

Universality of span 2-categories and the construction of 6-functor formalisms

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Cohomology theories

Grothendieck's theory of **six operations** manages the functoriality of cohomology:

$$\{\text{Geometric objects}\} \rightarrow D(\mathbb{Z}), \text{ Sp}, \dots, \quad X \mapsto H^*(X).$$

Better: work with the *full* chain complex (up to quasi-isomorphism), not just its cohomology.

Such theories can be **categorified**: they arise from a functor

$$\{\text{Geometric objects}\} \rightarrow \text{Cat}_{\infty}^{\text{st}}, \quad X \mapsto D(X).$$

$D(X)$ is the *category of coefficients for cohomology on X* .

Examples:

- $D(X) = \text{Shv}(X, D(\mathbb{Z}))$ on locally compact Hausdorff spaces,
- $D(X) = \text{Shv}_{\text{ét}}(X, D(\mathbb{Z})_{\ell\text{-tor}})$ on schemes over a field of characteristic $\neq \ell$.

Definition

Let $\mathcal{C}$ be an ∞ -category with finite limits (cartesian monoidal). A **coefficient system** is a lax symmetric monoidal functor

$$D^*: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}, \quad X \mapsto D(X)$$

such that each $D(X)$ is closed symmetric monoidal, and each pullback f^* admits a right adjoint f_* .

What lax symmetric monoidal gives us:

- External tensor products $-\boxtimes -: D(X) \times D(Y) \rightarrow D(X \times Y)$,
- Each $D(X)$ inherits a symmetric monoidal structure via $\Delta^* \circ \boxtimes$,
- Each f^* is symmetric monoidal.

Cohomology: For $f: X \rightarrow *$ and $\mathcal{F} \in D(X)$,

$$H^*(X, \mathcal{F}) := \Gamma_X(\mathcal{F}) := f_*(\mathcal{F}) \in D(*).$$

A coefficient system encodes various functorialities cleanly. In particular, it gives a clean formulation of duality:

Theorem (Poincaré/Verdier duality)

Let $f: X \rightarrow Y$ be a *proper* map which is a “manifold bundle” (smooth of relative dimension d). Then there is a **dualizing sheaf** $\omega_f \in D(X)$, locally of the form $\mathbb{1}[d]$, and an adjunction

$$f_* \dashv f^* \otimes \omega_f.$$

In words: the cohomology functor f_* is also (up to a twist) a *left* adjoint to f^* .

For $Y = *$: $H^*(X, \omega_X) \simeq H^*(X)^\vee$.

Exceptional pushforward

Problem: Properness is not a local property on X . (Open submanifolds of compact manifolds are usually non-compact!)

To prove Verdier duality locally, we need **compactly supported cohomology**: replace f_* by the **exceptional pushforward**

$$f_! : D(X) \rightarrow D(Y).$$

For $Y = *$:

$$H_c(X, \mathcal{F}) := (X \rightarrow *)_!(\mathcal{F}) \in D(*).$$

Verdier duality (general form)

For any smooth morphism $f : X \rightarrow Y$:

$$f_! \dashv f^* \otimes \omega_f.$$

Conditions on $f_!$

The functors $f_!$ must interact nicely with f^* and $\otimes$. Two key conditions:

(1) Base change: For every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ f' \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

an equivalence $g^* f_! \simeq f'_! h^*$.

(2) Projection formula: For $\mathcal{F} \in D(X)$, $\mathcal{F}' \in D(Y)$:

$$f_!(\mathcal{F} \otimes f^* \mathcal{F}') \simeq f_!(\mathcal{F}) \otimes \mathcal{F}'.$$

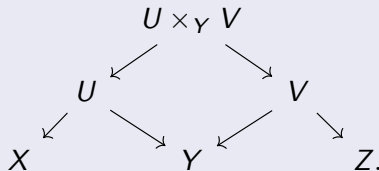
Moreover, these must hold **fully coherently** (compatible with all compositions). $\rightsquigarrow$ *span categories*.

The span ∞ -category

Construction

Let $E \subseteq \mathcal{C}$ be closed under composition and base change. The **span** ∞ -category $\text{Span}(\mathcal{C}, E)$:

- **Objects:** objects of $\mathcal{C}$,
- **Morphisms** $X \rightarrow Y$: spans $X \xleftarrow{f} U \xrightarrow{g} Y$ with $g \in E$,
- **Composition** via pullback:



The cartesian product on $\mathcal{C}$ equips $\text{Span}(\mathcal{C}, E)$ with a symmetric monoidal structure (not the cartesian one!).

3-functor and 6-functor formalisms

Definition

A **3-functor formalism** on $(\mathcal{C}, E)$ is a lax symmetric monoidal functor

$$D: \text{Span}(\mathcal{C}, E) \rightarrow \text{Cat}_\infty.$$

The inclusion $\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}(\mathcal{C}, E)$ (sending $f: X \rightarrow Y$ to the span $Y \xleftarrow{f} X \rightrightarrows X$) recovers the pullbacks f^* and pushforwards $g_!$.

Definition (Lurie, Gaitsgory–Rozenblyum, Liu–Zheng, Scholze, Mann)

A **6-functor formalism** is a 3-functor formalism in which f^* , $X \otimes -$, and $g_!$ all have right adjoints:

$$f^* \dashv f_*, \quad X \otimes - \dashv \underline{\text{Hom}}(X, -), \quad g_! \dashv g^!.$$

A 6-functor formalism encodes a huge amount of coherence. This makes them very powerful — and hard to construct!

Strategy: proper morphisms

Setting: Assume there is a class of *proper* morphisms $P \subseteq E \subseteq \mathcal{C}$ for which

$$p! \simeq p_*, \quad p \in P.$$

(E.g.: proper maps between schemes, or proper maps of manifolds.)

Advantage: Each p_* satisfies a universal property (it is right adjoint to p^*), so there are *canonical* comparison maps.

Goal: Given a coefficient system $D^*: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_\infty$ satisfying suitable conditions, extend it to a 3-functor formalism $D: \text{Span}(\mathcal{C}, P) \rightarrow \text{Cat}_\infty$.

The right framework is to work with $(\infty, 2)$ -categories.

Right P -adjointability

Definition

A lax sym. monoidal $D: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ is **right P -adjointable** if:

- 1 **Beck–Chevalley:** For every pullback square with $p \in P$,

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ p' \downarrow & \lrcorner & \downarrow p \\ Y' & \xrightarrow{h} & Y, \end{array}$$

the canonical map $g^* p'_* \xrightarrow{\sim} p_* h^*$ is an equivalence.

- 2 **Projection formula:** For $p \in P$: $p_*(A) \otimes B \xrightarrow{\sim} p_*(A \otimes p^* B)$.

Key point: In any $(\infty, 2)$ -category $\mathbb{E}$, adjunctions make sense. So right P -adjointability makes sense for $\mathcal{C}^{\text{op}} \rightarrow \mathbb{E}$.

The $(\infty, 2)$ -category $\text{Span}_2(\mathcal{C}, P)$

Construction

The $(\infty, 2)$ -category $\text{Span}_2(\mathcal{C}, P)$:

- **Objects:** objects of $\mathcal{C}$,
- **1-morphisms:** spans $X \xleftarrow{f} U \xrightarrow{p} Y$, with $p \in P$,
- **2-morphisms:** morphisms of spans (upward and right maps in P):

$$\begin{array}{ccccc} & & U & & \\ & f \swarrow & \uparrow & \searrow \in P & \\ X & & & & Y \\ & \nwarrow \in P & \uparrow & \nearrow \in P & \\ & & U' & & \end{array}$$

The diagram illustrates a 2-morphism between two spans. The top span has objects X and Y with a common object U above them, connected by arrows $f: X \leftarrow U$ and $\in P: U \rightarrow Y$. The bottom span has objects X and Y with a common object U' below them, connected by arrows $f': X \leftarrow U'$ and $\in P: U' \rightarrow Y$. A vertical arrow $\in P: U' \rightarrow U$ connects the two spans, representing a morphism in the 2-category.

Universal property of $\text{Span}_2(\mathcal{C}, P)$

Theorem (Gaitsgory–Rozenblyum; MacPherson; Stephanich)

The inclusion $\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}_2(\mathcal{C}, P)$ is right P -adjointable, and **universal** among right P -adjointable functors: for any right P -adjointable $D(-): \mathcal{C}^{\text{op}} \rightarrow \mathbb{E}$, there is a unique extension to a 2-functor

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & & \\ \downarrow & \searrow D(-) & \\ \text{Span}_2(\mathcal{C}, P) & \dashrightarrow_{\exists!} & \mathbb{E}. \end{array}$$

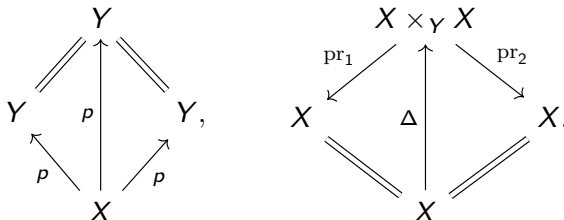
Consequence: To construct a 3-functor formalism extending D^* , it suffices to verify the Beck–Chevalley and projection formula conditions.

Proof sketch

Step 1: Check $\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}_2(\mathcal{C}, P)$ is right P -adjointable.

For $p: X \rightarrow Y$ in P , the right adjoint to $p^* = (Y \xleftarrow{P} X \xrightarrow{=} X)$ is $p_* = (X \xleftarrow{=} X \xrightarrow{P} Y)$.

Unit $\text{id}_Y \rightarrow p_* p^*$ and counit $p^* p_* \rightarrow \text{id}_X$:



Step 2: For the universal property, show $Y \mapsto \text{Hom}_{\text{Span}_2(\mathcal{C}, P)}(X, Y)$ is the free right P -adjointable functor.

Uses **parametrized category theory**: express the universal property as “freely adding parametrized limits” (co-Yoneda).

Étale maps and left adjoints

The case $p_! \simeq p_*$ (proper maps) is only half the story.

In practice there is also a class of **étale** (or open) maps $I \subseteq E \subseteq \mathcal{C}$ with the *dual* property:

$$i_! \dashv i^* \quad (i_! \text{ is a left adjoint to } i^*).$$

Examples: open immersions, étale maps of schemes, local homeomorphisms.

By **dualizing** the previous section, the functor $\mathrm{Span}(\mathcal{C}, I) \rightarrow \mathrm{Cat}_\infty$ encodes the left adjoints $i_!$.

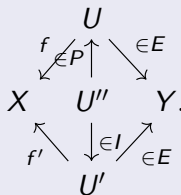
Challenge: How to *combine* both pieces into a single 6-functor formalism?

The $(\infty, 2)$ -category $\text{Span}_2(\mathcal{C}, E)_{I,P}$

Construction (C.–Lenz–Linskens)

The $(\infty, 2)$ -category $\text{Span}_2(\mathcal{C}, E)_{I,P}$:

- **Objects:** objects of $\mathcal{C}$,
- **1-morphisms:** spans $X \leftarrow U \xrightarrow{\in E} Y$,
- **2-morphisms:** *spans of spans*



Key point: 2-morphisms encode the adjunction data for both $i_!$ and p_* .

(I, P) -biadjointability

Definition

Let $I, P \subseteq E \subseteq \mathcal{C}$. A functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathbb{E}$ is (I, P) -**biadjointable** if:

- 1 **Left I -adjointable:** for $i \in I$, $F(i) = i^*$ has a left adjoint $i_!$ with Beck–Chevalley for pullbacks along I .
- 2 **Right P -adjointable:** for $p \in P$, $F(p) = p^*$ has a right adjoint p_* with Beck–Chevalley for pullbacks along P .
- 3 **Mixed Beck–Chevalley:** for any pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{i'} & \cdot \\ p' \downarrow & \lrcorner & \downarrow p \in P \\ \cdot & \xrightarrow{i \in I} & \cdot \end{array}$$

the canonical map $i_! p'_* \xrightarrow{\sim} p_* i'_!$ is an equivalence.

Main theorem

Theorem (C.–Lenz–Linskens)

Assume $I, P \subseteq E \subseteq \mathcal{C}$ are wide subcategories closed under base change, every map in E factors as $p \circ i$ with $p \in P$, $i \in I$, and maps in $I \cap P$ are truncated.

Then the inclusion

$$\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}_2(\mathcal{C}, E)_{I,P}$$

is the **initial (I, P) -biadjointable functor**.

Consequence: To construct a 6-functor formalism, it suffices to verify the three biadjointability conditions. All coherences come for free from the universal property.

Proof overview

Step 1. Verify directly that $\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}_2(\mathcal{C}, E)_{I,P}$ is (I, P) -biadjointable. Explicit computation of adjunctions.

Step 2. Abstract arguments produce the initial (I, P) -biadjointable $(\infty, 2)$ -category $\mathbb{B}(\mathcal{C}; I, P)$ and a comparison 2-functor

$$\mathbb{B}(\mathcal{C}; I, P) \rightarrow \text{Span}_2(\mathcal{C}, E)_{I,P}.$$

This is essentially surjective; reduce to checking equivalence on hom-categories.

Step 3. Reduce to identifying the **free** (I, P) -biadjointable functor:

$$Y \mapsto \text{Span}(E/Y, \text{morphisms in } P, \text{morphisms in } I).$$

Step 3a. Handle the special case $P \subseteq I$ via the theory of *semiadditive* ∞ -categories.

Step 3b. Reduce the general case to 3a by “freely adding the remaining right adjoints for P ”.

Key ingredient: contractibility of factorizations

A crucial geometric input in Step 3b:

Key Lemma

For every morphism $e: X \rightarrow Y$ in E , the ∞ -category

$$\mathrm{Fact}(e) = \{X \xrightarrow{i} Z \xrightarrow{p} Y \mid i \in I, p \in P, p \circ i \simeq e\}$$

is **weakly contractible**.

This says the factorization system (I, P) on E is “homotopy unique”.

Why it matters: When freely adding right adjoints for P on top of $\mathrm{Span}(\mathcal{C}, I)$, contractibility of $\mathrm{Fact}(e)$ ensures we do not over-identify or under-identify morphisms involving both I and P .

Semiadditive ∞ -categories

Step 3a uses the following classical model:

Lemma

An ∞ -category $\mathcal{C}$ is **semiadditive** iff it has finite products and finite coproducts that commute with each other.

The **free** semiadditive ∞ -category is $\text{Span}(\text{Fin})$. Given $\mathcal{C}$ semiadditive and $X \in \mathcal{C}$:

$$\text{Span}(\text{Fin}) \rightarrow \mathcal{C}, \quad I \mapsto \bigoplus_I X.$$

Backward spans use the *product* structure of $\bigoplus_I X$; forward spans use the *coproduct* structure.

This is the prototypical case of the main theorem (with $I = P =$ all maps of finite sets).

Summary

Main theorem: The inclusion $\mathcal{C}^{\text{op}} \hookrightarrow \text{Span}_2(\mathcal{C}, E)_{I,P}$ is the initial (I, P) -biadjointable functor.

Practical upshot: To construct a 6-functor formalism, it suffices to:

- 1 Have a coefficient system $D^*: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}_{\infty}$,
- 2 Verify Beck–Chevalley for I and P separately,
- 3 Verify the mixed Beck–Chevalley condition.

Applications:

- Étale sheaves with ℓ -adic coefficients on schemes,
- Sheaves on locally compact Hausdorff spaces,
- Motivic sheaves (with suitable I, P, E).

Thank you!