Linearization for real Lie groups

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1 Introduction

In Uli's talk, we saw the statement that if $f\colon X \to Y$ is a smooth submersion of real smooth manifolds, or more generally a representable submersion in $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$, then the map $f\colon \underline{X} \to \underline{Y}$ is Sp-smooth, giving rise to a dualizing object $f^!\,\mathbb{S} \in \mathrm{Shv}_{\text{ét}}(\underline{X};\mathrm{Sp}) \cong \mathrm{Shv}(X;\mathrm{Sp})$. The goal of this talk is to prove *Atiyah duality*: an explicit identification of this dualizing sheaf in terms of the relative tangent bundle $T_f = T_{X/Y}$ of f:

$$f^!\,\mathbb{S} \quad = \quad \mathbb{S}^{T_f}\,.$$

Here the right-hand side denotes the associated sphere bundle, which we'll also define.

2 Smoothness

Since Uli did not have the time to go into this, let me start by briefly proving that representable submersions in $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$ are actually Sp-smooth.

Theorem 2.1. *Let $f\colon X \to Y$ be a representable submersion in* $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$. *Then $f\colon \underline{X} \to \underline{Y}$ is* Sp-*smooth.*

Proof. Every atlas $g\colon M \twoheadrightarrow Y$ gives rise to a surjective map in the condensed Grothendieck topology on CondAn, and since smoothness can be checked after pullback to every light condensed animae we may reduce to the case of $Y = M$ a manifold. So also $X = N$ is a manifold. Since open embeddings are Sp-étale, they are in particular smooth. So we may check everything locally in both N and M. This allows us to reduce to the case of a projection $U \times \mathbb{R}^n \to U$. We may further reduce to $U \times \mathbb{R} \to U$, and then to just $f\colon\ \mathbb{R} \to *$.

Let's check all the three properties:

- First, need to check $f^!\colon \mathrm{Sp} \to \mathrm{Shv}(\mathbb{R};\mathrm{Sp})$ preserves colimits. The topology on $\mathbb{R}$ is generated by the open intervals $U = (a,b)$, so it suffices to show that $\mathrm{Hom}_{\mathrm{Shv}(\mathbb{R};\mathrm{Sp})}(h_U, f^!(-)) = \mathrm{Hom}_{\mathrm{Sp}}(f_!h_U, -)$ preserves colimits for every U, i.e. that $f_!h_U$ is compact. Using the fiber sequence $j_!j^* \to \mathrm{id} \to i_*i^*$ coming from the inclusion $U = (a,b) \hookrightarrow [a,b] \hookleftarrow \{a,b\}$, we may identify $f_!h_U$ with the relative cohomology $[a,b]$ with respect to its boundary, which is $\mathbb{S}[-1]$. This is invertible, hence compact.
- By semi-rigidity of $\mathrm{Shv}(\mathbb{R};\mathrm{Sp})$, it follows formally that the formation of $f^!$ commutes with base change.

- Now we check that $f^! \mathbb{1}$ is invertible. Let us first show that $f^! \mathbb{1}$ is a constant sheaf. For this, we use the addition map $\mu \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Note that we have two pullback squares as follows, with left vertical map either the projection or the addition map:

  $$\begin{tikzcd} \mathbb{R} \times \mathbb{R} \arrow[r, "\mathrm{pr}_2"] \arrow[d, shift right, "\mathrm{pr}_1"'] \arrow[d, shift left, "\mu"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R} \arrow[d, "f"] \\ \mathbb{R} \arrow[r, "f"] & *. \end{tikzcd}$$

  (The two squares are isomorphic under the shear map $\mathbb{R} \times \mathbb{R} \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}$, $(x, y) \mapsto (x + y, y)$.) It follows from base change that
  $$\mu^* f^! \mathbb{S} = \mathrm{pr}_2^! f^* \mathbb{S} = \mathrm{pr}_1^* f^! \mathbb{S} .$$
  But then if we pull back along the inclusion $\{0\} \times \mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}$, it follows that $f^! \mathbb{S}$ is pulled back along the composite $\mathbb{R} \to \{0\} \hookrightarrow \mathbb{R}$, and in particular it is a constant sheaf.

  So, in order to show that $f^! \mathbb{S}$ is invertible, it suffices to show that its cohomology $f_* f^! \mathbb{S}$ is invertible. And indeed, we have:
  $$f_* f^! \mathbb{S} \cong \mathrm{map}(\mathbb{S}, f_* f^! \mathbb{S}) = \mathrm{map}(f_! \mathbb{S}, \mathbb{S}) = \mathrm{map}(\mathbb{S}[-1], \mathbb{S}) = \mathbb{S}[1],$$
  where we used the calculation $f_! \mathbb{S} = \mathbb{S}[-1]$ from before.

This finishes the proof. $\square$

3 Locally constant sheaves

We need a brief digression into locally constant sheaves.

Recall that the constant sheaf functor $\mathrm{An} \hookrightarrow \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$ admits a left adjoint $|-| \colon \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}}) \to \mathrm{An}$, called the *shape* or *underlying anima*. It suffices to construct it on generators, i.e. on the euclidean spaces $\mathbb{R}^n$. But since constant sheaves are homotopy invariant, we just have $|\mathbb{R}^n| = *$.

Given $X \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$, we may regard both X and $|X|$ as a condensed anima, and we get a map $\underline{X} \to |X|$.

Proposition 3.1. *The pullback functor*
$$\mathrm{An}_{/|X|} = \mathrm{Shv}_{\text{ét}}(|X|) \to \mathrm{Shv}_{\text{ét}}(\underline{X})$$
is fully faithful. Its image consists of the locally constant sheaves, i.e. those sheaves $\mathcal{F}$ for which there is a cover $\{f_i \colon X_i \to X\}$ in $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$ for which each $f_i^* \mathcal{F}$ is constant.

Proof sketch. Both sides satisfy descent in X, so for fully faithfulness we may again reduce to euclidean spaces. Here the claim is again the homotopy invariance of sheaves.

If a sheaf is in the essential image, then in particular it becomes constant after pullback to every euclidean space, and in particular it is locally constant. For the converse, assume $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(X)$ is locally constant. Then it is locally on X contained in the subcategory $\mathrm{Shv}_{\text{ét}}(|X|)$. But since $X \mapsto |X|$ commutes with colimits, being contained in this subcategory is a local property. It follows that also $\mathcal{F} \in \mathrm{Shv}_{\text{ét}}(|X|)$, as desired. $\square$

Corollary 3.2. *Let G be a Lie group (over $\mathbb{R}$), with underlying group anima $|G|$. Then every étale sheaf on BG is locally constant, and*

$$\mathrm{Shv}_{\text{ét}}(\underline{BG}) \simeq \mathrm{An}_{/B|G|},$$

i.e. an étale sheaf on BG is an anima with an action by $|G|$.

Proof. Immediate, since $* \to BG$ is a cover and every sheaf on $*$ is constant. □

Remark 3.3. We in particular get that $\mathrm{Shv}_{\text{ét}}(B\,\mathbb{R}) \simeq \mathrm{An}$. While this seems like it would not give us interesting information, it turns out that we do actually get interesting behavior from this, in the form a J-homomorphism. This is one potential topic of an "applications" talk, if anyone is interested in giving that.

4 Sphere bundles

Recall that the description for the dualizing sheaf we are after is $f^!\,\mathbb{S} = \mathbb{S}^{T_f}$, the sphere bundle asssociated to the relative tangent bundle T_f. We will now properly define these sphere bundles.

Definition 4.1. Consider a stack $X \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$.

- A *vector bundle* on X is a map $f\colon V \to X$ obtained as a pullback of the universal vector bundle $\mathbb{R}^n/\mathrm{GL}_n(\mathbb{R}) \to B\,\mathrm{GL}_n(\mathbb{R})$ along some map of stacks $X \to B\,\mathrm{GL}_n(\mathbb{R})$:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{R}^n/\mathrm{GL}_n(\mathbb{R}) \arrow[d] \\ X \arrow[r] & B\,\mathrm{GL}_n(\mathbb{R}). \end{tikzcd}$$

- Let $f\colon V \to X$ be a vector bundle on X, and let $e\colon V \to X$ be its zero section (induced by $\{0\} \hookrightarrow \mathbb{R}^n$). We define the *sphere bundle* associated to V as

$$\mathbb{S}^V \quad := \quad e^* f^!\,\mathbb{S} \quad \in \quad \mathrm{Shv}_{\text{ét}}(\underline{X};\mathrm{Sp}).$$

Observation 4.2. If $f\colon V \to X$ is a vector bundle, the pullback to any manifold is a vector bundle in the ordinary sense. In particular, f is a representable submersion, hence cohomologically smooth. It follows that $f^!\,\mathbb{S}$ invertible, and thus so is $\mathbb{S}^V$.

Recall that every invertible sheaf is locally constant, so we see that $\mathbb{S}^V$ lies in the image of the functor

$$\mathrm{An}_{/|X|} \otimes \mathrm{Sp} = \mathrm{Shv}_{\text{ét}}(|X|;\mathrm{Sp}) \hookrightarrow \mathrm{Shv}_{\text{ét}}(\underline{X};\mathrm{Sp}).$$

We will now construct an explicit lift of $\mathbb{S}^V$ to $\mathrm{An}_{|X|}$. Let $j\colon V \setminus 0 \to V$ be the open complement to e, i.e. the pullback of $(\mathbb{R}^n \setminus 0)/\mathrm{GL}_n(\mathbb{R}) \to B\,\mathrm{GL}_n(\mathbb{R})$ along $X \to B\,\mathrm{GL}_n(\mathbb{R})$. This induces a morphism of animae $|V \setminus 0| \to |V|$, which in fact becomes a morphism in $\mathrm{An}_{/|X|}$. We let $|V|/|V \setminus 0|$ denotes its cofiber in this slice. Note that fiberwise this looks like $|\mathbb{R}^n|/|\mathbb{R}^n \setminus 0| = S^n$, so we're forming some kind of fiberwise one-point compactification of V. What we will show next is that we have

$$\mathbb{S}^V \quad \cong \quad \Sigma^\infty(|V|/|V \setminus 0|).$$

For this, we need to recall the shape functor:

Lemma 4.3. *Let $f\colon Y \to X$ be a fiber bundle in* $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$*, meaning that it is locally in X of the form $U \times F \to U$ for $U \in \mathrm{Man}_{\mathbb{R}}$ and $F \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$. Then:*

(1) *The pullback functor $f^*\colon \mathrm{Shv}(X; \mathrm{An}) \to \mathrm{Shv}(Y; \mathrm{An})$ admits a left adjoint $f_\natural\colon \mathrm{Shv}(Y; \mathrm{An}) \to \mathrm{Shv}(X; \mathrm{An})$, and these left adjoints satisfy base change.*

(2) *The left adjoint $f_\natural$ preserves locally constant sheaves, hence restricts to a left adjoint*

$$f_\natural\colon \mathrm{An}_{/|Y|} \to \mathrm{An}_{/|X|}$$

to the pullback map along $|f|\colon |Y| \to |X|$. (So it agrees with post-composition.)

Proof. For (1), we may construct the left adjoint locally on $\mathrm{Shv}(Y; \mathrm{An})$, and then extend by colimits. It thus suffices to construct this left adjoint locally in X, letting us reduce to the case $U \times F \to U$. There we have $\mathrm{Shv}(U \times F) = \mathrm{Shv}(U) \times \mathrm{Shv}(F)$, so we may reduce to $U = *$, i.e. for the map $F \to *$. Now, we may work locally on F, so reduce to the case $F = \mathbb{R}^n$. By induction, we then reduce to just $\mathbb{R} \to *$. Here we may argue as before: the left adjoint to $\mathrm{An} \to \mathrm{Shv}(\mathbb{R})$ is given on open intervals (a, b) by sending it to the point.

For (2), we may again work locally on X, reducing to $U \times F \to U$. Again using that $\mathrm{Shv}(U \times F) = \mathrm{Shv}(U) \times \mathrm{Shv}(F)$, we get that the left adjoint is given by tensoring with $\mathrm{An} \to \mathrm{Shv}(F)$, which clearly preserves locally constant sheaves (as it lands in constant sheaves). □

Corollary 4.4. *We have $\mathbb{S}^V \cong f_\natural e_* \mathbb{S}$.*

Proof. Since f is cohomologically smooth, we have $f^!(-) = f^*(-) \otimes f^! \mathbb{S}$, so by passing to left adjoints we get $f_\natural(-) = f_!(- \otimes f^! \mathbb{S})$. It then follows that

$$f_\natural e_* \mathbb{S} = f_!(f^! \mathbb{S} \otimes e_* \mathbb{S}) = f_! e_*(e^* f^! \mathbb{S})) = f_! e_!(e^* f^! \mathbb{S}) = e^* f^! \mathbb{S} = \mathbb{S}^V .$$ □

We now conclude:

Corollary 4.5. *There is an isomorphism $\mathbb{S}^V \cong \Sigma^\infty(|V|/|V \setminus 0|)$.*

Proof. Since j is an open embedding with closed complement e, we get a localization sequence

$$j_\natural \mathbb{S} \to \mathbb{S} \to e_* \mathbb{S} .$$

Applying $f_\natural$ to both sides, we get a cofiber sequence

$$(f \circ j)_\natural \mathbb{S} = f_\natural j_\natural \mathbb{S} \to f_\natural \mathbb{S} \to f_\natural e_* \mathbb{S} = \mathbb{S}^V .$$

But since $\mathbb{S} = \Sigma^\infty(S^0)$ and since $f_\natural$ and $(f \circ j)_\natural$ preserve locally constant sheaves, this cofiber sequence takes the form

$$\Sigma^\infty_+(|V \setminus 0|) \to \Sigma^\infty_+(|V|) \to \mathbb{S}^V .$$ □

5 Atiyah duality

We are now ready to prove Atiyah duality. Let us start with the following general observation:

Lemma 5.1. *Assume $f\colon X \to Y$ is some smooth map (in some six-functor formalism.) Then*

$$f^!\mathbb{1} \cong \delta^*\pi^!\mathbb{1},$$

where $\delta\colon X \to X\times_Y X$ is the diagonal and $\pi\colon X\times_Y X \to X$ is the first projection.

Proof. We have $f^!\mathbb{1} = \delta^*\pi_2^* f^!\mathbb{1} \cong \delta^*\pi^! f^*\mathbb{1}$, where we use that $f^!$ satisfies base change for the pullback square

$$\begin{tikzcd} X\times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

□

Recall: if $f\colon M \to N$ is a submersion, we have a relative tangent bundle $T_{M/N} := \ker(T_M \to f^*T_N) \in \mathrm{Vect}(M)$. This is compatible with base change and satisfies descent in the base N, so uniquely extends to representable submersions of stacks.

Theorem 5.2. *Let $f\colon X \to Y$ be a representable submersion in* $\mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$*, and let $T_f \to X$ be its relative tangent bundle. Then there is an isomorphism*

$$f^!\mathbb{S} \cong \mathbb{S}^{T_f}.$$

Proof. We make use of an important construction called the *deformation to the tangent bundle,* which will be discussed in detail in the next talk. We will use the following properties about this construction:

- There is a representable submersion $\Pi\colon D(X/Y) \to X\times\mathbb{R}$ equipped with a section Σ;
- Over $1 \in \mathbb{R}$, the fiber of Π is the map $\pi\colon X\times_Y X \to X$ with its diagonal section δ;
- Over $0 \in \mathbb{R}$, the fiber of Π is the map $T_f \to X$ with its zero section.

Now, since Π is Sp-smooth, the object $\Pi^!\mathbb{S} \in \mathrm{Shv}_{\text{ét}}(D(X/Y);\mathrm{Sp})$ is invertible, hence so is $\Sigma^*\Pi^!\mathbb{S} \in \mathrm{Shv}_{\text{ét}}(X\times\mathbb{R};\mathrm{Sp})$. In particular, this object is locally constant, so lies in the image of the functor

$$\mathrm{An}_{/|X|}\otimes\mathrm{Sp} \cong \mathrm{An}_{/|X|\times|\mathbb{R}|}\otimes\mathrm{Sp} \hookrightarrow \mathrm{Shv}_{\text{ét}}(X\times\mathbb{R};\mathrm{Sp}).$$

But then this provides an isomorphism

$$f^!\mathbb{S} = \delta^*\pi^!\mathbb{S} \quad\cong\quad 1^*\Sigma^*\Pi^!\mathbb{S} \quad\cong\quad 0^*\Sigma^*\Pi^!\mathbb{S} \quad\cong\quad e^*\pi^!\mathbb{S} = \mathbb{S}^V.$$

□

Next we discuss the classifying stacks of real Lie groups. Let $X \in \mathrm{Shv}(\mathrm{Man}_{\mathbb{R}})$ and let $G \to X$ be a group object in representable submersions over X. Write $f\colon BG \to X$ for the relative classifying stack, and let $e\colon X \to BG$ be the quotient map. Note: since e is a section of f which covers BG, and is cohomologically smooth, it follows that also f is cohomologically smooth.

Lemma 5.3. *Let $\mathfrak{g} \to X$ denote the Lie algebra of G, viewed as a vector bundle on X. Then there is a natural isomorphism*

$$e^* f^! \mathbb{S} \quad \cong \quad \mathbb{S}^{-\mathfrak{g}}.$$

Proof. As we will see next week, the construction $Y \mapsto D(Y/X)$ sends fiber products over X to fiber products over $X \times \mathbb{R}$. It follows that $D(G/X)$ is a Lie group over $X \times \mathbb{R}$. The fiber over $1 \in \mathbb{R}$ just recovers the original Lie group G. The fiber over $0 \in \mathbb{R}$ is by definition the Lie algebra $\mathfrak{g}$. (Note that a priori this carries some strange Lie group structure, but it follows from the Eckmann-Hilton argument that this structure is actually just given by fiberwise addition in the tangent spaces.)

Now consider the relative classifying stack $BD(G/X)$ over $X \times \mathbb{R}$. The map $BD(G/X) \to X \times \mathbb{R}$ is cohomologically smooth, and the same argument as in the previous proof allows us to deduce that

$$e^* f^! \mathbb{S} \cong (e')^* f'^! \mathbb{S},$$

where now $f' : B\mathfrak{g} \to X$ and $e' : X \to B\mathfrak{g}$ are defined analogously to f and e but now for the additive Lie group $\mathfrak{g}$ in place of G. To finish the proof, it thus remains to show that for any vector bundle V over X with maps $X \xrightarrow{e} BV \xrightarrow{f} X$ we have

$$e^* f^! \mathbb{S} \cong \mathbb{S}^{-V}.$$

Since we have $e^* f^! \mathbb{S} \otimes e^! \mathbb{S} \cong e^! f^! \mathbb{S} \cong \mathbb{S}$ by cohomological smoothness of e, it thus remains to show that $e^! \mathbb{S} = \mathbb{S}^V$, which is an instance of Lemma 5.1. □

We now deduce the following:

Theorem 5.4. *Let G be a real Lie group. Then for $f : BG \to *$ we get*

$$f^! \mathbb{S} \cong \mathbb{S}^{-\mathrm{ad}_G} \quad \in \quad \mathrm{Shv}_{\text{ét}}(BG; \mathrm{Sp}),$$

where ad_G denotes the adjoint representation of G, viewed as a vector bundle on BG.

Proof. By considering the conjugation action of G on itself, we may regard G as a G-equivariant group object, i.e. a group object in submersions over $X := BG$. The associated relative Lie algebra over BG is precisely the definition of ad_G. The maps e and f from the previous lemma specialize in this case to the maps

$$BG \xrightarrow{\Delta} BG \times BG \xrightarrow{\pi} BG,$$

so it follows from the previous lemma that $\mathbb{S}^{-\mathrm{ad}_G} \cong \Delta^* \pi^! \mathbb{S}$. But by Lemma 5.1 this is $f^! \mathbb{S}$. □

Remark 5.5. We used an identification of $BG \times BG$ with the relative classifying stack of G over BG associated to the adjoint action of G on itself. This may seem opaque. We offer the following perspective. For any groupoid X, there is a canonical functor

$$\mathrm{Ad}_X : X \to \mathrm{Grp}$$

to the category of groups. It sends a point $x \in X$ to the group $\mathrm{Aut}(x)$, and a map $\gamma : x \to y$ to the isomorphism $\mathrm{Aut}(x) \to \mathrm{Aut}(y)$ given by conjugation by γ. The composition with the functor from groups to groupoids given by taking the classifying space has a different description: it sends $x \in X$ to the component of X in which x lives. The identification is gotten by the obvious inclusion functor $B\,\mathrm{Aut}(x) \to X$ based at x.

In particular, if X is connected this composed functor to groupoids is constant with value X. On the other hand if X is also pointed, hence identifies with BG for a group G, then Ad_X by construction encodes the adjoint action of G on itself. Taking Grothendieck constructions, this provides the desired canonical identification. It is functorial in G and therefore passes to topoi.

Example 5.6. Suppose that G is a real Lie group which is compact. Then the diagonal of $p\colon BG \to *$ is Sp-proper (since it is locally isomorphic to projection off G). Thus by Definition 7.12 we get a natural transformation

$$p_! \to p_*.$$

On the other hand, passing to left adjoints in $f^! \simeq f^* \otimes f^! \mathbb{S}$ and using Theorem 10.9 shows that

$$p_!(-) \simeq p_\natural(- \otimes \mathbb{S}^{\mathrm{ad}_G}).$$

Thus we obtain a natural transformation

$$p_\natural(- \otimes \mathbb{S}^{\mathrm{ad}_G}) \to p_*(-)$$

of functors $\mathrm{Shv}_{\text{ét}}(BG; \mathrm{Sp}) \to \mathrm{Sp}$, relating the left and right adjoints to the pullback. But recall from 5.8 that $\mathrm{Shv}_{\text{ét}}(BG; \mathrm{Sp}) = \mathrm{Shv}_{\text{ét}}(B|G|; \mathrm{Sp})$ identifies with the ∞-category of spectra with $|G|$-action. In these terms $p_\natural$ stands for the $|G|$-orbit functor and p_* for the $|G|$-fixed point functor. Thus we recover the twisted transfer maps of [**ACD89**].

Note that the comparison of the two a priori different maps can be made by seeing they both satisfy the conditions of [**NS18**]. For the map constructed above, we can argue for this as follows: because $e\colon * \to */G$ is proper we have that $p_! \to p_*$ is an isomorphism on any e_*X, but by duality for e (a proper representable submersion) this is the same as $e_\natural X[d]$, hence one sees that $p_! \to p_*$ is an iso on compact objects.