is a covering sieve on Y. Then V is a covering sieve on X.

A *Grothendieck site* is a category C equipped with a Grothendieck topology.

Lemma 6.7. *Let C be a small category. Then there is a bijection between Grothendieck topologies on C in the sense of Definition 6.6 and Grothendieck topologies on* $\mathrm{PSh}(C)$ *in the sense of Definition 6.3.*

Proof. If τ is a Grothendieck topology on $\mathrm{PSh}(C)$, we consider the collections of sieves $U \hookrightarrow y(X)$ which are covering monomorphisms in $\mathrm{PSh}(C)$, i.e. are contained in τ. Conditions (1) and (2) are trivially satisfied. For (3), let $i\colon U \hookrightarrow y(Y)$ and $j\colon V \hookrightarrow y(X)$ be sieves such that i is a covering monomorphism and such that for every $f\colon y(Y) \to U$ the projection $y(Y) \times_{y(X)} V \hookrightarrow y(Y)$ is a covering monomorphism. Since τ is local by (b), it follows that also the projection $U \times_{y(X)} V \to U$ is a covering monomorphism, hence also $U \times_{y(X)} V \hookrightarrow U \hookrightarrow y(X)$ by (c), and thus also $V \hookrightarrow y(X)$ by (d).

Conversely, given a Grothendieck topology on C, we define the class of covering monomorphisms as those monomorphisms $U \hookrightarrow X$ for which the pullback to each representable presheaf is a covering sieve. We check that properties (a)-(d) are satisfied. (a) Note that τ^s is generated by the collection of covering sieves, which is small since C is small. (b) It is clear that τ is a local class. (c) For closure under composition, let $V \hookrightarrow U$ and $U \hookrightarrow X$ be two covering monomorphisms. To show that also $V \hookrightarrow X$ is a covering monomorphism, we may assume that $X = y(X')$ is representable, so that $U \hookrightarrow y(X')$ is a covering sieve. To get the same for $V \hookrightarrow y(X')$, it thus suffices to show that for all $f\colon Y \to X'$ in the sieve U, the pullback sieve $f^*V \hookrightarrow y(Y)$ is a covering sieve. But this holds since this map is a base change of $V \hookrightarrow U$, which was assumed to be in τ. (d) Finally, consider monomorphisms $V \overset{f}{\hookrightarrow} U \overset{g}{\hookrightarrow} X$ such that $gf \in \tau$. To show that also $g \in \tau$, we may again assume $X = y(X')$ is representable. Then $V \hookrightarrow y(X')$ is a covering sieve, so to show that also g is, it suffices to show that for all $h\colon Y \to X'$ in V the pullback $h^*U \hookrightarrow y(Y)$ is a covering sieve. But since $V \hookrightarrow U$ is a monomorphism, this map is simply the identity on $y(Y)$, hence a covering sieve.

It is immediate from the construction that these two assignments are inverse to each other, finishing the proof. □

Proposition 6.8 ([ABFJ24, Proposition 3.1.10])**.** *Let T be a topos. There is a bijection*

$$\{\text{Grothendieck topologies on } T\} \xrightleftharpoons[K\cap\mathrm{Mono}\leftarrow\!\shortmid K]{\tau\mapsto\tau^c} \{\text{monogenic congruences on } T\}\,.$$

Proof. First let τ be a Grothendieck topology on T. The congruence τ^c is monogenic, since $\tau \subseteq \mathrm{Mono}$ gives $\tau^c = \tau^m = (\tau \cap \mathrm{Mono})^m$, where the first equality uses Corollary 5.39. Moreover, we claim that $\tau^c \cap \mathrm{Mono} = \tau$. Since τ consists of monomorphisms, it suffices to show $\tau^c \cap \mathrm{Mono} \subseteq \tau$. Since τ is closed under base change, Proposition 5.29 gives $\tau^c = \tau^m = (\tau^{bc})^s = \tau^s$. Now let

$$E := \{f \in \mathrm{Ar}(T) \mid \mathrm{im}(f) \in \tau\}$$

be the class of *τ-coverings* (cf. [ABFJ24, Proposition 3.1.10]). We claim that E is saturated and contains τ, hence $\tau^s \subseteq E$. Clearly E contains all isomorphisms, and $\tau \subseteq E$ since $\mathrm{im}(m) = m$ for every monomorphism m. Let us verify the remaining conditions.

Base change. Since the epi-mono factorization is stable under base change and τ is closed under base change, E is closed under base change.

Composition. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in E, with epi-mono factorizations

$$X \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Im}(f) \xhookrightarrow{\mathrm{im}(f)} Y \overset{\mathrm{coim}(g)}{\twoheadrightarrow} \mathrm{Im}(g) \xhookrightarrow{\mathrm{im}(g)} Z,$$

so that $\mathrm{im}(f), \mathrm{im}(g) \in \tau$. Let $q\colon\ \mathrm{Im}(f) \to \mathrm{Im}(g)$ be the composite $\mathrm{coim}(g) \circ \mathrm{im}(f)$, with epi-mono factorization $q = j \circ e_q$ through some J:

$$\begin{tikzcd}
X \arrow[dr, two heads, "\mathrm{coim}(f)"'] & & Y \arrow[dr, two heads, "\mathrm{coim}(g)"'] & & Z \\
& \mathrm{Im}(f) \arrow[ur, hook, "\mathrm{im}(f)"] \arrow[dr, two heads, "e_q"'] & & \mathrm{Im}(g) \arrow[ur, hook, "\mathrm{im}(g)"] & \\
& & J \arrow[ur, hook, "j"] & &
\end{tikzcd}$$

Then $\mathrm{im}(gf) = \mathrm{im}(g) \circ j$, so it suffices to show $j \in \tau$. Pull back j along the effective epimorphism $\mathrm{coim}(g)\colon Y \twoheadrightarrow \mathrm{Im}(g)$ to obtain $\pi\colon M := Y \times_{\mathrm{Im}(g)} J \hookrightarrow Y$. Since $\mathrm{im}(f)$ factors through π (via the map $\mathrm{Im}(f) \to J$ and the inclusion $\mathrm{Im}(f) \hookrightarrow Y$), the composite $\mathrm{Im}(f) \hookrightarrow M \overset{\pi}{\hookrightarrow} Y$ equals $\mathrm{im}(f) \in \tau$. By axiom (d), it follows that $\pi \in \tau$. Since π is the pullback of j along an effective epimorphism and τ is a local class, we conclude $j \in \tau$. Finally, $\mathrm{im}(gf) = \mathrm{im}(g) \circ j \in \tau$ by axiom (c).

Colimits. First, E is closed under coproducts: since images commute with coproducts (as both the class of monomorphisms and the class of effective epimorphisms are closed under coproducts), we have $\mathrm{im}(\coprod f_i) = \coprod \mathrm{im}(f_i)$, and τ is closed under coproducts by locality. For a general colimit $f' : A' \to B'$ of a diagram $\{f_i : A_i \to B_i\}_{i \in I}$ in $\mathrm{Ar}(T)$, let $f : A \to B$ be the coproduct $\coprod_{i \in I} f_i$, so that $f \in E$. The canonical maps $a : A \twoheadrightarrow A'$ and $b : B \twoheadrightarrow B'$ are effective epimorphisms, and $f'a = bf$. Since b is an effective epimorphism (hence in E) and $f \in E$, the composition $bf = f'a$ lies in E by the above. Since a is an effective epimorphism, $\mathrm{im}(f'a) = \mathrm{im}(f')$, so $f' \in E$.

This establishes that E is saturated. If $m \in \tau^s \cap \mathrm{Mono}$, then $m \in E$, so $\mathrm{im}(m) = m$ lies in τ. Thus $\tau^c \cap \mathrm{Mono} = \tau^s \cap \mathrm{Mono} \subseteq \tau$.

Conversely, let K be a monogenic congruence and set $\tau_K := K \cap \mathrm{Mono}$. The congruence generated by τ_K is

$$\tau_K^c = \tau_K^m = (K \cap \mathrm{Mono})^m = K,$$

where the first equality uses Corollary 5.39, and the last is the monogenicity of K. We now verify that τ_K is a Grothendieck topology.

(a) Since τ_K is closed under base change (being $K \cap \mathrm{Mono}$), Proposition 5.29 gives $K = \tau_K^m = (\tau_K^{bc})^s = \tau_K^s$. In particular, τ_K^s is of small generation.

(b) The class τ_K is local:

- closure under base change is immediate from base-change stability of K and Mono;
- closure under small coproducts follows since K is saturated (hence closed under coproducts in $\mathrm{Ar}(T)$), and coproducts of monomorphisms are monomorphisms in a topos;
- for descent along effective epimorphisms, consider a pullback square as in Definition 2.46 with $f' \in \tau_K$. Then $f' \in K$, so $f \in K$ by locality of K. Also, since pullback along an effective epimorphism is conservative and preserves monomorphisms, f is monic. Hence $f \in K \cap \mathrm{Mono} = \tau_K$.

(c) If $f, g \in \tau_K$, then $f, g \in K$ and f, g are monomorphisms. Since K is saturated, $gf \in K$, and since monomorphisms are closed under composition, $gf \in \mathrm{Mono}$. Thus $gf \in \tau_K$.

(d) Let $V \overset{f}{\hookrightarrow} U \overset{g}{\hookrightarrow} X$ be monomorphisms with $gf \in \tau_K$. Since $gf \in K$, the image of gf in the quotient $q : T \to T/K$ is an isomorphism, and thus $q(g)$ admits a section. As q is left exact, $q(g)$ is a monomorphism, hence an isomorphism, so $g \in K$. Since g is monic, $g \in K \cap \mathrm{Mono} = \tau_K$.

Hence τ_K is a Grothendieck topology. Together with $\tau_K^c = K$ and $\tau^c \cap \mathrm{Mono} = \tau$, this shows the two assignments are inverse. □

6.1.2 Sheaves and Čech descent

We now introduce the category of sheaves on a Grothendieck site, and formulate its objects in terms of Čech descent.

Definition 6.9 (Sheaf category, [Lur09, Definition 6.2.2.6])**.** Let C be a category equipped with a Grothendieck topology τ. We denote by

$$\mathrm{Shv}_\tau(C) \quad \subseteq \quad \mathrm{PSh}(C)$$

the full subcategory of τ-sheaves. We denote the left exact left adjoint by

$$L_\tau\colon\ \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C).$$

The functor L_τ is known as the *sheafification functor*.

Example 6.10. Let X be a topological space. Then the poset $\mathrm{Open}(X)$ of open subsets admits a Grothendieck topology, called the *open covering topology*, for which the covering sieves of some $U \in \mathrm{Open}(X)$ are those sieves generated by open coverings $U = \bigcup_{i\in I} U_i$. We denote the resulting topos by

$$\mathrm{Shv}(X) \quad := \quad \mathrm{Shv}_{\mathrm{open}}(\mathrm{Open}(X)).$$

Definition 6.11. The *sheafified Yoneda functor* y_τ is the composite

$$C \overset{y}{\hookrightarrow} \mathrm{PSh}(C) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(C).$$

Notation 6.12. Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a collection of morphisms in a category C. The morphism $\bigsqcup_{i\in I} y(U_i) \to y(X)$ in $\mathrm{PSh}(C)$ factors as an effective epimorphism followed by a monomorphism:

$$\bigsqcup_{i\in I} y(U_i) \twoheadrightarrow U \hookrightarrow y(X).$$

We refer to the sieve $U \hookrightarrow y(X)$ as the *sieve generated by* $\mathcal{U}$.

Since limits and colimits in $\mathrm{PSh}(C)$ are computed pointwise, we see that this epi-mono factorization is also computed pointwise: for every $Z \in C$, the subanima

$$U(Z) \subseteq y(X)(Z) = \mathrm{Hom}_C(Z, X)$$

consists of those morphisms $Z \to X$ in C that factor through $U_i \to X$ for some $i \in I$.

If C carries a Grothendieck topology τ, we say that $\mathcal{U}$ is a *covering family* if the sieve it generates is a covering sieve.

Sheaves on C may equivalently be described in terms of *Čech descent*.

Definition 6.13 (Čech descent)**.** Let C be a category and let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a collection of morphisms. We will denote by $\check{C}_\bullet(\mathcal{U})$ the Čech nerve of the morphism

$$\bigsqcup_{i\in I} y(U_i) \to y(X)$$

in $\mathrm{PSh}(C)$. A presheaf $\mathcal{F} \in \mathrm{PSh}(C)$ is said to *satisfy Čech descent with respect to* $\mathcal{U}$ if the map

$$\mathcal{F}(X) = \mathrm{Hom}_{\mathrm{PSh}(C)}(y(X), \mathcal{F}) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}_n(\mathcal{U}), \mathcal{F})$$

is an equivalence. More concretely, if the relevant iterated fiber products of the U_i over X exist in C, then $\mathcal{F}$ satisfies Čech descent with respect to $\mathcal{U}$ if the diagram

$$\mathcal{F}(X) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_X U_j) \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots$$

is a limit diagram.

Proposition 6.14 (cf. [Lur09, Lemma 6.2.3.18]). *Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a collection of morphisms in a category C and let $U \hookrightarrow y(X)$ be the sieve generated by $\mathcal{U}$. Then a presheaf $\mathcal{F}$ on C satisfies Čech descent with respect to $\mathcal{U}$ if and only if it is local with respect to $U \hookrightarrow y(X)$.*

Proof. By definition, $U \hookrightarrow y(X)$ is obtained from the epi-mono factorization $\bigsqcup_{i\in I} y(U_i) \twoheadrightarrow U \hookrightarrow y(X)$. It follows that U is equivalent to the colimit of the Čech nerve $\check{C}(\mathcal{U})$ of $\mathcal{U}$, and we obtain for every $\mathcal{F} \in \mathrm{PSh}(C)$ an equivalence

$$\mathrm{Hom}_{\mathrm{PSh}(C)}(U, \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{PSh}(C)}(\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \check{C}_n(\mathcal{U}), \mathcal{F}) \simeq \lim_{[n]\in\mathbf{\Delta}} \mathrm{Hom}_{\mathrm{PSh}(C)}(\check{C}_n(\mathcal{U}), \mathcal{F}).$$

The claim now follows immediately. □

6.1.3 Effective epimorphisms in sheaf topoi

The effective epimorphisms in a sheaf topos $\mathrm{Shv}_\tau(C)$ can be characterized as those morphisms which *admit local sections*, in the following sense:

Definition 6.15. Let C be a category equipped with a Grothendieck topology τ. Let $f\colon X \to Y$ be a morphism in $\mathrm{Shv}_\tau(C)$ and assume that $Y = y(Y')$ lies in the image of the sheafified Yoneda functor $y\colon C \to \mathrm{Shv}_\tau(C)$. We say that f *admits local sections* if there exists a covering family $\{U_i \to Y'\}_{i\in I}$ of Y' such that the base change $f'\colon X \times_Y y(U_i) \to y(U_i)$ of f along the map $y(U_i) \to Y$ admits a section for every $i \in I$.

If $f\colon X \to Y$ is an arbitrary morphism in $\mathrm{Shv}_\tau(C)$, we say that f *admits local sections* if its base change along any map $y(Y') \to Y$ from a representable admits local sections.

Lemma 6.16. *A morphism $f\colon X \to Y$ in $\mathrm{Shv}_\tau(C)$ is an effective epimorphism if and only if it admits local sections.*

Proof. Since every map with a section is an effective epimorphism (Lemma 2.38) and effective epimorphisms form a local class, it is clear that any morphism admitting local sections is an effective epimorphism.

Conversely, assume $f\colon X \to Y$ is an effective epimorphism. Since effective epimorphisms are stable under base change, it suffices (by the definition of local sections) to treat the case where $Y = y(Y')$ is representable. Consider the epi–mono factorization in $\mathrm{PSh}(C)$:

$$X \twoheadrightarrow U_f \hookrightarrow y(Y').$$

Applying sheafification $L_\tau\colon \mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C)$ gives a factorization

$$X = L_\tau(X) \twoheadrightarrow L_\tau(U_f) \longrightarrow y(Y').$$

The first map is an effective epimorphism since L_τ preserves colimits, and the composite is f, hence also an effective epimorphism by assumption. By cancellation of effective epimorphisms, the map $L_\tau(U_f) \to y(Y')$ is an effective epimorphism. It is also a monomorphism, since L_τ is left exact. Hence it is an isomorphism.

Therefore the monomorphism $U_f \hookrightarrow y(Y')$ is inverted by L_τ. By Theorem 6.2, the class of morphisms inverted by L_τ is the congruence τ^c. So $U_f \hookrightarrow y(Y')$ lies in $\tau^c \cap \mathrm{Mono} = \tau$ by Proposition 6.8, i.e. it is a covering sieve on Y'. Unwinding the definition of U_f, this exactly says that f admits local sections. □

6.1.4 Morphisms of sites

Given a functor $u\colon C \to D$ between two Grothendieck sites, we may ask under what conditions u induces a topos morphism between the sheaf categories of C and D. There are two natural candidates for such a notion, called *continuous* and *cocontinuous* morphisms of sites.

Definition 6.17 (Continuous morphism)**.** Let (C,τ) and (D,τ') be Grothendieck sites. A functor $u\colon C \to D$ is called *continuous* if the restriction functor $u^*\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ preserves sheaves, i.e. restricts to a functor

$$u^*\colon \mathrm{Shv}_{\tau'}(D) \to \mathrm{Shv}_\tau(C).$$

We say that u is a *continuous morphism of sites* if this restriction is a morphism of topoi, i.e. its left adjoint is left exact. We denote the category of Grothendieck sites and continuous morphisms of sites by $\mathrm{Site}^{\mathrm{cont}}$. By definition, the assignment $(C,\tau) \mapsto \mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C)$ determines a contravariant functor

$$\mathrm{Shv}(-)\colon (\mathrm{Site}^{\mathrm{cont}})^{\mathrm{op}} \to \mathrm{Topos},$$

obtained by restricting the precomposition functoriality of the presheaf construction $\mathrm{PSh}(-)\colon \mathrm{Cat}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$.

Remark 6.18. The left adjoint of the restriction of u^* is given by the composite

$$\mathrm{Shv}_\tau(C) \hookrightarrow \mathrm{PSh}(C) \xrightarrow{u_!} \mathrm{PSh}(D) \xrightarrow{L_{\tau'}} \mathrm{Shv}_{\tau'}(D),$$

where $u_!$ is left Kan extension along u. In particular, any continuous functor u for which the left Kan extension functor $u_!$ is left exact is a continuous morphism of sites.

Note that $u_!$ is left exact whenever u admits a left adjoint, since then $u_!$ is given by restriction along that left adjoint.

The following criterion is often useful for checking continuity:

Lemma 6.19. *Let (C,τ) and (D,τ') be Grothendieck sites, and let $u\colon C \to D$ be a functor. Assume that every covering sieve $U \hookrightarrow y(X)$ of $X \in C$ is generated by a collection of morphisms $\{f_i\colon U_i \to X\}_{i\in I}$ in C satisfying the following conditions:*

- *Pullbacks along f_i exist in C and are preserved by $u\colon C \to D$;*
- *The sieve on $y(u(X))$ in D generated by the maps $u(f_i)$ is a covering sieve.*

Then u is a continuous functor.

Proof. Let $\mathcal{F}$ be a τ'-sheaf on D. We need to show that $u^*\mathcal{F}$ is a τ-sheaf on C. For a covering sieve $U \hookrightarrow y(X)$, let $\mathcal{U} = \{f_i\colon U_i \to X\}_{i\in I}$ be a collection of morphisms in C generating it satisfying the two conditions. Using Proposition 6.14 and the adjunction $u_! \dashv u^*$, we may equivalently show that the map

$$(u^*\mathcal{F})(X) \cong \mathrm{Hom}_{\mathrm{PSh}(D)}(y(u(X)),\mathcal{F}) \to \lim_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{PSh}(D)}(u_!(\check{C}_n(\mathcal{U})),\mathcal{F})$$

is an isomorphism. The first assumption on $\mathcal{U}$ guarantees that the simplicial object $u_!(\check{C}_n(\mathcal{U}))$ in $\mathrm{PSh}(D)$ agrees with the Čech nerve of the map $\bigsqcup_{i\in I} y(u(U_i)) \to y(u(X))$. The second assumption says that the associated sieve is a covering sieve. So a second application of Proposition 6.14 implies the claim. □

Corollary 6.20. *Let (C,τ) and (D,τ') be Grothendieck sites, and assume that the following two conditions are satisfied:*

- *The category C admits finite limits, and $u\colon C \to D$ preserves finite limits;*
- *The functor u preserves covering families.*

Then u is a continuous morphism of sites.

Proof. The left Kan extension functor $u_\sharp\colon \mathrm{PSh}(C) \to \mathrm{PSh}(D)$ is left exact by Proposition 2.43, as it is the colimit extension of the left exact functor $C \xrightarrow{u} D \hookrightarrow \mathrm{PSh}(D)$. Since u is a continuous functor by Lemma 6.19, the claim follows from Theorem 6.18. □

Example 6.21. Let $f\colon X \to Y$ be a continuous map of topological spaces. Then the preimage functor $f^{-1}\colon \mathrm{Open}(Y) \to \mathrm{Open}(X)$ preserves finite limits and preserves coverings, hence is a continuous morphism of sites. This gives rise to a functor $\mathrm{Open}(-)\colon \mathrm{Top}^{\mathrm{op}} \to \mathrm{Site}^{\mathrm{cont}}$. Composing it with the sheaf functor $(\mathrm{Site}^{\mathrm{cont}})^{\mathrm{op}} \to \mathrm{Topos}$, we obtain a functor

$$\mathrm{Shv}\colon \mathrm{Top} \to \mathrm{Topos}.$$

We now discuss a second mechanism by which a functor between Grothendieck sites induces a morphism of topoi, namely cocontinuity.

Definition 6.22 (Cocontinuous morphism)**.** Let (C,τ) and (D,τ') be Grothendieck sites. A functor $u\colon C \to D$ is called *cocontinuous morphism of sites* if the right Kan extension functor restricts to a functor

$$u_*\colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_{\tau'}(D).$$

In particular, u_* is a morphism of topoi, with left exact left adjoint given by the composite

$$\mathrm{Shv}_{\tau'}(D) \hookrightarrow \mathrm{PSh}(D) \xrightarrow{u^*} \mathrm{PSh}(C) \xrightarrow{L_\tau} \mathrm{Shv}_\tau(C).$$

If we denote by $\mathrm{Site}^{\mathrm{cocont}}$ the category of Grothendieck sites and cocontinuous morphisms, then the assignment $(C,\tau) \mapsto \mathrm{Shv}_\tau(C)$ defines a functor

$$\mathrm{Site}^{\mathrm{cocont}} \to \mathrm{Topos},$$

obtained by restricting the right Kan extension functoriality of the presheaf construction $\mathrm{PSh}\colon \mathrm{Cat} \to \mathrm{Pr}^{\mathrm{R}}$.

Let us make the cocontinuity condition on u more concrete.

Notation 6.23. Given an object $X \in C$ and a sieve $U \hookrightarrow y(u(X))$ of $u(X)$ in D, we define its *pullback along u* as the sieve $u^{-1}(U) \hookrightarrow y(X)$ of X obtained by forming the following pullback square in $\mathrm{PSh}(C)$:

$$\begin{tikzcd}
u^{-1}(U) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & u^*(U) \arrow[d, hook] \\
y(X) \arrow[r] & u^*(y(u(X)));
\end{tikzcd}$$

here the bottom map is the unit map $y(X) \to u^*u_!y(X) = u^*(y(u(X)))$. Note that $u^{-1}(U)$ consists of those morphisms $Z \to X$ in C for which the map $u(Z) \to u(X)$ lies in the sieve U.

Lemma 6.24. *A functor $u\colon C \to D$ is a cocontinuous morphism of sites if and only if for every $X \in C$ and every covering sieve $U \hookrightarrow y(u(X))$ of $u(X)$ in D, the pullback sieve $u^{-1}(U) \hookrightarrow y(X)$ is a covering sieve in C.*

Proof. By definition, u is a cocontinuous morphism if and only if the restriction functor $u^*\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ preserves the covering monomorphisms, or equivalently if for every covering sieve $V \hookrightarrow y(Y)$ in D the map $u^*(V) \hookrightarrow u^* y(Y)$ is a covering monomorphism in $\mathrm{PSh}(C)$. The latter condition means that for any map $f\colon y(X) \to u^* y(Y)$ in $\mathrm{PSh}(C)$ from a representable, the base change $u^*(V) \times_{u^* y(Y)} y(X) \hookrightarrow y(X)$. The map f corresponds to a map $u(X) \to Y$ in D, and pulling back the covering sieve $V \hookrightarrow y(Y)$ gives a covering sieve $U \hookrightarrow y(u(X))$. The claim now follows, since we have a pullback square

$$\begin{tikzcd}
u^{-1}(U) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & u^*(U) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & u^*(V) \arrow[d] \\
y(X) \arrow[r] & u^*(y(u(X))) \arrow[r] & u^*(y(Y)).
\end{tikzcd}$$

□

Corollary 6.25. *Let $v\colon (C, \tau) \hookrightarrow (D, \tau')$ be a functor between sites and assume that v admits a right adjoint $u\colon D \to C$. Then v is a cocontinuous morphism of sites if and only if u is a continuous morphism of sites. In this case, we have $v_* \simeq u^*$.*

Proof. The adjunction $v \dashv u$ induces an adjunction $v^* \dashv u^*$ on presheaf categories, showing that $v_* \simeq u^*$. By definition v is a cocontinuous morphism of sites if and only v_* preserves sheaves, while it follows from Theorem 6.18 that u is a continuous morphism of sites if and only if u^* preserves sheaves, showing the claim. □

6.2 Groupoid actions and principal bundles

In this section we discuss actions of groupoid objects in a topos T and prove that the category of such actions is equivalent to the slice over the classifying stack. We further recall the notion of a principal bundle and show that principal bundles are classified by maps into the classifying stack. Our treatment is based on [NSS15] and [SS21, Section 3.2], where the analogous situation for *group* actions is discussed.

Throughout this section, we fix a topos T and a groupoid object $\mathcal{G}\colon \mathbf{\Delta}^{\mathrm{op}} \to T$.

Remark 6.26. The material from this section was not actually discussed in Marc's course. I had written it as part of my PhD thesis, and decided to include the material here as well for easier reference. I have tried to convert all notations to the ones used in these notes, but there may be some leftover notations.

6.2.1 Actions by groupoid objects

We introduce the notion of an action of a groupoid object, generalizing the treatment of group actions in [SS21].

Definition 6.27. Let $\mathcal{G}$ be a groupoid object in T, and let $X \in T$ be an object. An *action* of $\mathcal{G}$ on X consists of the following data:

(1) a groupoid object $\mathcal{G} \ltimes X$ in T;

(2) an isomorphism $(\mathcal{G} \ltimes X)_0 \simeq X$;

(3) a cartesian natural transformation $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ of simplicial objects in T.

We let $\mathrm{Act}_{\mathcal{G}}(T) \subseteq \mathrm{Grpd}(T)_{/\mathcal{G}}$ denote the full subcategory spanned by the $\mathcal{G}$-actions in T.

Remark 6.28. Let $\mathcal{G} \ltimes X$ be an action of $\mathcal{G}$ on X. Since $(\mathcal{G} \ltimes X)_0 \simeq X$, the map $c\colon \mathcal{G} \ltimes X \to \mathcal{G}$ induces a structure map $c_0\colon X \to \mathcal{G}_0$. Since c is a cartesian natural transformation, there are cartesian squares

$$\begin{tikzcd}
\cdots \arrow[r] & (\mathcal{G} \ltimes X)_2 \arrow[r, "d_0"] \arrow[d, "c_2"] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{G} \ltimes X)_1 \arrow[r, "d_0"] \arrow[d, "c_1"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "c_0"] \\
\cdots \arrow[r] & \mathcal{G}_2 \arrow[r, "d_0"] & \mathcal{G}_1 \arrow[r, "d_0"] & \mathcal{G}_0,
\end{tikzcd}$$

and in particular we obtain isomorphisms

$$\begin{aligned}
&(\mathcal{G} \ltimes X)_1 \simeq \mathcal{G}_1 \times_{\mathcal{G}_0} X, \\
&(\mathcal{G} \ltimes X)_2 \simeq \mathcal{G}_2 \times_{\mathcal{G}_0} X \simeq \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \times_{\mathcal{G}_0} X,
\end{aligned}$$

and so on. Under these isomorphisms, the map $d_1\colon (\mathcal{G} \ltimes X)_1 \to (\mathcal{G} \ltimes X)_0$ corresponds to a map $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$, which we think of as the *action map*. The rest of the simplicial diagram $\mathcal{G} \ltimes X$ encodes the data witnessing that this action is unital and associative up to coherent homotopy.

Definition 6.29. For a groupoid object $\mathcal{G}$, we define its *classifying stack* as the geometric realization $\mathbf{B}\mathcal{G} := |\mathcal{G}|$.

If $\mathcal{G} \ltimes X$ is an action of a groupoid object $\mathcal{G}$ on an object $X \in T$, we define the *quotient* $X /\!\!/ \mathcal{G}$ as the geometric realization of $\mathcal{G} \ltimes X$:

$$X /\!\!/ \mathcal{G} \quad := \quad |\mathcal{G} \ltimes X| \quad \simeq \quad \operatorname{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} \mathcal{G}_n \times_{\mathcal{G}_0} X.$$

The cartesian transformation $\mathcal{G} \ltimes X \to \mathcal{G}$ induces a map $X /\!\!/ \mathcal{G} \to |\mathcal{G}| = \mathbf{B}\mathcal{G}$ on geometric realizations, and this defines a functor

$$- /\!\!/ \mathcal{G}\colon\ \mathrm{Act}_{\mathcal{G}}(T) \to T_{/\mathbf{B}\mathcal{G}}.$$

The following result shows that groupoid actions are classified by the classifying stack:

Proposition 6.30 (Classification of groupoid actions)**.** *The quotient functor is an equivalence of categories:*

$$- /\!\!/ \mathcal{G}\colon\ \mathrm{Act}_{\mathcal{G}}(T) \xrightarrow{\sim} T_{/\mathbf{B}\mathcal{G}}.$$

Proof. Recall from Lemma 2.33 that sending a groupoid $\mathcal{G}$ to the map $\mathcal{G}_0 \to \mathbf{B}\mathcal{G}$ defines an equivalence

$$\mathrm{Grpd}(T) \xrightarrow{\sim} \mathrm{EffEpi}(T)$$

between the categories of groupoid objects and effective epimorphisms in T. Moreover, it follows from descent that a morphism of groupoids $f\colon \mathcal{H} \to \mathcal{G}$ is cartesian if and only if the induced square

$$\begin{tikzcd} \mathcal{H}_0 \arrow[r] \arrow[d, "f_0"'] & \mathbf{B}\mathcal{H} \arrow[d, "\mathbf{B}f"] \\ \mathcal{G}_0 \arrow[r] & \mathbf{B}\mathcal{G} \end{tikzcd}$$

is a pullback square, giving an equivalence of categories $\mathrm{Grpd}(T)^{\mathrm{cart}} \xrightarrow{\sim} \mathrm{EffEpi}(T)^{\mathrm{cart}}$. Passing to slices over $\mathcal{G}$, this results in an equivalence

$$\mathrm{Act}_{\mathcal{G}}(T) \quad = \quad (\mathrm{Grpd}(T)^{\mathrm{cart}})_{/\mathcal{G}} \quad \xrightarrow{\sim} \quad (\mathrm{EffEpi}(T)^{\mathrm{cart}})_{/(\mathcal{G}_0 \to \mathbf{B}\mathcal{G})} \quad \xrightarrow{\sim} \quad T_{/\mathbf{B}\mathcal{G}}.$$

Here the last equivalence is induced by the target map $\mathrm{EffEpi}(T) \to T$, which is easily seen to be an equivalence whose inverse sends a morphism $B \to \mathbf{B}\mathcal{G}$ to the pullback square

$$\begin{tikzcd} B \times_{\mathbf{B}\mathcal{G}} \mathcal{G}_0 \arrow[r] \arrow[d, two heads] & \mathcal{G}_0 \arrow[d, two heads] \\ B \arrow[r] & \mathbf{B}\mathcal{G}. \end{tikzcd}$$

□

6.2.2 Free and transitive actions

Given a $\mathcal{G}$-action in T, one can define when the action is *free* or *transitive*. Actions which are both free and transitive will correspond to principal $\mathcal{G}$-bundles.

Definition 6.31 (Shear maps)**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object X in T. We define the *shear map* of $\mathcal{G} \ltimes X$ as the map

$$\mathrm{shear}_1 = (a, \mathrm{pr}_X)\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X \times X,$$

where $a\colon \mathcal{G}_1 \times_{\mathcal{G}_0} X \to X$ is the action map and pr_X is the projection onto X.

More generally, we define the *higher shear maps* as follows. Consider the commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, two heads] & X \arrow[d] \\ X /\!\!/ \mathcal{G} \arrow[r] & *. \end{tikzcd}$$

Regarding this as a morphism from $X \twoheadrightarrow X/\!\!/\mathcal{G}$ to $X \to *$ in $\mathrm{Ar}(T)$ and passing to Čech nerves, we obtain a morphism of groupoid objects

$$\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *).$$

Under the identifications of Remark 6.28, this may be displayed as follows:

$$\begin{tikzcd} \mathcal{G} \ltimes X \arrow[d, "\mathrm{shear}"] & = & \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathcal{G}_2 \times_{\mathcal{G}_0} X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\mathrm{shear}_2"] & \mathcal{G}_1 \times_{\mathcal{G}_0} X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\mathrm{shear}_1"] & X \arrow[d, equal, "\mathrm{shear}_0"] \\ \check{C}(X \to *) & = & \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \times X \times X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times X \arrow[r, shift left] \arrow[r, shift right] & X. \end{tikzcd}$$

Definition 6.32 (Free, transitive and regular actions)**.** Let $\mathcal{G} \ltimes X$ be an action of a groupoid object $\mathcal{G}$ on an object $X \in T$. We say that this action is

(1) *free* if its shear map shear_1 is a monomorphism;

(2) *transitive* if its shear map shear_1 is an effective epimorphism;

(3) *regular* if its shear map shear_1 is an isomorphism.

Note that $\mathcal{G} \ltimes X$ is regular if and only if it is both free and transitive.

In case the morphism $X \to *$ is an effective epimorphism in T, regularity of $\mathcal{G} \ltimes X$ can be expressed as the triviality of the quotient:

Lemma 6.33. *Let $X \in T$ be an object such that $X \to *$ is an effective epimorphism. Let $\mathcal{G} \ltimes X$ be an action of a groupoid object on X. Then the action is regular if and only if the quotient $X /\!\!/ \mathcal{G}$ is the terminal object of T.*

Proof. First assume that $X /\!\!/ \mathcal{G} \simeq *$. Then the map $X \twoheadrightarrow X /\!\!/ \mathcal{G}$ is equivalent to the map $X \to *$. It follows that the shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *)$ is an isomorphism, being defined by applying Čech nerves to these two equivalent morphisms.

Conversely, assume that $\mathcal{G} \ltimes X$ is regular and that $X \twoheadrightarrow *$ is an effective epimorphism. The map $X /\!\!/ \mathcal{G} \to *$ is obtained from the higher shear map $\mathrm{shear}\colon (\mathcal{G} \ltimes X) \to \check{C}(X \to *)$ by passing to colimits. It thus suffices to show that each of the higher shear maps $\mathrm{shear}_n\colon \mathcal{G}_n \times_{\mathcal{G}_0} X \to X^{n+1}$ is an isomorphism. We proceed by induction. The case $n = 0$ is trivial and the case $n = 1$ is assumed. For the inductive step, consider the diagram

$$\begin{tikzcd}
& \mathcal{G}_n \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_n"] \arrow[dl] \arrow[dd] & & X^{n+1} \arrow[dl] \arrow[dd] \\
\mathcal{G}_{n-1} \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_{n-1}", "\simeq"'] \arrow[dd] & & X^n \arrow[dd] & \\
& \mathcal{G}_1 \times_{\mathcal{G}_0} X \arrow[rr, "\mathrm{shear}_1", "\simeq"'] \arrow[dl] & & X^2 \arrow[dl] \\
X \arrow[rr, equal, "\mathrm{shear}_0"] & & X. &
\end{tikzcd}$$

Since the left and right faces are pullback squares, it follows that shear_n is an isomorphism as well. ◻

6.2.3 Principal bundles

Given a groupoid object $\mathcal{G}$ in T and an object $B \in T$, we obtain a notion of *principal $\mathcal{G}$-bundles over B* by applying the definition of regular actions to the slice topos $T_{/B}$. Observe that $\mathcal{G}$ gives rise to a groupoid $\mathcal{G} \times B$ in the slice $T_{/B}$ by applying the pullback functor $B \times -\colon T \to T_{/B}$. For every object $P \in T_{/B}$, there is an isomorphism

$$(\mathcal{G}_n \times B) \times_{\mathcal{G}_0 \times B} P \;\simeq\; \mathcal{G}_n \times_{\mathcal{G}_0} P \qquad \text{in } T_{/B},$$

and thus a $(\mathcal{G}\times B)$-action on P is the same as a $\mathcal{G}$-action on the underlying object of P together with a lift of the action groupoid along the forgetful functor $T_{/B} \to T$. For this reason, we will refer to a $(\mathcal{G}\times B)$-action in the slice $T_{/B}$ simply as a *$\mathcal{G}$-action over B*.

Definition 6.34 (Principal bundles)**.** Let $B \in T$ be an object and let $\mathcal{G}$ be a groupoid object in T.

(1) Let $p\colon P \to B$ be an object of $T_{/B}$ equipped with a $\mathcal{G}$-action over B. We say that p is a *formally principal $\mathcal{G}$-bundle over B* if this action is regular as a $(\mathcal{G}\times B)$-action in the slice $T_{/B}$, i.e. if the shear map

$$\mathrm{shear}_1\colon \mathcal{G}_1 \times_{\mathcal{G}_0} P \xrightarrow{\sim} P \times_B P$$

is an isomorphism.

(2) A formally principal $\mathcal{G}$-bundle $p\colon P \to B$ is called a *principal $\mathcal{G}$-bundle* if p is an effective epimorphism.

(3) We write

$$\mathrm{Bun}_{\mathcal{G}}(B) \quad\subseteq\quad \mathrm{Act}_{\mathcal{G}}(T_{/B})$$

for the full subcategory of principal $\mathcal{G}$-bundles over B.

Remark 6.35. For any morphism $f\colon B' \to B$ in T, the pullback functor $f^*\colon T_{/B} \to T_{/B'}$ preserves principal $\mathcal{G}$-bundles.

Lemma 6.36. *Let $p\colon P \twoheadrightarrow B$ be an effective epimorphism equipped with a $\mathcal{G}$-action over B, and let $P/\!\!/\mathcal{G} \in T_{/B}$ denote the quotient. Then p is a principal $\mathcal{G}$-bundle if and only if the canonical map $P/\!\!/\mathcal{G} \to B$ is an isomorphism.*

Proof. This is an instance of Lemma 6.33 applied in the slice topos $T_{/B}$. □

It follows from Lemma 6.36 that the forgetful functor from principal $\mathcal{G}$-bundles over B to $\mathcal{G}$-actions in T lands in the fiber over B of the quotient functor $-/\!\!/\mathcal{G}\colon\ \mathrm{Act}_{\mathcal{G}}(T) \to T$.

Proposition 6.37. *The forgetful functor induces an equivalence of categories*

$$\mathrm{Bun}_{\mathcal{G}}(B) \xrightarrow{\sim} \mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\}.$$

Proof. We construct an explicit inverse. Consider a $\mathcal{G}$-action $\mathcal{G} \ltimes P$ in T equipped with an isomorphism $B \xrightarrow{\sim} P/\!\!/\mathcal{G}$. The colimit cocone $(\Delta^{\mathrm{op}})^{\triangleright} \to T$ exhibiting $B \simeq P/\!\!/\mathcal{G}$ as a colimit of $\mathcal{G} \ltimes P$ adjoints over to provide a lift of the simplicial object $\mathcal{G}\ltimes P\colon \Delta^{\mathrm{op}} \to T$ to the slice $T_{/B}$. By adjunction, this lift comes equipped with a groupoid map to $\mathcal{G} \times B$, and since the forgetful functor $T_{/B} \to T$ preserves pullbacks this is a cartesian morphism of groupoids, thus defining a $(\mathcal{G}\times B)$-action in $T_{/B}$. This construction defines a functor

$$\mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\} \to \mathrm{Act}_{\mathcal{G}}(T_{/B}).$$

It follows directly from Lemma 6.36 that this functor factors through the subcategory $\mathrm{Bun}_{\mathcal{G}}(B)$ of principal $\mathcal{G}$-bundles over B. The two functors are clearly inverse to each other. □

We are now ready to state the main classification result:

Theorem 6.38 (Classification of principal bundles)**.** *Let $\mathcal{G}$ be a groupoid object in T and let $B \in T$ be an object. There is an equivalence of categories*

$$\mathrm{Hom}_T(B, \mathbf{B}\mathcal{G}) \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{G}}(B),$$

given on objects by sending a morphism $c\colon B \to \mathbf{B}\mathcal{G}$ to the principal $\mathcal{G}$-bundle $P \to B$ defined by the pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, two heads] & \mathcal{G}_0 \arrow[d, two heads] \\ B \arrow[r, "c"] & \mathbf{B}\mathcal{G}. \end{tikzcd}$$

Proof. Recall from Proposition 6.30 the equivalence $-/\!\!/\mathcal{G}\colon\ \mathrm{Act}_{\mathcal{G}}(T) \xrightarrow{\sim} T_{/\mathbf{B}\mathcal{G}}$, which induces an equivalence on fibers over T:

$$\mathrm{Act}_{\mathcal{G}}(T) \times_T \{B\} \ \simeq\ T_{/\mathbf{B}\mathcal{G}} \times_T \{B\} \ \simeq\ \mathrm{Hom}_T(B, \mathbf{B}\mathcal{G}).$$

The inverse sends a morphism $c\colon B \to \mathbf{B}\mathcal{G}$ to the base change $P = B \times_{\mathbf{B}\mathcal{G}} \mathcal{G}_0 \twoheadrightarrow B$ of the atlas $\mathcal{G}_0 \twoheadrightarrow \mathbf{B}\mathcal{G}$ along c. Combining with the equivalence from Proposition 6.37 finishes the proof. □

Corollary 6.39. *For every groupoid object $\mathcal{G}$ in T and every object $B \in T$, the category* $\mathrm{Bun}_{\mathcal{G}}(B)$ *is an anima.* □

6.3 Essential morphisms and shape theory

We introduce the notion of an *essential* morphism of topoi, closely following the presentation of Aizenbud and Carmeli [AC21, Section 3.1]. Essential morphisms are those for which the inverse image functor admits a further left adjoint, allowing one to define a notion of *shape* that measures the homotopy-theoretic complexity of a topos.

Definition 6.40 (Essential morphism)**.** A morphism of topoi $\varphi_*\colon T \to S$ is called *essential* if the inverse image functor $\varphi^*\colon S \to T$ admits a further left adjoint $\varphi_\sharp\colon T \to S$. We thus have an adjoint triple

$$\varphi_\sharp \quad\dashv\quad \varphi^* \quad\dashv\quad \varphi_*.$$

We call a topos T *essential* if the terminal morphism $\Gamma_*\colon T \to \mathrm{An}$ is essential.

Example 6.41. Every étale morphism of topoi is essential: given a topos T and an object $U \in T$, the functor $j^*\colon T \to T_{/U}$ given by $X \mapsto X \times U$ admits a left adjoint $j_\sharp = j_!\colon T_{/U} \to T$ given by the forgetful functor.

Example 6.42. For any small category C, the presheaf topos $\mathrm{PSh}(C)$ is essential. Indeed, the terminal morphism $\Gamma_*\colon\ \mathrm{PSh}(C) \to \mathrm{An}$ is given by taking the limit $\Gamma_*(F) = \lim_{c\in C} F(c)$, and its left adjoint $\Gamma^*\colon \mathrm{An} \to \mathrm{PSh}(C)$ sends an anima A to the constant presheaf $\underline{A}$. The functor Γ^* admits a further left adjoint $\Gamma_\sharp\colon\ \mathrm{PSh}(C) \to \mathrm{An}$ given by the colimit:

$$\Gamma_\sharp(F) \quad=\quad \mathrm{colim}_{c\in C}\, F(c).$$

For a general morphism of topoi φ_*, the inverse image functor φ^* does not necessarily admit a left adjoint. However, it always does after passing to pro-categories. This leads to the notion of the *relative shape* of a morphism of topoi. Recall that for a category C which admits finite limits, the *pro-category* of C is defined by

$$\mathrm{Pro}(C) \quad := \quad \mathrm{Ind}(C^{\mathrm{op}})^{\mathrm{op}} \quad \simeq \quad \mathrm{Fun}^{\mathrm{lex}}(C, \mathrm{An})^{\mathrm{op}}.$$

There is a fully faithful embedding $C \hookrightarrow \mathrm{Pro}(C)$ given by the Yoneda embedding $X \mapsto \mathrm{Hom}_C(X, -)$. An object of $\mathrm{Pro}(C)$ is called *pro-constant* (or *corepresentable*) if it lies in the essential image of this embedding.

Definition 6.43. Let $\varphi_* \colon T \to S$ be a morphism of topoi. We define the functor

$$\varphi_\sharp \colon T \to \mathrm{Pro}(S) \subseteq \mathrm{Fun}(S, \mathrm{An})^{\mathrm{op}}$$

by sending $X \in T$ to $\mathrm{Hom}_T(X, \varphi^*(-)) \colon S \to \mathrm{An}$.

Lemma 6.44. *Let $\varphi_* \colon T \to S$ be a morphism of topoi.*

(1) *The functor $\varphi_\sharp \colon T \to \mathrm{Pro}(S)$ preserves finite limits.*

(2) *The functor $\varphi_\sharp$ is left adjoint to the composite* $\mathrm{Pro}(S) \hookrightarrow \mathrm{Fun}(S, \mathrm{An})^{\mathrm{op}} \xrightarrow{\mathrm{ev}_{\varphi^*}} \mathrm{Fun}(T, \mathrm{An})^{\mathrm{op}}$, *where the second functor is induced by precomposition with φ^*.*

(3) *The morphism φ_* is essential if and only if $\varphi_\sharp$ factors through the embedding $S \hookrightarrow \mathrm{Pro}(S)$, in which case the factorization provides a left adjoint to φ^*.*

Proof. (1) The functor $\varphi_\sharp$ is the composite of the Yoneda embedding $T \hookrightarrow \mathrm{Fun}(T, \mathrm{An})^{\mathrm{op}}$ and the functor $\mathrm{Fun}(T, \mathrm{An})^{\mathrm{op}} \to \mathrm{Fun}(S, \mathrm{An})^{\mathrm{op}}$ induced by precomposition with φ^*. The Yoneda embedding preserves all limits, and precomposition with the left exact functor φ^* preserves finite limits.

(2) This follows from the fact that for $X \in T$ and $Y \in S$, we have

$$\mathrm{Hom}_{\mathrm{Pro}(S)}(\varphi_\sharp(X), Y) \quad \simeq \quad \varphi_\sharp(X)(Y) \quad = \quad \mathrm{Hom}_T(X, \varphi^*(Y)).$$

(3) The functor $\varphi_\sharp$ factors through S if and only if for every $X \in T$, the functor $\mathrm{Hom}_T(X, \varphi^*(-))$ is corepresentable. This is equivalent to φ^* admitting a left adjoint. □

Intuitively, the functor $\varphi_\sharp$ models the "homotopy type of the fibers" of the morphism φ_*. This intuition is formalized using shape theory.

Definition 6.45 (Relative shape)**.** Let $\varphi_* \colon T \to S$ be a morphism of topoi. We define the *relative shape* of φ_* as the pro-object

$$|\varphi| \quad := \quad \varphi_\sharp(*) \qquad \in \mathrm{Pro}(S).$$

Note that this pro-object represents the composite functor $\Gamma_T \circ \varphi^* \colon S \to \mathrm{An}$, where $\Gamma_T \colon T \to \mathrm{An}$ is the global sections functor of T.

We say that φ_* is of *constant shape* if $|\varphi|$ is pro-constant, i.e., if it is corepresentable by an object of S.

Definition 6.46 (Shape)**.** The *shape* $|T|$ of a topos T is the relative shape of the terminal morphism $\Gamma_*\colon T \to \mathrm{An}$:

$$|T| \quad := \quad |\Gamma_*| \quad \in \quad \mathrm{Pro}(\mathrm{An}).$$

We say that T *has constant shape* if the pro-anima $|T|$ is pro-constant. We say that an object $X \in T$ *has constant shape* if the slice topos $T_{/X}$ does.

Example 6.47. For a presheaf topos $\mathrm{PSh}(C)$, the shape is given by

$$|\mathrm{PSh}(C)| \quad \simeq \quad \mathrm{colim}_{c\in C} * \;\simeq\; |C|,$$

the geometric realization of C. In particular, $\mathrm{PSh}(C)$ has constant shape.

Example 6.48. Let X be a sufficiently nice topological space (e.g., locally compact Hausdorff). Then the shape $|\mathrm{Shv}(X)|$ of the sheaf topos on X recovers the classical *shape* of X in the sense of Borsuk. For spaces that are not locally contractible, the shape can differ from the weak homotopy type. For instance, the Warsaw circle has trivial homotopy groups but nontrivial shape.

Definition 6.49 (Locally of constant shape)**.** Let $\varphi_*\colon T \to S$ be a morphism of topoi. Given an object $U \in T$, we say that U *has constant shape relative to* φ_* if the composite morphism

$$T_{/U} \xrightarrow{j_*} T \xrightarrow{\varphi_*} S$$

is of constant shape.

We say that φ_* is *locally of constant shape* if there exists a collection of objects $\{U_i\}_{i\in I}$ which generates T under colimits such that each U_i has constant shape relative to φ_*.

A topos T is *locally of constant shape* if the terminal morphism $\Gamma_*\colon T \to \mathrm{An}$ is locally of constant shape.

The following proposition provides a key characterization of essential morphisms:

Proposition 6.50 (Aizenbud and Carmeli [AC21, Proposition 3.1.5])**.** *Let $\varphi_*\colon T \to S$ be a morphism of topoi and let $\{U_i\}_{i\in I}$ be a collection of objects which generates T under colimits. Then the following conditions are equivalent:*

1. *The morphism φ_* is essential;*
2. *Every object U_i has constant shape relative to φ_*.*

Proof. Recall from Example 6.41 that the étale morphism $j_*\colon T_{/U} \to T$ is essential. It follows that the relative shape of the composite $\varphi_* \circ j_*\colon T_{/U} \to S$ is the image of $U \in T_{/U}$ (viewed as the terminal object of $T_{/U}$) under the composite

$$T_{/U} \xrightarrow{j_\sharp} T \xrightarrow{\varphi_\sharp} \mathrm{Pro}(S).$$

Since $j_\sharp$ is the forgetful functor sending $(V \to U)$ to V, we see that the relative shape of $\varphi_* \circ j_*$ is equivalent to $\varphi_\sharp(U) \in \mathrm{Pro}(S)$.

If φ_* is essential, then $\varphi_\sharp$ factors through S, so $\varphi_\sharp(U)$ is pro-constant for all $U \in T$, and in particular for all U_i.

Conversely, assume that $\varphi_\sharp(U_i)$ is pro-constant for all $i \in I$. Since S is closed under colimits in $\mathrm{Pro}(S)$ and the collection $\{U_i\}$ generates T under colimits, it follows that $\varphi_\sharp \colon T \to \mathrm{Pro}(S)$ factors through S. By Lemma 6.44(3), this means φ_* is essential. □

Corollary 6.51 ([Lur17, Proposition A.1.8]). *A topos T is essential if and only if it is locally of constant shape.* □

Essential morphisms interact well with objects of constant shape:

Lemma 6.52. *Let $\varphi_* \colon T_1 \to T_2$ and $\psi_* \colon T_2 \to T_3$ be morphisms of topoi and assume that φ_* is essential. Let $U \in T_1$ be an object which has constant shape relative to the composite $\psi_* \circ \varphi_* \colon T_1 \to T_3$. Then the object $\varphi_\sharp(U) \in T_2$ has constant shape relative to ψ_*.*

Proof. The shape of $\varphi_\sharp(U) \in T_2$ relative to ψ_* is $\psi_\sharp(\varphi_\sharp(U)) \in \mathrm{Pro}(T_3)$. Since φ_* is essential, we have $\varphi_\sharp(U) \in T_2$, and the shape of U relative to $\psi_* \circ \varphi_*$ is

$$(\psi \circ \varphi)_\sharp(U) \;\simeq\; \psi_\sharp(\varphi_\sharp(U)).$$

By assumption, this is pro-constant, so $\varphi_\sharp(U)$ has constant shape relative to ψ_*. □

Corollary 6.53. *Let $\varphi_* \colon T_1 \to T_2$ be an essential morphism of topoi. Then $\varphi_\sharp \colon T_1 \to T_2$ sends objects of constant shape in T_1 to objects of constant shape in T_2.* □

6.4 Hypercovers

In Section 3.4, we defined a topos T to be *hypercomplete* if every ∞-connected morphism is an isomorphism, and showed that the ∞-connected and ∞-truncated morphisms form a factorization system. In this section, we give an alternative characterization of hypercompleteness in terms of *hypercovers*.

Recall that an effective epimorphism $f \colon X \twoheadrightarrow Y$ in a topos T may be thought of as a *cover* of Y. The Čech nerve $\check{C}(f)$ encodes all the iterated self-intersections $X^{\times_Y^n}$ of X over Y, and since f is an effective epimorphism this expresses Y as a colimit of these iterated self-intersections.

In certain situations, it is convenient to work with a more general notion of cover, known as a *hypercover*. A hypercover of Y also comes with an effective epimorphism $X \twoheadrightarrow Y$, but rather than simply using the iterated self-intersections $X^{\times_Y^n}$ one is allowed to further refine these at each stage: there is some object X_1 covering the intersection $X \times_Y X$, some object X_2 covering the triple intersections, and so forth. One can show that the colimit of any hypercover of Y admits a map to Y which is ∞-connected. In particular, if the topos T is hypercomplete then the hypercover expresses Y as a colimit. It turns out that the converse also holds:

Theorem 6.54. *Let T be a topos. The following conditions are equivalent:*

(1) *The topos T is hypercomplete.*

(2) *For every object $X \in T$, every hypercover of X is effective.*

We start by properly defining hypercovers and establishing some basic properties.

Notation 6.55. For each natural number $n \geq 0$, let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $\{[0], [1], \ldots, [n]\}$. If T is a topos, the restriction functor

$$(-)|_{\mathbf{\Delta}^{\leq n}} \colon \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T) \to \operatorname{Fun}((\mathbf{\Delta}^{\leq n})^{\mathrm{op}}, T)$$

admits a right adjoint given by right Kan extension along the inclusion $(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$. We let

$$\operatorname{cosk}_n \colon \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T) \to \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T)$$

denote the composition of the restriction functor with its right adjoint and refer to cosk_n as the *n-coskeleton functor*. A simplicial object $U_\bullet$ is called *n-coskeletal* if the unit map $U_\bullet \to \operatorname{cosk}_n(U_\bullet)$ is an isomorphism, or equivalently if $U_\bullet$ is a right Kan extension of its restriction to $(\mathbf{\Delta}^{\leq n})^{\mathrm{op}}$.

Definition 6.56 (Hypercover)**.** Let T be a topos. A simplicial object $U_\bullet \in \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T)$ is called a *hypercover* of T if, for each $n \geq 0$, the canonical map

$$U_n \to (\operatorname{cosk}_{n-1} U_\bullet)_n$$

is an effective epimorphism. We say that $U_\bullet$ is an *effective hypercover* if the colimit of $U_\bullet$ is a terminal object of T.

For an object $X \in T$, a *hypercover of X* is a hypercover of the slice topos $T_{/X}$.

Remark 6.57. The object $(\operatorname{cosk}_{n-1} U_\bullet)_n$ is known as the *n-th matching object* of $U_\bullet$. By the pointwise formula for right Kan extensions, it may be computed as a limit over the category $(\mathbf{\Delta}^{\leq n-1})_{/[n]}$. Since this category is finite, the object $(\operatorname{cosk}_{n-1} U_\bullet)_n$ is a finite limit of the objects $U_0, \ldots, U_{n-1}$.

Let us spell out the hypercover condition for small n:

- For $n = 0$, the (-1)-coskeleton of any simplicial object is constant at the terminal object, so $(\operatorname{cosk}_{-1} U_\bullet)_0 = *$. The condition thus says that $U_0 \to *$ is an effective epimorphism.
- For $n = 1$, the 0-coskeleton is determined by U_0, and one computes $(\operatorname{cosk}_0 U_\bullet)_1 = U_0 \times U_0$. The condition thus says that the map $(d_1, d_0) \colon U_1 \to U_0 \times U_0$ is an effective epimorphism.
- For $n = 2$, one computes $(\operatorname{cosk}_1 U_\bullet)_2 = U_1 \times_{U_0} U_1 \times_{U_0} U_1$, where the fiber products are taken along the appropriate face maps. The condition says that the map $U_2 \to U_1 \times_{U_0} U_1 \times_{U_0} U_1$ is an effective epimorphism.

Example 6.58. Let $f \colon X \twoheadrightarrow Y$ be an effective epimorphism in T. Then the Čech nerve $\check{C}(f)$, viewed as a simplicial object in $T_{/Y}$, is a hypercover of Y. Indeed, by Lemma 2.20 the simplicial object $\check{C}(f)$ is 0-coskeletal: for every $n \geq 0$, the matching map $\check{C}(f)_n \to (\operatorname{cosk}_{n-1} \check{C}(f))_n$ is an isomorphism.

In preparation of the proof of Theorem 6.54, we establish various basic properties of hypercovers.

Lemma 6.59. *Let $\varphi^* \colon S \to T$ be a morphism of logoi and let $U_\bullet$ be a hypercover of S. Then $\varphi^*(U_\bullet)$ is a hypercover of T.*

Proof. Since φ^* is left exact and the matching object $(\operatorname{cosk}_{n-1} U_\bullet)_n$ is a finite limit (Remark 6.57), we have $\varphi^*((\operatorname{cosk}_{n-1} U_\bullet)_n) \simeq (\operatorname{cosk}_{n-1} \varphi^*(U_\bullet))_n$. Since φ^* preserves effective epimorphisms, the result follows. $\square$

Corollary 6.60. *Let T be a topos with hypercompletion $L\colon T \to T^{\mathrm{hyp}}$. A simplicial object $U_\bullet$ of T is a hypercover of T if and only if $L(U_\bullet)$ is a hypercover of T^{hyp}.*

Proof. The "only if" direction is Lemma 6.59. For the converse, note that the unit map $U_n \to (\operatorname{cosk}_{n-1} U_\bullet)_n$ is an effective epimorphism if and only if its image under L is, since a morphism is an effective epimorphism if and only if it is so after applying L (the kernel of L consists of ∞-connected morphisms, which are in particular effective epimorphisms). □

Lemma 6.61. *Let $U \in T$ be an ∞-connected object. Then the constant simplicial object with value U is a hypercover of T.*

Proof. By Corollary 6.60, we may assume that T is hypercomplete. Then $U \simeq *$, so the constant simplicial object with value U is the terminal object of $\operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, T)$. Since the coskeleton functors preserve limits, the matching maps are all isomorphisms. □

Lemma 6.62 ([Lur09, Lemma 6.5.3.9])**.** *Let T be a topos and let $U_\bullet$ be an n-coskeletal hypercover of T. Then $U_\bullet$ is effective.*

Proof sketch. The details of the proof are intricate; we refer to [Lur09, Lemma 6.5.3.9] for a detailed argument. We only sketch the idea of Lurie's proof.

We proceed by induction on n. If $n = 0$, then $U_\bullet$ can be identified with (the underlying groupoid object of) the Čech nerve of the map $U_0 \to *$. Since $U_\bullet$ is a hypercover, this map is an effective epimorphism, so the Čech nerve is a colimit diagram and $|U_\bullet| \simeq *$.

Now assume $n > 0$ and let $V_\bullet = \operatorname{cosk}_{n-1} U_\bullet$. The unit map $f_\bullet\colon U_\bullet \to V_\bullet$ has the property that f_m is an isomorphism for $m < n$, and f_n (hence each f_m for $m \geq n$) is an effective epimorphism since $U_\bullet$ is a hypercover. Let $W^+_\bullet\colon \mathbf{\Delta}_+ \times \mathbf{\Delta} \to T$ be the Čech nerve of $f_\bullet$ (formed in the category of simplicial objects). By the inductive hypothesis, $|V_\bullet| \simeq *$, and since colimits in functor categories are computed pointwise, $|W^+_\bullet| \simeq *$.

Let $D_\bullet$ denote the diagonal of the bisimplicial object $W^+_\bullet|_{\mathbf{\Delta}\times\mathbf{\Delta}}$. Lurie shows that $U_\bullet$ is a retract of $D_\bullet$ in the category of semisimplicial objects. Since $\mathbf{\Delta}^{\mathrm{op}}$ is sifted, the diagonal functor is cofinal, and we have $|D_\bullet| \simeq |W^+_\bullet| \simeq *$. It follows that $|U_\bullet|$, being a retract of a terminal object, is itself terminal. □

Lemma 6.63. *Let $f_\bullet\colon U_\bullet \to V_\bullet$ be a morphism of simplicial objects in T. Suppose that $f_k\colon U_k \to V_k$ is an isomorphism for all $k \leq n$. Then the induced map $|f_\bullet|\colon |U_\bullet| \to |V_\bullet|$ is n-connected.*

Proof. By writing T as a left exact localization of a presheaf topos, and using that left exact localizations preserve colimits and n-connectivity, we reduce to the case $T = \mathrm{An}$. The result then follows from classical homotopy theory: the geometric realization of a map of simplicial spaces which is an isomorphism in degrees $\leq n$ is n-connected. □

Lemma 6.64. *Let T be a topos and let $U_\bullet$ be a hypercover of T. Then the canonical map $|U_\bullet| \to *$ is ∞-connected.*

Proof. We show that $|U_\bullet|$ is n-connected for every $n \geq 0$. Let $V_\bullet = \operatorname{cosk}_{n+1} U_\bullet$ and let $u_\bullet\colon U_\bullet \to V_\bullet$ be the unit map. Since $U_\bullet$ is a hypercover, the map u_m is an isomorphism for $m \leq n+1$. By Lemma 6.63, the induced map $|U_\bullet| \to |V_\bullet|$ is n-connected. By Lemma 6.62, $|V_\bullet| \simeq *$. It follows that $|U_\bullet|$ is n-connected. □

We are now ready to prove the main result of this section.

Proof of Theorem 6.54. (1) $\Rightarrow$ (2): Let $U_\bullet$ be a hypercover of $T_{/X}$. By Lemma 6.64, the canonical map $|U_\bullet| \to X$ is ∞-connected. Since T is hypercomplete, so is $T_{/X}$, hence this map is an isomorphism. Thus $U_\bullet$ is effective.

(2) $\Rightarrow$ (1): Let $f\colon U \to X$ be an ∞-connected morphism in T. Consider f as an object of $T_{/X}$ and let $f_\bullet$ be the constant simplicial object with value f. By Lemma 6.61, $f_\bullet$ is a hypercover of $T_{/X}$. By assumption, $f_\bullet$ is effective, meaning $|f_\bullet| \simeq X$ in $T_{/X}$. But $|f_\bullet| \simeq f$ since $f_\bullet$ is constant, so f is an isomorphism. This shows that T is hypercomplete. □

Remark 6.65. Theorem 6.54 is often useful in practice: to show that a topos T is hypercomplete, it suffices to verify that all hypercovers are effective. For sheaf topoi $\mathrm{Shv}(X)$ on a topological space X, this can often be checked using concrete open covers.

6.5 Localic topoi, Postnikov-completeness, and boundedness

Given an anima X, we may form the truncations $\tau_n X \in \mathrm{An}_{\leq n}$ for every $n \geq 0$. These truncations fit into the *Postnikov tower*

$$X \to \cdots \to \tau_{n+1}X \to \tau_n X \to \cdots \to \tau_0 X.$$

Moreover, every anima is canonically the limit of its Postnikov tower; in fact, the functor $\mathrm{An} \to \lim_n \mathrm{An}_{\leq n}$ sending X to the system $(\tau_n X)_n$ is an equivalence of categories.

The goal of this section is to study a closely related phenomenon in the world of topoi. Recall that every anima X gives rise to the slice topos $\mathrm{An}_{/X}$. We saw in Proposition 4.34 that the assignment $X \mapsto \mathrm{An}_{/X}$ defines a fully faithful functor

$$\mathrm{An} \hookrightarrow \mathrm{Topos}.$$

As we will see in this section, there are subcategories $\mathrm{Topos}_n \subseteq \mathrm{Topos}$ of 'n-localic topoi' for all n, and left adjoints $L_n\colon \mathrm{Topos} \to \mathrm{Topos}_n$. In the toposic situation, the comparison functor $\mathrm{Topos} \to \lim_n \mathrm{Topos}_n$ will no longer be an equivalence. Instead, it admits fully faithful left and right adjoints:

- The image of the fully faithful left adjoint $\lim_n \mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ is given by the *Postnikov-complete* topoi.
- The image of the fully faithful right adjoint $\lim_n \mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ is given by the *bounded* topoi.

We now properly define these notions. A great reference for this material is [BGH18, Section 3.2].

6.5.1 Localic topoi

We start by defining what it means for a topos to be n-localic.

Definition 6.66. Let $n \geq -1$. A topos T is called *n-localic* if for every topos S the induced map

$$\mathrm{Geom}(S,T) \xrightarrow{\sim} \mathrm{Geom}(S_{\leq n-1}, T_{\leq n-1}) := \mathrm{Fun}^{\mathrm{lex,colim}}(T_{\leq n-1}, S_{\leq n-1})$$

is an equivalence.

Notation 6.67. Let $\mathrm{Topos}_n \subseteq \mathrm{Topos}$ denote the full subcategory spanned by the n-localic topoi.

Proposition 6.68. (1) *For any topos T, there exists an $(n,1)$-category C with finite limits and a Grothendieck topology on C such that $T_{\leq n-1} \simeq \mathrm{Shv}_\tau(C)_{\leq n-1}$.*

(2) *The inclusion* $\mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$ *admits a left adjoint $L_n \colon \mathrm{Topos} \to \mathrm{Topos}_n$ such that $L_n T = \mathrm{Shv}_\tau(C)$ for all (C,τ) as in (1).*

Proof. (1) Consider a large enough cardinal κ such that T is κ-compactly generated, and let $C = (T_{\leq n-1})^\kappa$. We can arrange things in such a way that C has finite limits and the inclusion $C \hookrightarrow T$ preserves finite limits. Let τ be the Grothendieck topology on C in which a sieve over $X \in C$ is a covering sieve if and only if it contains a class of maps $Y_i \to X$ that is a joint effective epimorphism. Then the map $\mathrm{PSh}(C) \to T$ is left exact by Proposition 2.43 and factors as

$$\mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C) \to T.$$

On $(n-1)$-truncated objects, this induces functors

$$\mathrm{PSh}(C)_{\leq n-1} \to \mathrm{Shv}_\tau(C)_{\leq n-1} \to T_{\leq n-1}.$$

The second functor is an equivalence: it has a fully faithful right adjoint, and it is conservative as all maps inverted by $\mathrm{Shv}_\tau(C) \to T$ are ∞-connected and any ∞-connected map between $(n-1)$-truncated objects is an isomorphism. This finishes the proof of (1).

For (2), note that by Proposition 2.43, we have $\mathrm{Fun}_{\mathrm{Logos}}(\mathrm{PSh}(C), S) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(C,S)$, so that restriction along $C \to \mathrm{Shv}_\tau(C)$ induces a fully faithful functor

$$\mathrm{Fun}_{\mathrm{Logos}}(\mathrm{Shv}_\tau(C), S) \hookrightarrow \mathrm{Fun}^{\mathrm{lex}}(C,S),$$

whose image consists of those left exact functors $C \to S$ that send τ-covers to effective epimorphisms. Since C is n-truncated, the right-hand side is equivalent to $\mathrm{Fun}^{\mathrm{lex}}(C, S_{\leq n})$. But this is in turn equivalent to $\mathrm{Fun}^{\mathrm{lex,colim}}(\mathrm{Shv}_\tau(C)_{\leq n-1}, S_{\leq n-1})$. It thus follows that the map

$$\mathrm{Fun}_{\mathrm{Logos}}(\mathrm{Shv}_\tau(C), S) \to \mathrm{Fun}^{\mathrm{lex,colim}}(\mathrm{Shv}_\tau(C)_{\leq n-1}, S_{\leq n-1})$$

is an equivalence, showing that $\mathrm{Shv}_\tau(C)_{\leq n-1}$ is an n-localic topos. We also conclude that the canonical logos morphism $\mathrm{Shv}_\tau(C) \to T$, when regarded as a map $T \to \mathrm{Shv}_\tau(C)$ of topoi, exhibits $\mathrm{Shv}_\tau(C)$ as a left adjoint object to T under the inclusion $\mathrm{Topos}_n \hookrightarrow \mathrm{Topos}$. This finishes the proof. $\square$

Example 6.69. For $T = \mathrm{An}_{/X}$ we have $L_n(\mathrm{An}_{/X}) = \mathrm{An}_{/\tau_n X}$, providing a commutative diagram as follows:

$$\begin{tikzcd} \mathrm{An} \arrow[r, "\tau_n"] \arrow[d, hook, "X\mapsto \mathrm{An}_{/X}"'] & \mathrm{An}_{\leq n} \arrow[d, hook, "X\mapsto \mathrm{An}_{/X}"] \\ \mathrm{Topos} \arrow[r, "L_n"] & \mathrm{Topos}_n. \end{tikzcd}$$

So this is the sense in which L_n "generalizes" the truncation functor $\tau_n \colon \mathrm{An} \to \mathrm{An}_{\leq n}$.

Warning 6.70. If C is an $(n,1)$-category with finite limits, it is clear from the previous proposition that its presheaf category $\mathrm{PSh}(C)$ is n-localic. However, this *fails* if C is not assumed to have finite limits! For example, if C is the poset of open subsets of $\prod_{n\in\mathbb{N}}[0,1] = \mathbb{Q}$ that are homeomorphic to $[0,1)\times\mathbb{Q}$, then $\mathrm{PSh}(C)$ is not n-localic for any n.

Despite the previous warning, the presheaf category of an $(n,1)$-category is still n-localic *after hypercompletion*. The proof requires the following auxiliary result:

Lemma 6.71. *Let T be a hypercomplete topos and let $u\colon C \to T$ be a functor from a small category. Consider the following four statements:*

(1) *The restriction functor*

$$u^*\colon T \to \mathrm{PSh}(C), \qquad X \mapsto \mathrm{Hom}_T(u(-), X)$$

is fully faithful.

(2) *T is generated under colimits by the image of u.*

(3) *Every $X \in T$ is covered by the image of u, i.e. there exist objects $c_i \in C$ and an effective epimorphism $\bigsqcup_i u(c_i) \twoheadrightarrow X$.*

(4) *The counit $u_! u^*(X) \to X$ is ∞-connected for every $X \in T$.*

Then we have implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, if T is hypercomplete then all four statements are equivalent.

Proof. To see that (1) implies (2), let $u_!\colon \mathrm{PSh}(C) \to T$ be the left Kan extension of u. Since u^* is fully faithful, the counit $u_! u^* \to \mathrm{id}_T$ is an equivalence. As every presheaf is a colimit of representables and $u_!(y(c)) = u(c)$, we conclude that every object of T is a colimit of objects in the image of u.

For (2) $\Rightarrow$ (3), observe that the collection of objects receiving such an effective epimorphism is closed under colimits.

For (3) $\Rightarrow$ (4), note that by the pointwise formula for left Kan extensions we have

$$u_! u^*(X) \simeq \mathrm{colim}_{(c,\alpha)\in C_{/X}} u(c),$$

where the colimit is indexed by the category of pairs (c,α) with $c \in C$ and $\alpha : Y(c) \to u^*(X)$, or equivalently the category $C_{/X}$ of pairs $(c, \alpha\colon u(c) \to X)$. The counit map then corresponds to the canonical map

$$X' := \mathrm{colim}_{(c,\alpha)\in C_{/X}} u(c) \to X$$

induced by the maps α. To show this map is ∞-connected, we show it is n-connected by induction on n. The case $n = -1$ holds by assumption (1). Now consider the diagonal $X' \to X' \times_X X'$. Using descent, we may identify this with the map

$$\mathrm{colim}_{(c,\alpha),(d,\beta)\in C_{/X}}\ \mathrm{colim}_{(e,\gamma)\in C_{/u(c)\times_X u(d)}} u(e) \to \mathrm{colim}_{(c,\alpha),(d,\beta)\in C_{/X}} u(c)\times_X u(d).$$

By the induction hypothesis, the inner map

$$\mathrm{colim}_{(e,\gamma)\in C_{/u(c)\times_X u(d)}} u(e) \to u(c)\times_X u(d)$$

is $(n-1)$-connected for every (c, α) and (d, β). Hence the diagonal $X' \to X' \times_X X'$ is a colimit of $(n-1)$-connected maps, so is itself $(n-1)$-connected. This finishes the induction.

Finally, if T is hypercomplete and (4) is satisfied, then the counit map $u_! u^* \to \mathrm{id}$ is a natural isomorphism, and so u^* is fully faithful, giving (1). □

Lemma 6.72. *Let $n \geq -1$, and let T be a topos such that every object is covered by $(n-1)$-truncated objects. Then the map of topoi $T \to L_n(T)$ induces an equivalence on hypercompletions:*

$$T^{\mathrm{hyp}} \xrightarrow{\sim} (L_n T)^{\mathrm{hyp}}.$$

Proof. By Proposition 6.68, there is an $(n,1)$-category C with finite limits with a Grothendieck topology τ such that $L_n T \simeq \mathrm{Shv}_\tau(C)$. The construction of C shows that we may choose $C \subseteq T$ in such a way that it contains sufficiently many $(n-1)$-truncated objects of T, and using the assumption on T we can arrange C so that every object of T is covered by objects of C. Letting $u\colon C \hookrightarrow T$ denote the inclusion, we may consider the left Kan extension $u_!\colon \mathrm{PSh}(C) \to T$ and its right adjoint $u^*\colon T \to \mathrm{PSh}(C)$. By Lemma 6.71, the counit map $u_! u^* \to \mathrm{id}$ of this adjunction consists of ∞-connected maps, and in particular the composite

$$\mathrm{PSh}(C) \xrightarrow{u_!} T \twoheadrightarrow T^{\mathrm{hyp}}$$

is a left exact localization. In order to finish the proof, we need to show that the monogenic part of this localization is precisely the congruence τ^c generated by the Grothendieck topology τ. Equivalently, by Proposition 6.8, we need to show that a monomorphism $R \hookrightarrow y(X)$ in $\mathrm{PSh}(C)$ is inverted by this functor if and only if it is a covering sieve in C. But this is immediate from the choice of τ on C: the covering sieves in C are precisely those that are sent under $u_!$ to an effective epimorphism in T. □

6.5.2 Postnikov-completeness

Definition 6.73. The *Postnikov-completion* of a presentable category C is defined as the limit

$$\widehat{C} \quad := \quad \lim_n C_{\leq n}.$$

We say that C is *Postnikov-complete* if the canonical functor $\tau_*\colon C \to \widehat{C}, X \mapsto (\tau_n X)_n$ is an equivalence.

Remark 6.74. The functor τ_* admits a right adjoint $\lim_n C_{\leq n} \to C$ given by sending a system (X_n) of n-truncated objects to their limit $X = \lim_n X_n$. We see:

- τ_* is fully faithful if and only if the unit map $X \to \lim_n \tau_n X$ is an isomorphism for all $X \in C$, i.e. every object is the limit of its Postnikov tower. Note that this is necessary but not sufficient for τ_* to be an equivalence.
- τ_* is essentially surjective if and only if every system (X_n) satisfying the compatibility conditions arises as $(\tau_n X)$ for some $X \in C$.

Proposition 6.75. *If T is a topos, then $\widehat{T}$ is a topos. For any Postnikov-complete topos S, the morphism of topoi $\widehat{T} \to T$ induces an equivalence*

$$\mathrm{Geom}(S, \widehat{T}) \xrightarrow{\sim} \mathrm{Geom}(S, T).$$

Proof. It suffices to show that $\widehat{T}$ is a topos; the universal property then follows relatively easily. Note that $\widehat{T}$ is presentable, as a limit of presentable categories along colimit-preserving functors. Consider an object $X = (X_n)_n \in \lim_n T_{\le n}$. We then have a limit diagram

$$(\lim_n T_{\le n})_{/X} \to \cdots \to (T_{\le n+1})_{/X_{n+1}} \to (T_{\le n})_{/X_n} \to \dots$$

We may compare this sequence

$$\lim(\dots) \to \cdots \to (T_{/X_{n+1}})_{\le n} \xrightarrow{\tau^{\mathrm{rel}}_{n-1}\tau_n} (T_{/X_n})_{\le n-1},$$

where in the vertical direction we take the functors τ_n^{rel}. Moreover, each of the commutative squares admits a lift $\tau_n \colon (T_{/X_{n+1}})_{\le n} \to (T_{\le n})_{/X_n}$, and it follows that the two limits agree.

It thus remains to show that the assignment $X_n \mapsto (T_{/X_n})_{\le n-1}$ determines a limit-preserving functor $T^{\mathrm{op}}_{\le n} \to \mathrm{Cat}$. This holds because $(T_{/X})_{\le n-1} \simeq (T_{/\tau_n X})_{\le n-1}$. It then follows that sending an object X to $(\lim_n T_{\le n})_{/X} = \lim_n (T_{/X_n})_{\le n-1}$ also preserves limits. □

Lemma 6.76. *Every Postnikov-complete topos is hypercomplete.*

Proof. Let $f \colon X \to Y$ be an ∞-connected map in T. Since T is Postnikov-complete, the map f is the limit of the maps $\tau_n X \to \tau_n Y$, so it suffices to show that each of these maps is an isomorphism. The maps $X \to \tau_n X$ and $Y \to \tau_n Y$ are n-connected, and so is $f \colon X \to Y$. By right cancellation, it follows that $\tau_n X \to \tau_n Y$ is also n-connected. On the other hand, since $\tau_n X$ and $\tau_n Y$ are n-truncated, the maps $\tau_n X \to *$ and $\tau_n Y \to *$ are n-truncated. By left cancellation, it follows that $\tau_n X \to \tau_n Y$ is n-truncated as well. Thus $\tau_n X \to \tau_n Y$ is both n-connected and n-truncated, hence an isomorphism. □

Notation 6.77. We denote by $\mathrm{Topos}^{\mathrm{post}} \subseteq \mathrm{Topos}^{\mathrm{hyp}} \subseteq \mathrm{Topos}$ the full subcategories of Postnikov-complete and hypercomplete topoi. These two inclusions have right adjoints sending T to $T^{\mathrm{hyp}} = T_{\le\infty}$ and $T \mapsto \widehat{T}$.

6.5.3 Bounded topoi

Definition 6.78. A topos T is called *bounded* if $T \xrightarrow{\sim} \lim_n L_n T$. We write $\mathrm{Topos}^b \subseteq \mathrm{Topos}$ for the full subcategory of bounded topoi.

Lemma 6.79. *The functor* $L_n \colon \mathrm{Topos} \to \mathrm{Topos}_n$ *preserves (colimits and) filtered limits.*

Proof. Given a filtered diagram of topoi T_α, then its limit is computed in Cat. We have to show that the comparison map

$$L_n(\lim_\alpha T_\alpha) \to \lim_\alpha L_n T_\alpha$$

is an equivalence. Since both sides are n-localic topoi, it will suffice to show that it induces an equivalence on $(n-1)$-truncated objects, i.e. that the map

$$L_n(\lim_\alpha (T_\alpha))_{\le n-1} \to \lim_\alpha (L_n T_\alpha)_{\le n-1}$$

is an equivalence. But this holds since $(L_n T)_{\le n-1} = T_{\le n-1}$ for all T. □

Proposition 6.80. *The functor $L_*\colon \mathrm{Topos} \to \lim_n \mathrm{Topos}_n$ is a localization with adjoints on both sides.*

- *A topos T lies in the image of the left adjoint if and only if it is Postnikov complete.*
- *A topos T lies in the image of the right adjoint if and only if it is bounded.*

Proof. The right adjoint to L_* is given by sending $(T_n)_n$ to $\lim_n T_n$. By the lemma, we have $L_m(\lim_n T_n) = \lim_n L_m T_n = T_m$. This shows that this right adjoint is fully faithful, and that its image consists precisely of the bounded topoi.

Claim 6.81. The functor $L\colon \mathrm{Topos} \to \mathrm{Topos}^b$ admits a left adjoint given by $T \mapsto \widehat{T}$.

This concludes the proof, as the functor $T \to LT$ is an iso on n-truncated objects, hence will also become an isomorphism after completion.

To show the claim, let S be an arbitrary topos. We need to show that $T \to \widehat{T}$ induces an equivalence

$$\mathrm{Geom}(\widehat{T}, S) \xrightarrow{\sim} \mathrm{Geom}(T, LS),$$

where $LS := \lim_n L_n S$ is the bounded reflection of S. We have

$$\mathrm{Geom}(\widehat{T}, S) = \mathrm{Fun}^{\mathrm{lex,colim}}(S, \lim_n T_{\leq n}) \xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}(S, T_{\leq n}) \xrightarrow{\sim} \lim_n \mathrm{Fun}^{\mathrm{lex,colim}}(S_{\leq n}, T_{\leq n}).$$

But the right-hand side is equivalent to $\lim_n \mathrm{Geom}(T, L_n S) = \mathrm{Geom}(T, \lim_n L_n S) = \mathrm{Geom}(T, LS)$, which finishes the proof. $\square$

We may summarize this in the following diagram:

$$\begin{tikzcd}
\mathrm{Topos}^{\mathrm{post}} \arrow[dd, <->, "\sim"'] \arrow[dr, hook, shift left] & \\
& \mathrm{Topos} \arrow[ul, shift left, "(-)^\wedge"] \arrow[dl, shift left, "L"] \\
\mathrm{Topos}^b \arrow[ur, hook, shift left] &
\end{tikzcd}$$

6.6 Dimension theory

Recall that a CW-complex is said to be *of dimension n* if its highest dimensional cell has dimension n. We may say that an anima has dimension n if it can be represented by a CW-complex of dimension n, but not by one of dimension $n-1$. Perhaps surprisingly, the dimension of an anima X can be intrinsically described in terms of its associated slice topos $\mathrm{An}_{/X}$, by giving a general definition of the *dimension* of a topos. This also provides a convenient method to show Postnikov-completeness of a topos: We will see that every topos of locally finite dimension is Postnikov-complete.

Definition 6.82. Let $\varphi\colon S \to T$ be a morphism of topoi. We say that φ *has dimension $\leq d$* if for every n, φ sends n-connected maps to $(n-d)$-connected maps.

We say that T *has dimension $\leq d$* if the unique topos morphism $\Gamma = \mathrm{Hom}_T(*, -)\colon T \to \mathrm{An}$ has dimension $\leq d$.

Lemma 6.83. *A topos T has dimension $\leq d$ if and only if every $(d-1)$-connected object of T has a global section.*

Proof. For the 'only if'-direction, suppose T has dimension $\leq d$, and let X be a $(d-1)$-connected object. Then the unique map $X \to *$ is $(d-1)$-connected. By the dimension condition, the induced map $\Gamma(X) \to \Gamma(*) = *$ is (-1)-connected. A map to the terminal object being (-1)-connected means the domain is non-empty, so $\Gamma(X)$ is non-empty, i.e. X has a global section.

For the 'if'-direction, suppose every $(d-1)$-connected object of T has a global section. We need to show that for every n-connected map $f\colon X \to Y$ in T, the induced map $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is $(n-d)$-connected.

Let us start by showing it is an effective epimorphism whenever f is $(d-1)$-connected. An effective epimorphism of animae is characterized by having non-empty fibers over every point. Given any $y\colon * \to Y$ (i.e. any point $y \in \Gamma(Y)$), we need to show that the fiber is non-empty. The fiber over y is given by

$$\Gamma(X) \times_{\Gamma(Y)} \{y\} \;\simeq\; \mathrm{Hom}_T(*, X \times_Y *) \;=\; \Gamma(X_y),$$

where $X_y := X \times_Y *$ is the fiber of f over y. Since f is $(d-1)$-connected, the fiber X_y is a $(d-1)$-connected object. By our hypothesis, X_y has a global section, so $\Gamma(X_y)$ is non-empty, showing that $\Gamma(f)$ is an effective epimorphism.

We will now inductively show that in fact $\Gamma(f)\colon \Gamma(X) \to \Gamma(Y)$ is $(n-d)$-connected whenever f is n-connected. We already know it is an effective epimorphism, so by Theorem 3.20 it remains to show that its diagonal is $(n-d-1)$-connected. The diagonal identifies with $\Gamma(\Delta_f)$, where $\Delta_f\colon X \to X \times_Y X$ is the diagonal of f. Since f is n-connected, the diagonal Δ_f is $(n-1)$-connected, and so the claim follows from the induction hypothesis. □

Definition 6.84. We say that a topos T is *locally of dimension $\leq d$* (resp. *locally of finite dimension*) if it is generated under colimits by objects U such that the slice topos $T_{/U}$ has dimension $\leq d$ (resp. has some finite dimension).

Let us mention without proof some instances where we have bounds on the dimension.

Example 6.85. Let $X \in \mathrm{An}$, and assume that X is the image under $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ of a d-dimensional CW-complex. Then $\dim(\mathrm{An}_{/X}) \leq d$. The same works if X comes from a d-dimensional simplicial set under the localization functor $\mathrm{sSet} \to \mathrm{An}$.

In fact, Wall showed that for $d \neq 2$ the converse also holds true: if $\dim(\mathrm{An}_{/X}) \leq d$, then X admits a d-dimensional CW-model. For $d = 2$, Wall showed that $\dim(\mathrm{An}_{/X}) \leq 2$ if and only if X is a retract of some Y which has a 2-dimensional CW-model. He also showed that in this case one can always find a 3-dimensional CW-model for X. (Wall's formulation used as a definition for dimension the vanishing of cohomology with local coefficients. I believe Marc mentioned that this is equivalent to $\dim(\mathrm{An}_{/X}) \leq d$ but I didn't catch all details.)

Example 6.86. If C admits a terminal object 1, then $\mathrm{PSh}(C)$ has dimension ≤ 0. Indeed, in this case $\Gamma_*\colon \mathrm{PSh}(C) \to \mathrm{An}$ is given by evaluation at 1. The condition then says that if $\mathcal{F} \in \mathrm{PSh}(C)$ is (-1)-connected, i.e. if each $\mathcal{F}(X)$ is non-empty for $X \in C$, then $\mathcal{F}$ admits a global section. This is clear since $\mathrm{Hom}_{\mathrm{PSh}(C)}(*, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(*)$ is non-empty by assumption.

Example 6.87. The presheaf category $\mathrm{PSh}(\mathbb{N})$ on the poset of the natural numbers has dimension ≤ 1.

Example 6.88. Let X be a paracompact Hausdorff space. We say that X has *covering dimension* $\leq d$ if every open covering is refined by one with empty $(d+2)$-fold intersections. Then $\dim \mathrm{Shv}(X) \leq d$. This applies for example to every CW-complex of dimension $\leq d$.

Example 6.89. Let X be a spectral space of Krull dimension $\leq d$. Then $\mathrm{Shv}(X)$ has dimension $\leq d$.

Example 6.90. If X is a qcqs scheme, then

- $\dim(X_{\mathrm{Nis}}) \leq$ Krull dimension of X
- $\dim(X_{\mathrm{cdh}}) \leq$ Valuative dimension of X

Proposition 6.91. *Let T be a topos.*

(1) *If T is locally of finite dimension, then for all $X \in T$, the map $X \xrightarrow{\sim} \lim_n \tau_n X$ is an isomorphism. In particular, T is hypercomplete.*

(2) *If T is locally of dimension $\leq d$ for some d, then T is Postnikov-complete.*

Proof. (1) By definition, T is generated under colimits by objects of finite dimension. To check that $X \to \lim_n \tau_n X$ is an isomorphism, it suffices to apply $\mathrm{Hom}_T(U, -)$ where $U \in T$ is an object of finite dimension, say $\dim(T_{/U}) \leq d$. The transition map $X \to \tau_n X$ is n-connected. By definition of finite dimension, the map $\mathrm{Hom}_T(U, X) \to \mathrm{Hom}_T(U, \tau_n X)$ is $(n-d)$-connected. But as $n \to \infty$, also $(n-d) \to \infty$, so the induced map $\mathrm{Hom}_T(U, X) \to \lim_n \mathrm{Hom}_T(U, \tau_n X)$ is an isomorphism, as desired.

(2) For Postnikov completeness, we already know that the functor $T \to \widehat{T}$ is fully faithful (as the unit of the adjunction is an isomorphism by (1)). It remains to show that it is also essentially surjective. Consider $(X_n)_{n\in\mathbb{N}} \in \widehat{T}$, and let $X := \lim_n X_n$. We need to show that the map $X \to X_n$ induces an isomorphism $\tau_n(X) \to X_n$. (These maps form the counit maps of the adjunction.) Equivalently, we must show that the map $X \to X_n$ is n-connected for all n.

In order to do this, we will use our inductive characterization for n-connected maps. We will slightly rewrite our condition as follows:

For all n, there exists some $m \gg 0$ such that $X \to X_m$ is n-connected.

Note that this is clearly equivalent to the previous condition, since the map $X_m \to \tau_n(X_m) = X_n$ is always n-connected.

We will now consider the following more general condition. For a diagram $X_\bullet \colon \mathbb{N}^{\mathrm{op}} \to T$, we consider:

$$(*) \qquad \text{The map } X_{m+1} \to X_m \text{ is } n\text{-connected for } m \gg 0$$

We will show that this implies that the map $X \to X_m$ is n-connected for $m \gg 0$. To see this, note that the diagram $X \times_{X_\bullet} X$ also has property $(*)$. Note that we have $X = X \times_X X = \lim_n (X \times_{X_n} X)$. By induction, it thus suffices to show that there exists some m such that $X \to X_m$ is an effective epimorphism. Let n be such that the map $X_{k+1} \to X_k$ is $(d-1)$-connected for all $k \geq n$. By assumption on T, there exists some effective epimorphism $\bigsqcup_\alpha U_\alpha \to X_n$ with $\dim(T_{/U_\alpha}) \leq d$. By pulling back, we may assume without loss of generality that $\dim(T_{/X_n}) \leq d$. Now, in this case, the map

$$\mathrm{Hom}_{/X_n}(X_n, X_{n+k+1}) \to \mathrm{Hom}_{/X_n}(X_n, X_{n+k})$$

is (-1)-connected for all $k \geq 0$. It follows inductively that the map $\mathrm{id}_{X_n}\colon X_n \to X_n$ lifts to $X_n \to X_{n+k}$ for all k. By assembling all these sections, it follows that also the map $X \to X_n$ admits a section. In particular, it is an effective epimorphism. This finishes the proof. □

6.7 Coherence

We will now discuss the notion of *coherent topoi*. While this is a somewhat technical topic, it plays an important role in topos theory, and also shows up prominently in algebraic geometry, so it seems worth at least briefly discussing it. An alternative detailed treatment may be found in [BGH18, Chapter 3].

6.7.1 An overview

Before diving into the technical details, let us provide an overview of the results of this section.

The main concept introduced in this section is that of *coherent objects*. For intuition, the example to keep in mind is the coherent objects in An, which as we will see are precisely those animae X such that all the homotopy groups $\pi_i(X, x)$ are finite. We may think of coherence as some form of a strong "finiteness" condition on a topos/object on a topos. For example, we will prove the following compactness result below in Theorem 6.116:

Theorem (Compactness theorem)**.** Let $X \in T$ be a coherent object. Then $\tau_n X \in T_{\leq n}$ is compact for all $n \geq -1$.

Corollary 6.92. *For a coherent object $X \in T$, the functor $\mathrm{Ab}(T_{\leq 0}) \to \mathrm{Ab}^{\mathbb{Z}}, A \mapsto H^*(X; A)$ preserves filtered colimits.*

This theorem can be thought of as the underlying reason for various compactness results in the literature. For example, it is the reason that étale cohomology groups preserve filtered colimits: the étale topos is coherent.

While the theorem says that all truncations of a coherent object X are compact, the object X itself need not be compact! In particular, this notion of "finiteness" behaves quite differently from the other usual notion of finiteness that we are used to from homotopy theory, namely the subcategory $\mathrm{An}^{\mathrm{fin}} \subseteq \mathrm{An}$ of finite animae, obtained from the point by closing up under pushouts and finite coproducts.

There is another, quite unrelated, result that we will discuss, called the *completeness theorem*. In the higher categorical setting this is due to Lurie, but it follows a classical result for classical topoi due to Deligne.

Definition 6.93. A topos T is called *locally coherent* if every object $X \in T$ admits a cover $\bigsqcup_\alpha U_\alpha \twoheadrightarrow X$ by coherent objects.

Theorem (Completeness theorem, see Theorem 6.120)**.** Every locally coherent topos admits a surjection[1] from a topos of the form $\mathrm{An}_{/S}$ for some set[2] S.

Applying this result to an anima A, regarded as a topos by forming the slice $\mathrm{An}_{/A}$, this recovers the basic fact that there exists a surjection $S \twoheadrightarrow A$ from a set S.

A generic way to apply this theorem is to reduce suitable types of statements in locally coherent topoi to statements in sets/animae.

Example 6.94. If T is a hypercomplete locally coherent topos, we get a surjection $\mathrm{An}_{/S} \twoheadrightarrow T$, for which the pullback functor is conservative.

Example 6.95. A Kan fibration in any hypercomplete topos is a "realization-fibration", i.e. pullback along a Kan fibration commutes with geometric realizations: if $X_\bullet \to Z_\bullet$ is a Kan fibration and $Y_\bullet \to Z_\bullet$ is arbitrary, we have

$$\mathrm{colim}_n\, X_n \times_{Z_n} Y_n \xrightarrow{\sim} (\mathrm{colim}_n\, X_n) \times_{\mathrm{colim}_n Z_n} (\mathrm{colim}_n\, Y_n).$$

In the next couple of subsections, we will provide proper definitions and discuss the proofs of these theorems. We start in Section 6.7.2 with the definition of n-coherent objects for all n, and prove criteria for n-coherence. In Section 6.7.3 we define what it means for a site to be *finitary*, and show that the hypercomplete locally coherent topoi are precisely the hypercompletions of sheaves on finitary sites with pullbacks. In Section 6.7.4 we prove the compactness theorem and in Theorem 6.120 we prove the completeness theorem.

6.7.2 Coherent objects

The notion of coherence is closely related to the notion of quasi-compact objects in a topos:

Definition 6.96. An object $X \in T$ is called *quasi-compact* if any cover $\bigsqcup_i U_i \twoheadrightarrow X$ can be refined to a finite cover, i.e. there is a finite subset $J \subseteq I$ such that the map $\bigsqcup_{j \in J} U_j \to X$ is still an effective epimorphism.

We say a topos T is *quasi-compact* if $* \in T$ is quasi-compact.

One problem with the notion of quasi-compactness is that it is not closed under the formation of fiber products. We may think of the collection of coherent objects in a topos as the universal approximation of the collection of quasi-compact objects by a collection of objects closed under fiber products. As a first approximation, we may consider the following class of objects:

Definition 6.97. An object $X \in T$ is called *quasi-separated* if for any two maps $Y \to X$ and $Z \to X$ from quasi-compact objects Y and Z, also the fiber product $Y \times_X Z$ is quasi-compact.

The quasi-separatedness condition is very familiar for people working in algebraic geometry. However, it does not yet achieve our goal: there is in general no reason that quasi-separated objects are closed under base change. Now, in principle one could iterate this condition, and consider objects X such that the quasi-separated objects are closed under pullbacks, etcetera. However, what we would

[1] The notion of "surjection" of topoi was defined in Remark 5.66.

[2] Recall that by "set" we mean a 0-truncated anima.

like to argue here is that this is not the correct way to think about things. In fact, this notion of quasi-separated objects is not really well-behaved in an arbitrary topos; one might argue it is somewhat of a "coincidence" that it works in algebraic geometry. For example, in algebraic geometry it is true that $Y \times_X Z$ is quasi-compact for all Y, Z if and only if Δ_X is relatively quasi-compact, but this fails to hold in an arbitrary topos; the "if"-direction always holds, but the "only if"-direction fails: Δ_X being relatively quasi-compact is generally stronger. The reason this works in algebraic geometry is that you can cover any scheme by affine schemes, which are both quasi-compact and quasi-separated. After unwinding, you will find that the question reduces to the property that the product of two qcqs schemes over $\mathrm{Spec}(\mathbb{Z})$ is again qcqs. Note that this condition is precisely the next one in the hierarchy we hinted at before. More generally, this fact about Δ_X being relatively quasi-compact will hold in a topos T whenever the following conditions are satisfied:

- Every object in T is covered by qcqs objects.
- The terminal object of T is "2-coherent", meaning that products of qcqs objects are again qcqs.

Warning 6.98. The terminology on "coherence" can be somewhat confusing in the literature. For example, in SGA the word "coherent" is used in the following three different ways:

- There is a notion for an *object* $X \in T$ to be coherent; it just means that it is qcqs;
- There is a notion for a *morphism* $X \to Y$ to be coherent. Confusingly, this is *not* equivalent to the condition that the morphism $X \to Y$ is qcqs when regarded as an object in $T_{/Y}$. Since we would like to stick to the general convention that notions for maps are simply those for the associated object in the slice topos, we will follow Lurie to refer to this other definition by adding the adverb *relatively*, giving notions like "relatively quasi-compact" and "relatively qcqs".
- There is a notion for a *topos* to be coherent. This is the "correct" notion from the viewpoint of general topos theory. Marc mentioned he finds it quite impressive that Grothendieck managed to find the correct definition of coherence already so early on.

After all this preamble, let us now give a formal definition of coherent objects.

Definition 6.99. Let T be a topos. We will inductively define what it means for objects to be *n-coherent* for all $n \geq 0$:

- We say that an object $X \in T$ is 0-*coherent* if it is quasi-compact. (This is really a condition on the slice topos $T_{/X}$.)
- We say that T is *locally n-coherent* if any $X \in T$ has a cover by n-coherent objects.
- An object $X \in T$ is said to be *n-coherent* if $T_{/X}$ is $(n-1)$-coherent, locally $(n-1)$-coherent, and for all maps $Y \to X \leftarrow Z$ such that Y and Z are $(n-1)$-coherent, the pullback $Y \times_X Z$ is $(n-1)$-coherent.
- We say T is *n-coherent* if the terminal object is n-coherent.

Example 6.100. A topos is 1-coherent if it is locally quasi-compact, and qcqs. A topos T is 2-coherent if and only if it is qcqs, locally qcqs, and for all qcqs $X, Y \in T$ also $X \times Y$ is qcqs.

Definition 6.101. We say that $X \in T$ is *coherent* if it is n-coherent for all n. We say T is *coherent* if $* \in T$ is coherent and *locally coherent* if every object $X \in T$ admits a cover by coherent objects.

We denote the resulting subcategories of T by

$$T \supseteq \cdots \supseteq T^{n-\mathrm{coh}} \supseteq T^{(n+1)-\mathrm{coh}} \supseteq \cdots \supseteq \bigcap_n T^{n-\mathrm{coh}} =: T^{\mathrm{coh}}.$$

Example 6.102. One can show that an anima $X \in \mathrm{An}$ is n-coherent if and only if $\pi_i(X, x)$ is finite for all $i \leq n$; this can for example be proved using Proposition 6.107 below. Any anima is locally coherent (as we may cover X by points, which are coherent.)

Example 6.103. Let X be a locale. Then we may speak of elements of the locale being *quasi-compact*, which is equivalent to it being quasi-coherent in the associated sheaf topos $\mathrm{Shv}(X)$. The topos $\mathrm{Shv}(X)$ is 1-coherent if and only if every element of the locale is a union of quasi-compact elements, and the intersection of quasi-compact elements is again quasi-compact.

One can show that $\mathrm{Shv}(X)$ is 1-coherent if and only if it is coherent.

By the theorem from before, there is a surjection from a set, and it follows that the locale is spatial, i.e. is the locale of open subsets of a topological space. Moreover, this is necessarily a spectral space. Conversely, if X is a spectral space, then $\mathrm{Shv}(X)$ is coherent.

Definition 6.104. A map $f : Y \to X$ is called *relatively n-coherent* if for every $Z \to X$ with Z n-coherent, also $Z \times_X Y$ is n-coherent.

Warning 6.105. This is *not* the same as the condition that $f \in T_{/X}$ is n-coherent.

Lemma 6.106. *Let $f : Y \to X$ be a map and assume that X is $(n+1)$-coherent. Then f is relatively n-coherent if and only if Y is n-coherent.*

Proposition 6.107. *Let T be a topos, and suppose $T_0 \subseteq T$ is a full subcategory of n-coherent objects such that every $X \in T$ is covered by objects of T_0. Then the following conditions hold:*

- *A morphism $f : Y \to X$ is relatively n-coherent if and only if for every map $U \to X$ with $U \in T_0$ the pullback $U \times_X Y$ is n-coherent.*
- *An object $X \in T$ is $(n+1)$-coherent if and only if X is quasi-compact and for every pair of maps $U \to X \leftarrow V$ with $U, V \in T_0$, the pullback $U \times_X V$ is n-coherent.*

Corollary 6.108. *A topos T is locally coherent if and only if there exists a full subcategory $T_0 \subseteq T$ covering T which is closed under fiber products and such that every $X \in T_0$ is quasi-compact.*

Proof. The "only if" is clear: if T is locally coherent, we may take $T_0 := T^{\mathrm{coh}}$ the subcategory of coherent objects, which by design are closed under fiber products.

Conversely, we inductively use the proposition to show that $T_0 \subseteq T^{n\text{-coh}}$ for all n. In particular, T_0 consists of coherent objects. Since they cover T, we see that T is locally coherent by definition. $\square$

Remark 6.109. If T is not just locally coherent, but also coherent, then the collection of objects T_0 provided by the corollary may be assumed to contain the final object.

Proposition 6.110 (Coherence and connectivity)**.** *Consider a morphism $f\colon X \to Y$.*

(1) *If X is n-coherent and f is $(n-1)$-connected, then Y is n-coherent.*

(2) *If Y is n-coherent and f is n-connected, then X is n-coherent.*

(3) *If Y is n-coherent, f is $(n-2)$-truncated, and X is $(n-1)$-coherent, then X is n-coherent.*

Proof. We omit the proof, as it is an easy unwinding of definitions using the criteria for coherence and connectivity established before. ◻

Corollary 6.111. *Consider the unique factorization of $f\colon X \to Y$ as*

$$X \to \tau_{n-1}(X/Y) \to Y$$

into an $(n-1)$-connected map followed by an $(n-1)$-truncated map. If X is n-coherent and Y is m-coherent, then $\tau_{n-1}(X/Y)$ is $\max(n, m)$-coherent.

Proof. By Proposition 6.110(1), the object $\tau_{n-1}(X/Y)$ is n-coherent, so we are done if $n \geq m$. If $m > n$, we may inductively show that $\tau_{n-1}(X/Y)$ is $(n+k)$-coherent for all $k = 0, \ldots, m-n$. If we already know it is $(n+k-1)$-coherent, then using that Y is m-coherent and $\tau_{n-1}(X/Y) \to Y$ is $(n-1)$-truncated, we get by Proposition 6.110(3) that $\tau_{n-1}(X/Y)$ is $(n+k)$-coherent. ◻

Corollary 6.112. *Let T be an n-localic topos. If T is $(n+1)$-coherent, then T is coherent and locally coherent.*

Proof. By assumption, every object $X \in T$ is covered by objects from $T_{\leq n-1}$. Let $X \in T_{\leq n-1}$. By assumption on T, there exists an effective epimorphism $\bigsqcup_i U_i \twoheadrightarrow X$ where each U_i is n-coherent. Because X is $(n-1)$-truncated, this map descends to a map $\bigsqcup_i \tau_{n-1}(U_i) \twoheadrightarrow X$, which is still an effective epimorphism. By Proposition 6.110(1), each $\tau_{n-1}(U_i)$ is again n-coherent. We conclude that every $X \in T$ is covered by objects from $T^{n-\mathrm{coh}}_{\leq n-1}$.

Since T is $(n+1)$-coherent, it follows that $T^{n-\mathrm{coh}}_{\leq n-1}$ is closed under finite limits. By Corollary 6.108, it follows that T is both coherent and locally coherent. ◻

Corollary 6.113. *Assume that T is both bounded and coherent. Then T is locally coherent.*

Proof. We need to show that every $X \in T$ is covered by coherent objects. Without loss of generality, pick $X \in T_{\leq n}$. As before, we may find a surjection $\bigsqcup_i \tau_n(U_i) \twoheadrightarrow X$ where each U_i is $(n+1)$-coherent. By Proposition 6.110(1) and (3), it follows that each $\tau_n(U_i)$ is coherent. ◻

6.7.3 Finitary sites

Definition 6.114. A Grothendieck topology τ on a category C is called *finitary* if every τ-covering sieve contains a τ-covering sieve which is generated by a finite number of morphisms in C.

Proposition 6.115.

(1) *If (C, τ) is a finitary Grothendieck site and C admits fiber products, then* $\mathrm{Shv}_\tau(C)$ *is locally coherent. If C also admits a terminal object, then* $\mathrm{Shv}_\tau(C)$ *is also coherent.*

(2) *If T is hypercomplete and locally coherent, then there exists a finitary site (C, τ) with fiber products such that $T \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. If T is also coherent, then we can choose C to have a terminal object.*

Proof. The first part follows from the fact that every object in $\mathrm{Shv}_\tau(C)$ is covered by objects in the image of the Yoneda functor. (If the Grothendieck topology τ is subcanonical, then we can literally apply the result from before. But if it is not, the proof still goes through. Really, the result works for an arbitrary functor from some category; it does not need to be fully faithful.)

For (2), let $u \colon C \hookrightarrow T$ denote the inclusion. Since C is closed under fiber products, Proposition 2.43 shows that the left Kan extension

$$u_! \colon \mathrm{PSh}(C) \to T$$

is left exact. By definition of τ, all generating τ-covering sieves are sent to effective epimorphisms in T, so $u_!$ factors as

$$\mathrm{PSh}(C) \to \mathrm{Shv}_\tau(C) \to T.$$

On the other hand, Lemma 6.71 shows that the restriction functor $T \to \mathrm{PSh}(C)$ is fully faithful. Hence the induced functor $T \to \mathrm{Shv}_\tau(C)$ is also fully faithful, so the map $\mathrm{Shv}_\tau(C) \to T$ is a left exact localization. By definition, all generating τ-covering sieves are sent to effective epimorphisms in T, hence this localization is epigenic. We conclude that $T \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. □

6.7.4 Compactness of coherent objects

We will now show the compactness result for coherent objects mentioned before.

Theorem 6.116. *If $X \in T$ is n-coherent, then* $\mathrm{Hom}(X, -) \colon T^{\mathrm{op}}_{\leq n-1} \to \mathrm{An}$ *preserves filtered colimits, i.e. the object $\tau_{n-1}(X) \in T_{\leq n-1}$ is compact.*

Corollary 6.117. *If T is locally n-coherent, then $T_{\leq n-1}$ is compactly generated.*

Warning 6.118. An object X being coherent does *not* imply that it is compact.

Remark 6.119. The property of coherence is analogous to the property of "almost perfect" in the context of modules.

Proof of Theorem 6.116. By passing to slices, we may assume $X = *$. We prove the claim by induction on n. For $n = 0$, the object X is quasi-compact. It then follows essentially from the definition of quasi-compactness that $\mathrm{Hom}_T(X, -) \colon T^{\mathrm{op}}_{\leq -1} \to \mathrm{An}$ preserves filtered colimits.

Now consider a filtered category I and an object $Y \in \mathrm{Fun}(I, T_{\leq n-1})$. We have to show that the canonical comparison map

$$\beta_Y \colon \operatorname{colim}_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \operatorname{colim}_i Y_i)$$

is an isomorphism. We will show the following claim:

Claim: β_Y is an effective epimorphism.

Assuming this for a moment, we may conclude. Indeed, to show β_Y is an isomorphism, it suffices to show that it induces isomorphisms on Hom animae. Consider two points $x, y \in \mathrm{colim}_i \operatorname{Hom}(X, Y_i)$. We may lift them to some finite stage in I, and by replacing I by the slice under this object we may assume this is an initial object $0 \in I$. So we have $x, y\colon X \to Y_0$. The map induced by β_Y on Hom animae takes the form

$$\mathrm{colim}_i \{x\} \times_{\operatorname{Hom}_T(X,Y_i)} \{y\} \to \{x\} \times_{\operatorname{Hom}(X,\mathrm{colim}_i Y_i)} \{y\}.$$

But since the object $\{x\} \times_Y \{y\} \in \operatorname{Fun}(I, T_{\leq n-2})$ now lands in $(n-2)$-truncated objects, this is an isomorphism by the induction hypothesis.

Let us now prove the claim. Given a map $X \to \mathrm{colim}_i Y_i$, we need to show that it factors through some finite stage $Y_i \to \mathrm{colim}_i Y_i$. Since X is quasi-compact and locally $(n-1)$-coherent, there is an effective epimorphism $U \twoheadrightarrow X$ where U is $(n-1)$-coherent, satisfying the additional property that the composite $U \twoheadrightarrow X \to \mathrm{colim}_i Y_i$ factors through some Y_0, where $0 \in I$. (This uses that I is filtered.) Letting $U_\bullet := \check{C}_\bullet(U \to X)$ and $V_{i,\bullet} := \check{C}_\bullet(Y_0 \to Y_i)$, then the commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
X \arrow[r] & \mathrm{colim}_i Y_i
\end{tikzcd}$$

induces a map on Čech nerves of the form

$$U_\bullet \to \mathrm{colim}_i V_{i,\bullet}.$$

Since U is $(n-1)$-coherent and X is n-coherent, each U_m is $(n-1)$-coherent.

By induction on $m \leq n$, we will show that the map $\mathrm{sk}_m(U_\bullet) \to \mathrm{colim}_i \mathrm{sk}_m(V_{i,\bullet})$ factors through some $\mathrm{sk}_m(V_{i,\bullet}) \to \mathrm{colim}_i \mathrm{sk}_m(V_{i,\bullet})$, as semisimplicial objects. (Here sk_m denotes the m-skeleton.) With $m = n$, we then get

$$\tau_{n-1}|\mathrm{sk}_n(U_\bullet)| \xrightarrow{\sim} \tau_{n-1}(X)$$

and since $\operatorname{Hom}_T(X, Y_i) \cong \operatorname{Hom}_T(\tau_{n-1}(X), Y_i)$ we may conclude.

It suffices to show that for all m, the composite map

$$U_m \to U_\bullet[\partial\Delta^m] \to V_{i,\bullet}[\partial\Delta^m] \to \mathrm{colim}_i V_{i,\bullet}[\partial\Delta^m]$$

factors through some finite stage $V_{i,m}$. But this will be an instance of the induction hypothesis. Since each $V_{i,\bullet}$ is $(n-1)$-truncated, the map $V_{i,m} \to V_{i,\bullet}[\partial\Delta^m]$ is $(n-m-1)$-truncated, and since $n - m - 1 \leq n - 2$ we may apply the induction hypothesis in the slice topos $T_{/U_\bullet[\partial\Delta^m]}$ with U_m.

This finishes the proof. □

6.7.5 The completeness theorem

Recall that a morphism of logoi $\varphi\colon S \to T$ is called *surjective* if it detects effective epimorphisms (or equivalently if it detects ∞-connected maps). The goal of this section is to prove the following theorem:

Theorem 6.120 (Completeness theorem)**.** *If T is locally coherent, then there exists a surjection $S \to T$, where $S = \mathrm{An}^I$ for some (small) set I.*

Remark 6.121. For locales, this is exactly the result that says that coherent locales are spatial, i.e. the locale is a spectral space. For locally coherent classical topoi (in the classical sense), this recovers Deligne's completeness theorem.

There are several steps in the proof of this theorem which are interesting. Here is an outline of the proof:

(1) First, we reduce to the case where T is coherent. In this case, we saw that $T^{\mathrm{hyp}} \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$, where (C, τ) is a finitary site with finite limits.

(2) For every site (C, τ), there exists a surjection $\mathrm{Shv}_\rho(D) \to \mathrm{Shv}_\tau(C)$, where D is a poset. (This is called the *Diaconescu cover*. It is highly non-canonical.)

(3) If L is a 0-localic topos (i.e. a *locale*), then there exists a surjection $\mathrm{Shv}(\Lambda) \to L$ where Λ is a Boolean locale (i.e. a complete Boolean algebra).

(4) The locale $\mathrm{Shv}(\Lambda)$ is hypercomplete, so the functor $\mathrm{Shv}(\Lambda) \to L_0 \mathrm{Shv}_\rho(D)$ factors through the hypercompletion $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \simeq \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$.

(5) If (C, τ) is a finitary site with finite limits, then any map $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}_\tau(C)$ factors through $\mathrm{Shv}(\mathrm{Spec}(\Lambda))$, where $\mathrm{Spec}(\Lambda)$ is the spectrum of Λ, seen as a commutative ring.

(6) For any topological space X, the topos $\mathrm{Shv}(X)$ is covered by $\prod_{x \in X} \mathrm{An}$.

We illustrate the proof strategy with the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Shv}(\mathrm{Spec}(\Lambda)) \arrow[ddd, dashed, two heads, "(5)"] & \prod_{\mathrm{Spec}(\Lambda)} \mathrm{An} \arrow[l, two heads] \arrow[dd, dashed, two heads, "(6)"] & \\
\mathrm{Shv}(\Lambda) \arrow[urr] \arrow[r, dashed, "(4)"'] \arrow[dd, two heads, "(3)"] & \mathrm{Shv}_\rho(D)^{\mathrm{hyp}} \arrow[d] & & & \\
 & \mathrm{Shv}_\rho(D) \arrow[dl] \arrow[dr, two heads, "(2)"] & & \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \arrow[dl, two heads] \arrow[dr, "\simeq", "(1)"'] & \\
L_0 \mathrm{Shv}_\rho(D) & & \mathrm{Shv}_\tau(C) & & T^{\mathrm{hyp}}.
\end{tikzcd}$$

Assuming all these claims, we may give a proof of the theorem. We first consider $\mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \to T$, reducing us to $\mathrm{Shv}_\tau(C)$. Then we have some surjection $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, where D is a poset. Consider the 0-localic reflection $L_0 \mathrm{Shv}_\rho(D)$. There is a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow L_0 \mathrm{Shv}_\rho(D)$. Now, since $\mathrm{Shv}(\Lambda)$ is hypercomplete, this surjection will factor through $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \simeq \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$. All in all, we obtain the composite

$$\mathrm{Shv}(\Lambda) \to (\mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \to \mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C).$$

This then factors through a map $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, which is surjective by cancellation properties. There is a covering $\prod_{x\in\mathrm{Spec}(\Lambda)} \mathrm{An} \twoheadrightarrow \mathrm{Shv}(\mathrm{Spec}(\Lambda))$, and since its source is hypercomplete, the map to $\mathrm{Shv}_\tau(C)$ factors through $\mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. All in all, we obtain the desired surjection

$$\prod_{x\in\mathrm{Spec}(\Lambda)} \mathrm{An} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \simeq T^{\mathrm{hyp}} \twoheadrightarrow T.$$

In the remainder of this section, we will go through the details of this proof.

Boolean algebras

We recall some basic definitions regarding Boolean algebras.

Definition 6.122. A *Boolean algebra* Λ is a poset with finite joins $x \vee y$ and meets $x \wedge y$, where $\wedge$ distributes over $\vee$, and where every $x \in \Lambda$ has a complement x^c, i.e. an element satisfying $x \wedge x^c = \bot$ and $x \vee x^c = \top$.

Proposition 6.123. *Let Λ be a complete Boolean algebra. Then* $\mathrm{Shv}(\Lambda)$ *has dimension* ≤ 0*. In particular, it is Postnikov complete.*

Proof. Suppose that $X \in \mathrm{Shv}(\Lambda)$ is a (-1)-connected object with no global section. We will construct by transfinite induction a functor $\mathrm{Ord} \to \Lambda_{/X}$ which is strictly increasing. This will give a contradiction, finishing the proof.

Assume we have already constructed a functor $\varphi\colon \alpha \to \Lambda_{/X}$ for some ordinal α. We need to extend φ to a map $\alpha + 1 \to \Lambda_{/X}$. Consider $U := \mathrm{colim}_{\lambda<\alpha}\, \varphi(\lambda) \in \Lambda_{/X}$. Since X does not have a global section, $U \neq \top$. Define U' to be the complement of U, so that $U' \neq \bot$. Since X is (-1)-connected, we may write $U' = \bigvee_i U'_i$ with $U'_i \neq \bot$ such that there is a map $U'_i \to X$. Since $U \wedge U'_i = \bot$, we have $X(U'_i \wedge U) \xrightarrow{\sim} X(U'_i) \times X(U) \neq \emptyset$. This gives an extension of φ to $\alpha + 1$. □

To construct the functor (2) in the above diagram, we use the following:

Proposition 6.124. *For every* 0*-localic topos L, there exists a complete Boolean algebra Λ and a surjection* $\mathrm{Shv}(\Lambda) \twoheadrightarrow L$*.*

Proof. We have $L = \mathrm{Shv}(\mathcal{U})$ where $\mathcal{U}$ is some locale. We will show: for all $u < v$ in $\mathcal{U}$, there exists a complete Boolean algebra $\Lambda_{u,v} \neq 0$ and a lex colimit-preserving functor $[u, v] \to \Lambda_{u,v}$ (detecting that $u \neq v$). Then $\Lambda = \bigcup_{u<v} \Lambda_{u,v}$ works.

The key claim is: if $\mathcal{U} \neq *$, then there exists a left exact localization $\mathcal{U} \to \Lambda \neq 0$, where Λ is a complete Boolean algebra. We take

$$\Lambda = \{u \in \mathcal{U} \mid u = u^{cc}\},$$

where $u^c = \sup_{v\wedge u=\bot} v$. Then Λ contains both $\bot$ and $\top$, so is non-trivial. □

The Diaconescu cover

Proposition 6.125 (Diaconescu cover)**.** *Let C be a category with topology τ. Then there exists a surjection* $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$*, where D is a poset.*

We will prove this below. First, we state a corollary:

Corollary 6.126. *Let T be any topos. Then there exists a surjection* $\mathrm{Shv}(\Lambda) \twoheadrightarrow T$*, where Λ is a complete Boolean algebra.*

Proof. We may write $T^{\mathrm{hyp}} \simeq \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$ for some C. In light of the surjection $T^{\mathrm{hyp}} \twoheadrightarrow T$, we may assume $T = \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. By Proposition 6.125, there is a surjection $\mathrm{Shv}_\rho(D) \twoheadrightarrow \mathrm{Shv}_\tau(C)$, which induces a surjection $\mathrm{Shv}_\rho(D)^{\mathrm{hyp}} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}}$. Consider now the 0-localic reflection $L_0 \mathrm{Shv}_\rho(D)$ of this sheaf category. By Proposition 6.124, this receives a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow L_0 \mathrm{Shv}_\rho(D)$. Since $\mathrm{Shv}(\Lambda)$ is hypercomplete, this functor factors through the hypercompletion $(L_0 \mathrm{Shv}_\rho(D))^{\mathrm{hyp}}$, which is equivalent to $\mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$ by Lemma 6.72. It follows that the resulting functor $\mathrm{Shv}(\Lambda) \twoheadrightarrow \mathrm{Shv}_\rho(D)^{\mathrm{hyp}}$ is surjective. All in all, we obtain surjections

$$\mathrm{Shv}(\Lambda) \twoheadrightarrow (\mathrm{Shv}_\rho(D))^{\mathrm{hyp}} \twoheadrightarrow \mathrm{Shv}_\tau(C)^{\mathrm{hyp}} \simeq T^{\mathrm{hyp}} \twoheadrightarrow T$$

as desired. □

Proof of Proposition 6.125. We claim that there exists a poset D with a functor $f\colon D \to C$, such that:

(1) f is surjective on objects;

(2) for all $x \in D$, the induced functor $D_{/x} \to C_{/f(x)}$ is surjective.

To this end, let $\hat{C}$ be a simplicial set whose associated complete Segal anima is C. Define a poset P as follows:

$$P = \{(n, \sigma) \mid n \in \mathbb{N}, \sigma\colon \Delta^n \to \hat{C}^{\mathrm{op}}\},$$

where $(n, \sigma) \le (n', \sigma')$ if and only if $n \le n'$ and $\sigma = \sigma'|_{\Delta^{\{0,1,\dots,n\}}}$. We may define a functor $N(P) \to \hat{C}^{\mathrm{op}}$, which sends a sequence $(n_0, \sigma_0) \le \cdots \le (n_k, \sigma_k)$ to the composite $\Delta^k \xrightarrow{n_0,\dots,n_k} \Delta^{n_k} \xrightarrow{\sigma_k} \hat{C}^{\mathrm{op}}$. By passing to realizations, this gives a functor

$$D := P^{\mathrm{op}} \to C.$$

We prove that this satisfies (1) and (2). Surjectivity is clear. For stability under slices, consider $P_{(n,\sigma)/} \to (C^{\mathrm{op}})_{\sigma(n)/}$, and consider $\alpha \in (C^{\mathrm{op}})_{\sigma(n)/}$. This looks like

$$\sigma(0) \to \cdots \to \sigma(n) \xrightarrow{\alpha} C.$$

From now on, we use that we have f satisfying (1) and (2). We define a topology ρ on D as follows: a sieve $R \hookrightarrow y(x)$ is declared to be in ρ if for every $\alpha\colon x' \to x$ in D, the sieve $f(\alpha^*(R))$ is a τ-cover of $f(x)$ in C.

Claim: The composite

$$\mathrm{PSh}(C) \xrightarrow{f^*} \mathrm{PSh}(D) \to \mathrm{Shv}_\rho(D)$$

factors through $\mathrm{Shv}_\tau(C)$. To see this, let $R \hookrightarrow y(c)$ be a τ-covering sieve. We then have to show that the map $f^*(R) \hookrightarrow f^*(y(c))$ is ρ-effective epi. We may check this after pullback to any representable in D, so for a map $y(d) \to f^*(y(c))$ consider the pullback

$$\begin{tikzcd} S \arrow[r] \arrow[d] & y(d) \arrow[d] \\ f^*(R) \arrow[r] & f^*(y(c)). \end{tikzcd}$$

We need to show that $S \hookrightarrow y(d)$ is a ρ-covering, or equivalently that $f(S)$ τ-covers $f(d)$. Consider now the following square:

$$\begin{tikzcd} f_!(S) \arrow[r] \arrow[d] & f_!(y(d)) = y(f(d)) \arrow[d, "\varphi"] \\ R \arrow[r] & y(c). \end{tikzcd}$$

The pullback $\varphi^*(R) \hookrightarrow y(f(d))$ is a τ-sieve, and the top map factors through a map $f_!(S) \to \varphi^*(R)$. It thus remains to show that this map is an effective epimorphism. But given any element of $\varphi^*(R)$, assumption (2) guarantees that it is f applied to some element of d, but then by definition of S as a pullback square it is in fact an element of S. This finishes the construction of the functor $L_\rho f^* \colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_\rho(D)$.

We now have to show that the functor $L_\rho f^* \colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_\rho(D)$ detects effective epimorphisms. Let $X \to Y$ be a morphism in $\mathrm{Shv}_\tau(C)$ such that $L_\rho f^*(X \to Y)$ is an effective epimorphism. We need to show $X \to Y$ itself is effective epi. We may check this locally on C: we need to show that for all $y(c) \to Y$, the map $X \times_Y y(c) \to y(c)$ is a τ-covering. By assumption (1), we may assume $c = f(d)$ for some d, hence $Y(c) = f^*(Y)(d)$. Since $f^*(X) \to f^*(Y)$ is a ρ-effective epi by assumption, for the map $y(d) \to f^*(Y)$, there exists a ρ-covering $y(d_i) \to y(d)$ such that $y(d_i) \to f^*(Y)$ lifts to $y(d_i) \to f^*(X)$. But then the map $f(d_i) \to f(d) = c$ is a τ-covering (by definition of ρ) which refines $X \times_Y y(c) \to y(c)$. This finishes the proof. □

Factorization through the spectrum

Now, let Λ be a complete Boolean algebra. Then $\mathrm{Spec}(\Lambda)$ is a 0-dimensional spectral space. By some abstract theory we will treat as a black-box, there is an inclusion

$$u \colon \Lambda \hookrightarrow \mathrm{Open}(\mathrm{Spec}(\Lambda)),$$

which is the inclusion of the quasi-compact open (=clopen) subsets. This inclusion admits a left exact left adjoint

$$v \colon \mathrm{Open}(\mathrm{Spec}(\Lambda)) \to \Lambda,$$

which is given by sending U to $\bigvee_{u(\lambda)\subseteq U} \lambda$. (This is the smallest quasi-compact open containing U; a phrase that rarely makes sense in other contexts, but does make sense in this context.) This results in a left exact localization

$$\varphi_* \colon \mathrm{Shv}(\Lambda) \rightleftarrows \mathrm{Shv}(\mathrm{Spec}(\Lambda)) : \varphi^*,$$

where φ^* is the left Kan extension along v, while φ_* is left Kan extension along u, or equivalently precomposition with v. Note that φ_* is fully faithful.

Lemma 6.127. *For every effective epimorphism* $\bigsqcup_{i=1}^n Y_i \to X$ *in* $\mathrm{Shv}(\Lambda)$*, the map* $\bigsqcup_{i=1}^n \varphi_*(Y_i) \to \varphi_*(X)$ *is an effective epimorphism in* $\mathrm{Shv}(\mathrm{Spec}(\Lambda))$.

Proof. The category $\mathrm{Shv}(\mathrm{Spec}(\Lambda))$ is generated under colimits by the image of u: as $\mathrm{Spec}(\Lambda)$ is a spectral space, it is generated by the quasi-compact opens. Since we have

$$\varphi_*(Y_i) \times_{\varphi_*(X)} u(\lambda) = \varphi_*(Y_i \times_X \lambda),$$

we may assume that $X = \lambda \in \Lambda$. Now, consider the epi-mono factorization

$$Y_i \overset{u_i}{\twoheadrightarrow} \lambda_i \hookrightarrow \lambda.$$

Note that the map $\lambda_i \hookrightarrow \lambda$ is representable, because any subobject of a representable is representable (since Λ is precisely the subobjects of the terminal sheaf).

Since $\Lambda_{/\lambda_i}$ has dimension at most 0, the map u_i has a section s_i. So, we then have the composite

$$\bigsqcup_{i=1}^n u(\lambda_i) \xrightarrow{(s_i)} \bigsqcup_{i=1}^n \varphi_*(Y_i) \to u(\lambda),$$

which is an effective epimorphism in $\mathrm{Shv}(\mathrm{Spec}(\Lambda))$, because u preserves *finite* joins. But then it follows that also the second map is an effective epimorphism. $\square$

Proof of the completeness theorem

We can now finally deduce the proof of the main result.

Proof of Theorem 6.120. Let (C, τ) be a finitary site with finite limits. We have previously established that there exists a complete Boolean algebra Λ and a surjection $\mathrm{Shv}(\Lambda) \twoheadrightarrow \mathrm{Shv}_\tau(C)$. We claim that this factors as follows:

$$\mathrm{Shv}(\Lambda) \xrightarrow{\varphi_*} \mathrm{Shv}(\mathrm{Spec}(\Lambda)) \to \mathrm{Shv}_\tau(C).$$

Since C has finite limits, the given morphism of topoi $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}_\tau(C)$ corresponds to a morphism of logoi $\mathrm{Shv}_\tau(C) \to \mathrm{Shv}(\Lambda)$, which is obtained by left Kan extending some left exact functor $C \to \mathrm{Shv}(\Lambda)$. We may then consider the following composite:

$$C \to \mathrm{Shv}(\Lambda) \xrightarrow{\varphi_*} \mathrm{Shv}(\mathrm{Spec}(\Lambda)).$$

This is still left exact, and since φ_* preserves epimorphisms by Lemma 6.127, this gives a morphism of topoi $\mathrm{Shv}(\mathrm{Spec}(\Lambda)) \to \mathrm{Shv}_\tau(C)$. Moreover, since the composite $\mathrm{Shv}(\Lambda) \xrightarrow{\varphi_*} \mathrm{Shv}(\mathrm{Spec}(\Lambda)) \xrightarrow{\varphi^*} \mathrm{Shv}(\Lambda)$ is an identity (by fully faithfulness of φ_*), this restricts to the original $\mathrm{Shv}(\Lambda) \to \mathrm{Shv}_\tau(C)$. This finishes the proof. $\square$

6.7.6 Makkai completeness

For every locally coherent topos, we saw that there exists a surjection $\bigsqcup_S * \twoheadrightarrow T$. While the category Topos of topoi is not strictly speaking a topos itself, it in many ways behaves like one, so we may ask whether T can be recovered somehow from this map, similar to how one would usually form the Čech nerve.

Question 6.128. Can we recover T from its category of points?

The answer is yes, but only if we remember enough structure on the category $\mathrm{Pt}(T)$. (Just remembering $\mathrm{Pt}(T)$ itself is not going to work.) This is the classical *completeness theorem* of Makkai, generalized to higher topoi by Lurie. We may think of $\mathrm{Pt}(T)$ as the category of "models for our theory", e.g. if T is the theory of local rings, then $\mathrm{Pt}(T)$ is the category of local rings. So this is the sense in which this is a completeness theorem: it says we can recover a theory from the models of that theory.

Definition 6.129. An *ultracategory* is a category with ultraproducts, i.e., for every ultrafilter $\mathcal{U}$ on a set I, we are given a functor

$$P^{\mathcal{U}}\colon C^I \to C.$$

In more detail, let $\mathrm{Stone}^{\mathrm{free}}$ denote the category of the Stone–Čech compactifications of sets. Then the category $\mathrm{Cat}^{\mathrm{ultra}}$ of ultracategories is defined as a subcategory of $\mathrm{LaxFun}((\mathrm{Stone}^{\mathrm{free}})^{\mathrm{op}}, \mathrm{Cat})$ spanned by those lax functors F such that:

- We have $F(\beta I) \simeq \prod_I F(*)$.
- Given two morphisms $f\colon \beta I \to \beta J$ and $g\colon \beta J \to \beta K$, if f is induced from some $I \to J$, then $f^* \circ g^* \simeq (g \circ f)^*$.

Theorem 6.130 (Makkai completeness)**.** *The* 2*-functor* $\mathrm{Topos}^{\mathrm{coh},\flat} \to \mathrm{Cat}^{\mathrm{ultra}}$, $T \mapsto \mathrm{Pt}(T)$*, is fully faithful.*

Remark 6.131. There is a pullback square

$$\begin{tikzcd}
\mathrm{CondAn} \arrow[r] \arrow[d] & \mathrm{Cat}^{\mathrm{ultra}} \arrow[d] \\
\mathrm{An} \arrow[r] & \mathrm{Cat}.
\end{tikzcd}$$

Note how for An any lax functor is automatically strict.

6.8 Pretopoi

Pretopoi are a finitary version of topoi. There are several ways to interpret this slogan.

Definition 6.132. A category C is a *local pretopos* if the following conditions are satisfied:

(1) C admits fiber products;

(2) C admits finite coproducts, and they are van Kampen;

(3) C admits colimits of groupoid objects, which are effective and universal (or equivalently van Kampen).

A *pretopos* is a local pretopos with a final object.

Remark 6.133. In a local pretopos, we have the factorization system (effective epi, mono). Also the factorization system (n-connected, n-truncated) exists, but this is less obvious. It amounts to showing that n-truncation exists.

Definition 6.134. Let C be a local pretopos. The *effective epi topology* on C is defined as follows: a sieve on $X \in C$ is declared to be a covering sieve if it contains a *finite* family $(Y_i \to X)_i$ such that the map $\bigsqcup_i Y_i \to X$ is an effective epimorphism.

If C is small, we obtain a topos $\mathrm{Shv}(C)$ with respect to this topology.

Proposition 6.135. *If T is a topos, then T^{coh} is a local pretopos. In particular, if T is coherent, then T^{coh} is a pretopos.*

Proof. Since T satisfies conditions (1)–(3), it suffices to show that T^{coh} is closed under the following operations:

- Fiber products: this is clear from the definition of coherent objects;
- Finite coproducts: this follows by induction, reducing to the definition of quasi-compact objects;
- Colimits of groupoids: follows from descent property of coherence: given a groupoid $X_\bullet$ with colimit X, we may consider the pullback square

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ X_0 & \longrightarrow & X. \end{array}$$

  Since X_0, X_1 are coherent and $X_0 \twoheadrightarrow X$ is an effective epi, it follows from the properties of coherent objects we discussed that X is also coherent. □

Definition 6.136. A *prelogos* is just a pretopos. A *morphism of prelogoi* from S to T is a functor $f^*\colon S \to T$ which preserves finite limits, finite coproducts, and colimits of groupoids. This defines a category $\mathrm{Logos}^{\mathrm{pre}}$ of prelogoi. The category of *pretopoi* is defined as $\mathrm{Topos}^{\mathrm{pre}} := (\mathrm{Logos}^{\mathrm{pre}})^{\mathrm{op}}$.

Definition 6.137. We denote by $\mathrm{Topos}^{\mathrm{coh}} \subseteq \mathrm{Topos}$ the non-full subcategory spanned by the coherent topoi and the *coherent morphisms*: those morphisms of topoi $f\colon T \to S$ such that $f^*\colon S \to T$ preserves coherent objects.

There are functors

$$\mathrm{Topos}^{\mathrm{coh}} \to \mathrm{Topos}^{\mathrm{pre}}, \qquad T \mapsto T^{\mathrm{coh}}$$

and

$$\mathrm{Topos}^{\mathrm{pre}} \to \mathrm{Topos}^{\mathrm{coh}}, \qquad C \mapsto \mathrm{Shv}(C).$$

Definition 6.138. A pretopos C is called *bounded* if it is small and every $X \in C$ is truncated.

A pretopos C is *hypercomplete* if the following conditions are satisfied:

- It admits colimits of Kan simplicial objects;

- If $X_\bullet \to X$ is a trivial Kan fibration, then $|X_\bullet| \simeq X$.

Theorem 6.139. *There are equivalences*

$$\mathrm{Topos}^{\mathrm{coh},b} \simeq \mathrm{Topos}^{\mathrm{pre},b}, \qquad T \mapsto T^{\mathrm{coh}}_{<\infty}, \qquad C \mapsto \mathrm{Shv}(C)$$

and

$$\mathrm{Topos}^{\mathrm{coh},\mathit{loc\ coh},\mathrm{hyp}} \simeq \mathrm{Topos}^{\mathrm{pre},\mathrm{hyp}}, \qquad T \mapsto T^{\mathrm{coh}}, \qquad C \mapsto \mathrm{Shv}(C)^{\mathrm{hyp}}.$$

6.9 Compactly assembled categories and exponentiability

In this section we characterize the exponentiable topoi, i.e. those topoi T for which the exponential S^T exists for all topoi S. The answer is provided by the notion of compactly assembled categories: a topos is exponentiable if and only if it is compactly assembled as a presentable category. We begin by studying finitary functors as an auxiliary notion, then introduce compactly assembled categories and establish their connection to classifying topoi, before proving the characterization of exponentiable topoi.

We write $\mathrm{Fun}^\kappa(C,T) \subseteq \mathrm{Fun}(C,T)$ for the full subcategory of κ-filtered-colimit-preserving functors.

Proposition 6.140. *Let C be an accessible category with κ-filtered colimits, let T be a topos. Then* $\mathrm{Fun}^\kappa(C,T)$ *is a topos.*

Proof. We may write $C = \mathrm{Ind}_\lambda(C_0)$ for some small category C_0, and some regular cardinal $\lambda \gg \kappa$. It then follows that

$$\mathrm{Fun}^\kappa(C,T) \subseteq \mathrm{Fun}^\lambda(C,T) \simeq \mathrm{Fun}(C_0,T).$$

Note that $\mathrm{Fun}(C_0,T)$ is a topos. Moreover, $\mathrm{Fun}^\kappa(C,T)$ is closed under colimits and finite limits. It remains to show that $\mathrm{Fun}^\kappa(C,T)$ is accessible. We will do this by cutting it out of $\mathrm{Fun}(C_0,T)$ by imposing a small set of conditions.

Note that a functor $F \in \mathrm{Fun}^\lambda(C,T)$ preserves κ-filtered colimits if and only if F preserves colimits indexed by κ-filtered posets. Using $\lambda \gg \kappa$, we may write every κ-filtered poset as a λ-filtered colimit of λ-small κ-filtered colimits, so the condition is equivalent to F preserving λ-small κ-filtered colimits.

Now, let A be a λ-small poset. Since C is accessible and A is small, the functor category $\mathrm{Fun}(A,C)$ is again accessible, hence there exists a set of functors $\alpha\colon A \to C$ that generates $\mathrm{Fun}(A,C)$ under κ-filtered colimits for some large $\kappa' \gg \lambda$, hence in particular under λ-filtered colimits. We then have a pullback square

$$\begin{tikzcd} \mathrm{Fun}^\kappa(C,T) \arrow[r] \arrow[d] & \mathrm{Fun}^\lambda(C,T) \arrow[d] \\ \prod_\alpha T \arrow[r, "\Delta"] & \prod_\alpha \mathrm{Ar}(T), \end{tikzcd}$$

where the right vertical functor sends F to the collection of maps $(\mathrm{colim}_A F \circ \alpha \to F(\mathrm{colim}_A \alpha))$. This finishes the proof. ◻

Remark 6.141. The category $\mathrm{Fun}^\kappa(C,T)$ has a universal property: for every topos S, there is a natural equivalence

$$\mathrm{Geom}(\mathrm{Fun}^\kappa(C,T),S) \simeq \mathrm{Fun}^\kappa(C,\mathrm{Geom}(T,S)).$$

Both sides are naturally a full subcategory of $\mathrm{Fun}(C \times S, T)$, and one can check that in both cases we precisely get those functors $C \times S \to T$ that preserve κ-filtered colimits in the left variable, and preserve finite limits and colimits in the second variable.

Corollary 6.142. *The functor* $\mathrm{Pt}\colon \mathrm{Topos} \to \mathrm{Cat}^{\mathrm{acc},\kappa}$ *admits a (2-categorical) left adjoint*

$$\mathrm{Cat}^{\mathrm{acc},\kappa} \to \mathrm{Topos}, \qquad C \mapsto \mathrm{Fun}^{\kappa}(C, \mathrm{An}).$$

We now introduce the central notion of this section. A category is called compactly assembled if it satisfies certain retract conditions with respect to ind-categories. The following proposition provides three equivalent characterizations.

Proposition 6.143. *Let C be a category with filtered colimits. Then the following are equivalent:*

(1) *There exists a small category C_0 such that C is a retract of* $\mathrm{Ind}(C_0)$ *in* Cat^{ω} *(the category of categories with filtered colimits).*

(2) *There exists a small category C_0 such that there exists an adjunction $C \rightleftarrows \mathrm{Ind}(C_0)$ in* Cat^{ω} *such that $C \hookrightarrow \mathrm{Ind}(C_0)$ is fully faithful.*

(3) *C is accessible, and the "colimit functor"* $\mathrm{colim}\colon \mathrm{Ind}(C) \to C$ *(defined as the left adjoint to $y\colon C \hookrightarrow \mathrm{Ind}(C)$) admits a further left adjoint $y'\colon C \hookrightarrow \mathrm{Ind}(C)$.*

Definition 6.144. A category satisfying the equivalent conditions of Proposition 6.143 is called *compactly assembled*.

Proof of Proposition 6.143. Note that (2) clearly implies (1), since the unit of the adjunction exhibits the map $\mathrm{Ind}(C_0) \to C$ as a retraction of $C \hookrightarrow \mathrm{Ind}(C_0)$.

For (3) $\implies$ (2), we write $\mathrm{Ind}(C) = \bigcup_{\lambda > \kappa} \mathrm{Ind}(C^{\lambda})$, where C is κ-accessible. Since C^{κ} is small, we have $y'(C^{\kappa}) \subseteq \mathrm{Ind}(C^{\lambda})$ for some λ. Since C is generated under colimits by C^{κ} and y' preserves colimits, this implies that $y'(C) \subseteq \mathrm{Ind}(C^{\lambda})$. But then the functor $y'\colon C \to \mathrm{Ind}(C^{\lambda})$ is a left adjoint to $\mathrm{colim}\colon \mathrm{Ind}(C^{\lambda}) \to C$, giving (2).

Finally, we prove that (1) $\implies$ (3). The assumption gives us functors

$$C \xrightarrow{u} \mathrm{Ind}(C_0) \xrightarrow{v} C$$

satisfying $vu = \mathrm{id}_C$. We may now consider the idempotent $uv\colon \mathrm{Ind}(C_0) \to \mathrm{Ind}(C_0)$. We may then write

$$C = \lim(\cdots \to \mathrm{Ind}(C_0) \xrightarrow{uv} \mathrm{Ind}(C_0) \xrightarrow{uv} \mathrm{Ind}(C_0)),$$

and in particular C is a limit of accessible categories, hence itself accessible. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Ind}(C) \arrow[r, "u"] \arrow[d, "\mathrm{colim}"] & \mathrm{Ind}(\mathrm{Ind}(C_0)) \arrow[r, "v"] \arrow[d, "\mathrm{colim}"] & \mathrm{Ind}(C) \arrow[d, "\mathrm{colim}"] \\
C \arrow[r, "\hat{u}"] & \mathrm{Ind}(C_0) \arrow[r, "\hat{v}"] & C.
\end{tikzcd}$$

The middle vertical map admits a left adjoint $y'\colon \mathrm{Ind}(C_0) \hookrightarrow \mathrm{Ind}(\mathrm{Ind}(C_0))$.

Now, we show that the left adjoint $y'\colon C \hookrightarrow \mathrm{Ind}(C)$ exists. We may do this pointwise, so consider some $X \in C$. The map $\mathrm{Hom}(X, \mathrm{colim}(-))$ is a retract of the representable copresheaf

$\mathrm{Hom}(\hat{v}y'u(X), -)$ in $\mathrm{Fun}(\mathrm{Ind}(C), \mathrm{An})$. Since $\mathrm{Ind}(C)$ is idempotent complete, this means that $\mathrm{Hom}(X, \mathrm{colim}(-))$ is representable. Hence there exists some $y'(X) \in \mathrm{Ind}(C)$, which then provides the desired left adjoint object to X. $\square$

Corollary 6.145. *Let C be a presentable category. Then C is compactly assembled if and only if it is a retract in* Pr^{L} *of a compactly generated category.*

The key property of compactly assembled categories is that they admit classifying topoi for sheaves valued in them. We establish this through an equivalence relating functors out of $\mathrm{Fun}^{\omega}(C, \mathrm{An})$ to functors from C^{op}.

Proposition 6.146. *Let C be compactly assembled, and let D be presentable. Then there is an equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), D) \simeq \mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D),$$

where the RHS means those functors that preserve cofiltered limits.

Proof sketch. If $C = \mathrm{Ind}(C_0)$ is compactly generated, we have $\mathrm{Fun}^{\omega}(C, \mathrm{An}) = \mathrm{Fun}(C_0, \mathrm{An}) = \mathrm{PSh}(C_0^{\mathrm{op}})$, whereas $\mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D) = \mathrm{Fun}(C_0^{\mathrm{op}}, D)$. We would like to say: the claim now follows as every C is a retract of a compactly generated one. However, for this reasoning to work, we need to find a way to write down the comparison map for the compactly generated case in a way that does not rely on picking an identification $C = \mathrm{Ind}(C_0)$.

To this end, consider the category

$$\mathrm{Fun}^{+}(C, \mathrm{An}) := \mathrm{Fun}^{\omega}(C, \mathrm{An}) \cup \mathrm{Fun}^{\mathrm{rep}}(C, \mathrm{An}) \subseteq \mathrm{Fun}(C, \mathrm{An}).$$

Consider a functor $F\colon \mathrm{Fun}^{+}(C, \mathrm{An}) \to D$. Then, if $C = \mathrm{Ind}(C_0)$, one can show that:

- F is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}$ if and only if $F|_{\mathrm{Fun}^{\omega}}$ preserves colimits.
- F is right Kan extended from Fun^{ω} if and only if $F|_{\mathrm{Fun}^{\mathrm{rep}}}$ preserves cofiltered limits.

Using the adjunction $C \rightleftarrows \mathrm{Ind}(C_0)$, we then deduce that the following are equivalent:

- F is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}$ and $F|_{\mathrm{Fun}^{\mathrm{rep}}}$ preserves cofiltered limits.
- F is right Kan extended from Fun^{ω} and $F|_{\mathrm{Fun}^{\omega}}$ preserves colimits.

It follows that both sides $\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), D)$ and $\mathrm{Fun}_{\omega}(C^{\mathrm{op}}, D)$ agree with the full subcategory of $\mathrm{Fun}(\mathrm{Fun}^{+}(C, \mathrm{An}), D)$ satisfying these two equivalent conditions. $\square$

Corollary 6.147. *Let T be a topos and C a presentable category which is compactly assembled. Then the equivalence*

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), T) \simeq \mathrm{Fun}_{\omega}(C^{\mathrm{op}}, T)$$

restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{Logos}}(\mathrm{Fun}^{\omega}(C, \mathrm{An}), T) \simeq \mathrm{Fun}^{\mathrm{R}}(C^{\mathrm{op}}, T).$$

Remark 6.148. Recall that the category $\mathrm{Fun}^{\mathrm{R}}(C^{\mathrm{op}}, T)$ is the Lurie tensor product $C \otimes T$ in Pr^{L}, which in turn may be written as $\mathrm{Fun}^{\mathrm{R}}(T^{\mathrm{op}}, C)$. This category is also known as the category of *C-valued sheaves on T*. If one is a bit more careful about the functoriality of this equivalence in T, then we see: the topos $\mathrm{Fun}^{\omega}(C, \mathrm{An})$ is the classifying topos for C-valued sheaves.

We have thus shown that if C is compactly assembled, then a classifying topos for C-valued sheaves exists. It turns out that the converse is also true:

Proposition 6.149. *Let C be a presentable category. Then a classifying topos for C-valued sheaves exists if and only if C is compactly assembled.*

Proof. It remains to show the "only if" part. Let E be such a classifying topos, and write it as a left exact localization $i_*\colon E \hookrightarrow \mathrm{PSh}(D)$ of a presheaf category. This induces a geometric morphism $i_*\colon \mathrm{Shv}_C(E) \hookrightarrow \mathrm{Shv}_C(\mathrm{PSh}(D))$. Let $F_{\mathrm{univ}} \in \mathrm{Shv}_C(E)$ be the universal C-valued sheaf on E. By universality, there exists a morphism of topoi $f\colon \mathrm{PSh}(D) \to E$ satisfying $i_*(F) = f^*(F)$. By fully faithfulness of i_* we have $F \cong i^* i_*(F) \cong i^* f^*(F) = (f \circ i)^*(F)$. By universality again, it follows that $f \circ i \cong \mathrm{id}_E$. In particular, E is a retract of $\mathrm{PSh}(D)$ in Topos. By applying $\mathrm{Pt}(-)$, it then follows that $C \cong \mathrm{Shv}_C(\mathrm{An}) \cong \mathrm{Fun}^{\mathrm{R}}(\mathrm{An}, E)$ is a retract of $\mathrm{Pt}(\mathrm{PSh}(D)) = \mathrm{Fun}_{\mathrm{Topos}}(\mathrm{An}, \mathrm{PSh}(D))$ in Cat^{ω}, proving the claim. □

Remark 6.150. Let C be a compactly assembled category. Then there is an equivalence $\mathrm{Pt}(\mathrm{Fun}^{\omega}(C, \mathrm{An})) \simeq C$, where $X \in C$ corresponds to the evaluation functor $\mathrm{ev}_X\colon \mathrm{Fun}^{\omega}(C, \mathrm{An}) \to \mathrm{An}$.

Lemma 6.151. *The functor* $\mathrm{Cat}^{\mathrm{ca}} \to \mathrm{Topos}$ *given by* $C \mapsto \mathrm{Fun}^{\omega}(C, \mathrm{An})$ *is a fully faithful* 2*-functor.*

Proof. By Corollary 6.142, the functor $\mathrm{Pt}\colon \mathrm{Topos} \to \mathrm{Cat}^{\mathrm{acc},\omega}$ admits a left adjoint given by $C \mapsto \mathrm{Fun}^{\omega}(C, \mathrm{An})$. The unit of this adjunction provides a map $C \to \mathrm{Pt}(\mathrm{Fun}^{\omega}(C, \mathrm{An}))$, which by Remark 6.150 is an equivalence when C is compactly assembled. Therefore the restriction of this adjunction to $\mathrm{Cat}^{\mathrm{ca}}$ exhibits $\mathrm{Cat}^{\mathrm{ca}} \to \mathrm{Topos}$ as fully faithful. □

Having established that compactly assembled categories have classifying topoi, we now turn to the main application: characterizing exponentiable topoi. Recall that the exponential S^T in Topos, if it exists, represents the functor $U \mapsto \mathrm{Hom}_{\mathrm{Topos}}(U \otimes T, S)$.

Let S and T be topoi. If the *exponential* (internal hom) in Topos exists, it will be denoted S^T. In other words, this is a topos S^T equipped with a morphism of topoi $S^T \otimes T \to S$ which induces an equivalence

$$\mathrm{Hom}_{\mathrm{Topos}}(U, S^T) \simeq \mathrm{Hom}_{\mathrm{Topos}}(U \otimes T, S).$$

Observation 6.152. If S^T exists, it is automatically a 2-categorical exponential: for every topos U, we have another topos $\mathrm{Ar}(U)$, and we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Geom}(U, S^T)) &\cong \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U), S^T) \\
&\cong \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U) \otimes T, S) \\
&\cong \mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Ar}(U \otimes T), S) \\
&\cong \mathrm{Hom}_{\mathrm{Cat}}([1], \mathrm{Geom}(U \otimes T, S)).
\end{aligned}$$

Remark 6.153. We have $\mathrm{Pt}(S^T) \simeq \mathrm{Geom}(T, S)$.

Definition 6.154. We say that a topos T is *exponentiable* if S^T exists for all S.

Theorem 6.155. *A topos T is exponentiable if and only if it is compactly assembled.*

Proof. For the "only if" direction, assume that T is exponentiable. To show it is compactly assembled, consider some compactly assembled category C and consider $S := \mathrm{Fun}^{\omega}(C, \mathrm{An})$. By assumption, the exponential S^T exists. Moreover, we have

$$\mathrm{Geom}(U, S^T) \cong \mathrm{Geom}(U \otimes T, S) \cong \mathrm{Shv}_C(U \otimes T) \cong \mathrm{Shv}_{C \otimes T}(U).$$

In particular, taking $C = \mathrm{An}$, we see that S^T classifies T-valued sheaves. By Proposition 6.149, it follows that T is compactly assembled.

For the "if" direction, assume T is compactly assembled. We need to show that S^T exists for every topos S. We may write S as a pullback in Topos of a square of the form

$$\begin{tikzcd} S \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & C, \end{tikzcd}$$

where A, B and C are presheaf categories on categories with finite limits. (Write S as the localization of a presheaf category, and then we may write a localization as a pullback by using the arrow category.)

Hence we may assume that $S = \mathrm{PSh}(D)$, where D has finite limits. Hence we get

$$S = \mathrm{PSh}(D) \cong \mathrm{Fun}^{\omega}(\mathrm{Ind}(D^{\mathrm{op}}), \mathrm{An}).$$

Setting $C := \mathrm{Ind}(D^{\mathrm{op}})$, we see that an exponential S^T, if it exists, is a classifying topos for $(C \otimes T)$-valued sheaves. Since $C \otimes T$ is compactly assembled, Proposition 6.149 implies that such a classifying topos exists. □

6.10 Topoi of parametrized objects

We introduce the category $\int_{\mathrm{An}} C$ of An-parametrized objects of a category C and discuss under what conditions on C this is a topos. This is closely related to categories that look like "categories of ∞-connected objects in a topos". This material may also be found in [Hoy19b]. It is best treated as mostly a curiosity. The material will not be used anywhere else, and readers may wish to skip this section.

6.10.1 Categories of An-parametrized objects

Definition 6.156. Let C be a category, not necessarily small. We define the functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ as the cartesian unstraightening of the functor

$$\mathrm{Fun}(-, C)\colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}, \qquad A \mapsto \mathrm{Fun}(A, C).$$

We refer to $\int_{\mathrm{An}} C$ as the category of An-*parametrized objects of* C. Note that an object of $\int_{\mathrm{An}} C$ is a pair (A, X), where $A \in \mathrm{An}$ is an anima and $X\colon A \to C$ is an A-indexed family of objects of C. A morphism $(A, X) \to (B, Y)$ consists of a morphism of animae $f\colon A \to B$ and a map $X \to f^*Y$ in $\mathrm{Fun}(A, C)$, where $f^*Y := Y \circ f\colon A \to C$.

The fiber of $\int_{\mathrm{An}} C$ over $A \in \mathrm{An}$ is $\mathrm{Fun}(A, C)$. By taking $A = *$, we in particular obtain an inclusion $C \hookrightarrow \int_{\mathrm{An}} C$. Note that this is fully faithful, since $\{*\} \hookrightarrow \mathrm{An}$ is fully faithful.

While we gave an explicit construction of $\int_{\mathrm{An}} C$, we will show next that it can also be characterized via the following two universal properties:

(a) For arbitrary C, the category $\int_{\mathrm{An}} C$ is obtained from C by freely adding anima-indexed colimits;

(b) If C admits weakly contractible colimits, then $\int_{\mathrm{An}} C$ admits all small colimits, the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ preserves weakly contractible colimits, and $\int_{\mathrm{An}} C$ is the free category with a functor from C satisfying these two properties.

We start with property (a), which is relatively straightforward to prove.

Lemma 6.157. *The category $\int_{\mathrm{An}} C$ admits anima-indexed colimits, and the functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ preserves anima-indexed colimits.*

Proof. Consider a functor $F = (A_\bullet, X_\bullet)\colon I \to \int_{\mathrm{An}} C$ for which we wish to construct a colimit, where I is an anima. Consider the colimit $A := \mathrm{colim}_{i \in I} A_i$ of the underlying animae, thus extending $A_\bullet$ to a colimit cocone $\overline{A}_\bullet\colon I^\triangleright \to \mathrm{An}$. Denote by $f_i\colon A_i \to A$ the maps in the colimit cocone. The diagram F is a lift of $A_\bullet\colon I \to \mathrm{An}$, corresponding to a section of the cartesian fibration $I \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I$. Since I is an anima, this section is automatically a cartesian section, and it thus defines an object of the category

$$\Gamma_I^{\mathrm{cart}}(I \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I) \quad\simeq\quad \lim_{i \in I^{\mathrm{op}}} \mathrm{Fun}(A_i, C) \quad\simeq\quad \mathrm{Fun}(\mathrm{colim}_i A_i, C) \quad\simeq\quad \mathrm{Fun}(A, C).$$

This defines an object $X \in \mathrm{Fun}(A, C)$, hence an object $(A, X) \in \int_{\mathrm{An}} C$. Applying a similar reasoning to the diagram $\overline{A}_\bullet\colon I^\triangleright \to \mathrm{An}$, we see that

$$\Gamma_{I^\triangleright}^{\mathrm{cart}}(I^\triangleright \times_{\mathrm{An}} \int_{\mathrm{An}} C \to I^\triangleright) \quad\simeq\quad \lim_{i \in (I^\triangleright)^{\mathrm{op}}} \mathrm{Fun}(A_i, C) = \mathrm{Fun}(A, C),$$

so that the object X gives rise to a cartesian section $(\overline{A}_\bullet, \overline{X}_\bullet)\colon I^\triangleright \to \int_{\mathrm{An}} C$ lifting $\overline{A}_\bullet$, which essentially by construction extends $X_\bullet$. We will show that this is a colimit cocone in $\int_{\mathrm{An}} C$, which simultaneously establishes both claims of the lemma.

To this end, consider another object $(B, Y) \in \int_{\mathrm{An}} C$. We need to show that top map in the following commutative diagram is an equivalence:

$$\begin{tikzcd}
\mathrm{Hom}_{\int_{\mathrm{An}} C}((A, X), (B, Y)) \arrow[r] \arrow[d, "\pi"'] & \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\int_{\mathrm{An}} C}((A_i, X_i), (B, Y)) \arrow[d, "\pi"] \\
\mathrm{Hom}_{\mathrm{An}}(A, B) \arrow[r, "\simeq"] & \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{An}}(A_i, B).
\end{tikzcd}$$

The bottom map in this diagram is an equivalence by definition of A as a colimit. Therefore, it suffices to check that the square induces equivalences on vertical fibers over every map of animae

$f\colon A \to B$. Letting $f_i\colon A_i \to A \xrightarrow{f} B$ denote the composite maps for all i, this map on fibers takes the following form:

$$\mathrm{Hom}_{\mathrm{Fun}(A,C)}(X, f^*Y) \to \lim_{i\in I^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{Fun}(A_i,C)}(X_i, f_i^*Y).$$

But this map is an isomorphism, since the pullback functors along the maps $A_i \to A$ induce an equivalence $\mathrm{Fun}(A, C) \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{Fun}(A_i, C)$, and the restriction of X to A_i is X_i by construction. This finishes the proof. $\square$

Lemma 6.158. *Let D be a category admitting anima-indexed colimits. Then the inclusion $D \hookrightarrow \int_{\mathrm{An}} D$ admits a left adjoint* $\mathrm{colim}\colon \int_{\mathrm{An}} D \to D$*, given on objects by sending (A, X) to* $\mathrm{colim}_{a\in A} X_a$*.*

Proof. Since left adjoints may be constructed objectwise, it will suffice to show that for every $Y \in D$ there is a natural equivalence

$$\mathrm{Hom}_{\int_{\mathrm{An}} D}((A, X), (*, Y)) \xrightarrow{\sim} \mathrm{Hom}_D(\mathrm{colim}_{a\in A} X_a, Y).$$

But this is clear: since there is a unique map $A \to *$ in An, the left-hand side simplifies to $\mathrm{Hom}_{\mathrm{Fun}(A,D)}(X, \mathrm{const}_Y)$, which by the very definition of colimits in D is also the right-hand side. $\square$

Lemma 6.159. *Let* $\mathrm{Cat}^{\mathrm{An\text{-}colim}}$ *denote the subcategory of* Cat *consisting of the categories with anima-indexed colimits and functors preserving anima-indexed colimits. Then the construction $C \mapsto \int_{\mathrm{An}} C$ defines a left adjoint to the inclusion* $\mathrm{Cat}^{\mathrm{An\text{-}colim}} \hookrightarrow \mathrm{Cat}$*, with unit and counit given by the functors*

$$i\colon C \hookrightarrow \int_{\mathrm{An}} C \qquad \text{and} \qquad \mathrm{colim}\colon \int_{\mathrm{An}} D \to D,$$

respectively.

Proof. We check that the two triangle identities are satisfied. The first one takes the form

$$\begin{tikzcd} D \arrow[r, hook, "i"] \arrow[dr, equal] & \int_{\mathrm{An}} D \arrow[d, "\mathrm{colim}"] \\ & D. \end{tikzcd}$$

This commutes via the counit map $\mathrm{colim} \circ i \to \mathrm{id}_D$, which is an isomorphism as i is fully faithful. The second triangle identity amounts to the claim that for every object $(A, X) \in \int_{\mathrm{An}} C$, the canonical map $\mathrm{colim}_{a\in A}(\{a\}, X_a) \to (A, X)$ in $\int_{\mathrm{An}} C$ is an isomorphism, which as observed before is an immediate consequence of the computation of colimits in $\int_{\mathrm{An}} C$. $\square$

We may now deduce the first claimed universal property of $\int_{\mathrm{An}} C$:

Proposition 6.160. *Let C and D be categories and assume that D admits anima-indexed colimits. Then restriction along $i\colon C \hookrightarrow \int_{\mathrm{An}} C$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{An\text{-}colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}(C, D).$$

Proof. By the Yoneda lemma, we may check this after applying $\mathrm{Hom}_{\mathrm{Cat}}(E, -)$ for all $E \in \mathrm{Cat}$. Observe that $\mathrm{Fun}(E, D)$ still admits anima-indexed colimits, which are computed pointwise in D. Under currying-uncurrying, the map in question then translates to the map

$$\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{An\text{-}colim}}}(\int_{\mathrm{An}} C, \mathrm{Fun}(E, D)) \to \mathrm{Hom}_{\mathrm{Cat}}(C, \mathrm{Fun}(E, D))$$

given by restriction along $i\colon C \hookrightarrow \int_{\mathrm{An}} C$. This is an isomorphism by the adjunction from the previous lemma. □

We now move on to the second universal property of $\int_{\mathrm{An}} C$, which is more subtle; I believe Marc said that he hasn't seen this in the literature before. For the proof, we will make use of the category $\mathrm{PSh}(C)$ of *small presheaves* on C, i.e. the full subcategory of $\mathrm{PSh}(C)$ generated by the representable presheaves under small colimits. The inclusion $C \hookrightarrow \mathrm{PSh}(C)$ is the universal functor from C into a category with small colimits. (Recall that $\mathrm{PSh}(C) = \mathrm{PSh}(C)$ whenever C is small, but not for arbitrary C.) The small presheaves may be characterized as those $\mathcal{F}$ for which the category of elements $\mathrm{El}(\mathcal{F}) = C_{/\mathcal{F}} \subseteq \mathrm{PSh}(C)_{/\mathcal{F}}$ admits a final functor from a small category.

Proposition 6.161. *Let C be a category admitting small weakly contractible colimits.*

(1) *The inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ preserves weakly contractible colimits.*

(2) *The inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ uniquely extends to a left adjoint $L\colon \mathrm{PSh}(C) \to \int_{\mathrm{An}} C$;*

(3) *The right adjoint $R\colon \int_{\mathrm{An}} C \hookrightarrow \mathrm{PSh}(C)$ is fully faithful, exhibiting $\int_{\mathrm{An}} C$ as a Bousfield localization of* $\mathrm{PSh}(C)$*. In particular, $\int_{\mathrm{An}} C$ admits small colimits.*

(4) *For a category D with small colimits, a colimit-preserving functor $F\colon \mathrm{PSh}(C) \to D$ inverts L-local maps if and only if its restriction $C \to D$ preserves weakly contractible colimits.*

Proof. (1) Consider a functor $X_\bullet\colon I \to C$, where I is weakly contractible, and set $X := \mathrm{colim}_{i \in I} X_i$. We have to show that the colimit of this diagram in $\int_{\mathrm{An}} C$ is $(*, x)$. This amounts to the claim that for every object $(B, Y) \in \int_{\mathrm{An}} C$, the canonical map

$$\mathrm{Hom}_{\int_{\mathrm{An}} C}((*, X), (B, Y)) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_{\int_{\mathrm{An}} C}((*, X_i), (B, Y))$$

is an isomorphism. Since this map lives over $\mathrm{Hom}_{\mathrm{An}}(*, B) = B$, it suffices to show that the induced map on fibers over each $b \in B$ is an isomorphism. But this map on fibers is the map $\mathrm{Hom}_C(X, Y_b) \to \lim_{i \in I^{\mathrm{op}}} \mathrm{Hom}_C(X_i, Y_b)$, which is an isomorphism by construction of X.

(2) We will show that the left Kan extension of the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ along the Yoneda embedding $C \hookrightarrow \mathrm{PSh}(C)$ exists. Given a small presheaf $\mathcal{F}$, the pointwise formula for left Kan extensions tells us that it suffices to show that in $\int_{\mathrm{An}} C$ we may form the colimit of the functor

$$\mathrm{El}(\mathcal{F}) = C_{/\mathcal{F}} \xrightarrow{\mathrm{fgt}} C \hookrightarrow \int_{\mathrm{An}} C.$$

By assumption on $\mathcal{F}$, the category $\mathrm{El}(\mathcal{F})$ comes with a final functor $I \to \mathrm{El}(\mathcal{F})$ from some small category I. It will thus suffice to show that for any small category I and any functor $X\colon I \to C$ the colimit of $I \to \int_{\mathrm{An}} C$ exists. We do this in two steps:

- Consider the map $p\colon I \to |I|$ to the geometric realization of I. We will show that the Kan extension of X along p exists in $\int_{\mathrm{An}} C$. Since p is a cofinal functor, its relative slices are weakly contractible by Quillen's Theorem A. By part (1), this means that the Kan extension of X along p exists in C, and that it is preserved by the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$.
- The colimit $\operatorname{colim}_i X_i$ in $\int_{\mathrm{An}} C$ is now computed as the colimit of the left Kan extension $p_\sharp X\colon |I| \to \int_{\mathrm{An}} C$, which exists since $|I|$ is an anima and we established the existence of anima-indexed colimits in Lemma 6.157.

By general nonsense, it follows that the left Kan extension $L\colon \mathrm{PSh}(C) \to \int_{\mathrm{An}} C$ preserves all colimits. A right adjoint R to L is given explicitly as follows:

$$R\colon \int_{\mathrm{An}} C \to \mathrm{PSh}(C), \qquad R((A,X))(Y) := \mathrm{Hom}_{\int_{\mathrm{An}} C}((*,Y),(A,X)).$$

The functor $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ induces a map $\mathrm{Hom}_{\int_{\mathrm{An}} C}((*,Y),(A,X)) \to \mathrm{Hom}_{\mathrm{An}}(*,A) \cong A$, whose fiber over $a \in A$ is $\mathrm{Hom}_C(Y, X_a)$, so we obtain isomorphisms

$$R((A,X))(Y) \;\cong\; \operatorname{colim}_{a\in A} \mathrm{Hom}_C(Y,X_a) \;\cong\; \operatorname{colim}_{a\in A} y(X_a)(Y).$$

In particular, we see that $R((A,X)) \cong \operatorname{colim}_{a\in A} y(X_a)$ is a small colimit of representables, so that it is indeed contained in $\mathrm{PSh}(C)$.

(3) To show that R is fully faithful, we must show that for every object $(A,X) \in \int_{\mathrm{An}} C$, the counit $LR(A,X) \to (A,X)$ is an isomorphism. But this is clear from the computation $R(A,X) \cong \operatorname{colim}_{a\in A} y(X_a)$ from before, and the fact that $(A,X) \cong \operatorname{colim}_{a\in A}(*,X_a)$ by the explicit description of anima indexed colimits from Lemma 6.157.

(4) If F inverts all L-local maps, then F in particular inverts the canonical map $\operatorname{colim}_{i\in I} y(X_i) \to y(\operatorname{colim}_{i\in I} x_i)$ for every weakly contractible diagram $X\colon I \to C$, so that $F(y(\operatorname{colim}_i X_i)) \cong F(\operatorname{colim}_i y(X_i)) \cong \operatorname{colim}_i F(y(X_i))$. Conversely, assume $F \circ y\colon C \to D$ preserves weakly contractible colimits, or equivalently F inverts the map $\operatorname{colim}_{i\in I} y(X_i) \to y(\operatorname{colim}_{i\in I} x_i)$ for every weakly contractible diagram $X\colon I \to C$. We wish to show that F inverts the L-local maps. It will suffice to show that F inverts the unit map $\mathcal{F} \to RL(\mathcal{F})$ for all $\mathcal{F} \in \mathrm{PSh}(C)$. By picking a final map $I \to \mathrm{El}(\mathcal{F})$ from a small category I, we may write $\mathcal{F}$ as the colimit over I of the functor $I \to \mathrm{El}(\mathcal{F}) = C_{/\mathcal{F}} \to C \xhookrightarrow{y} \mathrm{PSh}(C)$. As before, we may first left Kan extend it along $p\colon I \to |I|$, which lets us write $\mathcal{F}$ as an $|I|$-indexed colimit of colimits of the form $\operatorname{colim}_{i\in I_x} y(X_i)$ for contractible categories $I_x = \mathrm{fib}_x(I \to |I|)$. On the other hand, we saw that $L(\mathcal{F})$ is the colimit of $I \to C_{/\mathcal{F}} \to C \hookrightarrow \int_{\mathrm{An}} C$, and we saw that the left Kan extension along $I \to |I|$ is still contained in C. We conclude that we may write $RL(\mathcal{F})$ as an $|I|$-indexed colimit of objects of the form $y(\operatorname{colim}_{i\in I_x} X_i)$. Finally, we see that the unit map $\mathcal{F} \to RL(\mathcal{R})$ is an $|I|$-indexed colimit of maps of the form $\operatorname{colim}_{i\in I} y(X_i) \to y(\operatorname{colim}_{i\in I} x_i)$. Since F preserves colimits and inverts these maps, we conclude that it also inverts the unit maps, as claimed. □

Lemma 6.162. *A category C admits small colimits if and only if it admits anima-indexed colimits and weakly contractible colimits. Similarly, if C and D are categories with small colimits, then a functor $F\colon C \to D$ preserves small colimits if and only if it preserves both anima-indexed colimits and weakly contractible colimits.*

Proof. Assume that C admits anima-indexed colimits and weakly contractible colimits. We need to show that it admits a colimit for an arbitrary diagram $X\colon I \to C$, where I is a small category. To this end, consider the localization functor $p\colon I \to |I|$ inverting all morphisms in I. As discussed before, the finality of p implies that the relative slices of p are weakly contractible, so that the left Kan extension $p_\sharp X\colon |I| \to C$ of X along p exists in C. Moreover, the colimit of $p_\sharp X$ exists in C since $|I|$ is an anima. Since Kan extensions compose, we conclude that the colimit of X exists in C. The argument for preservation of colimits is entirely analogous. □

We are now ready to establish the second universal property of $\int_{\mathrm{An}} C$. To formulate this precisely, let us denote by $\mathrm{Cat}^{\mathrm{w.c.colim}}$ the subcategory of Cat spanned by the categories admitting weakly contractible colimits and those functors that preserve weakly contractible colimits. Note that it contains the category $\mathrm{Cat}^{\mathrm{colim}}$ of categories with colimits and colimit-preserving functors.

Proposition 6.163. *The adjunction* $\mathrm{Cat} \rightleftarrows \mathrm{Cat}^{\mathrm{An\text{-}colim}}$ *from Lemma 6.159 restricts to an adjunction*

$$\mathrm{Cat}^{\mathrm{w.c.colim}} \rightleftarrows \mathrm{Cat}^{\mathrm{colim}}.$$

In particular, if C is a category with small weakly contractible colimits and D is a category with small colimits, then restriction along the inclusion $C \hookrightarrow \int_{\mathrm{An}} C$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{w.c.colim}}(C, D).$$

Proof. We saw in Proposition 6.160 that restriction along the inclusion induces an equivalence

$$\mathrm{Fun}^{\mathrm{An\text{-}colim}}(\int_{\mathrm{An}} C, D) \xrightarrow{\sim} \mathrm{Fun}(C, D).$$

By part (1) of Proposition 6.161, this restriction functor restricts to a fully faithful functor

$$\mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D) \hookrightarrow \mathrm{Fun}^{\mathrm{w.c.colim}}(C, D).$$

For essential surjectivity, let $F\colon C \to D$ be a functor preserving weakly contractible colimits, and consider its colimit-preserving extension $\mathrm{PSh}(C) \to D$. By part (4) of Proposition 6.161, this inverts the L-local maps in $\mathrm{PSh}(C)$, and so descends to a colimit-preserving functor $\int_{\mathrm{An}} C \to D$ extending F. This finishes the proof. □

6.10.2 Categories of objects parametrized by topoi

The construction of $\int_{\mathrm{An}} C$ may be generalized, replacing An by an arbitrary topos.

Definition 6.164. Let T be a cocomplete category and let C be a category admitting contractible colimits. We define the *category of T-parametrized objects of C* as the tensor product

$$\int_T C \quad := \quad T \otimes \int_{\mathrm{An}} C$$

in the category $\mathrm{Cat}^{\mathrm{colim}}$ of cocomplete categories (using that $\int_{\mathrm{An}} C$ admits colimits by Proposition 6.161).

Corollary 6.165. *The functor $T \times C \hookrightarrow T \times \int_{\mathrm{An}} C \to \int_T C$ is universal among functors $F\colon T \times C \to D$ into a cocomplete category D which preserve all colimits in the first variable and weakly contractible colimits in the second variable.*

Proof. By definition, the tensor product $\int_T C = T \otimes \int_{\mathrm{An}} C$ comes equipped with a functor from $T \times \int_{\mathrm{An}} C$ satisfying the universal property that for every other cocomplete category D, precomposition with this functor defines an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\int_T C, D) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim,colim}}(T \otimes \int_{\mathrm{An}} C, D) = \mathrm{Fun}^{\mathrm{colim}}(T, \mathrm{Fun}^{\mathrm{colim}}(\int_{\mathrm{An}} C, D)).$$

The claimed universal property for $\int_T C$ thus follows immediately from the universal property of $\int_{\mathrm{An}} C$ established in Proposition 6.163. ◻

The cartesian fibration $\pi\colon \int_{\mathrm{An}} C \to \mathrm{An}$ preserves all colimits by Proposition 6.161, hence induces an analogous functor $\pi_T\colon \int_T C \to T$ by tensoring it with T in $\mathrm{Cat}^{\mathrm{colim}}$. We will now identify the fiber of π_T over some $A \in T$ with the category of "A-parametrized objects of C", defined as the limit-preserving functors $X\colon T_{/A}^{\mathrm{op}} \to C$.

Proposition 6.166. *Let T and C be presentable categories. Then the functor $\pi_T\colon \int_T C \to T$ is a bicartesian fibration, which is classified by both of the following functors:*

$$T \to \mathrm{Cat}, \qquad A \mapsto T_{/A} \otimes C,$$

and

$$T^{\mathrm{op}} \to \mathrm{Cat}, \qquad A \mapsto \mathrm{Fun}^{\mathrm{lim}}(T_{/A}^{\mathrm{op}}, C).$$

Proof sketch. Note that the first functor takes the form $T \to \mathrm{Pr}^{\mathrm{L}}$, hence its cocartesian unstraightening is always a bicartesian fibration whose cartesian straightening is the corresponding functor $T^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{R}}$ obtained by passing to right adjoints, which by Lurie's explicit formula for tensor products in Pr^{L} is given by $A \mapsto \mathrm{Fun}^{\mathrm{lim}}(T_{/A}^{\mathrm{op}}, C)$. It thus suffices to prove the first claim.

Since T is presentable, we may write it as a left Bousfield localization of a presheaf category $\mathrm{PSh}(D)$ for some small category D. We then need to show that the functor $T \otimes \int_{\mathrm{An}} C \to \int_T T_{/(-)} \otimes C$ is an equivalence. By writing the left-hand side as a Bousfield localization of $\mathrm{PSh}(D) \otimes \int_{\mathrm{An}} C$ and the right-hand side as a Bousfield localization of $\int_{\mathrm{PSh}(D)} \mathrm{PSh}(D)_{/-} \otimes C$, we may reduce to $T = \mathrm{PSh}(D)$. This reduction is useful, but one still needs a nontrivial argument in the presheaf case; we omit those details here. ◻

6.10.3 Loci

We now turn to the problem of characterizing when parametrized objects form a topos.

Question 6.167. Given a topos T and a category C, when is the T-parametrization $\int_T C$ a topos?

This is interesting because of examples like $\int_T \mathrm{Sp} = \mathrm{Exc}^1(\mathrm{An}_*^{\mathrm{fin}}, T)$, which is a topos. To answer this, we introduce the concept of a *locus*.

Definition 6.168. A *locus* is an accessible category with pullbacks and van Kampen weakly contractible colimits.

Lemma 6.169. *Let C and C' be accessible categories with pullbacks and weakly contractible colimits, and let $G\colon C' \to C$ be a conservative functor which preserves pullbacks and weakly contractible colimits. If C is a locus, then also C' is a locus.*

Proof. Consider a diagram $X_\bullet\colon I \to C'$, with I weakly contractible, and set $X := \operatorname{colim}_{i\in I} X_i$. We need to show that the top adjunction in the following diagram is an adjoint equivalence:

$$\begin{tikzcd} C'_{/X} \arrow[r, shift left] \arrow[d, "G"'] & \lim_{i\in I^{\mathrm{op}}} C'_{/X_i} \arrow[l, shift left] \arrow[d, "G"] \\ C_{/G(X)} \arrow[r, shift left] & \lim_{i\in I^{\mathrm{op}}} C_{/G(X_i)} . \arrow[l, shift left] \end{tikzcd}$$

By assumption, the bottom adjunction is an adjoint equivalence. Moreover, since G preserves pullbacks and weakly contractible colimits, the vertical maps (are defined and) commute with both of the adjoint functors. It then follows from conservativity of G that both the unit and counit of the top adjunction are isomorphisms, proving the claim. $\square$

Corollary 6.170. *If C and C' are loci, and $F\colon C' \to C$ preserves pullbacks and weakly contractible colimits, then also the fibers of F are loci.*

Proof. For an object $c \in C$, the fiber $F^{-1}(c)$ is the pullback of $C' \xrightarrow{F} C \leftarrow *$. Since C, C' are accessible, so is the fiber, as the inclusion $\mathrm{Acc} \hookrightarrow \mathrm{Cat}$ preserves limits. Pullbacks and weakly contractible colimits in the fiber are computed in C' (as both F as well as the inclusion $c\colon * \to C$ preserve them). Since $F^{-1}(c) \to C'$ is conservative, the claim then follows from the previous lemma. $\square$

Corollary 6.171. *If C is a locus, then all slices $C_{/X}$ and $C_{X/}$ are loci.*

Proof. Note that $\mathrm{Ar}(C)$ is a locus, so the claim follows from the previous corollary. $\square$

Example 6.172. Let C be a locus with a final object, and let A be a small category with finite colimits and a final object. Then the subcategory

$$\mathrm{Exc}^{[m,n]}(A, C) \subseteq \mathrm{Fun}(A, C)$$

of n-reduced m-excisive functors is a locus. Indeed, this is the fiber of the functor $P_{n-1}\colon \mathrm{Exc}^m(A, C) \to \mathrm{Exc}^{n-1}(A, C)$.

For example, we have $\mathrm{Exc}^{[n,1]}(A, C) = \mathrm{Exc}^n_*(A, C)$.

Lemma 6.173. *Every topos is a locus. Conversely, a locus T is a topos if and only if it admits a strictly initial object.*

Proof. It is clear that every topos is a locus admitting a strictly initial object. Conversely, if T is a locus admitting a strictly initial object, then it in particular admits colimits, so T is presentable. Moreover, the functor $T^{\mathrm{op}} \to \mathrm{Cat}$, $X \mapsto T_{/X}$ preserves weakly contractible limits (T a locus) and the terminal object (as $\emptyset \in T$ strictly initial, see Example 2.11), hence all limits. $\square$

Lemma 6.174. *Let C be an accessible category with weakly contractible colimits. If C is stable, then it is a locus.*

Proof. Let C be a stable accessible category with weakly contractible colimits. It has all finite limits, so in particular pullbacks. We need to check the van Kampen condition for weakly contractible colimits. Equivalently, we must check that weakly contractible colimits are both universal and effective. For universality, consider a map $Y \to X = \mathrm{colim}_{i \in I} X_i$ with I weakly contractible, and consider for every $i \in I$ the pullback square

$$\begin{tikzcd} X_i \times_X Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X_i \arrow[r] & Y. \end{tikzcd}$$

Since pullback squares agree with pushout squares, they are closed under colimits, so the induced square

$$\begin{tikzcd} \mathrm{colim}_{i \in I} X_i \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{colim}_{i \in I} X_i \arrow[r, "\sim"] & X \end{tikzcd}$$

is still a pullback square. In particular, the top map is an isomorphism. For effectivity, we argue similar: for a cartesian transformation $Y_\bullet \to X_\bullet$, we may for fixed $i \in I$ take the colimit over all $j \in I_{i/}$ of the following pullback squares to obtain the claim:

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_j \arrow[d] \\ X_i \arrow[r] & X_j. \end{tikzcd}$$

□

Lemma 6.175. *Let T be a topos. Then $T^{\geq \infty}$ is a locus in which every map is an effective epimorphism.*

Proof. The subcategory $T^{\geq \infty}$ of T spanned by ∞-connected objects is closed under colimits and finite limits in T by Proposition 3.36. Since weakly contractible colimits in T are van Kampen, $T^{\geq \infty}$ inherits this condition.

For any $X \in T^{\geq \infty}$, the map $X \to *$ is ∞-connected by definition. It follows from left cancellation that any map $X \to Y$ is ∞-connected, so in particular an effective epimorphism in T. Since $T^{\geq \infty}$ is closed under finite limits and geometric realizations, it is then also an effective epimorphism in $T^{\geq \infty}$. □

The following is our main theorem of this section:

Theorem 6.176. *Let C be a category with pullbacks and weakly contractible colimits. Then the following conditions are equivalent:*

(1) *The category C is a locus;*

(2) *The category $\int_{\mathrm{An}} C$ is a topos;*

(3) *For every topos T, the category $\int_T C$ is a topos.*

Moreover, the following conditions are equivalent to each other:

(4) *The category C is a locus in which every map is an effective epimorphism;*

(5) *There exists a topos T such that $C = T^{\geq\infty}$.*

(6) *There exists a topos T such that $T_{\leq\infty} = \mathrm{An}$ and $T^{\geq\infty} = C$.*

Remark 6.177. We may think of the categories in (4-6) as "nilpotent thickenings of the point". They include all stable categories.

Proof. We momentarily assume the equivalence between (1)-(3) and deduce the equivalence between (4)-(6). It is clear that (6) implies (5) and that (5) implies (4) (by the previous lemma). So assume (4). We define $T = \int_{\mathrm{An}} C$. By part (2), this is a topos. Moreover, one can show that $T_{\leq\infty} = \mathrm{An}$. Moreover, we see that $T^{\geq\infty} = C$.

For the equivalence of (1)-(3), first note that (2) is equivalent to (3) since $\int_T C = T \otimes \int_{\mathrm{An}} C$, where the right-hand side is the product of topoi if $\int_{\mathrm{An}} C$ is a topos. For (2) $\implies$ (1), note that $C = \mathrm{fib}_*(\int_{\mathrm{An}} C \to \mathrm{An})$, so the claim follows from Corollary 6.170. It remains to show that (1) implies (2). As C is accessible, also $\int_{\mathrm{An}} C$ is accessible. (While this statement seems to be in the literature only in the presentable case, the argument for the accessible case is similar, according to Marc.) It also has colimits, so it is in fact presentable. It remains to show that colimits in $\int_{\mathrm{An}} C$ are van Kampen.

Consider a diagram $(X_\bullet, E_\bullet)\colon I \to \int_{\mathrm{An}} C$, where $X_i \in \mathrm{An}$ and $E_i \in \mathrm{Fun}(X_i, C)$. We need to show that the functor $\mathrm{colim}\colon \mathrm{Fun}^{\mathrm{cart}}(I, \int_{\mathrm{An}} C)_{/(X_\bullet,E_\bullet)} \to (\int_{\mathrm{An}} C)_{/(X,E)}$ is an equivalence. Note that this functor fits in a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{cart}}(I, \int_{\mathrm{An}} C)_{/(X_\bullet,E_\bullet)} \arrow[r] \arrow[d] & (\int_{\mathrm{An}} C)_{/(X,E)} \arrow[d] \\
\mathrm{Fun}^{\mathrm{cart}}(I, \mathrm{An})_{/X_\bullet} \arrow[r] & \mathrm{An}_{/X}.
\end{tikzcd}$$

The vertical two maps are cartesian fibrations and the bottom map is an equivalence, so it will suffice to show that the top map induces equivalences on fibers.

Note that $(\int_{\mathrm{An}} C)_{/(X,E)} \simeq \lim_{x\in X}(\int_{\mathrm{An}} C)_{/(*,E_x)}$, so it will suffice to prove the claim when $X = *$. In that case $\mathrm{An}_{/X} \simeq \mathrm{An}$. Given $Y \in \mathrm{An}$, the map on fibers over Y is identified with a limit over $y \in Y$ of copies of the following map

$$\lim_{i\in I^{\mathrm{op}}} \mathrm{Fun}(X_i, C)_{/E_i} \to C_{/E},$$

and it remains to show this is an equivalence. But by reriting the left-hand side as $\lim_{x\in \mathrm{El}(X_\bullet)} C_{/E_\bullet} \to C_{/E}$, this follows from van Kampen condition on weakly contractible colimits in C, since $\mathrm{El}(X_\bullet)$ is weakly contractible when $X = *$. □

6.10.4 A generalization

The category $\int_{\mathrm{An}} C$ of parametrized objects in C may be seen as a special case of a more general construction $P_K^{K'}(C)$ one can do. The question of when $\int_{\mathrm{An}} C$ is a topos can then be generalized to the question of when $P_K^{K'}(C)$ is a topos. This should mostly be treated as just a curiosity and will not reappear later in the notes. Some of the discussion here is related to Rezk's article [Rez25].

Notation 6.178. Let $K \subseteq K'$ be two classes of small categories. Given a category C with K-indexed colimits, we may form a functor

$$C \to \mathcal{P}_K^{K'}(C)$$

which is universal among K-colimit preserving functors into a category with K'-indexed colimits.

Remark 6.179. Assume for simplicity that C is small. If K' consists of all small categories and K of none, $\mathcal{P}^{\mathrm{all}}(C) = \mathrm{PSh}(C)$ is simply the presheaf category. For arbitrary K (but still $K' = \mathrm{all}$), we may identify $\mathcal{P}_K^{\mathrm{all}}(C)$ with the localization of $\mathrm{PSh}(C)$ at the class of maps

$$\{\alpha_X\colon\ \mathrm{colim}_{i\in I}\, y(X_i) \to y(\mathrm{colim}_{i\in I}\, X_i) \mid I \in K, X\colon I \to C\}\ :$$

for every cocomplete category D, a colimit-preserving functor $\mathrm{PSh}(C) \to D$ is determined by its restriction to C, and it factors through the localization if and only if it inverts each α_X, which is exactly the condition that the restriction preserve K-indexed colimits.

It follows from Yoneda that we may identify $\mathcal{P}_K^{\mathrm{all}}(C)$ with the full subcategory of $\mathrm{PSh}(C)$ consisting of those presheaves $F\colon C^{\mathrm{op}} \to \mathrm{An}$ which send K-indexed colimits in C to limits in An.

Example 6.180. Let K' be the class of all small categories and let K be the class of the weakly contractible categories. If C is a category with weakly contractible colimits, the category $P_{\mathrm{wc}}^{\mathrm{all}}(C)$ is universal among functors $C \to C'$ into a cocomplete category C' which preserve weakly contractible colimits. Since this is also precisely the universal property of the category $\int_{\mathrm{An}} C$ of An-parametrized objects in C established in Proposition 6.163, we obtain an equivalence

$$\int_{\mathrm{An}} C \xrightarrow{\sim} \mathcal{P}_{\mathrm{wc}}^{\mathrm{all}}(C).$$

In the case where K' consists of the weakly contractible categories, we saw in Proposition 6.160 that $\int_{\mathrm{An}} C = \mathcal{P}_{\mathrm{wc}}^{\mathrm{all}}(C)$ may alternatively be described as $\mathcal{P}^{\mathrm{animae}}(C)$, i.e. the category obtained from C by freely adding anima-indexed colimits. This is in some sense saying that anima-indexed colimits are "complementary" to weakly contractible colimits. Let us give some ad hoc terminology and notation for this behavior:

Notation 6.181. Let K and K' be two classes of small categories. We write $K \perp K'$, and say that *K is complementary to K'*, if for every category C with K'-indexed colimits, the categories $\mathcal{P}^K(C)$ and $\mathcal{P}_{K'}^{\mathrm{all}}(C)$ define the same full subcategory of $\mathrm{PSh}(C)$ (or of the category of small presheaves, if C is large), under the identifications from Remark 6.179.

Equivalently, the unique K-colimit-preserving extension

$$\mathcal{P}^K(C) \to \mathcal{P}_{K'}^{\mathrm{all}}(C)$$

of $C \to \mathcal{P}_{K'}^{\mathrm{all}}(C)$ is an equivalence.

Warning 6.182. This relation is *not* symmetric: K being complementary to K' does not imply that K' is complementary to K.

Lemma 6.183. *If $K \perp K'$, then the following two conditions are satisfied:*

(1) *For every category D, the unique $(K \cup K')$-colimit-preserving extension $\mathcal{P}^{K\cup K'}(D) \to \mathcal{P}^{\mathrm{all}}(D)$ of $D \to \mathcal{P}^{\mathrm{all}}(D)$ is an equivalence. Equivalently,* $\mathrm{PSh}(D)$ *is generated by representables under K-colimits and K'-colimits.*

(2) *In* An*, K-indexed colimits commute with K'-indexed limits.*

Proof. By comparing universal properties, we see that $\mathcal{P}^K(\mathcal{P}^{K'}(D)) \xrightarrow{\sim} \mathcal{P}^{K\cup K'}(D)$ and $\mathcal{P}^{\mathrm{all}}_{K'}(\mathcal{P}^{K'}(D)) \xrightarrow{\sim} \mathcal{P}^{\mathrm{all}}(D)$, so (1) follows by taking $C = \mathcal{P}^{K'}(D)$. For (2), let $I \in K'$, $J \in K$, and let $X\colon I \times J \to \mathrm{An}$ be a diagram. We must show that the canonical map $\mathrm{colim}_{j\in J} \lim_{i\in I} X_{i,j} \to \lim_{i\in I} \mathrm{colim}_{j\in J} X_{i,j}$ is an isomorphism. Let $C := \mathrm{PSh}(I^{\mathrm{op}})$, and for each $j \in J$ let $d_j \in C$ be the presheaf $i \mapsto X_{i,j}$. Also let

$$u := \mathrm{colim}_{i\in I^{\mathrm{op}}}\, y(i) \quad \in \quad C \qquad \text{and} \qquad F := \mathrm{colim}_{j\in J}\, y(d_j) \quad \in \quad \mathrm{PSh}(C).$$

By the large-C variant of Remark 6.179 (using small presheaves), $\mathcal{P}^{\mathrm{all}}_{K'}(C)$ is a full subcategory of $\mathrm{PSh}(C)$. Using the assumption $K \perp K'$, this full subcategory agrees with $\mathcal{P}^K(C) \subseteq \mathrm{PSh}(C)$, hence is closed under K-indexed colimits. Since each $y(d_j)$ belongs to it, it follows that $F \in \mathcal{P}^{\mathrm{all}}_{K'}(C)$, and thus F sends the I^{op}-indexed colimit $u = \mathrm{colim}_{i\in I^{\mathrm{op}}}\, y(i)$ in C to an I-indexed limit in An:

$$F(u) \simeq \lim_{i\in I} F(y(i)).$$

Now compute both sides using pointwise colimits and Yoneda:

$$F(u) \simeq \mathrm{colim}_{j\in J}\, \mathrm{Hom}_C(u, d_j) \simeq \mathrm{colim}_{j\in J} \lim_{i\in I} \mathrm{Hom}_C(y(i), d_j) \simeq \mathrm{colim}_{j\in J} \lim_{i\in I} X_c i, j,$$

while

$$\lim_{i\in I} F(y(i)) \simeq \lim_{i\in I} \mathrm{colim}_{j\in J}\, \mathrm{Hom}_C(y(i), d_j) \simeq \lim_{i\in I} \mathrm{colim}_{j\in J} X_{i,j}.$$

This proves (2).

□

Example 6.184. The following are examples of complementary classes:

- {Filtered categories} $\perp$ {Finite categories}
- {Sifted categories} $\perp$ {Finite sets}
- {Weakly contractible categories} $\perp$ {Empty category}
- {All small categories} $\perp$ $\emptyset$
- Another example not recorded in the literature is {Animae} $\perp$ {Weakly contractible categories}.
- Subexample: {n-groupoids} $\perp$ {Weakly n-connected categories}
- {Distilled categories} $\perp$ {Pushout diagram}.

The question about $\int_{\mathrm{An}} C$ being a topos now has the following natural generalization:

Question 6.185. Let $K \perp K'$ be complementary and let C be a category which has K'-indexed colimits which are van Kampen. Is $\mathcal{P}^K(C) = \mathcal{P}^{\mathrm{all}}_{K'}(C)$ necessarily a topos?

Marc says he doesn't know whether this is true in general. But it is true when K is one of the following classes of categories:

- Weakly contractible categories. In this case, $\mathcal{P}^{\mathrm{w.c.}}(C)$ is also known as $\mathcal{P}_{\emptyset}(C)$, the full subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ spanned by those functors F satisfying $F(\emptyset) = *$. It is readily verified that both weakly contractible colimits and the initial object are van Kampen, so that $\mathcal{P}_{\emptyset}(C)$ is a topos.
- Sifted categories. In this case, $\mathcal{P}^{\mathrm{sifted}}(C)$ is also known as $\mathcal{P}_{\Sigma}(C)$, the subcategory of $\mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ spanned by the finite-product-preserving functors. This is in fact a sheaf category, hence a topos; see [BH21, Lemma 2.4] for a proof.
- All categories. In this case, $\mathcal{P}^{\mathrm{all}}(C)$ is also known as the presheaf category $\mathrm{PSh}(C)$, which is a topos.
- All animae. In this case, $\mathcal{P}^{\mathrm{animae}}(C) = \int_{\mathrm{An}} C$, so this is a topos by Theorem 6.176.
- Probably also for n-groupoids.

However, it seems to Marc that this is *not* true when $K = \{\text{Filtered categories}\}$. You would need to show that if C is a category with finite colimits which are van Kampen, then $\mathrm{Ind}(C)$ is a topos. Although Marc does not know an explicit counterexample, he expects this to fail in general.

7 Geometry

This final chapter gives an outlook on geometric applications of the theory developed earlier. We begin with the topos-theoretic formulation of scheme theory, then introduce Lurie's language of *geometries* and explain how it leads to the formalism of *fractured topoi*. Next we indicate how six-functor formalisms give rise to fractured topoi. We end with two examples: analytic stacks in the sense of Clausen–Scholze and Gestalten in the sense of Scholze–Stefanich.

As this is an outlook chapter, the treatment is brief and most proofs are omitted. Several statements below are included as a roadmap and are intentionally stated informally.

7.1 Schemes

We begin with the topos-theoretic formulation of schemes. The point is to isolate the features that will later be abstracted by Lurie's notions of geometry and fractured topos.

Definition 7.1. A *ringed topos* is a pair (T, O) where T is a topos and O is a sheaf of static rings on T (i.e. just commutative rings in the classical sense).

Definition 7.2. A *Zariski scheme* is a ringed topos (T, O) which is locally isomorphic to $\mathrm{Spec}(R)_{\mathrm{Zar}}$, i.e. there exists a covering family $\{U_i \to *\}_{i\in I}$ in T such that for every $i \in I$ the induced ringed topos $(T_{/U_i}, O|_{U_i})$ is isomorphic to $\mathrm{Spec}(R_i)_{\mathrm{Zar}}$ for some ring R_i. A *morphism of Zariski schemes* $(T, O_T) \to (S, O_S)$ is a pair $(f, f^\sharp)$ consisting of a morphism of topoi $f : T \to S$ together with a morphism of sheaves of static rings $f^\sharp : O_S \to f_* O_T$ which is *local*, in the sense that it detects units: for every map $A \to B$ induced by $f^\sharp$ on sections, the following square is a pullback square:

$$\begin{tikzcd} A^\times \arrow[r] \arrow[d] & B^\times \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

This results in a category $\mathrm{Sch}_{\mathrm{Zar}}$.

Definition 7.3. An *étale scheme* is a ringed topos (T, O) that is locally isomorphic to $\mathrm{Spec}(R)_{\text{ét}}$. We similarly obtain a category $\mathrm{Sch}_{\text{ét}}$.

Example 7.4.
- A classical scheme is exactly a 0-localic Zariski scheme.
- A classical Deligne–Mumford stack is exactly a 1-localic étale scheme.
- If (T, O) is a Zariski scheme, and $X \in T$, then $(T_{/X}, O|_{T_{/X}})$ is again a Zariski scheme.

Remark 7.5. Étale schemes are also called *Deligne–Mumford ∞-stacks*.

The following are some standard facts from scheme theory, translated to the language of topoi:

- The global section functor

$$\mathrm{LocRingTopos} \to \mathrm{CRing}^{\mathrm{op}}, \qquad (T, \mathcal{O}) \mapsto \mathcal{O}(T)$$

  admits a fully faithful right adjoint

$$\mathrm{CRing}^{\mathrm{op}} \to \mathrm{LocRingTopos}, \qquad R \mapsto \mathrm{Spec}(R)_{\mathrm{Zar}}.$$

- The functor

$$\mathrm{Sch}_{\mathrm{Zar}} \to \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{op}}), \qquad (T, \mathcal{O}) \mapsto \mathrm{Hom}(\mathrm{Spec}(-)_{\mathrm{Zar}}, (T, \mathcal{O}))$$

  is fully faithful. Moreover, a Zariski sheaf X is in the image if and only if there exists an effective epimorphism $\bigsqcup_i U_i \xrightarrow{f_i} X$ where U_i is representable and f_i is representable by an étale map (of ringed topoi).

- The category $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$ is the classifying topos for local rings.

We have made a number of different definitions here. These constructions fit into a common pattern, which we will now discuss. We first list some more facts about scheme theory which are a bit less standard.

Definition 7.6. A morphism of Zariski schemes is called *étale* if its underlying morphism of topoi is an étale map. (Beware that this notion of étale is totally unrelated to the previous notion of étale schemes! This is unfortunate terminology.) For example, every map which is locally in source/target an open immersion of schemes is an étale map.

We denote the resulting wide subcategory of $\mathrm{Sch}_{\mathrm{Zar}}$ by

$$\mathrm{Sch}_{\mathrm{Zar}}^{\text{ét}} := \mathrm{Sch}_{\mathrm{Zar}} \times_{\mathrm{Topos}} \mathrm{Topos}^{\text{ét}} \subseteq \mathrm{Sch}_{\mathrm{Zar}}\,.$$

Proposition 7.7. *The category* $\mathrm{Sch}_{\mathrm{Zar}}^{\text{ét}}$ *is (up to size issues) a topos. In fact, there is an equivalence*

$$\mathrm{Sch}_{\mathrm{Zar}}^{\text{ét}} \simeq \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{ad,op}}),$$

where $\mathrm{CRing}^{\mathrm{ad}}$ *is the wide subcategory of* CRing *spanned by the localization maps of the form* $R \to R[s^{-1}]$.

Observation 7.8. For every topos T, the category $\mathrm{Shv}_{\mathrm{CRing}}(T)$ admits a factorization system: every map $A \to B$ factors uniquely as

$$A \to A[S^{-1}] \to B,$$

where the second map is a local map. This factorization is functorial in T.

Based on these results and observations, we see that the topos $E = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$ has the following additional structure:

(1) Left Kan extension along the inclusion $\mathrm{CRing}^{\mathrm{fp,ad,op}} \hookrightarrow \mathrm{CRing}^{\mathrm{fp,op}}$ induces a (non-full) faithful functor

$$E^{\mathrm{corp}} := \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,ad,op}}) \hookrightarrow \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}}) = E$$

whose image is also a topos. We refer to this subcategory E^{corp} as the subcategory of *corporeal objects*.

(2) There exists a wide subcategory E^{ad} of E consisting of the "representable étale maps".

(3) The functor $\mathrm{Geom}(-, E)\colon \mathrm{Topos}^{\mathrm{op}} \to \mathrm{Cat}$ factors as

$$\mathrm{Topos}^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{fact}} \to \mathrm{Cat},$$

where $\mathrm{Cat}^{\mathrm{fact}}$ is the category of categories equipped with a factorization system.

(4) The category $\mathrm{Topos}_{//E}$ of ringed topoi admits a wide subcategory

$$(\mathrm{Topos}_{//E})^{\mathrm{loc}} \subseteq \mathrm{Topos}_{//E}$$

consisting of the *local* morphisms of ringed topoi.

The guiding point is that each of these four structures determines the others, and each characterizes the subcategory of schemes.

In the next two sections we will consider abstract versions of this story: Lurie's notion of a *geometry* which encapsulates the idea of having affine test objects with morphisms playing the role of open inclusions and the covers they generate, and the resulting notion of a *fractured topos*.

7.2 Geometries

Classical algebraic geometry is built from affine schemes, basic open immersions, and covers by basic opens. Lurie's notion of a geometry abstracts exactly this pattern:

$$\begin{aligned} \text{affine test objects} &\rightsquigarrow \text{objects of a category } C, \\ \text{basic open immersions} &\rightsquigarrow \text{admissible morphisms}, \\ \text{covers by basic opens} &\rightsquigarrow \text{admissible covers}. \end{aligned}$$

For the applications below, we will use the following slightly site-theoretic formulation.

Definition 7.9. Let C be a category with finite limits. An *admissibility structure* is a wide subcategory $C^{\mathrm{ad}} \subseteq C$ of *admissible morphisms* satisfying the following properties:

- C^{ad} is closed under base change in C;
- C^{ad} is left cancellable: given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$, if g and gf are admissible morphisms, then so is f;
- admissible morphisms are closed under retracts.

Definition 7.10. Let (C, C^{ad}) be a category with an admissibility structure. A topology τ is said to be *compatible with* C^{ad} if every τ-covering sieve contains a τ-covering sieve generated by admissible maps. We refer to such triples

$$\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$$

as *geometric sites*. Covering families generated by admissible morphisms will be called *admissible covers*. If C is small and idempotent complete, we will also call such a triple a *geometry*.

7.2.1 Zariski geometries

The basic example is the *classical Zariski geometry*

$$\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}} := (\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}},$$

where $\mathrm{CRing}^{\mathrm{fp}}$ denotes the category of finitely presented ordinary commutative rings. Since $\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}}$ is the opposite of a category of rings, we will describe its morphisms in terms of the corresponding maps of rings in the opposite direction.

The admissible morphisms in $\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}}$ correspond to principal localizations

$$R \longrightarrow R[f^{-1}], \qquad f \in R.$$

Geometrically, this is the inclusion of the principal open

$$D(f) = \mathrm{Spec}(R[f^{-1}]) \hookrightarrow \mathrm{Spec}(R).$$

A finite family

$$\{R \to R[f_i^{-1}]\}_{i \in I}$$

generates a covering sieve precisely when the elements f_i generate the unit ideal in R.

Let us spell out why this is a geometry. Finite limits in $\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}}$ are finite colimits of finitely presented rings. Base change of admissible morphisms follows from the formula

$$S \otimes_R R[f^{-1}] \simeq S[\varphi(f)^{-1}]$$

for a map $\varphi\colon R \to S$. Left cancellability says that if a principal localization $R \to R[g^{-1}]$ factors through another principal localization $R \to R[f^{-1}]$, then the map $R[f^{-1}] \to R[g^{-1}]$ is already a principal localization: since f is already invertible in $R[g^{-1}]$, we have $R[g^{-1}] \cong R[g^{-1}][f^{-1}] \cong R[f^{-1}][g^{-1}]$. Closure under retracts follows from the universal property of localization: a retract of the functor that imposes invertibility of f again imposes invertibility of a single element.

There is also a *spectral Zariski geometry*

$$\mathcal{G}^{\mathrm{sp}}_{\mathrm{Zar}} := (\mathrm{CAlg}^{\omega})^{\mathrm{op}},$$

where CAlg^{ω} denotes the category of compact commutative $\mathbb{E}_\infty$-rings. The admissible morphisms are again principal localizations

$$R \longrightarrow R[f^{-1}], \qquad f \in \pi_0(R),$$

and a finite family $\{R \to R[f_i^{-1}]\}_{i \in I}$ is a cover precisely when the elements f_i generate the unit ideal in $\pi_0(R)$. The verification is the same as in the classical case, using the corresponding localization formula for commutative $\mathbb{E}_\infty$-rings.

7.2.2 Other examples

Here are some other geometries obtained by changing the category of affine test objects, the admissible morphisms, or the topology:

- *Derived Zariski geometry*: the analogous Zariski geometry based on compact animated rings.
- *Classical étale geometry*: take $C = (\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}}$, take τ to be the étale topology, and let C^{ad} be the class of morphisms corresponding to étale maps of rings.
- *Derived étale geometry*: the analogous version based on compact animated rings.
- *Spectral étale geometry*: replace ordinary rings by compact commutative $\mathbb{E}_\infty$-rings and use the spectral étale topology.
- *Differential geometry*: use finitely presented C^∞-rings, with admissible maps encoding open immersions.

7.2.3 Structured topoi

Let $\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$ be a geometry and let T be a topos. A *$\mathcal{G}$-structure* on T is a left exact functor

$$\mathcal{O}\colon C \to T$$

such that for every admissible cover $\{U_i \to X\}_{i\in I}$, the induced map

$$\coprod_{i\in I} \mathcal{O}(U_i) \longrightarrow \mathcal{O}(X)$$

is an effective epimorphism in T. We denote by $\mathrm{Str}_{\mathcal{G}}(T)$ the full subcategory of $\mathrm{Fun}^{\mathrm{lex}}(C, T)$ spanned by the $\mathcal{G}$-structures.

In the classical Zariski case, the equivalence

$$\mathrm{Ind}((\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}})^{\mathrm{op}}) \simeq \mathrm{CRing}$$

identifies a left exact functor $\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}} \to \mathrm{An}$ with an ordinary (static) commutative ring A. Under this identification, the functor associated to A sends a finitely presented ring S to

$$\mathrm{Hom}_{\mathrm{CRing}}(S, A).$$

The Zariski cover condition says that, whenever $f_1, \ldots, f_n \in S$ generate the unit ideal, the map

$$\coprod_i \mathrm{Hom}_{\mathrm{CRing}}(S[f_i^{-1}], A) \longrightarrow \mathrm{Hom}_{\mathrm{CRing}}(S, A)$$

is surjective. In concrete terms: for every map $\varphi\colon S \to A$, at least one of the elements $\varphi(f_i) \in A$ must be invertible.

This condition is equivalent to A being a local ring. Indeed, if A is local, then not all of the elements $\varphi(f_i)$ can lie in the maximal ideal, so one must be invertible. Conversely, the displayed condition implies that the nonunits in A form an ideal, which must then be the unique maximal ideal. All in all, we see that

$$\mathrm{Str}_{\mathcal{G}^{\mathrm{cl}}_{\mathrm{Zar}}}(\mathrm{An}) \simeq \{\text{ordinary local rings}\}.$$

In the spectral Zariski case, the same argument starts from

$$\mathrm{Ind}((\mathcal{G}^{\mathrm{sp}}_{\mathrm{Zar}})^{\mathrm{op}}) \simeq \mathrm{CAlg}$$

and identifies a left exact functor $\mathcal{G}^{\mathrm{sp}}_{\mathrm{Zar}} \to \mathrm{An}$ with a commutative ring spectrum R. The cover condition says that if $f_1, \ldots, f_n \in \pi_0(S)$ generate the unit ideal, then for every map $\varphi\colon S \to R$ at least one of the elements $\varphi(f_i) \in \pi_0(R)$ is invertible. Equivalently, $\pi_0(R)$ is a local ring. Thus

$$\mathrm{Str}_{\mathcal{G}^{\mathrm{sp}}_{\mathrm{Zar}}}(\mathrm{An}) \simeq \{\text{local commutative ring spectra}\}.$$

The upshot of this discussion is that Zariski structures provide a point-free version of the usual locality condition on stalks. Indeed, given a Zariski structure on a topos T, pulling back along any point $x^*\colon T \to \mathrm{An}$ of the topos produces a local commutative ring spectrum. If T admits enough points, then a Zariski structure is nothing but a sheaf of rings on T such that all stalks are local. If T does not admit enough points, then Zariski structures are the correct replacement.

7.2.4 Local morphisms of structures

Let $\mathcal{O}, \mathcal{O}' \in \mathrm{Str}_{\mathcal{G}}(T)$ be two $\mathcal{G}$-structures. A natural transformation

$$\alpha\colon \mathcal{O} \to \mathcal{O}'$$

is called *local* if for every admissible morphism $U \to X$ in C, the square

$$\begin{array}{ccc} \mathcal{O}(U) & \longrightarrow & \mathcal{O}'(U) \\ \downarrow & & \downarrow \\ \mathcal{O}(X) & \longrightarrow & \mathcal{O}'(X) \end{array}$$

is a pullback square in T.

For the classical Zariski geometry and $T = \mathrm{An}$, this recovers the usual notion of a local homomorphism. A map of Zariski structures corresponds to a map of local rings $A \to B$. Testing locality on an admissible map $S \to S[f^{-1}]$ says that the square

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CRing}}(S[f^{-1}], A) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(S[f^{-1}], B) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{CRing}}(S, A) & \longrightarrow & \mathrm{Hom}_{\mathrm{CRing}}(S, B) \end{array}$$

is a pullback. In other words, an element of A is invertible if and only if its image in B is invertible. For local rings this is equivalent to the classical condition that the preimage of the maximal ideal of B is the maximal ideal of A. The spectral case is identical after passing to π_0: a map $R \to R'$ of local commutative $\mathbb{E}_\infty$-rings is local precisely when $\pi_0(R) \to \pi_0(R')$ is local in the classical sense.

7.3 Fractured topoi

A geometry produces two closely related sheaf topoi: the topos of all sheaves on the affine test objects, and the topos obtained by restricting to admissible morphisms. Lurie's notion of *fractured topos* abstracts the structure retained by such a pair. In the Zariski case, this is the structure behind the four equivalent descriptions encountered above: corporeal objects, admissible morphisms, factorization systems, and local morphisms of structured topoi.

Definition 7.11. A *fractured topos* is a topos E equipped with a subcategory $j_!\colon E^{\mathrm{corp}} \hookrightarrow E$ satisfying the following three properties:

(1) The category E^{corp} admits fiber products and $j_!$ preserves them.

(2) The functor $j_!$ admits a right adjoint j^* which is conservative and preserves colimits.

(3) For every morphism $U \to V$ in E^{corp}, the commutative square

$$\begin{tikzcd} j_! j^* j_! U \arrow[r] \arrow[d] & j_! j^* j_! V \arrow[d] \\ j_! U \arrow[r] & j_! V \end{tikzcd}$$

is a pullback square.

We refer to objects in E^{corp} as *corporeal objects*. For our discussion, it will be convenient to add the following fourth axiom, which is satisfied in all examples:

(4*) Every retract in E of a corporeal object is corporeal.

Let (E, E^{corp}) be a fractured topos. The following properties hold:

- The category E^{corp} is a topos, and the right adjoint $j_*\colon E^{\mathrm{corp}} \to E$ to j^* is a morphism of topoi.
- For every object $X \in E^{\mathrm{corp}}$, the functor $j_!\colon E^{\mathrm{corp}}_{/X} \to E_{/j_!(X)}$ is fully faithful.
- The category E is generated under colimits by corporeal objects.

We will now discuss how to construct fractured topoi.

7.3.1 Fractures from geometries

Construction 7.12. Let $(C, C^{\mathrm{ad}}, \tau)$ be a geometric site. Then the category C^{ad} inherits a topology τ, in which the covering sieves are those sieves which generate covering sieves in C. Then the inclusion $j\colon C^{\mathrm{ad}} \hookrightarrow C$ defines a functor $j_!\colon \mathrm{Shv}_\tau(C^{\mathrm{ad}}) \hookrightarrow \mathrm{Shv}_\tau(C)$. This admits a right adjoint $j^*\colon \mathrm{Shv}_\tau(C) \to \mathrm{Shv}_\tau(C^{\mathrm{ad}})$.

Theorem 7.13. *The functor* $j_!\colon \mathrm{Shv}_\tau(C^{\mathrm{ad}}) \hookrightarrow \mathrm{Shv}_\tau(C)$ *is a fractured topos.*

Example 7.14. Every geometry listed in Section 7.2 gives a fractured topos by this construction. For instance, the classical Zariski geometry gives the pair

$$\mathrm{Shv}_{\mathrm{Zar}}((\mathrm{CRing}^{\mathrm{fp,ad}})^{\mathrm{op}}) \hookrightarrow \mathrm{Shv}_{\mathrm{Zar}}((\mathrm{CRing}^{\mathrm{fp}})^{\mathrm{op}}),$$

which is the fractured topos underlying the topos-theoretic description of ordinary schemes above.

Definition 7.15. Let (E, E^{corp}) be a fractured topos. A morphism in E is said to be *admissible* if its base change to any $X \in E^{\mathrm{corp}}$ is in E^{corp}. This determines a wide subcategory E^{ad} of E.

Definition 7.16. Let E be a topos. An admissibility structure E^{ad} is called *local* if the following conditions are satisfied:

- The class E^{ad} is a local class of maps.
- For every $X \in E$, the slice $E^{\mathrm{ad}}_{/X} := (E^{\mathrm{ad}})_{/X}$ is presentable, and the (fully faithful) inclusion $E^{\mathrm{ad}}_{/X} \hookrightarrow E_{/X}$ preserves colimits. (In particular, it admits a right adjoint $\rho_X \colon E_{/X} \to E^{\mathrm{ad}}_{/X}$.)

Given a local admissibility structure on E, we say that an object $X \in E$ is *corporeal* if the functor $\rho_X \colon E_{/X} \to E^{\mathrm{ad}}_{/X}$ preserves colimits. We denote by

$$E^{\mathrm{corp}} \subseteq E^{\mathrm{ad}}$$

the full subcategory spanned by the corporeal objects. We say that E^{ad} is *geometric* if, moreover, E is generated under colimits by corporeal objects.

Theorem 7.17. *Let E be a topos. The assignments $(E, E^{\mathrm{corp}}) \mapsto (E, E^{\mathrm{ad}})$ and $(E, E^{\mathrm{ad}}) \mapsto (E, E^{\mathrm{corp}})$ determine an equivalence of posets*

$$\{\textit{Fractured structures on } E\} \simeq \{\textit{Geometric admissibility structures on } E\}.$$

7.3.2 Local maps

Consider a fractured topos (E, E^{corp}). Consider two morphisms of topoi $f, g \colon T \to E$, and let $\alpha \colon f^* \to g^*$ be a natural transformation. We say that α is *local* if for every morphism $h \colon U \to V$ in E^{corp}, the following square is a pullback square:

$$\begin{array}{ccc} f^*U & \xrightarrow{\alpha} & g^*U \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ f^*V & \xrightarrow{\alpha} & g^*V. \end{array}$$

This determines a collection of maps in $\mathrm{Geom}(T, E)$ which is the right class of a factorization system on $\mathrm{Geom}(T, E)$. We denote by

$$(\mathrm{Topos}_{//E})^{\mathrm{loc}} \subseteq \mathrm{Topos}_{//E}$$

the wide subcategory consisting of those morphisms corresponding to maps $S \to T$ with a local natural transformation α of morphisms $S \to E$.

Example 7.18. For $E = \mathrm{PSh}(\mathrm{CRing}^{\mathrm{fp,op}})$, the lax slice $\mathrm{Topos}_{//E}$ is the category of ringed topoi. For $E = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,op}})$, the slice $\mathrm{Topos}_{//E}$ is the full subcategory of the previous one spanned by the locally ringed topoi. Here, the subcategory $E^{\mathrm{corp}} = \mathrm{Shv}_{\mathrm{Zar}}(\mathrm{CRing}^{\mathrm{fp,ad,op}})$ corresponds to the subcategory $(\mathrm{Topos}_{//E})^{\mathrm{loc}}$ consisting of the *local* morphisms of locally ringed topoi.

We now explain how to recover the notion of schemes from the notion of local morphisms.

Theorem 7.19. *Let (E, E^{corp}) be a fractured topos. Consider the functor*

$$E \to \mathrm{Fun}(\mathrm{Topos}^{\mathrm{loc}}_{//E}, \mathrm{An}), \qquad X \mapsto \Gamma^X : (f : T \to E) \mapsto \Gamma(f^*(X)).$$

(1) *If X is corporeal, then Γ^X is representable by the pair $(E^{\mathrm{ad}}_{/X}, E^{\mathrm{ad}}_{/X} \hookrightarrow E_{/X} \to E)$.*

(2) *The resulting functor*

$$\{\textit{Full subcat of } E^{\mathrm{corp}}\} \xrightarrow{\mathrm{Re}} \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

is fully faithful.

(3) *A morphism $f : Y \to X$ in E is admissible if and only if for every base change $Y' \to X'$ of f, if $\Gamma^{X'}$ is representable by some pair $(T, \mathcal{O}_T)$, then $\Gamma^{Y'}$ is representable by some $(S, \mathcal{O}_S)$ which is étale over $(T, \mathcal{O}_T)$.*

Combining these statements, it follows that the subcategory $\mathrm{Topos}^{\mathrm{loc}}_{//E} \subseteq \mathrm{Topos}_{//E}$ determines the fracture structure on E.

7.3.3 Schemes

In Section 7.1, we provided a topos-theoretic perspective on the classical notion of a scheme. We may extend this definition to an arbitrary geometric site $\mathcal{G} = (C, C^{\mathrm{ad}}, \tau)$.

Construction 7.20. Consider the fractured topos $E = \mathrm{Shv}_\tau(C)$, $E^{\mathrm{corp}} := \mathrm{Shv}_\tau(C^{\mathrm{ad}})$. By Theorem 7.19, we obtain a functor

$$\Gamma\colon \mathrm{Topos}^{\mathrm{loc}}_{//E} \to \mathrm{Pt}(\mathrm{PSh}(C))^{\mathrm{op}} \simeq \mathrm{Pro}(C),$$

which admits a right adjoint

$$\mathrm{Spec}_{\mathcal{G}}\colon \mathrm{Pro}(C) \to \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

given by sending a functor $f : T \to E$ to the composite

$$C \hookrightarrow \mathrm{PSh}(C) \twoheadrightarrow \mathrm{Shv}_\tau(C) = E \xrightarrow{f^*} T \xrightarrow{\Gamma} \mathrm{An},$$

which can be checked directly (omitted here) to be left exact. If τ is subcanonical, this functor is fully faithful.

Definition 7.21. We define the subcategory

$$\mathrm{Sch}_{\mathcal{G}} \quad \subseteq \quad \mathrm{Topos}^{\mathrm{loc}}_{//E}$$

of *$\mathcal{G}$-schemes* as the full subcategory on those objects that are locally in the image of $\mathrm{Spec}_{\mathcal{G}}$. In other words, this is the essential image of the inclusion

$$\mathrm{Shv}_\tau(\mathrm{Pro}(C)^{\mathrm{ad}}) \hookrightarrow \mathrm{Shv}_\tau(\mathrm{Pro}(C)).$$

Question 7.22. Does this category depend only on the fractured topos (E, E^{corp}), or is the choice of geometric site important?

7.4 Geometries from six-functor formalisms

Six-functor formalisms, developed systematically in [Sch25], provide a categorification of classical cohomology theories. The key insight relevant to our discussion is that every six-functor formalism on a category with an admissibility structure canonically determines a topology, the D-topology, and hence gives rise to a geometric site and a fractured topos.

Definition 7.23. Let (C, C^{ad}) be an admissibility structure. A *presentable six-functor formalism* is a lax symmetric monoidal functor $D\colon \operatorname{Span}(C, C^{\mathrm{ad}}) \to \mathrm{Pr}$.

Definition 7.24. The *D-topology* on C is generated by those families $(X_i \to X)_i$ of morphisms in C^{ad} such that the functor $D^!$ satisfies descent along every base change of the family. (This is called "universal !-descent".) We denote the resulting Grothendieck topology on C by τ_D.

Proposition 7.25. *The triple $(C, C^{\mathrm{ad}}, \tau_D)$ is a geometric site.*

Proof. This holds essentially by the definition of τ_D. □

Corollary 7.26. *The pair $(\mathrm{Shv}_{\tau_D}(C), \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}}))$ is a fractured topos. In particular, we obtain a local admissibility structure $\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}}$ on $\mathrm{Shv}_{\tau_D}(C)$.*

Recall that "local admissibility structure" includes a clause about locality in the target, but also about locality in the source.

Example 7.27.

- Any map in C^{ad} is admissible in $\mathrm{Shv}_{\tau_D}(C)$.
- Any map which is *representably* in C^{ad} (i.e. after pullback to any representable it is in C^{ad}) is admissible.

Proposition 7.28 (Mann–Scholze)**.** *The six-functor formalism D extends uniquely to a six-functor formalism on $(\mathrm{Shv}_{\tau_D}(C), \mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}})$ such that both functors*

$$D^*|_{\mathrm{Shv}_{\tau_D}(C)^{\mathrm{op}}}\colon \mathrm{Shv}_{\tau_D}(C)^{\mathrm{op}} \to \mathrm{Cat} \qquad \textit{and} \qquad D^!|_{(\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}})^{\mathrm{op}}}\colon (\mathrm{Shv}_{\tau_D}(C)^{\mathrm{ad}})^{\mathrm{op}} \to \mathrm{Cat}$$

preserve limits.

Proof. See [Man22, Appendix A.5]. □

In fact, there is the following refinement of this result. In the previous result, the extensions are defined on some monogenic localization of the presheaf category $\mathrm{PSh}(C)$. As we will see, there is a more general (non-monogenic) localization on which these extensions can be defined.

Construction 7.29. Consider the functor $D^!|_{\mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})^{\mathrm{op}}}\colon \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})^{\mathrm{op}} \to \mathrm{Cat}$. Let Σ_D be the largest congruence contained in the preimage of isomorphisms by this functor. More explicitly:

$$\Sigma_D \quad = \quad \{f \mid \text{every base change of every } \Delta^n f \text{ is a } D^!\text{-isomorphism}\}.$$

Note that $\Sigma_D \subseteq \mathrm{Conn}_\infty$, since the $D^!$-isomorphisms are contained in the effective epimorphisms, by the very definition of the D-topology.

We now define the category $\mathrm{Shv}^+_{\tau_D}(C)$ via the following pullback square:

$$\begin{tikzcd}
\mathrm{Shv}^+_{\tau_D}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Shv}_{\tau_D}(C) \arrow[d, "j^*"] \\
\mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})[\Sigma_D^{-1}] \arrow[r] & \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}}).
\end{tikzcd}$$

The top functor is still a left exact localization. The left vertical map admits a left adjoint which exhibits the pair $(\mathrm{Shv}^+_{\tau_D}(C), \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})[\Sigma_D^{-1}])$ as a fractured topos.

Remark 7.30. There is a formalism of *fractured localization*: the localization of a fractured topos, which inherits a new fractured topos structure. The fractured topos $(\mathrm{Shv}^+_{\tau_D}(C), \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})[\Sigma_D^{-1}])$ is an example of this construction.

Lemma 7.31. *The six-functor formalism descends further to a functor*

$$D\colon\ \mathrm{Span}(\mathrm{Shv}^+_{\tau_D}(C), \mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})[\Sigma_D^{-1}]) \to \mathrm{Pr}^{\mathrm{L}}.$$

Remark 7.32. We are implicitly using here the following general argument in a six-functor formalism: if $f\colon X \to Y$ is a map in $\mathrm{Shv}_{\tau_D}(C^{\mathrm{ad}})$ such that all base changes are $D^!$-isomorphisms, then f is also a D^*-isomorphism. (There is an easy argument using self-duality in the span category: given an admissible morphism $Y \to X$, then $D(Y)$ is a self-dual $D(X)$-module, and the duality exchanges f^* and $f_!$.)

Similarly, universal !-descent implies universal $*$-descent.

7.5 Analytic stacks

In this section, we discuss analytic stacks in the sense of Clausen–Scholze as an example of the preceding formalism.

Definition 7.33. A *condensed anima* is a hypercomplete sheaf on the site CHaus of compact Hausdorff spaces equipped with the topology in which a finite collection of maps $\{X_i \to X\}_i$ is declared to be a covering if and only if the map $\bigsqcup_i X_i \twoheadrightarrow X$ is surjective. Equivalently, it is a hypercomplete sheaf on the site ProFin of profinite sets, equipped with the topology inherited from CHaus.

In practice, we will work with *light condensed animae*: hypercomplete sheaves on the site of *countably* profinite sets (i.e. profinite sets presented by countable cofiltered diagrams). We will be denoting this by

$$\mathrm{CondAn} := \mathrm{Shv}(\{\text{Countably profinite sets}\})^{\mathrm{hyp}}.$$

In this setting, we take "ring" to mean "animated commutative ring".

Definition 7.34. An *analytic ring* is a pair $A = (A^{\triangleleft}, \mathrm{Mod}_A)$ consisting of a condensed ring $A^{\triangleleft}$ together with a reflective $\iota\colon \mathrm{Mod}_A \hookrightarrow \mathrm{Mod}_{A^{\triangleleft}}$ (with left adjoint $L\colon \mathrm{Mod}_{A^{\triangleleft}} \to \mathrm{Mod}_A$) satisfying the following conditions:

- Mod_A is closed under colimits (also limits, but this is automatic).

- Mod_A is closed under the internal hom $\underline{\mathrm{Hom}}(M, -)$ for every $M \in \mathrm{Mod}_{A^\triangleleft}$.
- The functor $\iota L \colon \mathrm{Mod}_{A^\triangleleft} \to \mathrm{Mod}_{A^\triangleleft}$ preserves connective objects with respect to the canonical t-structure.
- We have $A^\triangleleft \in \mathrm{Mod}_A$.

Remark 7.35. In the original definition of analytic rings, this was defined in the "non-light" setting. In this case, there is an additional condition one needs to put: the Frobenius map $A \to A/p$ is a map of analytic rings for every prime p. In the light setting, this condition turns out to be automatic. (It is not known whether this condition is automatic in the "non-light" setting.)

Definition 7.36. A map $f \colon A \to B$ of analytic rings is a map $A^\triangleleft \to B^\triangleleft$ satisfying the condition that the pushforward functor $\mathrm{Mod}_{B^\triangleleft} \to \mathrm{Mod}_{A^\triangleleft}$ restricts to a functor

$$f_* \colon \mathrm{Mod}_B \to \mathrm{Mod}_A.$$

We denote its left adjoint by

$$f^* \colon \mathrm{Mod}_A \to \mathrm{Mod}_B,$$

which is given by first including into $\mathrm{Mod}_{A^\triangleleft}$, then applying base change to $\mathrm{Mod}_{B^\triangleleft}$, and then reflecting back into Mod_B. We denote the resulting category of analytic rings by AnRing.

Definition 7.37. A map $f \colon A \to B$ of analytic rings is said to be:

- *proper* if the functor f_* induces an equivalence

$$\mathrm{Mod}_B \xrightarrow{\sim} \mathrm{Mod}_{B^\triangleleft} \times_{\mathrm{Mod}_{A^\triangleleft}} \mathrm{Mod}_A.$$

- an *open immersion* if the functor f^* admits a further left adjoint

$$f_! \colon \mathrm{Mod}_B \to \mathrm{Mod}_A$$

  which satisfies the projection formula: $f_!(f^*(M) \otimes_B N) \cong M \otimes_A f_!(N)$.

- !*-able* if the map of analytic rings $(B^\triangleleft, \mathrm{Mod}_{B^\triangleleft} \times_{\mathrm{Mod}_{A^\triangleleft}} \mathrm{Mod}_A) \to (B, \mathrm{Mod}_B)$ is an open immersion.

Lemma 7.38. *A map f is !-able if and only if f factors as an open map followed by a proper map.*

Proposition 7.39. *There is a presentable six-functor formalism*

$$D \colon \mathrm{Span}(\mathrm{AnRing}^{\mathrm{op}}, \mathrm{AnRing}^{!,\mathrm{op}}) \to \mathrm{Pr}, \qquad A \mapsto \mathrm{Mod}_A.$$

Proof. This may be obtained as a consequence of, for example, [CLL25]. □

Definition 7.40. The category of *analytic stacks* is defined to be

$$\mathrm{AnStck} := \mathrm{Shv}^+_{\tau_D}(\mathrm{AnRing}^{\mathrm{op}}).$$

Remark 7.41. The original definition of AnStck given in the lectures by Clausen and Scholze turned out to have a mistake; the above is the corrected definition.

7.6 Gestalten

We finish the lecture course by giving a brief outlook on the recent framework of *Gestalten* by Scholze and Stefanich [Sch26]. This section is intentionally schematic and should be read as an informal outlook rather than a self-contained formal treatment.

Construction 7.42. There are presentably symmetric monoidal categories $n\,\mathrm{Pr}$ for all n, satisfying

$$0\,\mathrm{Pr} = \mathrm{An}, \qquad 1\,\mathrm{Pr} = \mathrm{Pr}\,.$$

Heuristically, for $n \geq 1$, we may think of $(n+1)\,\mathrm{Pr}$ more or less as $\mathrm{Mod}_{n\,\mathrm{Pr}}(\mathrm{Pr})$ (up to some size issues one needs to take care of).

Definition 7.43. A *Stefanich ring* is a categorical spectrum $A = (A_0, A_1, A_2, \dots)$, i.e. we have $A_n \in \mathrm{CAlg}(n\,\mathrm{Pr})$ for every n and there are isomorphisms $A_n \cong \mathrm{End}_{A_{n+1}}(\mathbb{1})$. This defines a category StRing.

Definition 7.44. The category of *Gestalten* is defined as

$$\mathrm{Gest} := \mathrm{StRing}^{\mathrm{op}}.$$

The following is an informal version of a result proved by Scholze and Stefanich.

Theorem 7.45 (Scholze–Stefanich)**.** *Up to various issues, the category* Gest *is a topos.*

Construction 7.46. For every n, there is a fully faithful functor

$$\mathrm{CAlg}(n\,\mathrm{Pr})^{\mathrm{op}} \hookrightarrow \mathrm{Gest}.$$

Indeed, any $A_n \in \mathrm{CAlg}(n\,\mathrm{Pr})$ admits a canonical "delooping" $A_{n+1} := \mathrm{Mod}_{A_n}(n\,\mathrm{Pr})$, and repeating this we obtain a Gestalt. Since the assignment $A_n \mapsto \mathrm{Mod}_{A_n}(n\,\mathrm{Pr})$ is fully faithful, so is the functor $\mathrm{CAlg}(n\,\mathrm{Pr})^{\mathrm{op}} \to \mathrm{Gest}$.

Definition 7.47. A Gestalt A is called *n-affine* if it lies in the image of this fully faithful functor. Equivalently, A satisfies $A_{k+1} \cong \mathrm{Mod}_{A_k}(k\,\mathrm{Pr})$ for all $k \geq n$.

Example 7.48. The 0-affine Gestalten are precisely the commutative monoids:

$$\mathrm{CMon}(\mathrm{An}) = \mathrm{CAlg}(\mathrm{An}) \hookrightarrow \mathrm{Gest}.$$

Since $\mathrm{Pr}_{\mathrm{st}} \subseteq \mathrm{Pr}$ is fully faithful, we similarly get a fully faithful functor

$$\mathrm{CAlg}(\mathrm{Sp}) \hookrightarrow \mathrm{Gest}.$$

Remark 7.49. Note that objects in the image of CAlg(Sp) are 1-affine, but not 0-affine in the absolute sense. To fix this, we can also work "relatively to the sphere": we define the category $\mathrm{Gest}_{\mathbb{S}}$ as the slice

$$\mathrm{Gest}_{\mathbb{S}} := \mathrm{Gest}_{/\mathrm{Spec}(\mathrm{Sp})}.$$

Note that the forgetful functor $\mathrm{Gest}_{\mathbb{S}} \to \mathrm{Gest}$ is fully faithful, by fully faithfulness of $\mathrm{Pr}_{\mathrm{st}} \hookrightarrow \mathrm{Pr}$. For every ring spectrum $R \in \mathrm{CAlg}(\mathrm{Sp})$, its associated Gestalt is 0-affine relative to the sphere $\mathbb{S}$.

Gestalten from six-functor formalisms (informal sketch)

Consider a presentable six-functor formalism

$$D\colon \operatorname{Span}(C, C^{\mathrm{ad}}) \to \mathrm{Pr}\,.$$

For every $X \in C$, we obtain a new six-functor formalism

$$D_X\colon \operatorname{Span}(C^{\mathrm{ad}}_{/X}) \to \mathrm{Pr}\,.$$

This gives rise to a "2-category of kernels" $K_D(X)$, defined as

$$K_D(X) := \operatorname{Mod}_{D_X}(\operatorname{Span}(C^{\mathrm{ad}}_{/X}), \mathrm{Pr}) \quad \in \quad 2\,\mathrm{Pr}\,.$$

This 2-category of kernels is functorial in X, and is ambidextrous on the subcategory $C^{\mathrm{ad,op}}$. In particular, one obtains a six-functor formalism

$$K_D\colon \operatorname{Span}(C, C^{\mathrm{ad}}) \to 2\,\mathrm{Pr}\,.$$

This functor is a delooping of D, in the sense that

$$\operatorname{End}_{K_D(X)}(\mathbb{1}) \cong D(X)$$

for all $X \in C$.

By iterating this construction, we obtain a functor

$$C \to \mathrm{Gest}, \qquad X \mapsto [X]_D := (\operatorname{End}_{D(X)}(\mathbb{1}), D(X), K_D(X), K_D^2(X), \dots).$$

Definition 7.50. We say that D *satisfies Künneth* if the functor

$$D^*\colon C^{\mathrm{ad}} \to \operatorname{CAlg}(\mathrm{Pr})^{\mathrm{op}}$$

preserves pullbacks.

In this case, one can show that $K_D^n(X) \cong \operatorname{Mod}_{D(X)}(n\,\mathrm{Pr})$. In particular, $[X]_D$ is a 1-affine Gestalt.

Example 7.51. The six-functor formalism D on $\mathrm{AnRing}^{\mathrm{op}}$ we constructed before satisfies Künneth.

Proposition 7.52. *The functor* $[-]_D\colon C^{\mathrm{ad}} \to \mathrm{Gest}$ *preserves pullbacks.*

Proposition 7.53. *A collection of maps* $(X_i \to X)_i$ *is a* D*-cover in* C *if and only if the map* $\bigsqcup_i [X_i]_D \to [X]_D$ *is an effective epimorphism in* Gest*.*

Combining these two results, we see that the functor $[-]_D\colon C^{\mathrm{ad}} \to \mathrm{Gest}$ extends uniquely to a left exact colimit-preserving functor

$$F\colon \operatorname{Shv}_{\tau_D}(C^{\mathrm{ad}}) \to \mathrm{Gest}.$$

By considering the class $F^{-1}(\mathrm{Iso})$, we have an epigenic localization of $\operatorname{Shv}_{\tau_D}(C^{\mathrm{ad}})$. Essentially by definition of Σ_D we get an inclusion

$$F^{-1}(\mathrm{Iso}) \subseteq \Sigma_D.$$

In particular, the functor F further extends to $\operatorname{Shv}_{\tau_D}(C^{\mathrm{ad}})[\Sigma_D^{-1}] \to \mathrm{Gest}$.

Remark 7.54. In the case of analytic rings, Scholze claims in his lecture notes that this inclusion is an equality. (Marc expects this to hold in general.)

A Factorization systems and saturated classes

In this appendix, we collect the necessary background on factorization systems and saturated classes. These tools are used throughout the text to construct the epi-mono factorization, to define modalities like n-connected maps, and to ensure the existence of quotients by congruences.

A.1 Factorization systems

We begin with the definition of orthogonality.

Definition A.1. Let C be a category. A morphism $l\colon A \to B$ is said to be *left orthogonal* to a morphism $r\colon X \to Y$, written $l \perp r$, if the following square is a pullback square:

$$\begin{tikzcd}
\mathrm{Hom}_C(B,X) \arrow[r, "-\circ l"] \arrow[d, "r\circ -"'] & \mathrm{Hom}_C(A,X) \arrow[d, "r\circ -"] \\
\mathrm{Hom}_C(B,Y) \arrow[r, "-\circ l"] & \mathrm{Hom}_C(A,Y).
\end{tikzcd}$$

Equivalently, for every commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\
B \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

the space of dashed fillers making both triangles commute is contractible. In this case, we also say that r is *right orthogonal* to l.

Given a collection of morphisms S, we denote by $S^{\perp}$ the collection of all morphisms that are right orthogonal to every morphism in S, and by ${}^{\perp}S$ the collection of those that are left orthogonal to every morphism in S.

Definition A.2. A *factorization system* on a category C consists of two classes of morphisms L and R of C satisfying the following two conditions:

(1) *Existence of factorization*: Every morphism f in C admits a factorization $f \simeq r \circ l$, where $l \in L$ and $r \in R$.

(2) *Orthogonality*: Every morphism $l \in L$ is left orthogonal to every morphism $r \in R$.

The classes of maps in a factorization system satisfy various useful closure properties.

Proposition A.3. *Let (L, R) be a factorization system on C.*

(1) *We have $L = {}^{\perp}R$ and $R = L^{\perp}$.*

(2) *The classes L and R are closed under composition and contain all equivalences.*

(3) *The classes L and R are closed under retracts.*

(4) *The class R is closed under base change (pullbacks).*

(5) *The class L is closed under cobase change (pushouts).*

(6) *The class R has the left cancellation property: if $g \circ f \in R$ and $g \in R$, then $f \in R$.*

(7) *The class L has the right cancellation property: if $g \circ f \in L$ and $f \in L$, then $g \in L$.*

(8) *If C admits small limits, the class R is closed under small limits in $\mathrm{Ar}(C)$.*

(9) *If C admits small colimits, the class L is closed under small colimits in $\mathrm{Ar}(C)$.*

Proof. Left to the reader. □

Lemma A.4. *Let (L, R) be a factorization system on C. Any morphism $f : X \to Y$ which is both in L and in R is an isomorphism.*

Proof. Consider the commutative square:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Since $f \in L$ (as the left vertical map) and $f \in R$ (as the right vertical map), the definition of orthogonality implies there is a unique filler g. One immediately checks that g provides an inverse to f. □

Lemma A.5. *Let (L, R) be a factorization system on C, and consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.*

(1) *Assume C admits pullbacks. If $gf \in R$ and $\Delta_g \in R$, then $f \in R$.*

(2) *Assume C admits pushouts. If $gf \in L$ and $\nabla_f \in L$, then $g \in L$.*

Proof. We prove (1); the proof for (2) is dual. We may factor f as the composite

$$X \xrightarrow{(\mathrm{id}_X, f)} X \times_Z Y \xrightarrow{\mathrm{pr}_Y} Y.$$

The second map is a base change of $gf : X \to Z$ along g, hence lies in R because $gf \in R$. The first map is a base change of the diagonal $\Delta_g : Y \to Y \times_Z Y$ along the map $f \times \mathrm{id}_Y : X \times_Z Y \to Y \times_Z Y$. Since $\Delta_g \in R$, it follows that $(\mathrm{id}_X, f) \in R$. Since R is closed under composition, we conclude $f \in R$. □

Lemma A.6 ([ABFJ22, Lemma 3.1.8]). *Let C be a category which admits pullbacks and let (L, R) be a factorization system on C. Then for every object $X \in C$ the slice $C_{/X}$ admits a factorization system in which a morphism lies in the left/right class if and only if the underlying morphism in C does.* □

Lemma A.7. *Let (L, R) be a factorization system on C.*

(1) *Let $\mathrm{Ar}^R(C) \subseteq \mathrm{Ar}(C)$ denote the full subcategory spanned by the morphisms in R. Then the inclusion $\mathrm{Ar}^R(C) \hookrightarrow \mathrm{Ar}(C)$ admits a left adjoint. Given a morphism $f : A \to B$ with unique factorization $A \xrightarrow{l} C \xrightarrow{r} B$, this left adjoint sends f to r.*

(2) *Similarly, the inclusion $\mathrm{Ar}^L(C) \hookrightarrow \mathrm{Ar}(C)$ of the morphisms in L admits a right adjoint, sending f to l.*

Proof. By symmetry, it suffices to prove statement (1). It suffices to construct the left adjoint pointwise. We claim that the morphism

$$\begin{tikzcd} A \arrow[r, "l"] \arrow[d, "f"'] & C \arrow[d, "r"] \\ B \arrow[r, equal] & B \end{tikzcd}$$

in $\mathrm{Ar}(C)$ is a counit morphism exhibiting $r \in \mathrm{Ar}^R(C)$ as a right adjoint object to f under the inclusion. Indeed, for any other morphism r', the map

$$\mathrm{Hom}_{\mathrm{Ar}(C)}(r, r') \to \mathrm{Hom}_{\mathrm{Ar}(C)}(f, r')$$

is an equivalence as a consequence of the fact that $l \perp r'$. □

A.2 Saturated classes

In practice, factorization systems are often constructed by specifying a set of generators for the left class.

Definition A.8. Let L be a class of morphisms of a presentable category C. We say that L is *saturated* if it contains all isomorphisms, is closed under composition, and the subcategory of $\mathrm{Ar}(C)$ spanned by the morphisms in L is closed under colimits.

Remark A.9. Given a factorization system (L, R) on C, it immediately follows from Proposition A.3 that L is saturated.

Note that the intersection of any collection of saturated classes is again saturated. This allows us to make the following definition:

Definition A.10. Let Σ be a class of morphisms in C. The *saturation* of Σ, denoted Σ^s, is the smallest saturated class containing Σ. We say that a saturated class L is *of small generation* if $L = \Sigma^s$ for some *set* of morphisms Σ.

The following proposition provides the bridge between saturated classes and factorization systems.

Proposition A.11 (Lurie, HTT, Proposition 5.5.5.7). *Let C be a presentable category and let L be a saturated class of small generation. Then the pair $(L, L^\perp)$ forms a factorization system on C.*

A.3 Strongly saturated classes

When studying localizations of presentable categories, we require a stronger stability property.

Definition A.12. Let L be a class of morphisms of a presentable category C. We say L is *strongly saturated* if it is saturated and closed under 2-out-of-3.

As with saturated classes, we say a strongly saturated class L is *of small generation* if it is the smallest strongly saturated class containing some *set* of morphisms Σ. The following two results identify these classes with kernels of accessible Bousfield localizations.

Definition A.13. Let $\varphi\colon C \to D$ be a cocontinuous functor between presentable categories. We define its *kernel* as the collection of morphisms in C that is inverted by φ:

$$\ker(\varphi) \quad := \quad \{f\colon X \to Y \in \mathrm{Ar}(C) \mid \varphi(f)\colon \varphi(X) \xrightarrow{\sim} \varphi(Y)\}.$$

Proposition A.14 (Lurie, HTT, Proposition 5.5.4.16)**.** *Let $\varphi\colon C \to D$ be a cocontinuous functor between presentable categories. Then* $\ker(\varphi)$ *is strongly saturated and of small generation.*

Proposition A.15 (Lurie, HTT, Proposition 5.5.4.15)**.** *Let C be a presentable category and let Σ be a strongly saturated class of small generation.*

(1) *The inclusion* $\mathrm{Loc}_\Sigma(C) \hookrightarrow C$ *of the Σ-local objects in C admits an accessible left adjoint* $L\colon C \to \mathrm{Loc}_\Sigma(C)$.

(2) *The category* $\mathrm{Loc}_\Sigma(C)$ *is presentable.*

(3) *We have* $\ker(L) = \Sigma$*, i.e. a morphism in C is inverted by L if and only if it lies in Σ.*

B Limits and colimits in categories of sections

In this appendix, we record some basic facts about the existence of limits and colimits in categories of sections of cartesian and cocartesian fibrations used various times in the text.

Proposition B.1. *Let K be a category. Let $F\colon I \to \mathrm{Cat}$ be a functor satisfying the following two conditions:*

(1) *Each category $F(i)$ admits K-indexed colimits;*

(2) *Each functor $F(i) \to F(j)$ induced by a morphism $i \to j$ in I preserves K-indexed colimits.*

Then the category $\Gamma_I(\int_I^{\mathrm{cc}} F)$ of sections of the cocartesian unstraightening $\int_I^{\mathrm{cc}} F \to I$ admits K-indexed colimits, and the evaluation functors $\mathrm{ev}_i\colon \Gamma_I(\int_I^{\mathrm{cc}} F) \to F(i)$ preserve K-indexed colimits for all $i \in I$.

Proof. Let us abbreviate $\int_I F := \int_I^{\mathrm{cc}} F$. Consider the full subcategory of $\mathrm{Cat}_{/I}$ spanned by those functors $J \to I$ for which the functor category $\mathrm{Fun}_{/I}(J, \int_I F)$ admits K-indexed colimits, and that these are computed pointwise. The statement of the proposition is saying that this category contains (the identity on) I. Observe that this subcategory is closed under colimits: if we have $J \simeq \mathrm{colim}_\alpha J_\alpha$, then we see that $\mathrm{Fun}_{/I}(J, \int_I F) \simeq \lim_\alpha \mathrm{Fun}_{/I}(J_\alpha, \int_I F)$. By assumption, each of these categories in the limit admit K-indexed colimits, and since these are computed pointwise we see that also all transition functors in this diagram preserve K-indexed colimits. It follows that also the limit admits K-indexed colimits, giving the claim. Since $\mathrm{Cat}_{/I}$ is generated under colimits by the maps $[1] \to I$ (since Cat is generated by $[1]$), we are reduced to proving the claim for $[1] \to I$. In this case, we have $\mathrm{Fun}_{/I}([1], \int_I F) \simeq \Gamma_{[1]}([1] \times_I \int_I F)$, so we are reduced to the case $I = [1]$. In this case, $\Gamma_{[1]}(\int_{[1]} F)$ sits in a pullback square as follows:

$$\begin{tikzcd}
\Gamma_{[1]}(\int_{[1]} F) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ar}(F(1)) \arrow[d, "s"] \\
F(0) \arrow[r] & F(1).
\end{tikzcd}$$

The assumptions on F guarantee that this pullback is taken inside categories with K-indexed colimits, and we conclude that also $\Gamma_{[1]}(\int_{[1]} F)$ admits K-indexed colimits. $\square$

Corollary B.2. *Consider a functor $F\colon I \to \mathrm{Cat}$ such that each $F(i)$ admits K-indexed limits and each $F(i) \to F(j)$ preserves K-indexed limits. Then the category $\Gamma_I(\int_I^{\mathrm{ct}} F)$ of sections of the cartesian unstraightening of F admits K-indexed limits, and the functors $\mathrm{ev}_i\colon \Gamma_I(\int_I^{\mathrm{ct}} F) \to F(i)$ preserve K-indexed limits for all $i \in I$.*

Proof. This is an instance of the previous result applied to $i \mapsto F(i)^{\mathrm{op}}$, using that $(\int_I^{ct} F)^{\mathrm{op}} \simeq \int^{\mathrm{ct}} F^{\mathrm{op}}$. ◻

In an analogous way, one proves:

Proposition B.3. *Consider a functor $F\colon I \to \mathrm{Cat}$ such that each $F(i)$ admits K-indexed limits and each $F(i) \to F(j)$ preserves K-indexed limits. Then $\Gamma_I(\int^{\mathrm{cc}} F)$ admits K-indexed limits, which are computed pointwise. If F instead lands in categories with K-indexed colimits, then $\Gamma_I(\int^{\mathrm{ct}} F)$ admits K-indexed colimits, which are computed pointwise.* ◻

Remark B.4. As a consequence of [Lur09, Proposition 5.1.2.2], it is not actually needed to assume the functors $F(i) \to F(j)$ preserve the relevant (co)limits, as the required transition maps in the expected (co)limits already exist. However, the proof in this case is necessarily more complicated. The key statement is [Lur09, Lemma 5.1.2.1].

Bibliography

[ABFJ18] M. Anel, G. Biedermann, E. Finster, and A. Joyal. “Goodwillie’s calculus of functors and higher topos theory”. *J. Topol.* 11.4 (2018), pp. 1100–1132.

[ABFJ20] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. “A generalized Blakers-Massey theorem”. *J. Topol.* 13.4 (2020), pp. 1521–1553.

[ABFJ22] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. “Left-exact localizations of ∞-topoi. I: Higher sheaves”. *Adv. Math.* 400 (2022). Id/No 108268, p. 64.

[ABFJ24] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. “Left-exact localizations of ∞-topoi. II: Grothendieck topologies”. *J. Pure Appl. Algebra* 228.3 (2024), p. 63.

[ABFJ25] Mathieu Anel, Georg Biedermann, Eric Finster, and André Joyal. *Left-exact localizations of ∞-topoi III: The acyclic product*. Preprint, arXiv:2308.15573. 2025. URL: https://arxiv.org/abs/2308.15573.

[AC21] Avraham Aizenbud and Shachar Carmeli. “Relative de Rham theory on Nash manifolds”. *arXiv preprint arXiv:2105.12417* (2021).

[Ada74] Jiri Adamek. “Free algebras and automata realizations in the language of categories”. *Commentat. Math. Univ. Carol.* 15 (1974), pp. 589–602.

[AJ21] Mathieu Anel and André Joyal. “Topo-logie”. *New spaces in mathematics. Formal and conceptual reflections*. Cambridge: Cambridge University Press, 2021, pp. 155–257.

[BGH18] Clark Barwick, Saul Glasman, and Peter Haine. *Exodromy*. Preprint, arXiv:1807.03281. 2018. URL: https://arxiv.org/abs/1807.03281.

[BH21] Tom Bachmann and Marc Hoyois. *Norms in motivic homotopy theory*. Vol. 425. Astérisque. Paris: Société Mathématique de France (SMF), 2021.

[CLL25] Bastiaan Cnossen, Tobias Lenz, and Sil Linskens. *Universality of span 2-categories and the construction of 6-functor formalisms*. Preprint, arXiv:2505.19192. 2025. URL: https://arxiv.org/abs/2505.19192.

[Hoy19a] Marc Hoyois. *On Quillen’s plus construction*. Unpublished note, available at https://hoyois.app.uni-regensburg.de/papers/acyclic.pdf. 2019.

[Hoy19b] Marc Hoyois. “Topoi of parametrized objects”. *Theory Appl. Categ.* 34 (2019), pp. 243–248.

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009.

[Lur17] Jacob Lurie. “Higher algebra”. *https://www.math.ias.edu/~lurie/papers/HA.pdf* (2017).

[Lur18] Jacob Lurie. “Spectral Algebraic Geometry”. *under construction (version dated February 2018), www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf* (2018).

[Man22] Lucas Mann. “A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry”. *arXiv preprint arXiv:2206.02022* (2022).

[McL90] Colin McLarty. “The uses and abuses of the history of topos theory”. *Br. J. Philos. Sci.* 41.3 (1990), pp. 351–375.

[NSS15] Thomas Nikolaus, Urs Schreiber, and Danny Stevenson. “Principal ∞-bundles: general theory”. *J. Homotopy Relat. Struct.* 10.4 (2015), pp. 749–801.

[Rez25] Charles Rezk. “Free colimit completion in ∞-categories”. *Homology Homotopy Appl.* 27.1 (2025), pp. 275–292.

[Sch25] Peter Scholze. *Six-Functor Formalisms*. Preprint, arXiv:2510.26269. 2025. URL: https://arxiv.org/abs/2510.26269.

[Sch26] Peter Scholze. *Geometry and Higher Category Theory*. Lecture notes. Available at: https://people.mpim-bonn.mpg.de/scholze/Gestalten.pdf. 2026.

[SS21] Hisham Sati and Urs Schreiber. “Equivariant principal infinity-bundles”. *arXiv preprint arXiv:2112.13654* (2021).

[SW25] Christian Sattler and David Wärn. *Note on confluent colimits*. Available from https://www.cse.chalmers.se/~sattler/docs/confluent/new-2025.txt. 2025.

[Uem25] Taichi Uemura. *Colimits in the ∞-category of ∞-topoi and étale morphisms*. Preprint, arXiv:2506.10431] (2025). 2025.

[Wär25] David Wärn. *Path spaces of pushouts*. Preprint, arXiv:2402.12339. 2025. URL: https://arxiv.org/abs/2402.12339.