The six operations in topology

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1 Introduction

After having had two talks on symplectic geometry, we are going to switch gears: we will turn our attention to the formalism of *six operations in topology*. We may think of this formalism as a "categorification" of the cohomology $H^*(X;R)$ of a space X with coefficients in some (commutative) ring R. We will discuss this in much more detail in the talk, but here is a rough picture:

- For every space X, there is an ∞-category $\mathrm{Shv}(X;R)$ of $D(R)$-*valued sheaves on* X. This is a derived version of the classical notion of sheaves on X valued in R-modules.
- Given a sheaf $\mathcal{F} \in \mathrm{Shv}(X;R)$, we may perform a construction called the *total sections*, which is just the value of $\mathcal{F}$ at the entire space X:
$$\Gamma(X;\mathcal{F}) \quad := \quad \mathcal{F}(X) \quad \in \quad D(R).$$
- There is a designated sheaf $\underline{R} \in \mathrm{Shv}(X;R)$, called the *constant sheaf on* R. (It is the sheafification of the constant presheaf.) The cohomology of X is then recovered as the cohomology of the total sections of $\underline{R}$:
$$H^*(X;R) \quad \cong \quad H^*(\Gamma(X;\underline{R})).$$
- Given a sheaf $\mathcal{F} \in \mathrm{Shv}(X;R)$, there is another construction, called the *sections with compact support*, denoted $\Gamma_c(X;\mathcal{F}) \in D(R)$. Its cohomology recovers the compactly supported cohomology of X:
$$H^*_c(X;R) \quad \cong \quad H^*(\Gamma_c(X;\underline{R})).$$

The advantage of categorifying cohomology is that the ∞-categories $\mathrm{Shv}(X;R)$ carry a lot of structure which cannot immediately be formulated at the level of cohomology. It is similar to the step of passing from Betti numbers to cohomology groups:

$$\begin{array}{cc}
\mathrm{Shv}(X;R) & \text{categorical invariant} \\
\rightsquigarrow \Gamma(X;-) & \\
H^*(X;R) & \text{algebraic invariant} \\
\rightsquigarrow \mathrm{rk} & \\
b^n(X;R) & \text{numerical invariant}
\end{array}$$

Think about Poincaré duality: while this is visible to some extent at the level of Betti numbers, it becomes a much more powerful statement when formulated at the level of (co)homology groups, where we can talk about group isomorphisms and have the contravariant functoriality of cohomology available. Going up one step further to the categorical level buys us even more: we can talk about adjunctions between the homology/cohomology functors, and have symmetric monoidal structures available that interact nicely with them. All this structure, and the relations between them, can be assembled into something called a *six-functor formalism*. The goal of this talk is to explain how to construct all this structure in the case of $\mathrm{Shv}(X;R)$. In the next talk, we will then use this to explore a modern formulation of Poincaré duality: a statement at the categorical level whose 'cohomological shadow' is classical (twisted) Poincaré duality.

2 Derived ∞-categories

The modern incarnation of the formalism of six operations heavily uses the theory of ∞-categories. While there is no time to give a proper introduction to this theory, I want to give motivation for why this theory is so powerful for studying things like cohomology, and give some indications for how to work with it.

As a starting observation, let us note that the cohomology groups $H^*(X;R)$ arise from some more fundamental object: the cochain complex $C^*(X;R)$. Now, while chain complexes are a priori quite rigid algebraic objects, we are often only interested in studying them *up to chain homotopies*: we wish to think of two maps $f, g\colon C_\bullet \to D_\bullet$ of chain complexes as "the same" if there exists a chain homotopy $H\colon f \simeq g$ between them. This is actually not an isolated situation: in many other situations (like in algebraic topology) we encounter a notion of 'homotopy' between maps. And it doesn't stop here: given two different homotopies, we may ask for *homotopies between homotopies*, and so on and so forth. It may seem unclear at this point why we care about all these higher homotopies, but let me just say for now that it has turned out to be very useful in homotopy theory to change our entire mindset about what 'equal' means, and *always* ask for homotopies rather than strict equalities:

> **Fundamental Principle of Homotopy Theory**: When expressing that two entities are 'equal', we must always specify *how* they are equal by providing a homotopy/isomorphism between them.

If we take this principle seriously and start trying to do category theory, we see we are naturally lead to considering *higher category theory*:

$$\infty\text{-category theory} \quad = \quad \text{category theory} \quad + \quad \text{homotopy theory.}$$

In an ∞-category we may still talk about *objects* and *morphisms*, but we may no longer talk about *strict equality* of morphisms. Instead, we have *homotopies* $H\colon f \simeq g$ between morphisms. And given two homotopies $H, K\colon f \simeq g$, we have *homotopies between homotopies*. And so on and so forth.

While it can take some time to get used to this theory, the bottom line is that a lot of it works very similarly to ordinary category theory.

Warning 2.1. One thing to note is that we may no longer speak of the Hom *sets*, since in a set it is possible to say whether two elements are 'equal' or not, and we shouldn't be able to say this in an ∞-category. Instead, the morphisms in C form a *Hom anima* $\mathrm{Hom}_C(x, y)$. You may think of an anima like a set, in that it encodes some collection of objects, except that in an anima we aren't allowed to say that 'x = y'; instead, the equalities between x and y are themselves structures that assemble into a new anima $\{x = y\}$.

What is so great about ∞-categories is that they allow for a more powerful version of the usual notion of 'short exact sequence' in homological algebra. Recall that for a sequence

$$A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C$$

of abelian groups, with i injective, p surjective and $pi = 0$, we have that A is the kernel of p if and only if C is the cokernel of i. If this holds, we call this sequence a *short exact sequence*. In an ∞-category, we must replace this as follows:

- First, instead of $pi = 0$ we must ask for a homotopy $pi \simeq 0$.
- Second, instead of talking about the *kernel* of p, we must talk about the *fiber* $\mathrm{fib}(p)$ of p: it comes with a map $\mathrm{fib}(p) \xrightarrow{i'} B$ and a homotopy $pi' \simeq 0$, and is in fact *universal* with respect to such data.
- Similarly, we must replace the *cokernel* of i with the *cofiber* $\mathrm{cofib}(i)$, which has the dual universal property.

Definition 2.2. Let C be an ∞-category. We say it is *stable* if:

(1) It admits a *zero object* $0 \in C$, i.e. for every X there are unique maps $X \to 0$ and $0 \to X$;

(2) For every sequence

$$A \xrightarrow{i} B \xrightarrow{p} C$$

with a homotopy $pi \simeq 0$, we have that A is the fiber of p if and only if C is the cofiber of i.

We refer to the sequences in part (2) as *exact sequences*. The main thing to note in this definition is that we no longer require i to be injective and p to be surjective!! This is really true for *any* maps i and p. As a consequence, we no longer have to worry about left and right exactness:

Lemma 2.3. *For a functor $F\colon C \to D$ between stable ∞-categories, we have:*

$$F \text{ is left exact} \quad\Longleftrightarrow\quad F \text{ is exact} \quad\Longleftrightarrow\quad F \text{ is right exact}.$$

Proof. The functor F is left exact if and only if it preserves finite products and fibers. It is exact if it preserves direct sums and exact sequences. And it is right exact if it preserves finite coproducts and cofibers. But by stability, we have:

- For $X, Y \in C$, we have $X \times Y = X \oplus Y = X \coprod Y$.
- A sequence $X \to Y \to Z$ is a fiber sequence iff it is an exact sequence iff it is a cofiber sequence. □

Now, with all this preamble, we can define the derived ∞-category of a commutative ring.

Definition 2.4. Let R be a commutative ring. We define the ∞-category $D(R)$ as

$$D(R) \quad := \quad \mathrm{Ch}(R)[\text{quasi-isomorphism}^{-1}],$$

i.e. the universal ∞-category equipped with a functor from the 1-category $\mathrm{Ch}(R)$ of chain complexes of R-modules that inverts the quasi-isomorphisms.

Proposition 2.5. *The ∞-category $D(R)$ is stable.*

The cofiber of $f_\bullet\colon C_\bullet \to D_\bullet$ in $D(R)$ is computed using the familiar cone construction $C(f)_n := C_{n-1} \oplus D_n$.

3 Sheaf categories

Let us now define the sheaf ∞-categories $\mathrm{Shv}(X; R)$ and study their properties.

Definition 3.1. Let X be a topological space and let R be a commutative ring. A functor $\mathcal{F}\colon \mathrm{Open}(X)^{\mathrm{op}} \to D(R)$ is called a *sheaf* if for every collection of opens $\{U_i\}_{i\in I}$ with union U, the map

$$\mathcal{F}(U) \quad\to\quad \lim\left(\textstyle\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j) \rightrightarrows\!\!\!\to \prod_{i,j,k\in I} \mathcal{F}(U_i \cap U_j \cap U_k) \rightrightarrows\!\!\!\rightrightarrows \cdots \right)$$

is an isomorphism. We denote by

$$\mathrm{Shv}(X; R) \quad\subseteq\quad \mathrm{PSh}(X; R) \;:=\; \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, D(R))$$

the full subcategory spanned by the sheaves.

The sheaf condition roughly speaking says that we can define elements of $\mathcal{F}(U)$ by defining them locally on U and checking that they glue appropriately. While the usual sheaf condition for sheaves in a 1-category would just be an equalizer diagram, the fact that in the ∞-categorical setting our ‘equalities’ have been replaced by data means that there are higher gluing conditions for this data, which is why we need the entire cosimplicial diagram. For nice enough spaces, the sheaf condition is equivalent to the following three conditions:

- $\mathcal{F}(\emptyset) = *$;

- $\mathcal{F}(U \cup V) \simeq \mathcal{F}(U) \times \mathcal{F}(U \cap V)\mathcal{F}(V)$;
- If U is a union of a sequence $U_0 \subseteq U_1 \subseteq U_2 \subseteq \dots$, then $\mathcal{F}(U) \xrightarrow{\sim} \lim_n \mathcal{F}(U_n)$.

Warning 3.2. The ∞-category $\mathrm{Shv}(X; R)$ is generally different from the derived ∞-category $\mathrm{Shv}(X; \mathrm{Mod}_R)$ of sheaves on X with values in the 1-category of R-modules: there is a fully faithful functor $D(\mathrm{Shv}(X; \mathrm{Mod}_R)) \hookrightarrow \mathrm{Shv}(X; R)$, but the essential image consists only of the so-called 'hypersheaves', i.e. those for which isomorphisms can be checked on stalks. See Appendix A for some more details.

If X is paracompact and has finite covering dimension (for example, if M is a manifold) then the two agree.

Any presheaf can universally be turned into a presheaf via a process called 'sheafification':

Proposition 3.3 (Lurie)**.** *The inclusion* $\mathrm{Shv}(X; R) \hookrightarrow \mathrm{PSh}(X; R)$ *admits a left adjoint*

$$L\colon \mathrm{PSh}(X; R) \to \mathrm{Shv}(X; R).$$

Remark 3.4. The ∞-category $D(R)$ is symmetric monoidal, where the tensor product $- \otimes^L -$ is given by the *derived tensor product* of chain complexes (computed by taking the tensor product of projective resolutions of the chain complexes). The ∞-category $\mathrm{PSh}(X; R)$ inherits a symmetric monoidal structure, defined via $(\mathcal{F} \otimes^{\mathrm{pre}} \mathcal{G})(U) := \mathcal{F}(U) \otimes^L \mathcal{G}(U)$. The subcategory of sheaves then inherits a symmetric monoidal structure by sheafifying:

$$\mathcal{F} \otimes \mathcal{G} \qquad := \qquad L(\mathcal{F} \otimes^{\mathrm{pre}} \mathcal{G}).$$

Definition 3.5. Let $f\colon X \to Y$ be a continuous map of topological spaces. We define the *pushforward functor* f_* as

$$f_*\colon \mathrm{Shv}(X; R) \to \mathrm{Shv}(Y; R), \qquad \mathcal{F} \mapsto f_*\mathcal{F}, \qquad (f_*\mathcal{F})(U) := \mathcal{F}(f^{-1}(U)).$$

Proposition 3.6. *The functor f_* admits a left exact left adjoint* $f^*\colon \mathrm{Shv}(Y; R) \to \mathrm{Shv}(X; R)$.

Proof. The left adjoint is given by the composite

$$\mathrm{Shv}(Y; R) \hookrightarrow \mathrm{PSh}(Y; R) \xrightarrow{f^{-1}} \mathrm{PSh}(X; R) \xrightarrow{L} \mathrm{Shv}(X; R),$$

where f^{-1} is given by left Kan extension along $\mathrm{Open}(Y) \to \mathrm{Open}(X), U \mapsto f^{-1}(U)$. □

The functor f^* is called the *pullback functor.*

Proposition 3.7. *Let $j\colon U \hookrightarrow X$ be an open embedding. Then the functor* $j^*\colon \mathrm{Shv}(X; R) \to \mathrm{Shv}(U; R)$ *admits a further left adjoint*

$$j_!\colon \mathrm{Shv}(U; R) \to \mathrm{Shv}(X; R).$$

Proof. This is given by the composite

$$\mathrm{Shv}(U; R) \hookrightarrow \mathrm{PSh}(U; R) \xrightarrow{j_!} \mathrm{PSh}(X; R) \xrightarrow{L} \mathrm{Shv}(X; R),$$

where the middle functor is left Kan extension along the inclusion $\mathrm{Open}(U)^{\mathrm{op}} \hookrightarrow \mathrm{Open}(X)^{\mathrm{op}}$. (It is determined by the fact that it sends the free R-module sheaf on $V \subseteq U$ to the free R-module sheaf on V, now regarded as an open subset of X.) □

If a space X is the union of a closed subspace and its complement, we may recover a sheaf on X from its restrictions to both of these subspaces:

Theorem 3.8 (Localization theorem)**.** *Let $i\colon Z \hookrightarrow X$ be a closed subspace, and let $j\colon U \subseteq X$ be its open complement. Then the functors $i^*\colon \operatorname{Shv}(X;R) \to \operatorname{Shv}(Z;R)$ and $j^*\colon \operatorname{Shv}(X;R) \to \operatorname{Shv}(U;R)$ are jointly conservative. In fact, for any $\mathcal{F} \in \operatorname{Shv}(X;R)$ there is an exact sequence*

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

Proof idea. The proof proceeds as follows:

- Show that the pushforward functor $i_*\colon \operatorname{Shv}(Z;R) \to \operatorname{Shv}(X;R)$ commutes with all colimits.
- Reduce the claim to a statement about sheaves of animae.
- Use that $\operatorname{Shv}(X)$ is generated under colimits by the sheaves $y(V)$ represented by open subsets $V \subseteq X$, so that it suffices to prove the claim for $\mathcal{F} = y(V)$.
- Compute that $j_! j^* y(V) \cong j_! y(U \cap V) = y(U \cap V)$ (by definition of j^* and $j_!$) while $i_* i^* y(V) \cong y(U \cup V)$ (as $W \cap Z \subseteq V \cap Z$ if and only if $W \subseteq U \cup V$)). The statement then essentially boils down to the claim that the square
$$\begin{tikzcd} y(U \cap V) \arrow[r] \arrow[d] & y(V) \arrow[d] \\ y(U) \arrow[r] & y(U \cap V) \end{tikzcd}$$
is a pushout square, which is just a special case of the sheaf condition. □

4 Proper pushforwards

We have previously seen various functors on sheaves:

- The tensor product $\mathcal{F} \otimes \mathcal{G}$ and its right adjoint $\underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})$
- The pullback functor f^* and its right adjoint f_*.

We now come to the last two operations: the *proper pushforward* $f_!$ and its right adjoint $f^!$.

Definition 4.1. Let $f\colon X \to Y$ be a continuous map between locally compact Hausdorff spaces. We define the functor $f_!\colon \operatorname{Shv}(X;R) \to \operatorname{Shv}(Y;R)$ by leting the sheaf $f_!\mathcal{F}$ take value on $V \in \operatorname{Open}(Y)$ given by the formula

$$(f_!\mathcal{F})(V) \quad := \quad \operatorname{colim}_{\substack{K \subseteq f^{-1}(V) \\ K \to V \text{ proper}}} \operatorname{fib}\Big(\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V) \setminus K)\Big),$$

where the colimit runs over all subspaces K of $f^{-1}(V)$ such that the composite $K \hookrightarrow f^{-1}(V) \to V$ is a proper map.

The condition on K is more or less saying that K is a family of compact spaces over V. Note that an element of $\mathrm{fib}(\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V) \setminus K))$ is an element which vanishes on the complement of K, i.e. it is a section with *support in* K. Taking the colimit over all K is then essentially saying that we allow all elements that have a support that is proper over V.

Specializing to the case $Y = *$ gives rise to the functor

$$\Gamma_c \colon \mathrm{Shv}(X; R) \to \mathrm{Shv}(*; R) \simeq D(R)$$

mentioned in the introduction:

Definition 4.2. Given a sheaf $\mathcal{F} \in \mathrm{Shv}(X; R)$, we define the *compactly supported sections of* $\mathcal{F}$ *over* U as

$$\Gamma_c(X; \mathcal{F}) \quad := \quad \mathrm{colim}_{K \subseteq X} \, \mathrm{fib}(\mathcal{F}(X) \to \mathcal{F}(X \setminus K)) \quad \in \quad D(R),$$

where the colimit ranges over all compact subsets $K \subseteq X$.

Notice that for $U \subseteq V$, any compact subset of U is also a compact subset of V, giving a map $\Gamma_c(U; \mathcal{F}) \to \Gamma_c(V; \mathcal{F})$. This defines a functor

$$\mathbb{D}(\mathcal{F}) \colon \mathrm{Open}(X) \to D(R).$$

One can show that this is a *cosheaf*, meaning that it is a sheaf when regarded as a functor $\mathrm{Open}(X)^{\mathrm{op}} \to D(R)^{\mathrm{op}}$.

Theorem 4.3 (Verdier duality, Lurie)**.** *Let X be a locally compact Hausdorff space. Then the functor $\mathbb{D} \colon \mathrm{Shv}(X; R) \to \mathrm{CoShv}(X; R)$ is an equivalence of ∞-categories.*

Remark 4.4. Since cosheaves are simply sheaves with values in $D(R)^{\mathrm{op}}$, we have pullback and pushforward functors

$$f^* \colon \mathrm{CoShv}(Y; R) \rightleftarrows \mathrm{CoShv}(X; R) : f_*.$$

One can show that $f_!$ is explicitly given by the composite

$$\mathrm{Shv}(X; R) \xrightarrow[\sim]{\mathbb{D}} \mathrm{CoShv}(X; R) \xrightarrow{f_*} \mathrm{CoShv}(Y; R) \xrightarrow[\sim]{\mathbb{D}^{-1}} \mathrm{Shv}(Y; R).$$

Indeed, one can observe that $f_!\mathcal{F}$ satisfies the formula

$$\Gamma_c(V; f_!\mathcal{F}) \cong \Gamma_c(f^{-1}(V); \mathcal{F})$$

for all open $V \subseteq X$, which is the defining property of the above composite.

Lemma 4.5. *Assume that $f \colon X \to Y$ is a proper map. Then $f_! = f_*$.*

Proof. In this case, the entire subset $K = f^{-1}(V)$ is proper over V, and the colimit simply reduces to

$$(f_!\mathcal{F})(V) \simeq \mathrm{fib}\Big(\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(\emptyset)\Big) \simeq \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V). \qquad \square$$

Lemma 4.6. *Assume that $j \colon U \hookrightarrow X$ is an open embedding. Then $j_!$ is the left adjoint functor from Theorem 3.7.*

Proof. By passing to right adjoints, it will suffice to show that $j^! \cong j^*$. But by definition of $j^!$, we have $\Gamma_c(V; j^!\mathcal{F}) \cong \Gamma_c(V; \mathcal{F})$ for any open $V \subseteq U$, and this clearly agrees with $\Gamma_c(V; j^*\mathcal{F})$, since $j^*\mathcal{F}(V \setminus K) = \mathcal{F}(V \setminus K)$ for all compact $K \subseteq V$. $\square$

5 The proper base change theorem

Now that we have the six operations ($f^* \dashv f_*, \otimes \dashv \underline{\mathrm{Hom}}, f_! \dashv f^!$) it is time to talk about the relations they satisfy. The relation between f^* and $\otimes$ is easy to describe:

Proposition 5.1 (Symmetric monoidality)**.** *For any $f\colon X \to Y$, the functor $f^*\colon \mathrm{Shv}(Y;R) \to \mathrm{Shv}(X;R)$ is symmetric monoidal:*

$$f^*(\mathcal{F} \otimes \mathcal{G}) \quad \cong \quad f^*\mathcal{F} \otimes f^*\mathcal{G}.$$

Proof. Immediate from the formulas for $\otimes$ and f^*. □

The interaction between f^* and $\otimes$ with $f_!$ is more difficult:

Proposition 5.2 (Proper base change)**.** *Consider a pullback square*

$$\begin{tikzcd} X' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd}$$

Then there is a natural isomorphism

$$g^* f_! \xrightarrow{\sim} f'_! h^*$$

of functors $\mathrm{Shv}(X;R) \to \mathrm{Shv}(Y;R)$.

Proposition 5.3 (Proper projection formula)**.** *For a map $f\colon X \to Y$ between locally compact Hausdorff spaces, and sheaves $\mathcal{F} \in \mathrm{Shv}(X;R)$ and $\mathcal{G} \in \mathrm{Shv}(Y;R)$, there is a natural isomorphism*

$$f_!(\mathcal{F} \otimes f^*\mathcal{G}) \quad \cong \quad f_!(\mathcal{F}) \otimes \mathcal{G}.$$

Let us start by reducing Proper Base Change to the Proper Projection Formula. We will use the following as a black box:

Proposition 5.4 (See e.g. [Aok23, Corollary 1.10])**.** *For a pullback square as above, we have an equivalence of ∞-categories*

$$\mathrm{Shv}(X';R) \quad \cong \quad \mathrm{Shv}(X;R) \otimes_{\mathrm{Shv}(Y;R)} \mathrm{Shv}(Y';R).$$

Here the right-hand side denotes the relative tensor product in the ∞-category of symmetric monoidal presentable ∞-categories. We unfortunately don't have time to explain what it means, but in essence we wish to think of $\mathrm{Shv}(X;R)$ as analogous to a 'commutative ring' with 'multiplication' given by the tensor product; the RHS then resembles the pushout in commutative rings.

Proof Proper Base Change, assuming Proper Projection Formula. We may factor f into a composite $X \hookrightarrow \overline{X} \xrightarrow{p} Y$, where the inclusion is open and p is compact. In this way, we may assume that either f is an open embedding or f is a proper map. The case for f an open embedding is clear: both sides preserve colimits and on the generators $j(V)$ for $V \subseteq X$ open both sides result in $j(g^{-1}(V))$. It thus remains to prove the case for f proper.

In this case, we use the Proper Projection Formula to conclude that the right adjoint $f_*\colon \mathrm{Shv}(X;R) \to \mathrm{Shv}(Y;R)$ is $\mathrm{Shv}(Y;R)$-linear. In particular, tensoring it with $\mathrm{Shv}(Y';R)$ over $\mathrm{Shv}(Y;R)$ gives another adjunction

$$\mathrm{Shv}(X;R) \rightleftarrows \mathrm{Shv}(X';R).$$

Unwinding definitions, this is saying that the commutative square

$$\begin{tikzcd}
\mathrm{Shv}(Y;R) \arrow[rr, "f^*"] \arrow[d, "g^*"'] & & \mathrm{Shv}(X;R) \arrow[d, "h^*"] \\
\mathrm{Shv}(Y';R) \arrow[r, "f'^*"] & \mathrm{Shv}(X';R) \cong & \mathrm{Shv}(X;R) \otimes_{\mathrm{Shv}(Y;R)} \mathrm{Shv}(Y';R)
\end{tikzcd}$$

is horizontally right adjointable, i.e. that the Beck-Chevalley map $g^* f_! \xrightarrow{\sim} f'_! h^*$ is an isomorphism. □

Proof Proper Projection Formula. Again, we factor f as an open embedding followed by a proper map to reduce to either two cases.

If f is an open embedding, both functors preserve colimits, so we may check the property on generators: let $\mathcal{F} = j(U)$ for $U \subseteq X$ and$\mathcal{G} = j(V)$ for $V \subseteq Y$. Then both sides agree with $j_!(U \cap f^{-1}(V))$.

If f is proper, we proceed via the following steps:

- The functor $f_! = f_*$ preserves colimits. To show this, one needs an alternative description of $\mathrm{Shv}(X;R)$ in terms of *K-sheaves*, which unfortunately we do not have time to get into right now. So we will just assume this as a black box.
- As a consequence, we may reduce to the case $\mathcal{G} = j(V)$, where $V \subseteq Y$ is open. In this case, we have $f^*\mathcal{G} = j'(V')$, where $V' := f^{-1}(V)$ and $j'\colon V' \hookrightarrow X$ is the inclusion.
- A general fact is that we have $\mathcal{F} \otimes j(V) \otimes j_! j^* \mathcal{F}$ for every sheaf $\mathcal{F}$: both sides commute with colimits in $\mathcal{F}$ and the claim is clear on generators.
- All in all, we have the following computation:

$$f_*(\mathcal{F} \otimes f^*\mathcal{G}) = f_*(\mathcal{F} x j'_!(V')) = f_* j'_! j'^* \mathbb{F} = j_! f'_* j'^* \mathcal{F} = j_! j^* f_* \mathcal{F} = f_* \mathcal{F} \otimes j_! V = f_* \mathcal{F} \times \mathcal{G}.$$

This finishes the proof. □

A Sheaves versus hypersheaves

The goal of this appendix is to give a proof sketch of the claim of Theorem 3.2. This was listed in the program description for this talk, but because of time reasons didn't make it into the talk. The argument is taken from [Sch23, Proposition 7.1].

Definition A.1. Given a sheaf $\mathcal{F} \in \mathrm{Shv}(X;R)$ and a point $x \in X$, the *stalk* $\mathcal{F}_x \in D(R)$ is the object $i_x^* \mathcal{F}$, i.e. the image of $\mathcal{F}$ under the pullback functor $i_x^*\colon \mathrm{Shv}(X;R) \to \mathrm{Shv}(\{x\};R) \cong D(R)$ induced by the inclusion $i_x\colon \{x\} \hookrightarrow X$.

Definition A.2. Let X be a topological space. We denote by $\mathrm{HypShv}(X;R)$ the localization of $\mathrm{Shv}(X;R)$ at those morphisms $\mathcal{F} \to \mathcal{G}$ that induce isomorphisms $\mathcal{F}_x \to \mathcal{G}_x$ at all stalks.

Proposition A.3. *The composite functor* $\mathrm{Ch}(\mathrm{Shv}(X;\mathrm{Mod}_R)) \to \mathrm{Shv}(X;R) \to \mathrm{HypShv}(X;R)$ *sends every quasi-isomorphism to an isomorphism, and the induced functor*

$$D(\mathrm{Shv}(X;\mathrm{Mod}_R)) \to \mathrm{HypShv}(X;R)$$

is an equivalence of ∞-categories, compatible with the t-structures on both sides.

Proof sketch. A morphism of hypersheaves is an isomorphism if and only if it induces isomorphisms on all stalks. So, given a quasi-isomorphism $C_\bullet \to D_\bullet$ of chain complexes in $\mathrm{Shv}(X;\mathrm{Mod}_R)$, we need to show that the induced map of hypercomplete sheaves is an isomorphism on stalks. Now, one needs to use the fact that the hypercompletion functor does not change stalks, essentially by definition. So the claim follows from the observation that the map $C_\bullet \to D_\bullet$ induces quasi-isomorphisms on stalks, since the stalk functor i_x^* is exact hence preserves quasi-isomorphisms.

To show that the induced functor $D(\mathrm{Shv}(X;\mathrm{Mod}_R)) \to \mathrm{HypShv}(X;R)$ is an equivalence, we observe that both the source and the target are generated by the sheaves $j_!\underline{R}$ freely generated by the opens $j\colon U \hookrightarrow X$. In particular, if we can show this functor is fully faithful, it will automatically be essentially surjective as well, as the generators $j_!\underline{R}$ lie in the image. For fully faithfulness, it will now suffice to show that for every $\mathcal{F} \in D(\mathrm{Shv}(X;\mathrm{Mod}_R))$ the map

$$\mathrm{Hom}_{D(\mathrm{Shv}(X;\mathrm{Mod}_R))}(j_!\underline{R},\mathcal{F}) \to \mathrm{Hom}_{\mathrm{HypShv}(X;R)}(j_!\underline{R},\widetilde{\mathcal{F}})$$

is an isomorphism, where $\widetilde{\mathcal{F}}$ is the associated hypersheaf. By adjunction, this agrees with the map

$$\mathcal{F}(U) \to \widetilde{\mathcal{F}}(U).$$

But up to quasi-isomorphism we may replace $\mathcal{F}$ by a complex which is *K-injective*, meaning that $\mathrm{Hom}_{\mathrm{Ch}(\mathrm{Shv}(X;\mathrm{Mod}_R))}(-,\mathcal{F})$ inverts all quasi-isomorphisms. And for such a K-injective complex, the hypersheafification is the identity, in the sense that $\widetilde{\mathcal{F}} = \mathcal{F}$. Roughly speaking, the argument is that any hypercover induces a resolution of $j_!\underline{R}$ for all $j\colon U \hookrightarrow X$, and then mapping out of that resolution into $\mathcal{F}$ turns it into an isomorphism by the K-injectivity assumption. This gives the claim. □

References

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