Introduction to Higher Algebra

Lecture course at University of Regensburg
Summer term 2025

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Regensburg, August 19, 2025

Contents

Introduction

As the title indicates, this course provides an introduction to *Higher Algebra*: the study of algebraic structures, such as rings, modules, and algebras, within the framework of higher category theory. I will assume the reader is familiar with the framework of higher category theory and stable homotopy theory; see here for my lecture notes from last semester.

Recall that the main conceptual difference between ordinary mathematics and homotopy theory lies in the treatment of *equality*, which is no longer treated as merely a property but becomes *structure*: When we say two things are "the same", we should provide an explicit isomorphism or homotopy witnessing this sameness. Applying this principle to algebraic structures leads naturally to the idea of *homotopy coherent* algebraic structures. Here, relations like associativity $(a \cdot b) \cdot c \simeq a \cdot (b \cdot c)$ hold only up to a specified homotopy. But this is just the beginning: these homotopies themselves must satisfy higher coherence relations, which in turn are witnessed by homotopies-between-homotopies, ad infinitum. The language of ∞-categories provides the necessary tools to manage this infinite hierarchy of coherence data rigorously and effectively.

In the previous semester, we have already encountered some examples of this principle in action. For instance, we defined commutative monoids in an ∞-category C with finite products as product-preserving functors $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$, where $\mathrm{Span}(\mathrm{Fin})$ is the span category of finite sets. The 'trick' employed by this definition is to find the appropriate ∞-category (in this case $\mathrm{Span}(\mathrm{Fin})$) that captures the algebraic structure together with all its possible coherences, so that the functor M maps all this structure to coherence data in C. This led to the notion of a symmetric monoidal ∞-category as a commutative monoid in the ∞-category Cat_∞. Building on this, we defined *commutative algebras* in a symmetric monoidal ∞-category $(C, \otimes, 1)$ as symmetric monoidal functors $A\colon \mathrm{Fin} \to C$, where Fin is the category of finite sets equipped with its cocartesian monoidal structure.

In this course, our aim is to develop this framework of higher algebra in much greater detail. Our primary motivation will be the study of commutative algebra in the ∞-category Sp of spectra. In more detail, we wish to:

- Upgrade the tensor product of spectra introduced last semester to a symmetric monoidal structure on Sp;
- Define associative and commutative ring spectra;
- Define modules over these ring spectra;
- Construct homotopical analogues of classical algebraic operations, such as relative tensor products ($M \otimes_R N$), localization, and completion;

- Study homotopical versions of familiar module properties, like flatness, projectivity, and perfection;

- Develop the basics of deformation theory for commutative ring spectra, including the cotangent complex.

To achieve these goals, we will rely heavily on the formalism of ∞-*operads*; developing this formalism will be the objective of the first part of this course. Heuristically, an operad provides a way to encode families of operations and their compositions. For each natural number n, an operad specifies an anima (or set/space in the classical setting) $\mathcal{O}(n)$ of possible n-ary operations. The operad structure then dictates how these operations can be composed—for example, how plugging an n-ary operation into one input of a k-ary operation yields an $(n + k - 1)$-ary operation. Crucially, this composition must satisfy coherence conditions like associativity and unitality. In the ∞-categorical setting, these conditions must hold up to coherent higher homotopies, making the precise formulation somewhat subtle.

Structure of the course

This course will have two parts:

- In Part I, we provide an extensive development of the framework of ∞-operads;

- In Part II, we use this framework to study the 'higher algebra' of ring spectra.

Part I is organized as follows:

- We start in Chapter 1 by introducing the classical notion of a (colored) operad and providing a characterization that we can use to define ∞-operads in the ∞-categorical setting.

- Chapters 2 and 3 provide two important pieces of background material. In Chapter 2, we will introduce the notion of *cartesian* and *cocartesian fibrations*, and their relation to functors into Cat_∞ via the *straightening/unstraightening correspondence*. In Chapter 3, we will formally define span categories $\mathrm{Span}_{L,R}(\mathcal{C})$ and study their basic properties.

- Actual operad theory starts in Chapter 4. We work with a definition of ∞-operads different from that of Lurie, replacing the category Fin_* of finite sets with the span category $\mathrm{Span}(\mathrm{Fin})$. (The resulting ∞-categories of ∞-operads are equivalent to each other.) We also associate a *multimorphism operad* $\mathcal{M}_\mathcal{C}$ to every symmetric monoidal ∞-category $\mathcal{C}$, and discuss how this results in notions of lax symmetric monoidal functors, symmetric monoidal subcategories, and symmetric monoidal Bousfield localizations.

- In Chapter 5, we introduce two fundamental examples of symmetric monoidal structures: the *cartesian* monoidal structure $(\mathcal{C}, \times)$ whenever $\mathcal{C}$ admits finite product, and the *cocartesian* monoidal structure $(\mathcal{C}, \amalg)$ whenever $\mathcal{C}$ admits finite coproducts.

- Building on the cartesian monoidal structure on An, we explain in Chapter 6 how to obtain symmetric monoidal structures on the ∞-categories An_*, CMon(An), CGrp(An) and Sp of pointed animae, commutative monoids, commutative groups and spectra, respectively. We will also see that each of the comparison functors

$$(\mathrm{An}, \times) \xrightarrow{(-)_+} (\mathrm{An}_*, \wedge) \xrightarrow{F^{\mathrm{CMon}}} (\mathrm{CMon}(\mathrm{An}), \otimes) \xrightarrow{(-)^{\mathrm{grp}}} (\mathrm{CGrp}(\mathrm{An}), \otimes) \hookrightarrow (\mathrm{Sp}, \otimes)$$

are canonically symmetric monoidal.

Part II is organized as follows:

- We start in Chapter 8 with a general discussion of algebras and modules in symmetric monoidal ∞-categories: the free-forgetful adjunction on module categories, the relative tensor product, restriction and extension of scalars, and endomorphism algebras.
- In Chapter 9, we study the module categories $\mathrm{Mod}_R := \mathrm{Mod}_R(\mathrm{Sp})$ in more detail. Section 9.1 introduces the notion of a *t-structure* on a stable ∞-category, which makes clear the connection between homological algebra. In Section 9.2 and Section 9.3 we introduce higher analogues of projective and flat modules, and in Section 9.4 we introduce the spectral analogue of 'perfect complexes' in homological algebra. Finally, we discuss the notion of localizations of ring spectra and their modules in Section 9.5.
- The course ends with Chapter 10 about deformation theory of commutative ring spectra. We assign to every module M over a commutative ring spectrum A a *split square-zero extension* $A \ltimes M$ and use that to define *derivations* of A with values in M. This leads to the notion of the *cotangent complex* L_A, which corepresents the animae of derivations. We then introduce general square-zero extensions $A' \to A$, and show that the maps $\tau_{\leq n}A \to \tau_{\leq n-1}A$ in the Postnikov tower of a commutative connective ring spectrum A are square-zero extensions. We will use this to prove the following classification of étale algebras over A:

Theorem. Let A be a connective ring spectrum. Then the functor $\pi_0 \colon \mathrm{CAlg}_A \to \mathrm{CAlg}^{\heartsuit}_{\pi_0 A}$ induces an equivalence between the ∞-category $\mathrm{CAlg}^{\text{ét}}_A$ of étale A-algebras and the ordinary category $(\mathrm{CAlg}^{\heartsuit}_{\pi_0 A})^{\text{ét}}$ of étale $\pi_0(A)$-algebras.

Acknowledgments

Most of the theory on ∞-operads and commutative algebra of ring spectra treated in this course is, either directly or indirectly, taken from Lurie's book *Higher Algebra* [Lur17].

I would like to thank Jan Steinebrunner, Rune Haugseng, Sil Linskens and Christoph Winges for various helpful conversations regarding the basic theory of ∞-operads.

I thank Alissa Doggwiler for pointing out various typos.

For Part I, I have taken inspiration from the note on ∞-operads by Haugseng [Hau23]. See also Harpaz [Har19] for an alternative treatment of ∞-operads.

The structure of Part II is based on Gepner's excellent introductory article [Gep20].

Conventions

While I will mostly follow the conventions from Lurie's book *Higher Algebra*, there are the following notable differences:

- All definitions are 'synthetic' in nature, in the sense that they do not make reference to a specific model for ∞-categories;
- I will say 'anima' instead of 'space' to refer to ∞-groupoids.
- The notation $\mathrm{Hom}_C(x, y)$ denotes the Hom anima in an ∞-category C. If C is stable, I write $\mathrm{map}_C(x, y)$ for the mapping spectrum.
- Instead of defining ∞-operads in terms of ∞-categories over the category Fin_* of finite pointed sets, I wish to work with ∞-categories over the *span category* $\mathrm{Span}(\mathrm{Fin})$, for reasons explained in Section 1.4. Similarly, commutative monoids in an ∞-category C are defined as product-preserving functors $\mathrm{Span}(\mathrm{Fin}) \to C$ rather than functors $\mathrm{Fin}_* \to C$ satisfying the Segal condition. This is just a matter of convention: the resulting theories are equivalent.
- I wish to notationally distinguish an ∞-operad $\mathcal{O}$ from its total ∞-category $\mathcal{O}^{\otimes}$: in this text, an ∞-operad $\mathcal{O}$ consists of a pair $(\mathcal{O}^{\otimes}, p_{\mathcal{O}})$ where $\mathcal{O}^{\otimes}$ is an ∞-category and $p_{\mathcal{O}} \colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ is a functor satisfying various properties.
- I define symmetric monoidal ∞-categories as commutative monoids in Cat_∞, i.e., as product-preserving functors $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$. In particular, I will notationally distinguish a symmetric monoidal ∞-category C from its associated ∞-operad. Combined with the previous point, this means we have the following three objects:
  - The symmetric monoidal ∞-category $C \colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$;
  - The associated ∞-operad, which I will denote by $\mathcal{M}_C$ and refer to as the *multimorphism operad of* C. In the literature, this ∞-operad is usually not given a name.
  - The total ∞-category $C^{\otimes} := (\mathcal{M}_C)^{\otimes}$ of the ∞-operad $\mathcal{M}_C$.

  These are related as follows: the ∞-category $C^{\otimes}$ comes equipped with a cocartesian fibration $p_C \colon C^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ which straightens to C, and the ∞-operad $\mathcal{M}_C$ is the pair $(C^{\otimes}, p_C)$.
- Given an ∞-operad $\mathcal{O}$, I will denote the multimorphism animae of $\mathcal{O}$ as $\mathcal{O}(\{x_i\}_{i\in I}; y)$ rather than $\mathrm{Mul}_{\mathcal{O}}(\{x_i\}_{i\in I}; y)$.
- Operads will always be written in caligraphic/cursive font, like $\mathcal{A}ssoc$ and $\mathcal{C}omm$. In particular, the little n-disks operad is $\mathcal{E}_n$, and the left module operad is $\mathcal{LM}od$. Day convolution operads are written $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$, and the internal hom in Op_∞ with respect to the BV-tensor product is denoted $\mathcal{A}lg(\mathcal{O}, \mathcal{P})$.

Part I

The theory of ∞-operads

1 Operads: the heuristics

Before diving into abstract theory, we will discuss the sort of structure an ∞-operad is aiming to capture.

1.1 Non-colored operads

Let us start by thinking through how to describe some algebraic structure on an object A of a symmetric monoidal category $(C, \otimes, \mathbb{1})$. Often, such algebraic structures are given in terms of a collection of *n-ary operations* for natural numbers $n \geq 0$, i.e., maps of the form $A^{\otimes n} \to A$ from the n-fold tensor product of A to itself. For example, if A is an associative algebra object in C, we have the multiplication map $\cdot \colon A^{\otimes 2} \to A, (x, y) \mapsto xy$. Also the unit is of this form: when $n = 0$, an n-fold tensor product of A is by convention the monoidal unit $\mathbb{1} \in C$, and we often want to treat the unit element $1 \in A$ as a 0-ary operation $1 \colon \mathbb{1} \to A$.

Observe that operations may be combined to obtain more of them. For example, assume that for natural numbers $k, n_1, \ldots, n_k \geq 0$ we are given operations on A of the form

$$P \colon A^{\otimes k} \to A, \qquad Q_i \colon A^{\otimes n_i} \to A, \quad i = 1, \ldots, k.$$

Then we may combine them into an n-ary operation $P(Q_1, \ldots, Q_k)$ for $n := \sum_{i=1}^{k} n_i$ by forming the following composite:

$$A^{\otimes n} \cong \bigotimes_{i=1}^{k} A^{\otimes n_i} \xrightarrow{\bigotimes_{i=1}^{k} Q_i} \bigotimes_{i=1}^{k} A = A^{\otimes k} \xrightarrow{P} A.$$

We will refer to the operation $P(Q_1, \ldots, Q_k)$ as the *composition* of P with $(Q_1, \ldots, Q_k)$. In similar fashion we may also always consider the *identity operation* $\mathrm{id} \colon A \to A, a \mapsto a$. Finally, we may permute the inputs of an n-ary operation by any permutation $\sigma \in \Sigma_n$:

$$(P\sigma)(x_1, \ldots, x_n) := P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

If we apply this process to our associative algebra A, we find that there are precisely $n!$ possible ways of constructing an n-ary operation: for every permutation $\sigma \in \Sigma_n := \mathrm{Aut}_{\mathrm{Set}}(\{1, \ldots, n\})$, we obtain a multiplication map $m_\sigma \colon A^{\otimes n} \to A$ given by $m_\sigma(x_1, \ldots, x_n) := x_{\sigma(1)} \cdots x_{\sigma(n)}$.

The notion of a (non-colored, symmetric, non-enriched) *operad* is trying to axiomatize the collections of possible operations one may have, together with their various compositions and permutations:

Definition 1.1.1 ([May72], [BV73]). A *non-colored operad* $\mathcal{O}$ consists of:

- A set $O(n)$ of n-ary operations, for every $n \geq 0$;
- A *composition map*

$$- \circ -\colon O(k) \times \prod_{i=1}^{k} O(n_i) \to O(n_1 + \cdots + n_k), \quad (P, Q_1, \ldots, Q_k) \mapsto P(Q_1, \ldots, Q_k)$$

  for all natural numbers $k, n_1, \ldots, n_k \geq 0$;
- An *identity operation* $\mathrm{id} \in O(1)$.
- Composition is *unital* and *associative*:

$$P(\mathrm{id}, \ldots, \mathrm{id}) = P = \mathrm{id}(P),$$
$$R(P_1(Q_1^1, \ldots, Q_{k_1}^1), \ldots, P_l(Q_1^l, \ldots, Q_{k_l}^l)) = (R(P_1, \ldots, P_l))(Q_1^1, \ldots, Q_{k_1}^1, Q_1^2, \ldots, Q_{k_1}^l).$$

- The set $O(n)$ comes equipped with a right action by the symmetric group Σ_n. The composition law in O should be suitably compatible with this permutation action; See Exercise A.1 for details.

The following are some examples of non-colored operads:

Example 1.1.2. The *associative operad $\mathcal{A}ssoc$* has $\mathcal{A}ssoc(n) := \Sigma_n$, i.e., there is one n-ary operation for every permutation $\sigma \in \Sigma_n$;

Example 1.1.3. The *commutative operad $Comm$* has $Comm(n) := *$, i.e., there is a *single* n-ary operation for every $n \geq 0$;

Example 1.1.4. There is the *trivial operad $\mathcal{T}riv$*, whose only operation is the identity operation:

$$\mathcal{T}riv(1) = \{\mathrm{id}\}, \qquad \mathcal{T}riv(n) = \emptyset \quad \text{for} \quad n \neq 1.$$

Example 1.1.5. The *pointed operad $\mathcal{E}_0$* has only one non-identity operation:

$$\mathcal{E}_0(1) = \{\mathrm{id}\}, \qquad \mathcal{E}_0(0) = \{\eta\}, \qquad \mathcal{E}_0(n) = \emptyset \quad \text{for} \quad n \geq 2.$$

Example 1.1.6. For every field there is an operad called the *Lie operad*, denoted $\mathcal{L}ie$, which captures the operations present in a Lie algebra. The set $\mathcal{L}ie(n)$ is a certain subset of the free Lie algebra $L(x_1, \ldots, x_n)$ generated by n elements $x_1, \ldots, x_n$, namely the subset of the 'multilinear' elements: formal sums of iterated Lie brackets that use each variable x_i exactly once.

Example 1.1.7. For every $k \geq 0$, there is a topological operad[1] called the *little k-cubes operad*, often denoted $\mathcal{E}_k$. Its n-ary operations are the so-called 'rectilinear' embeddings $\bigsqcup_{i=1}^{n} [0,1]^k \to [0,1]^k$ of a disjoint union of n copies of the k-dimensional cube into the k-dimensional cube. Here 'rectilinear' means that on each of the n components it is given by an affine function, i.e., maps of the form

$$[0,1]^k \to [0,1]^k, \quad (x_i)_{i=1}^k \mapsto (a_i x_i + b_i)_{i=1}^k,$$

which allow us to scale and shift the cube but not rotate or distort it in any other way. The composition in $\mathcal{E}_k$ is given by composition of rectilinear embeddings.

[1] As opposed to the other examples, this operad is *topological*, in that we need to remember the topology on $O(n)$.

Remark 1.1.8. For $n = 1$, there is an operad morphism $\mathcal{E}_1 \to \mathcal{A}ssoc$ such that for every $n \geq 0$ the map $\mathcal{E}_1(n) \to \mathcal{A}ssoc(n)$ is a homotopy equivalence, see Exercise A.6. In particular, after passing from sets/topological spaces to animae (as we will do when working with ∞-operads) there will no longer be a distinction between $\mathcal{E}_1$ and $\mathcal{A}ssoc$. In more classical setups, for example when working with model categories, the homotopically correct operad to work with is often $\mathcal{E}_1$. This is the reason why in older literature you will often find the phrase '$\mathcal{E}_1$-algebras' in places where we would say 'associative algebra'.

Example 1.1.9 (Endomorphism operad)**.** For an object A of an symmetric monoidal category C, we may form the *endomorphism operad* $\mathcal{E}nd_C(A)$, given by $\mathcal{E}nd_C(A)(n) := \mathrm{Hom}_C(A^{\otimes n}, A)$. Composition in $\mathcal{E}nd_C(A)$ is given by composition in C.

Given the structure an operad has, it is not difficult to guess the correct notion of *morphisms* between them: if $\mathcal{O}$ and $\mathcal{P}$ are operads, then an operad morphism $f\colon \mathcal{O} \to \mathcal{P}$ is a collection of maps $f(n)\colon \mathcal{O}(n) \to \mathcal{P}(n)$ that preserves the identity operation, the composition of operations and the permutation of operations.

Definition 1.1.10. Let $\mathcal{O}$ be a non-colored operad and let C be a symmetric monoidal category. We define an *$\mathcal{O}$-algebra in C* as a pair (A, f_A) consisting of an object A of C together with an operad morphism $f_A\colon \mathcal{O} \to \mathcal{E}nd_C(A)$. In particular, f_A consists of an n-ary operation $P_A\colon A^{\otimes n} \to A$ for every $P \in \mathcal{O}(n)$ satisfying the condition that composition of operations in $\mathcal{O}$ corresponds to the composition of operations on A.

Exercise 1.1.11. Show that:

- $\mathcal{T}riv$-algebras are simply objects of C;
- $\mathcal{E}_0$-algebras are 'pointed objects' of C, i.e., objects A equipped with a 'unit map' $\eta\colon \mathbb{1} \to A$. (This should not be confused with the notion of 'pointed objects' in a category with terminal object $*$.)
- $\mathcal{C}omm$-algebras are objects A equipped with maps $\eta\colon \mathbb{1} \to A$ and $m\colon A \otimes A \to A$ which are unital, associative and commutative.
- $\mathcal{A}ssoc$-algebras are similar but without commutativity.

Remark 1.1.12. Note that $\mathcal{C}omm$ is the *terminal* operad: there is precisely one operad map $\mathcal{O} \to \mathcal{C}omm$ given by the unique maps $\mathcal{O}(n) \to \mathcal{C}omm(n) = *$. Also note that $\mathcal{T}riv$ is the *initial* operad: there is precisely one operad map $\mathcal{T}riv \to \mathcal{O}$ sending the unique operation id of $\mathcal{T}riv$ to the identity operation $\mathrm{id} \in \mathcal{O}(1)$.

1.2 Colored operads

While the notion of non-colored operads introduced previously is elegant, it is insufficient for our purposes. To illustate this, let us discuss the algebraic structure of an associative algebra A acting on some module M. To encode such an action, we need a map in C of the form

$$\mathrm{act}\colon A \otimes M \to M, \qquad (a, m) \mapsto am,$$

and its source is not a tensor-power of a single object. More generally, given a collection of objects (A_x) in C we would like to have some notion of operad which can deal with operations of the form

$$P\colon A_{y_1} \otimes \cdots \otimes A_{y_n} \to A_x$$

for certain choices of indices $y_1, \ldots, y_n, x$. This leads to the following 'multicolored' version of Definition 1.1.1:

Definition 1.2.1. A *colored operad* (or *multicategory*) $\mathcal{O}$ consists of the following structure:

- There is a set $\mathcal{O}^{\simeq}$ of *colors* (sometimes also called the *objects* of $\mathcal{O}$);
- For colors $x, y_1, \ldots, y_n \in \mathcal{O}^{\simeq}$, there is a set $\mathcal{O}((y_1, \ldots, y_n); x)$ of *multimorphisms* from $(y_1, \ldots, y_n)$ to x. We refer to x as the *output color* and to the y_i as the *input colors*;
- If we are additionally given colors $z^i_j \in \mathcal{O}^{\simeq}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, there should be a *composition map*

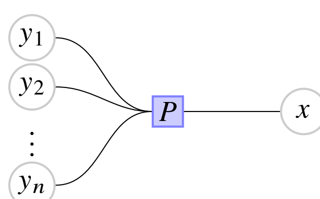
$$\circ\colon \mathcal{O}((y_1, \ldots, y_n); x) \times \prod_{i=1}^{n} \mathcal{O}((z^i_1, \ldots, z^i_{k_i}); y_i) \to \mathcal{O}((z^1_1, \ldots, z^k_{k_n}); x).$$

- For every color x there is an identity operation $\mathrm{id}_x \in \mathcal{O}((x); x)$;
- Composition should be unital and associative like before;
- For every permutation $\sigma \in \Sigma_n$ there is a 'permutation isomorphism'

$$\mathcal{O}((y_1, \ldots, y_n); x) \xrightarrow{\sim} \mathcal{O}((y_{\sigma(1)}, \ldots, y_{\sigma(n)}); x)$$

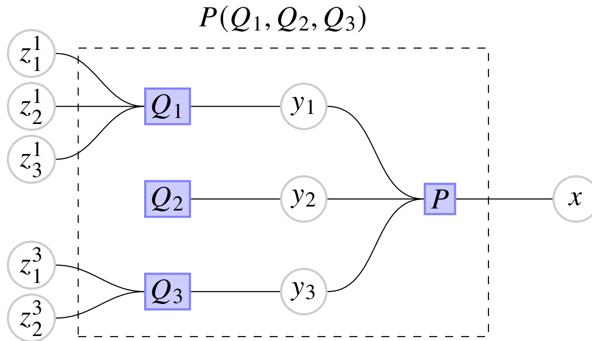
  which is compatible with the multiplication in the group Σ_n and is furthermore compatible with composition in $\mathcal{O}$. We leave a precise formulation of this condition to the reader; see Exercise A.1.

We may pictorially think of a multimorphism P from $(y_1, \ldots, y_n)$ to x as follows:[2]

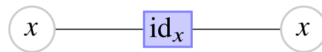


[2]This figure is inspired by [BS22, Figure 1].

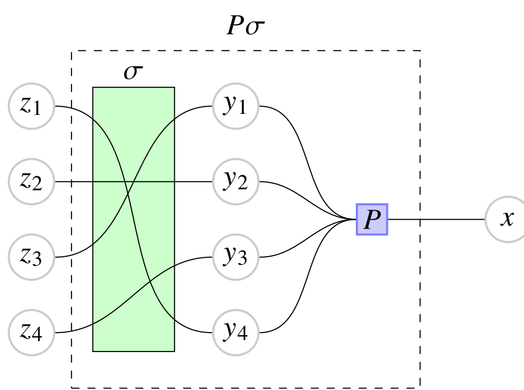
The composition of multimorphisms may then be illustrated as follows (with $n = k_1 = 3$, $k_2 = 0$ and $k_3 = 2$):



The identity multimorphism will be pictured as follows:



Finally, we have the permutation operation on multimorphisms, which we may picture as follows (with $z_i := y_{\sigma(i)}$):



Definition 1.2.2. If $\mathcal{P}$ is another colored operad, then a *morphism* $f\colon \mathcal{O} \to \mathcal{P}$ of colored operads consists of a map of sets $f\colon \mathcal{O}^{\simeq} \to \mathcal{P}^{\simeq}$ on colors and for all colors $x, y_1, \ldots, y_n \in \mathcal{O}^{\simeq}$ a map

$$f\colon \mathcal{O}((y_1, \ldots, y_n); x) \to \mathcal{P}((fy_1, \ldots, fy_n); fx)$$

on multimorphisms, in such a way that f preserves identity operations, composition of operations and permutations.

Let us go through some examples of colored operads:

Example 1.2.3 (From non-colored to colored)**.** Every non-colored operad $\mathcal{O}$ may be regarded as a colored operad with a single object: we take $\mathcal{O}^{\simeq} := \{*\}$ and we set

$$\mathcal{O}(\underbrace{(*, \ldots, *)}_{n \text{ times}}; *) := \mathcal{O}(n).$$

From now on, we will abuse notation and regard non-colored operads as colored operads in this way. In particular, we obtain colored operads $\mathcal{A}ssoc$, $\mathcal{C}omm$, $\mathcal{L}ie$ and $\mathcal{E}_k$.

Warning 1.2.4. While $\mathcal{C}omm$ is still terminal as a colored operad, $\mathcal{T}riv$ is no longer initial as a colored operad: the initial colored operad has an empty set of colors.

Example 1.2.5 (From colored to non-colored)**.** Conversely, for every colored operad $\mathcal{O}$ and every color $x \in \mathcal{O}^{\simeq}$ we obtain a non-colored operad $\mathcal{O}_x$ by setting

$$\mathcal{O}_x(n) := \mathcal{O}(\underbrace{(x, \dots, x)}_{n \text{ times}}; x).$$

The operad $\mathcal{O}_x$ comes with an obvious 'inclusion map' $\mathcal{O}_x \hookrightarrow \mathcal{O}$ of colored operads.

Example 1.2.6 (Multimorphism operad)**.** Let C be a symmetric monoidal category. We define a colored operad $\mathcal{M}_C$, called the *multimorphism operad*[3] *of* C, as follows:

- The colors of $\mathcal{M}_C$ are the objects of C.
- For objects $x, y_1, \dots, y_n$ of C, the multimorphisms in $\mathcal{M}_C$ are given by maps from the tensor product:
$$\mathcal{M}_C((y_1, \dots, y_n); x) \quad := \quad \mathrm{Hom}_C(y_1 \otimes \cdots \otimes y_n; x);$$
- The identity operations are given by the identity maps $\mathrm{id}_x \in \mathrm{Hom}_C(x, x)$;
- The composition of operations is given by the composition in C;
- The permutation operations are induced by the isomorphisms $y_1 \otimes \cdots \otimes y_n \simeq y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}$ coming from the symmetry isomorphisms in C.

Note that for an object A of C, the non-colored operad $(\mathcal{M}_C)_A$ obtained by applying Example 1.2.5 is precisely the endomorphism operad $\mathcal{E}nd_C(A)$ introduced in Example 1.1.9.

Example 1.2.7 (Module operad)**.** There is the *module operad* $\mathcal{M}od$ which has two colors a and m, so that $\mathcal{M}od^{\simeq} = \{a, m\}$. The operations are given as follows:

- If the output color is a, the operations should encode a commutative algebra structure, so we only want to allow operations all of whose input colors are a as well:
$$\mathcal{M}od((x_1, \dots, x_n); a) := \begin{cases} * & \text{if } x_j = a \text{ for all } j = 1, \dots, n, \\ \emptyset & \text{otherwise,} \end{cases}$$
- If the output color is m, the operations should encode the action of the commutative algebra on a module, so we only want operations that have precisely one input color that is m, while all others are a:
$$\mathcal{M}od((x_1, \dots, x_n); m) := \begin{cases} * & \text{if for some } i \text{ we have } x_i = m \text{ and } x_j = a \text{ for } j \neq i, \\ \emptyset & \text{otherwise.} \end{cases}$$

[3] This operad is not usually given a name in the literature. Some sources refer to it as the 'colored endomorphism operad of C', generalizing the terminology from Example 1.1.9. The terminology 'multimorphism operad' was suggested to me by Jan Steinebrunner.

The composition is uniquely determined.

Note that restricting the operations to the color a gives the commutative operad, while restricting to the operations to the color m gives the trivial operad:

$$\mathcal{M}od_a \cong \mathcal{C}omm \qquad\qquad \text{and} \qquad\qquad \mathcal{M}od_m \cong \mathcal{T}riv.$$

Example 1.2.8 (Left module operad)**.** There is also a non-commutative version of the module operad, capturing the structure of a left module over an associative algebra. We will denote it by $\mathcal{LM}od$ and call it the *left module operad*. It again has two colors, $\mathcal{LM}od^{\simeq} = \{a, m\}$. The operations are defined as follows:

- If the output color is a, we again only want operations all of whose input colors are a, but we should allow them to be permuted:
$$\mathcal{LM}od((x_1, \ldots, x_n); a) := \begin{cases} \Sigma_n & \text{if } x_j = a \text{ for all } j = 1, \ldots, n, \\ \emptyset & \text{otherwise.} \end{cases}$$

- If the output color is m, we should again only allow operations such that precisely one of the inputs is m. To allow for permutations of the other inputs, let us write $I_m := \{i \in \{1, \ldots, n\} \mid x_i = m\}$ for the set of indices corresponding to module inputs, and let $I_a := \{j \in \{1, \ldots, n\} \mid x_j = a\}$ be the set of indices corresponding to algebra inputs. We then set
$$\mathcal{LM}od((x_1, \ldots, x_n); m) := \begin{cases} \Sigma_{I_a} & \text{if } |I_m| = 1 \text{ (which implies } |I_a| = n - 1), \\ \emptyset & \text{if } |I_m| \neq 1. \end{cases}$$
Here Σ_{I_a} denotes the set of permutations of the index set I_a. Note that this set has cardinality $(n - 1)!$.

The composition laws and the action of the symmetric group Σ_n (which permutes input positions $(x_1, \ldots, x_n)$ and acts on the operations accordingly) are defined in the natural way. We will leave the details to the curious reader.

As in the commutative case, restricting operations to the color a recovers the underlying algebra operad, while restricting to the color m yields the trivial operad:

$$\mathcal{LM}od_a \cong \mathcal{A}ssoc \qquad \text{and} \qquad \mathcal{LM}od_m \cong \mathcal{T}riv.$$

Just as in the non-colored case, we may define algebras over operads in terms of operad morphisms:

Definition 1.2.9. Let $\mathcal{O}$ be a colored operad and let C be a symmetric monoidal category. An *$\mathcal{O}$-algebra in C* is a morphism $\mathcal{O} \to \mathcal{M}_C$ of colored operads.

Example 1.2.10. For specific instances of $\mathcal{O}$, the $\mathcal{O}$-algebras often have specialized names:

- For $\mathcal{O} = \mathcal{A}ssoc$ these are called *associative algebras*;

- For $\mathcal{O} = \mathcal{C}omm$ they are *commutative algebras*;

- For $\mathcal{O} = \mathcal{E}_k$ they are called *$\mathcal{E}_k$-algebras*;

- For $\mathcal{O} = \mathcal{M}od$, an $\mathcal{O}$-algebra is called a *module*. Restricting along the inclusion $\mathcal{C}omm \hookrightarrow \mathcal{M}od$ gives the *underlying commutative algebra* of the module, generically denoted A. Restricting along the inclusion $\mathcal{T}riv \hookrightarrow \mathcal{M}od$ gives the *underlying object*, generically denoted M.
- Similarly, $\mathcal{O} = \mathcal{LM}od$ encodes *left modules*: pairs (A, M) where A is an *associative* algebra rather than a commutative one.

1.3 Encoding the coherences: the envelope of an operad

The notion of ∞-operad we are after is a homotopical version of the notion of colored operad introduced in the previous section: the set $\mathcal{O}^{\simeq}$ of colors should be replaced by an anima, and similarly the sets $\mathcal{O}((y_1, \ldots, y_n); x)$ of multimorphisms should be replaced by animae. This leads to some subtleties regarding higher coherences: the unitality and associativity conditions required in Definition 1.2.1 should now become additional *data*, and this data should again satisfy further higher compatibilities. Writing down all the required structure explicitly would be impractical.

Fortunately, there is a way to circumvent this difficulty: the unitality, associativity and permutation laws required in an operad $\mathcal{O}$ can be reformulated in terms of a certain symmetric monoidal category $\mathrm{Env}(\mathcal{O})$ called the *envelope* of the operad. This alternative description of operads can immediately be adapted to the ∞-categorical setting, thus providing a definition of ∞-operads. We now define this envelope construction in the classical setting.

We start with an observation regarding the formulation of colored operads: while we have formulated the notion of multimorphisms only for ordered tuples $(y_1, \ldots, y_n) \in (\mathcal{O}^{\simeq})^n$, it is actually more natural to formulate them for *unordered tuples*:

Definition 1.3.1. An *unordered tuple* of the set $\mathcal{O}^{\simeq}$ is a pair (I, x) where I is a finite set and where $x \colon I \to \mathcal{O}^{\simeq}$ is a family of colors x_i indexed by I. We will often denote such unordered tuples by $\{x_i\}_{i \in I}$ for short.

Given two unordered tuples $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, we can form their *unordered concatenation*

$$\{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J} \quad := \quad \{z_k\}_{k\in K}, \qquad K := I \sqcup J, \quad z_i := x_i, \quad z_j := y_j.$$

More generally, if we are given a finite family $\{x_i^j\}_{i\in I_j}$ of unordered tuples for $j \in J$, then their unordered concatenation is $\{z_k\}_{k\in K}$ with $K := \bigsqcup_{j\in J} I_j$ and $z_{(j,i)} := x_i^j$.

Convention 1.3.2. Given a color $x \in \mathcal{O}^{\simeq}$ and an unordered tuple $\{y_i\}_{i\in I}$, we can define the set

$$\mathcal{O}(\{y_i\}_{i\in I}; x)$$

of multimorphisms from $\{y_i\}_{i\in I}$ to x by fixing once and for all some bijection $I \cong \{1, \ldots, n\}$. Under this convention, the composition map in the operad $\mathcal{O}$ has the form

$$\circ \colon \mathcal{O}(\{y_i\}_{i\in I}; x) \times \prod_{i\in I} \mathcal{O}(\{z_j^i\}_{j\in J_i}; y_i) \to \mathcal{O}(\{z_j^i\}_{(i,j)\in \sqcup_{i\in I} J_i}); x).$$

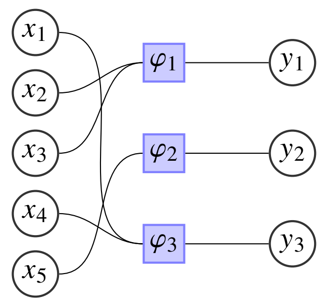
The permutation isomorphism then says that for every bijection $\sigma \colon J \xrightarrow{\cong} I$, there is an isomorphism

$$\mathcal{O}(\{y_i\}_{i\in I}; x) \xrightarrow{\cong} \mathcal{O}(\{y_{\sigma(j)}\}_{j\in J}; x\}).$$

Henceforth, we use the unordered notation for colored operads.

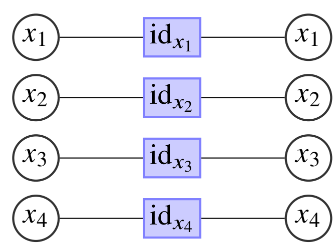
Definition 1.3.3. Let O be a colored operad. We define a symmetric monoidal category $\mathrm{Env}(O)$, called the *envelope* of O.

- The objects of $\mathrm{Env}(O)$ are unordered tuples $\{x_i\}_{i\in I}$ of colors of O;
- A morphism $\Phi\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $\mathrm{Env}(O)$ consists of a morphism of finite sets $f\colon I \to J$ together with a multimorphism $\varphi_j \in O(\{x_i\}_{i\in I_j}; y_j)$ for every $j \in J$. Here we write $I_j := f^{-1}(j)$ for the preimage of j under f. We will visualize this data as follows:



Here the function f is indicated by the lines from left to right, so that $f(1) = f(4) = 3$, $f(2) = f(3) = 1$ and $f(5) = 2$.

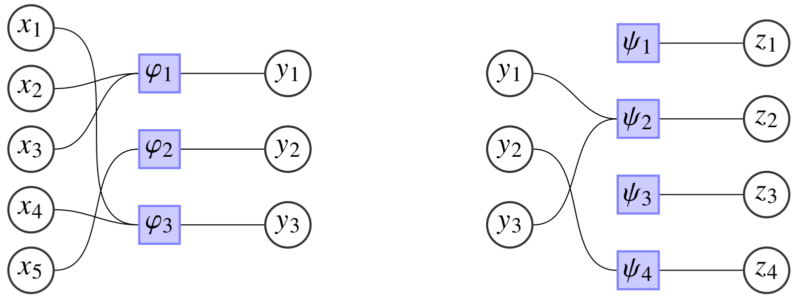
- The identity of $\{x_i\}_{i\in I}$ is given by taking $f = \mathrm{id}_I\colon I \to I$, and taking $\varphi_j = \mathrm{id}_{x_j} \in O(\{x_i\}_{i\in\{j\}}; x_j)$:



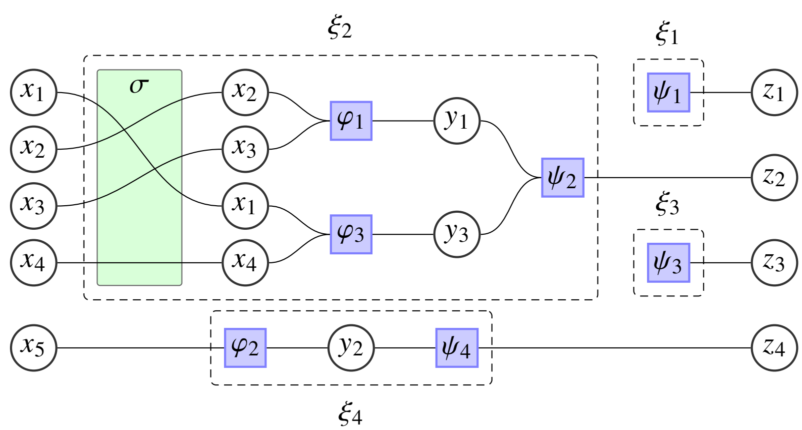
- The composition of $\Phi = (f, \varphi_j)\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ and $\Psi = (g, \psi_k)\colon \{y_j\}_{j\in J} \to \{z_k\}_{k\in K}$ is given by $\Psi \circ \Phi = (h, \xi_k)$ where $h := g \circ f\colon I \to K$ is the composition of g and f, and for every $k \in K$ the multimorphism $\xi_k \in O(\{x_i\}_{i\in I_k}; z_k)$ is the image of $(\psi_k, (\varphi_j)_{j\in J_k})$ under the composition map
$$\circ\colon O(\{y_j\}_{j\in J_k}; z_k) \times \prod_{j\in J_k} O(\{x_i\}_{i\in I_j}; y_j) \to O(\{x_i\}_{i\in I_k}; z_k);$$
here we are implicitly using that $I_k = (g\circ f)^{-1}(k) = f^{-1}(g^{-1}(k))$ may be written as the disjoint union of the sets $I_j = f^{-1}(j)$ for $j \in J_k = g^{-1}(k)$.
- The fact that composition in $\mathrm{Env}(O)$ is unital and associative precisely amounts to the unitality and associativity conditions for colored operads.

The category $\mathrm{Env}(O)$ admits a symmetric monoidal structure given by unordered concatenation of tuples.

The composition in $\mathrm{Env}(O)$ may look somewhat difficult when written in formulas, but it becomes clear what one needs to do if one visually represents the morphisms using diagrams. For illustration, let us examine the composition of the following two morphisms in $\mathrm{Env}(O)$:



First, we determine the composite map $h := g \circ f \colon \{1, \ldots, 5\} \to \{1, \ldots, 4\}$; in this case, this is given by $h(1) = h(2) = h(3) = h(4) = 2$ and $h(5) = 4$. Note that the preimages $h^{-1}(1)$ and $h^{-1}(3)$ are empty, so that ξ_1 and ξ_3 will be nullary operations. The multimorphism ξ_2 will have x_1, x_2, x_3 and x_4 as input colors, and is obtained by composing ψ_2 with (φ_1, φ_3), where we need to remember which of the four inputs should go into φ_1 and which ones go into φ_3. Finally, the multimorphism ξ_4 has only x_5 as input color, and is simply the composite $\psi_4 \circ \varphi_2$. The resulting composite morphism can be illustrated as follows:



Note: we draw the permutation σ here to match the ordered tuple convention used in Section 1.2, since since the nodes must be drawn in some fixed order on the page. When working with unordered tuples, the σ would not appear: the composition of multimorphisms simply remembers that its inputs are indexed by the disjoint union of $\{1, 3\}$ and $\{2, 4\}$ without fixing any order between them.

Example 1.3.4. The envelope $\mathrm{Env}(\mathcal{C}omm)$ of the commutative operad is the category Fin of finite sets. The symmetric monoidal structure is given by disjoint union.

Exercise 1.3.5. Compute the envelope $\mathrm{Env}(\mathcal{A}ssoc)$ of the associative operad; see Exercise A.9.

Exercise 1.3.6. Show that the assignment $O \mapsto \mathrm{Env}(O)$ is functorial: every morphism of operads $O \to \mathcal{P}$ induces a functor $\mathrm{Env}(O) \to \mathrm{Env}(\mathcal{P})$. In particular, the unique operad map $O \to \mathcal{C}omm$ induces a symmetric monoidal functor

$$\mathrm{Env}(O) \to \mathrm{Env}(\mathcal{C}omm) \simeq \mathrm{Fin}$$

to the category of finite sets.

Exercise 1.3.7. Show that the assignment $O \mapsto \mathrm{Env}(O)$ is left adjoint to the assignment $C \mapsto \mathcal{M}_C$: for an operad O and a symmetric monoidal category C, there is a bijection between operad morphisms $O \to \mathcal{M}_C$ and symmetric monoidal functors $\mathrm{Env}(O) \to C$.

In particular, commutative algebras $Comm \to \mathcal{M}_C$ correspond to symmetric monoidal functors $\mathrm{Fin} \to C$, which was precisely the definition of commutative algebra we gave last semester!

The nice feature about the envelope construction is that it allows us to formulate the unitality and associativity conditions for composition of multimorphisms in O in terms of unitality and associativity of morphisms in an actual category. In particular, this could potentially provide a convenient way to generalize the definition of operads to the ∞-categorical setting. The main question we would then need to address is: what structure do we need on a symmetric monoidal category C to guarantee that it is of the form $\mathrm{Env}(O)$ for some operad O? We identify the three crucial properties that $\mathrm{Env}(O)$ has:

(1) The envelope comes equipped with the symmetric monoidal functor $p\colon \mathrm{Env}(O) \to \mathrm{Fin}$ from Exercise 1.3.6. The colors of O are precisely those objects that are sent to a one-point set $\{i\}$ under this functor.

(2) Given an object $X \in \mathrm{Env}(O)$ with $p(X) = I$, we may uniquely write X as an I-tuple $\{x_i\}_{i\in I}$ for some colors $x_i \in p^{-1}(\{i\})$. Similarly, the morphisms in the fiber $p^{-1}(I)$ are I-tuples of maps $\{f_i : x_i \to y_i\}_{i\in I}$. In other words, the fiber over I is the I-fold product of the fiber over the one-point set.

(3) The morphisms into an arbitrary tuple $\{y_j\}_{j\in J}$ are completely determined by morphisms into 1-tuples: If we consider two unordered tuples $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$, then the fiber of the induced map

$$\mathrm{Hom}_{\mathrm{Env}(O)}(\{x_i\}_{i\in I}, \{y_j\}_{j\in J}) \xrightarrow{p} \mathrm{Hom}_{\mathrm{Fin}}(I, J)$$

over some map $f : I \to J$ is given by the product

$$\prod_{j\in J} \mathrm{Hom}_{\mathrm{Env}(O)}(\{x_i\}_{i\in I_j}, y_j).$$

One can verify that these three properties precisely characterize the envelopes of operads: a symmetric monoidal category C equipped with a symmetric monoidal functor $p\colon C \to \mathrm{Fin}$ is of the form $\mathrm{Env}(O)$ for some O if and only if it satisfying the corresponding analogues of conditions (2) and (3):

(2') For every finite set I, the tensor product in C defines an equivalence

$$\prod_{i\in I} C_{\{i\}} \xrightarrow{\sim} C_I,$$

where C_I is the fiber of p over $I \in \mathrm{Fin}$;

(3') For every finite collection $(I_j)_{j\in J}$ of finite sets and every collection of objects $X_j \in p^{-1}(I_j)$ and $y_j \in p^{-1}(\{j\})$, the commutative square

$$\begin{tikzcd}
\prod_{j\in J} \mathrm{Hom}_C(X_j, y_j) \arrow[r, "\bigotimes_{j\in J}"] \arrow[d] & \mathrm{Hom}_C(\bigotimes_{j\in J} X_j, \bigotimes_{j\in J} y_j) \arrow[d, "p"] \\
* \arrow[r] & \mathrm{Hom}_{\mathrm{Fin}}(\bigsqcup_{j\in J} I_j, J)
\end{tikzcd}$$

is a pullback square, where the bottom map hits the map $\bigsqcup_{j\in J} I_j \to \bigsqcup_{j\in J} * = J$ induced by the maps $I_j \to *$.

Exercise 1.3.8. Given a symmetric monoidal category C with $p\colon C \to \mathrm{Fin}$ satisfying (2') and (3'), construct a colored operad O and construct an equivalence $C \xrightarrow{\sim} \mathrm{Env}(O)$.

1.4 Writing concatenations as products

The characterization of envelopes of operads obtained in the previous section is neat, and immediately generalizes to the ∞-categorical setting. Nevertheless, this is not the definition of ∞-operads that usually appears in the literature. One inconvenience about this characterization is that it requires us to provide a structure of a symmetric monoidal ∞-category, which is a task substantially more involved than for ordinary categories since we can no longer 'just write it down by hand'. It would be convenient if we could avoid reliance on symmetric monoidal structures.

Fortunately, this is possible: there is a variation of the envelope construction in which the concatenation of tuples becomes the *categorical product*, and hence does not need to be recorded as additional data. In order to motivate this variant, let us recall how we dealt with a similar problem last semester when treating commutative monoids. Recall that we defined a commutative monoid as a product-preserving functor $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$ from the *span category of finite sets*. (See Chapter 3 below for a recollection on span categories). While all the relevant data about addition in M seems to be contained in the right-pointing maps, the left-pointing maps are crucial as well since they guarantee that the disjoint unions in Fin become products in the span category, hence get sent to products in C. We may think of this as a trade-off:

- Either we consider commutative *algebras*, which are symmetric monoidal functors $\mathrm{Fin} \to C$ and in particular rely on having a symmetric monoidal structure;
- Or we consider commutative *monoids*, which no longer require any symmetric monoidal structure, but come at the cost of having to add left-pointing morphisms to the source category that enforce cartesian products.

This construction can be generalized to arbitrary colored operads O. The construction will be somewhat informal due to the fact that span categories are (2,1)-categories rather than (1,1)-categories, but it will provide the crucial intuition for our definition of ∞-operads in Chapter 4.

Construction 1.4.1. Let O be a colored operad. We define a category $O^{\otimes}$ as follows:

- The objects of $O^{\otimes}$ are unordered tuples $\{x_i\}_{i\in I}$;

- A morphism $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $\mathcal{O}^{\otimes}$ consists of a span
$$I \xleftarrow{f} K \xrightarrow{g} J$$
of finite sets together with a multimorphism $\varphi_j \in \mathcal{O}(\{x_{f(k)}\}_{k\in K_j}; y_j)$ for every $j \in J$. Here we write $K_j := g^{-1}(j)$ for the preimage of j under g.

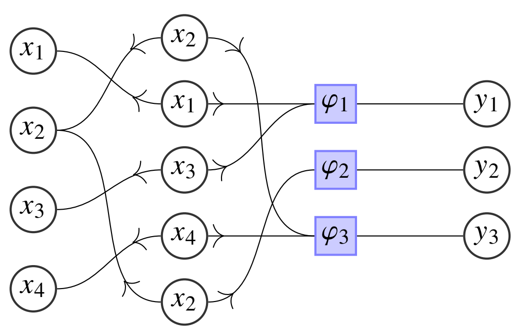
- The identity of $\{x_i\}_{i\in I}$ is given by taking $f = g = \mathrm{id}_I \colon I \to I$, and taking $\varphi_j = \mathrm{id}_{x_j} \in \mathcal{O}(\{x_i\}_{i\in\{j\}}; x_j)$.

- The composition of $(f, g, \varphi_j)\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ and $(h, k, \psi_l)\colon \{y_j\}_{j\in J} \to \{z_l\}_{l\in L}$ is given by $(f \circ \mathrm{pr}_K, k \circ \mathrm{pr}_M, \xi_l)$ where pr_K and pr_M are the projection maps in the following pullback diagram
$$\begin{tikzcd}
 & & K \times_J M \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right, "f\circ \mathrm{pr}_K"'] \arrow[ddrr, bend left, "k\circ \mathrm{pr}_M"] & & \\
 & K \arrow[dl, "f"'] \arrow[dr, "g"] & & M \arrow[dl, "h"'] \arrow[dr, "k"] & \\
I & & J & & L,
\end{tikzcd}$$
and for every $l \in L$ the multimorphism $\xi_l \in \mathcal{O}(\{x_{f(k)}\}_{(k,m)\in(K\times_J M)_l}; z_l)$ is the image of $(\psi_l, (\varphi_j)_{j\in J_l})$ under the composition map
$$\circ\colon \mathcal{O}(\{y_{h(m)}\}_{m\in M_l}; z_l) \times \prod_{j\in M_l} \mathcal{O}(\{x_{f(k)}\}_{k\in K_{h(m)}}; y_{h(m)}) \to \mathcal{O}(\{x_{f(k)}\}_{(k,m)\in K\times_J M_l}; z_l).$$

- As in $\mathrm{Env}(\mathcal{O})$, the fact that composition in $\mathcal{O}^{\otimes}$ is unital and associative precisely amounts to the unitality and associativity conditions in $\mathcal{O}$.

Exercise 1.4.2. Show that $\mathcal{O}^{\otimes}$ is well-defined and is functorial in $\mathcal{O}$. Argue that it admits finite products given by the unordered concatenation of tuples.

This time, morphisms in $\mathcal{O}^{\otimes}$ may be visualized as follows:

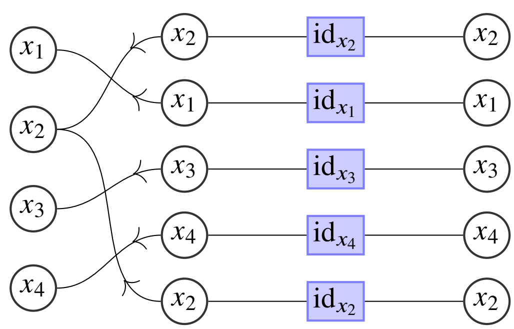


The first part of the data consists of two maps $f\colon \{1,\dots,5\} \to \{1,\dots,4\}$ and $g\colon \{1,\dots,5\} \to \{1,2,3\}$; in this case f is given by $f(1) = f(5) = 2$, $f(2) = 1$, $f(3) = 3$ and $f(4) = 4$, while g is given by $g(2) = g(3) = 1$, $g(5) = 2$ and $g(1) = g(4) = 3$. In particular, the preimages of 1, 2 and 3 under g are $\{2,3\}$, $\{5\}$ and $\{1,4\}$, respectively. The second part of the data consists of multimorphisms $\varphi_1\colon (x_{f(2)}, x_{f(3)}) \to y_1$, $\varphi_2\colon (x_{f(5)}) \to y_2$ and $\varphi_3\colon (x_{f(1)}, x_{f(4)}) \to y_3$.

Exercise 1.4.3. Make a visualization of how composition in $\mathcal{O}^{\otimes}$ works, generalizing the picture for $\mathrm{Env}(\mathcal{O})$ by adding in the left-pointing arrows. How is the process of forming the pullback $K \times_J M$ reflected in the visualization?

For $\mathcal{O} = \mathcal{C}omm$ we get $\mathcal{C}omm^{\otimes} = \mathrm{Span}(\mathrm{Fin})$. Since the construction of $\mathcal{O}^{\otimes}$ is functorial in $\mathcal{O}$, we obtain a functor
$$p \colon \mathcal{O}^{\otimes} \to \mathcal{C}omm^{\otimes} = \mathrm{Span}(\mathrm{Fin})$$
which preserves finite products. As before, we may ask: can we characterize those categories over $\mathrm{Span}(\mathrm{Fin})$ that are of the form $\mathcal{O}^{\otimes}$ for some operad? Note that $\mathcal{O}^{\otimes}$ still satisfies conditions (2') and (3') from the previous section, which may now be formulated in terms of the categorical product and no longer require us to remember a symmetric monoidal structure. In fact, condition (3') becomes automatic, due to the nature of categorical products. But (3') is not strong enough anymore: it only determines the maps in $\mathcal{O}^{\otimes}$ that live over the *right-pointing* maps in $\mathrm{Span}(\mathrm{Fin})$, and it does not tell us what to do over a general span $I \leftarrow K \to J$.

To reduce the general case to the special case, consider an object $\{x_i\}_{i\in I}$ of $\mathcal{O}^{\times}$ and consider a morphism $f \colon K \to I$ of finite sets. We may then form the morphism $\widetilde{f} \colon \{x_i\}_{i\in I} \to \{x_{f(k)}\}_{k\in K}$ in $\mathcal{O}^{\otimes}$ whose underlying span of finite sets is $I \xleftarrow{f} K \xrightarrow{=} K$ and whose multimorphisms $\varphi_k \in \mathcal{O}(\{x_{f(k)}\}, x_{f(k)})$ are the identity morphisms $\mathrm{id}_{x_{f(k)}}$. This morphism $\widetilde{f}$ may be visualized as follows:



This morphism has the special feature that it is *p-cocartesian* in the following sense:

(3'') Consider objects $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ in $\mathcal{O}^{\otimes}$, and consider a span $I \xleftarrow{f} K \xrightarrow{g} J$. Given a morphism $\widetilde{h} \colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ in $\mathcal{O}^{\otimes}$ lifting the given span[4], there exists a unique morphism $\widetilde{g} \colon \{x_{f(k)}\}_{k\in K} \to \{y_j\}_{j\in J}$ which lifts the span $K \xleftarrow{=} K \xrightarrow{g} J$ and satisfies $\widetilde{h} = \widetilde{g} \circ \widetilde{f}$:

$$\begin{tikzcd}
 & \{x_{f(k)}\}_{k\in K} \arrow[dr, dashed, "\exists!\widetilde{g}"] & \\
\{x_i\}_{i\in I} \arrow[ur, "\widetilde{f}"] \arrow[rr, "\widetilde{h}"] & & \{y_j\}_{j\in J}.
\end{tikzcd}$$

Indeed, the lift $\widetilde{h}$ is encoded by certain multimorphisms $\varphi_j \in \mathcal{O}(\{x_{f(k)}\}_{k\in K_j}; y_j)$, and this is precisely the data we need for defining the lift $\widetilde{g}$.

In a sense, we see that these morphisms $\widetilde{f}$ in $\mathcal{O}^{\otimes}$ living over left-pointing spans do not really *contribute* very much apart from encoding the categorical product structure. For this reason, they

[4]This means that the image of $\widetilde{h}$ under p is the given span $I \xleftarrow{f} K \xrightarrow{g} J$.

are called *inert morphisms*. In contrast, the morphisms in $\mathcal{O}^{\otimes}$ living over *right*-pointing spans are the ones that encode the actual operad structure, and are called the *active morphisms*.

Exercise 1.4.4. Regarding $\{x_i\}_{i\in I}$ as the I-indexed product of the objects x_i and similarly for $\{x_{f(k)}\}_{k\in K}$ (as in Exercise 1.4.2), show that the morphism $\widetilde{f}\colon \{x_i\}_{i\in I} \to \{x_{f(k)}\}_{k\in K}$ is the map whose s-th component is the projection map onto $x_{f(k)}$.

The upshot of this discussion is that we now have a description of colored operads that immediately generalizes to the ∞-categorical setting.

Definition 1.4.5. A Span(Fin)-∞-*operad* is a pair $\mathcal{O} = (\mathcal{O}^{\otimes}, p_{\mathcal{O}})$ consisting of an ∞-category $\mathcal{O}^{\otimes}$ equipped with a functor $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ satisfying the following conditions:

(1) The ∞-category $\mathcal{O}^{\otimes}$ admits finite products, and the functor $p_{\mathcal{O}}$ preserves finite products;

(2) For every finite set I, the product in $\mathcal{O}^{\otimes}$ defines an equivalence

$$\prod_{i\in I} \mathcal{O}^{\otimes}_{\{i\}} \xrightarrow{\sim} \mathcal{O}^{\otimes}_{I}.$$

Here we write $\mathcal{O}^{\otimes}_{I}$ for the fiber of $p_{\mathcal{O}}$ over the set I.

(3) For a morphism $f\colon J \to I$ in Fin and objects $X_i \in \mathcal{O}^{\otimes}$, the map $\widetilde{f}\colon \prod_{i\in I} X_i \to \prod_{j\in J} X_{f(j)}$ whose j-th component is the projection to $X_{f(j)}$ is *p-cocartesian*. This means that for every $Y \in \mathcal{O}^{\otimes}$ the following square is a pullback square:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}^{\otimes}}(\prod_{j\in J} X_{f(j)}, Y) \arrow[r, "-\circ\widetilde{f}"] \arrow[d, "p_{\mathcal{O}}"'] & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\prod_{i\in I} X_i, Y) \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(p_{\mathcal{O}}(\prod_{j\in J} X_{f(j)}), p_{\mathcal{O}}(Y)) \arrow[r, "-\circ p_{\mathcal{O}}(\widetilde{f})"] & \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(p_{\mathcal{O}}(\prod_{i\in I} X_i), p_{\mathcal{O}}(Y)).
\end{tikzcd}$$

Warning 1.4.6. In this course, I will take Definition 1.4.5 as the principal definition of ∞-operad. In particular, I will simply say ∞-*operad* instead of *Span(Fin)-∞-operad*. However, this is *not* the usual definition of ∞-operad found in Higher Algebra [Lur17]. In the usual definition, one instead works with the subcategory Span(Fin, inj, all) of the span category in which the left-pointing morphisms are injective. (Recall that this subcategory is equivalent to the category Fin_* of finite pointed sets.) This subcategory has the advantage that it is an actual $(1,1)$-category as opposed to a $(2,1)$-category. However, it has an important disadvantage: concatenation of tuples is no longer simply a product, so the conditions on $\mathcal{O}^{\otimes}$ need to be formulated in a more complicated way.

We will eventually show that our definition of ∞-operads agrees with the usual one from Lurie [Lur17]: using the envelope construction for ∞-operads, it will be a consequence of the equivalence between functors $\mathrm{Fin}_*^{\mathrm{op}} \to \mathrm{Cat}_\infty$ satisfying the Segal condition and the product-preserving functors $\mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$.

Before studying ∞-operads in more detail, we will need to introduce some more background material: in the next chapter we will introduce the notion of cartesian/cocartesian fibrations and their relation to functors into Cat_∞ via the *straightening/unstraightening correspondence*, and in the chapter after that we will formally define span categories $\mathrm{Span}_{L,R}(\mathcal{C})$ and study their basic properties. We return to ∞-operads in Chapter 4.

2 Cocartesian fibrations and (un)straightening

An important notion in ∞-category theory is that of a *cocartesian fibration*. Roughly speaking, a cocartesian fibration is a functor $p\colon E \to C$ whose fibers $E_x := E \times_C \{x\}$ vary covariantly in x, in the sense that for every morphism $f\colon x \to y$ in C we obtain a *cocartesian transport functor* $f_!\colon E_x \to E_y$. These functors are fully coherent in f, in the sense that they assemble into a functor $\mathrm{Str}(p)\colon C \to \mathrm{Cat}_\infty$, given on objects by sending x to E_x and on morphisms by sending f to $f_!$. This provides a powerful way to construct functors into Cat_∞, known as the *straightening/unstraightening correspondence*. For example, it will allow us to formally define the Hom-functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$ of an ∞-category C.

2.1 Cartesian and cocartesian fibrations

We start with some basic definitions regarding (co)cartesian fibrations.

Definition 2.1.1. Consider a functor $p\colon E \to C$.

(1) A morphism $\varphi\colon e \to e'$ in E is called *p-cocartesian* if, for every other object e'' in E, the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_E(e', e'') \arrow[r, "- \circ \varphi"] \arrow[d, "p"'] & \mathrm{Hom}_E(e, e'') \arrow[d, "p"] \\
\mathrm{Hom}_C(pe', pe'') \arrow[r, "- \circ p(\varphi)"] & \mathrm{Hom}_C(pe, pe'')
\end{tikzcd}$$

is a pullback square. Informally, this means that for every solid diagram of the form

$$\begin{tikzcd}
 & e' \arrow[dr, dashed, "\exists!"] & & & & pe' \arrow[dr] & \\
e \arrow[ur, "\varphi"] \arrow[rr] & & e'' & \overset{p}{\rightsquigarrow} & pe \arrow[ur, "p\varphi"] \arrow[rr] & & pe''
\end{tikzcd}$$

in E for which its image in C has been completed to a triangle, there exists a unique map $e' \to e''$ that forms a commutative triangle in E and projects to the given one in C.

(2) We say that p is a *cocartesian fibration* if for every morphism $f\colon x \to y$ in C and every object $e \in E_x$ there exists a p-cocartesian morphism $\varphi\colon e \to e'$ satisfying $p(\varphi) = f$. We refer to such a morphism φ as a *p-cocartesian lift of f*.

(3) Let $p\colon E \to C$ and $p'\colon E' \to C$ be two cocartesian fibrations. A functor $F\colon E \to E'$ over C, i.e., a commutative triangle

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & C, & \end{tikzcd}$$

is called *cocartesian over C* if it sends p-cocartesian morphisms in E to p'-cocartesian morphisms in E'.

(4) If C is a small ∞-category, we denote by

$$\mathrm{Cocart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$$

the non-full subcategory spanned by the cocartesian fibrations $p\colon E \to C$ and the cocartesian functors between them.

Cocartesian lifts are unique if they exist:

Lemma 2.1.2. *Let $p\colon E \to C$ be a functor and consider a morphism $f\colon x \to y$ in C. For an object $e \in E_x$, let $L(e, f) \subseteq E_{e/} \times_{C_{x/}} \{f\}$ denote the full subcategory spanned by the p-cocartesian lifts $\varphi\colon e \to e'$ of f. Then $L(e, f)$ is either empty or contractible.*

Proof. Assume that $L(e, f)$ is non-empty, so that there exists a p-cocartesian lift of f. In particular, the functor $L(e, f) \to *$ is essentially surjective. We show that it is also fully faithful, i.e., that the Hom animae in $L(e, f)$ are contractible. Consider two cocartesian lifts $\varphi_1\colon e \to e'_1$ and $\varphi_2\colon e \to e'_2$ of f. Then the Hom anima in $L(e, f)$ between φ_1 and φ_2 is computed as the fiber over id_f of the following map:

$$\mathrm{Hom}_{E_{e/}}(\varphi_1, \varphi_2) \to \mathrm{Hom}_{C_{x/}}(p\varphi_1, p\varphi_2) = \mathrm{Hom}_{C_{x/}}(f, f).$$

But the two Hom animae in the slices $E_{e/}$ and $C_{x/}$ may themselves be computed as the horizontal fibers in the following commutative square:

$$\begin{tikzcd} \mathrm{Hom}_E(e'_1, e'_2) \arrow[r, "-\circ\varphi_1"] \arrow[d, "p"'] & \mathrm{Hom}_E(e, e'_2) \arrow[d, "p"] \\ \mathrm{Hom}_C(y, y) \arrow[r, "-\circ f"] & \mathrm{Hom}_C(x, y). \end{tikzcd}$$

But since φ_1 is p-cocartesian, this square is a pullback square, so the two horizontal fibers are equivalent. It follows that $\mathrm{Hom}_{L(e,f)}(\varphi_1, \varphi_2) = *$, as desired. □

Notation 2.1.3. Given an object $e \in E_x$, we may sometimes denote a cocartesian lift of $f\colon x \to y$ by $\varphi\colon e \to f_!e$, and refer to $f_!e$ as the *cocartesian transport of e along f*. By Lemma 2.1.2, this is well-defined.

Exercise 2.1.4. For a functor $p\colon E \to C$, consider the full subcategory

$$\mathrm{Ar}^{\mathrm{cocart}}(E) \subseteq \mathrm{Fun}([1], E)$$

spanned by the p-cocartesian morphisms.

(1) Show that the functor

$$\mathrm{Ar}^{\mathrm{cocart}}(E) \to E \times_{p,C,\mathrm{ev}_0} \mathrm{Fun}([1], C), \quad (\varphi\colon e \to e') \mapsto (e, p\varphi)$$

is fully faithful.

(2) Deduce that p is a cocartesian fibration if and only if this functor is an equivalence.

(3) Use this equivalence to show that for every morphism $f\colon x \to y$, the assignment $e \mapsto f_! e$ from Notation 2.1.3 can be enhanced to a cocartesian transport functor

$$f_!\colon E_x \to E_y.$$

The definition of cocartesian fibrations may be dualized:

Definition 2.1.5. For a functor $p\colon E \to C$, we say that a morphism $\varphi\colon e \to e'$ in E is *p-cartesian* if for every third object e'' in E the commutative square

$$\begin{tikzcd} \mathrm{Hom}_E(e'', e) \arrow[r, "\varphi\circ-"] \arrow[d, "p"'] & \mathrm{Hom}_E(e'', e') \arrow[d, "p"] \\ \mathrm{Hom}_C(pe'', pe) \arrow[r, "p\varphi\circ-"] & \mathrm{Hom}_C(pe'', pe') \end{tikzcd}$$

is a pullback square. We say p is a *cartesian fibration* if for every morphism $f\colon x \to y$ and every $e' \in E_y$ there exists a p-cartesian lift $\varphi\colon e \to e'$ of f. We similarly obtain a notion of a *cartesian functor over C*, and this leads to a (non-full) subcategory

$$\mathrm{Cart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}.$$

Remark 2.1.6. Note that a functor $p\colon E \to C$ is a cartesian fibration if and only if its opposite $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to C^{\mathrm{op}}$ is a cocartesian fibration.

Let us go through a basic example of a (co)cartesian fibration to see these definitions in action:

Example 2.1.7. Let C be an ∞-category, let $E := \mathrm{Fun}([1], C)$ be its arrow category, and consider the evaluation functor $p := \mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$. An object of E is given by a morphism in C, which for emphasis we will write vertically: $\left(\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix}\right)$. A morphism $\varphi\colon \left(\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ \downarrow \\ Y' \end{smallmatrix}\right)$ in E is given by a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & Y'. \end{tikzcd}$$

We claim:

(1) The morphism φ in E is p-cocartesian if and only if g is an equivalence. In particular, ev_1 is *always* a cocartesian fibration: given $e = \left(\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix}\right)$ and $f\colon p(\left(\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix}\right)) = Y \to Y'$ we may always form the square with $X' := X$ and $g = \mathrm{id}_X$.

(2) The morphism φ in E is p-cartesian if and only if the square is a pullback square (a.k.a. a *cartesian* square). In particular, ev_1 is a cartesian fibration if and only if C admits pullbacks.

I will leave it to the reader to spell out a detailed proof of these claims, see Exercise A.12. But let's at least see what these claims are heuristically saying.

Claim (1) says that for every third object $\left(\begin{smallmatrix} X'' \\ \downarrow \\ Y'' \end{smallmatrix}\right)$ of E and every solid diagram below, there exists a unique dashed arrow making the diagram commute:

$$\begin{tikzcd}
 & X' \arrow[dr, dashed] \arrow[dd] & \\
X \arrow[ur, "g"] \arrow[rr] \arrow[dd] & & X'' \arrow[dd] \\
 & Y' \arrow[dr] & \\
Y \arrow[ur, "f"] \arrow[rr] & & Y''.
\end{tikzcd}$$

If g is an equivalence, we may invert it and it is clear that the dashed filler is unique. Conversely, if φ is p-cocartesian, then by taking $X'' = X$ we obtain a morphism $g' \colon X' \to X$ satisfying $g'g \cong \mathrm{id}_X$, and by taking $X'' = X'$ we then use uniqueness to get that $gg' = \mathrm{id}_{X'}$. (Alternatively, we use that there always exists a p-cocartesian lift with g an equivalence so the claim follows by uniqueness of p-cocartesian lifts.)

Claim (2) says that for every third object $\left(\begin{smallmatrix} X'' \\ \downarrow \\ Y'' \end{smallmatrix}\right)$ of E and every solid diagram below, there exists a unique dashed arrow making the diagram commute:

$$\begin{tikzcd}
 & X \arrow[dr, "g"] \arrow[dd] & \\
X'' \arrow[ur, dashed] \arrow[rr] \arrow[dd] & & X' \arrow[dd] \\
 & Y \arrow[dr, "f"] & \\
Y'' \arrow[ur] \arrow[rr] & & Y'.
\end{tikzcd}$$

This is a reformulation of the condition that the original square is a pullback square.

Example 2.1.8. The forgetful functor $\mathrm{VectBund} \to \mathrm{Top}, (X, E) \mapsto X$ is a cartesian fibration. See Exercise A.13 for details.

Lemma 2.1.9. *Show that cocartesian fibrations are closed under composition and pullback.*

Proof. This is left as an exercise, see Exercises A.14 and A.15 □

Exercise 2.1.10. Show that $C \to *$ is both a cartesian and cocartesian fibration. Deduce that any projection functor $\mathrm{pr}_D \colon C \times D \to D$ is both a cartesian and cocartesian fibration, and that a morphism $(f \colon c \to c', g \colon d \to d')$ is pr_D-cocartesian (or pr_D-cartesian) if and only if f is an equivalence.

Lemma 2.1.11 (Cancellation of cocartesian morphisms)**.** *Let $p \colon E \to C$ be a functor and let $f \colon x \to y$ and $g \colon y \to z$ be morphisms in E such that f is p-cocartesian. Then g is p-cocartesian if and only if gf is p-cocartesian.*

Proof. For every other object $w \in E$, we consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_E(z,w) \arrow[r, "-\circ g"] \arrow[d, "p"'] & \mathrm{Hom}_E(y,w) \arrow[r, "-\circ f"] \arrow[d, "p"', "p"] & \mathrm{Hom}_E(x,w) \arrow[d, "p"] \\
\mathrm{Hom}_C(pz,pw) \arrow[r, "-\circ p(g)"] & \mathrm{Hom}_C(py,pw) \arrow[r, "-\circ p(f)"] & \mathrm{Hom}_C(px,pw).
\end{tikzcd}$$

Since f is p-cocartesian, the right square is a pullback square. It follows from the pasting law for pullback squares that the left square is a pullback if and only if the outer square is a pullback. This proves the claim. □

Exercise 2.1.12. Let $p\colon E \to C$ be a functor and let $\varphi\colon e \to e'$ be a p-cocartesian morphism. Then φ is an isomorphism if and only if $p(\varphi)$ is an isomorphism.

2.1.1 Left and right fibrations

Proposition 2.1.13. *Let $p\colon E \to C$ be a cocartesian fibration. Then the following three conditions are equivalent:*

(1) The functor p is conservative, i.e., a morphism f in E is an isomorphism as soon as $p(f)$ is an isomorphism in C;

(2) For each $x \in C$, the fiber $E_x := \{x\} \times_C E$ is an anima;

(3) Every morphism in E is p-cocartesian.

Proof. It is clear that (1) implies (2): every morphism in the fiber E_x is sent to the identity morphism id_x, hence is an isomorphism by conservativity of p.

Assuming (2), consider a morphism $\psi\colon e \to e''$ in E. We need to show that ψ is p-cocartesian. Since p is a cocartesian fibration, there exists a p-cocartesian morphism $\varphi\colon e \to e'$ lifting $p(\psi)$. Since φ is p-cocartesian, there then exists a unique morphism $\xi\colon e' \to e''$ fitting in the following commutative diagram lifting the degenerate one in C:

$$\begin{tikzcd}
& e' \arrow[dr, dashed, "\xi"] & & & & pe'' \arrow[dr, equal] & \\
e \arrow[ur, "\varphi"] \arrow[rr, "\psi"] & & e'' & \arrow[r, squiggly, "p"] & pe \arrow[ur, "p\psi"] \arrow[rr, "p\psi"] & & pe''.
\end{tikzcd}$$

But then the morphism ξ lies in the fiber $E_{pe''}$, which is an anima, so ξ is an isomorphism. In particular ξ is p-cocartesian, and hence so is $\psi = \xi \circ \varphi$.

Finally, assume that (3) holds, and consider a morphism $\varphi\colon e \to e'$ in E such that $p\varphi$ is invertible. We need to show that φ is invertible. By assumption, there is an inverse $g\colon pe' \to pe$ of $p\varphi$. We may then find a p-cocartesian lift $\psi\colon e' \to e''$ of g. But then $\varphi \circ \psi$ and $\psi \circ \varphi$ are cocartesian lifts of the respective identity maps on pe and pe'. By uniqueness of cocartesian lifts (Lemma 2.1.2) these composites are isomorphisms. It follows that φ is an isomorphism with inverse ψ. □

Definition 2.1.14. A functor $p\colon E \to C$ is called a *left fibration* if it is a cocartesian fibration satisfying the equivalent conditions (1), (2) and (3) from the proposition. This gives rise to a full subcategory

$$\mathrm{LFib}(C) \subseteq \mathrm{Cocart}(C).$$

Note that $\mathrm{LFib}(C)$ is also a full subcategory of $(\mathrm{Cat}_\infty)_{/C}$ (in contrast to $\mathrm{Cocart}(C)$).

Dually, p is called a *right fibration* if it is a cartesian fibration which satisfies the equivalent conditions (1), (2) and (3'): Every morphism in E is cartesian. We write $\mathrm{RFib}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$ for the resulting full subcategory.

Lemma 2.1.15. *Left fibrations are closed under composition and pullback.*

Proof. Follows from Lemma 2.1.9. □

Proposition 2.1.16. *Let C be an ∞-category. For every object $x \in C$, the functor $t\colon C_{x/} \to C$ sending a morphism $x \to y$ to its target y is a left fibration. Dually, the source functor $s\colon C_{/x} \to C$ is a right fibration.*

Proof. We prove the case for $t\colon C_{x/} \to C$; the other case is dual. Consider a morphism in $C_{x/}$, which takes the form of a commutative triangle

$$\begin{tikzcd}
 & x \arrow[dl, "g"'] \arrow[dr, "h"] & \\
y \arrow[rr, "f"] & & z.
\end{tikzcd}$$

To show that this morphism is cocartesian with respect to the target functor $t\colon C_{x/} \to C$, we have to show that for every third object $x \to w$, the top square in the following commutative diagram is a pullback square:

$$\begin{tikzcd}
\mathrm{Hom}_{C_{x/}}(z,w) \arrow[rr, "-\circ f"] \arrow[dr, "t"] \arrow[dd] & & \mathrm{Hom}_{C_{x/}}(y,w) \arrow[dr, "t"] \arrow[dd] & \\
 & \mathrm{Hom}_C(z,w) \arrow[rr, "-\circ f"] \arrow[dd, "-\circ h"] & & \mathrm{Hom}_C(y,w) \arrow[dd, "-\circ g"] \\
* \arrow[rr, equal] \arrow[dr] & & * \arrow[dr] & \\
 & \mathrm{Hom}_C(x,w) \arrow[rr, equal] & & \mathrm{Hom}_C(x,w).
\end{tikzcd}$$

But this follows from the pasting law for pullback squares, since the bottom square and left/right faces are pullback squares. □

2.2 Straightening/unstraightening

Let $p\colon E \to C$ be a cocartesian fibration. For every object x of C, we may consider the fiber E_x of p over x. For a morphism $f\colon x \to y$ in C, there is a *cocartesian transport* functor $f_!\colon E_x \to E_y$. Furthermore, given another morphism $g\colon y \to z$, there is a natural isomorphism

$$g_! \circ f_! \cong (g \circ f)_!\colon E_x \to E_z.$$

Morally speaking, this construction should assemble into a functor

$$\mathrm{Str}^{\mathrm{cc}}(p)\colon C \to \mathrm{Cat}_\infty, \quad x \mapsto E_x, \quad f \mapsto f_!.$$

This is precisely the content of the straightening/unstraightening correspondence:

Theorem 2.2.1 (Straightening/Unstraightening, Lurie [Lur09b], Hebestreit, Heuts, and Ruit [HHR21])**.** *Let C be a small ∞-category. Then there exist equivalences of ∞-categories*

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty)$$

and

$$\mathrm{Str}^{\mathrm{ct}}\colon \mathrm{Cart}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty).$$

Furthermore, these equivalences are natural[1] *in C, in the sense that for every functor $F\colon C \to D$ the following two squares commute:*

$$\begin{tikzcd} \mathrm{Cocart}(D) \arrow[r, "\mathrm{Str}^{\mathrm{cc}}"] \arrow[d, "F^*"'] & \mathrm{Fun}(D, \mathrm{Cat}_\infty) \arrow[d, "-\circ F"] \\ \mathrm{Cocart}(C) \arrow[r, "\mathrm{Str}^{\mathrm{cc}}"] & \mathrm{Fun}(C, \mathrm{Cat}_\infty) \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathrm{Cart}(D) \arrow[r, "\mathrm{Str}^{\mathrm{ct}}"] \arrow[d, "F^*"'] & \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat}_\infty) \arrow[d, "-\circ F^{\mathrm{op}}"] \\ \mathrm{Cart}(C) \arrow[r, "\mathrm{Str}^{\mathrm{ct}}"] & \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty). \end{tikzcd}$$

Finally, the composites

$$\mathrm{Cat}_\infty \simeq \mathrm{Cocart}(*) \xrightarrow{\mathrm{Str}^{\mathrm{cc}}} \mathrm{Fun}(*, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$$
$$\mathrm{Cat}_\infty \simeq \mathrm{Cart}(*) \xrightarrow{\mathrm{Str}^{\mathrm{ct}}} \mathrm{Fun}(*, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$$

are (isomorphic to) the identity functors.

This is one of the most fundamental and most important results in the theory of ∞-categories. The theorem was first proved in the model of quasicategories by Jacob Lurie; the proof has since been substantially simplified by [HHR21]. Since this result is so fundamental to the theory, it may also be taken as an axiom, which is what we will do in these notes.

Given a cocartesian fibration $p\colon E \to C$, we refer to the functor $\mathrm{Str}^{\mathrm{cc}}(p)\colon C \to \mathrm{Cat}_\infty$ as its *(cocartesian) straightening*. The ∞-categories $\mathrm{Str}^{\mathrm{cc}}(p)(x)$ for $x \in C$ are precisely the fibers of p. In particular, $\mathrm{Str}^{\mathrm{cc}}(p)$ lands in the subcategory $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ if and only if all fibers are animae, which by Proposition 2.3.6 is equivalent to p being a left fibration. The dual discussion also holds for right fibrations, and we obtain:

Corollary 2.2.2. *The straightening equivalences* $\mathrm{Str}^{\mathrm{cc}}$ *and* $\mathrm{Str}^{\mathrm{ct}}$ *restrict to equivalences*

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{LFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{An}) \qquad \textit{and} \qquad \mathrm{Str}^{\mathrm{ct}}\colon \mathrm{RFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Note that if C is an anima, then every functor $E \to C$ is both a cocartesian and a cartesian fibration. It is a left/right fibration if and only if also E is an anima. This gives the following immediate consequence:

[1] See Remark 2.2.6 for a more refined version of the naturality.

Corollary 2.2.3. *Let X be an anima. Then there is an equivalence of ∞-categories*

$$\mathrm{Str}\colon \mathrm{An}_{/X} \xrightarrow{\sim} \mathrm{Fun}(X, \mathrm{An}).$$

Exercise 2.2.4. A morphism $f\colon Y \to X$ of animae is called a *covering* if all its fibers are sets, i.e., if all the higher homotopy groups of all fibers vanish. Letting $\mathrm{Cov}(X) \subseteq \mathrm{An}_{/X}$ denote the full subcategory spanned by the coverings, deduce that the straightening equivalence induces an equivalence

$$\mathrm{Cov}(X) \xrightarrow{\sim} \mathrm{Fun}(\Pi_1(X), \mathrm{Set})$$

between the category of covering spaces over X and the category of functors $\Pi_1(X) \to \mathrm{Set}$. Explain how this provides a high-tech version of the classical classification of covering spaces.

Notation 2.2.5. We will denote the inverses to the straightening functors $\mathrm{Str}^{\mathrm{cc}}$ and $\mathrm{Str}^{\mathrm{ct}}$ by

$$\mathrm{Un}^{\mathrm{cc}}\colon \mathrm{Fun}(C, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cocart}(C) \qquad \text{and} \qquad \mathrm{Un}^{\mathrm{ct}}\colon \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\sim} \mathrm{Cart}(C),$$

and refer to these as (cocartesian/cartesian) *unstraightening*. Given a functor $F\colon C \to \mathrm{Cat}_\infty$, we may describe its unstraightening $\mathrm{Un}^{\mathrm{cc}}(F)$ informally as follows:

- An object in $\mathrm{Un}^{\mathrm{cc}}(F)$ is a pair (x, a), where x is an object of C and a is an object of $F(x)$;
- A morphism in $\mathrm{Un}^{\mathrm{cc}}(F)$ from (x, a) to (y, b) is a pair (f, φ), where $f : x \to y$ is a morphism in C and $\varphi\colon f_!(a) \to b$ is a morphism in $F(y)$. Here we write $f_!$ for the functor $F(f)\colon F(x) \to F(y)$ induced by f using the functoriality of F.
- Composition of $(f, \varphi)\colon (x, a) \to (y, b)$ and $(g, \psi)\colon (y, b) \to (z, c)$ is defined as $(g \circ f, \psi \circ g_!(\varphi))$.

The functor $\mathrm{Un}^{\mathrm{cc}}(F) \to C$ is given by $(x, a) \mapsto x$ and $(f, \varphi) \mapsto f$. A dual description holds for $\mathrm{Un}^{\mathrm{ct}}(F)$.

Remark 2.2.6. In Theorem 2.2.1 we only formulated a ‘morphismwise’ version of functoriality of the straightening equivalences. Let us remark on how to formulate a fully functorial version.

For this, we will need to assume the existence of an even bigger universe CAT_∞ of *large* ∞-categories, and assume that Cat_∞ is an object of CAT_∞. Applying cartesian straightening to the cartesian fibration $\mathrm{ev}_1\colon \mathrm{Fun}([1], \mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty$ results in a contravariant functor

$$\mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{CAT}_\infty, \quad C \mapsto (\mathrm{Cat}_\infty)_{/C}, \quad (F\colon C \to D) \mapsto (F^*\colon (\mathrm{Cat}_\infty)_{/D} \to (\mathrm{Cat}_\infty)_{/C}).$$

Since cocartesian fibrations are closed under pullback, this restricts to a functor

$$\mathrm{Cocart}(-)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{CAT}_\infty.$$

Similarly, the assignment $C \mapsto \mathrm{Fun}(C, \mathrm{Cat}_\infty)$ defines a functor

$$\mathrm{Fun}(-, \mathrm{Cat}_\infty)\colon \mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{CAT}_\infty,$$

where functoriality is given by precomposition. The fully coherent version of the straightening equivalence would now ask for a natural equivalence

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{Cocart}(-) \xrightarrow{\sim} \mathrm{Fun}(-, \mathrm{Cat}_\infty)$$

of functors $\mathrm{Cat}_\infty^{\mathrm{op}} \to \mathrm{CAT}_\infty$.

2.3 Fiberwise criteria

We obtain the following useful consequence of straightening-unstraightening:

Theorem 2.3.1. *Consider a cocartesian functor*

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & C, & \end{tikzcd}$$

where p and p' are cocartesian fibrations between small ∞-categories. Then F is an equivalence of ∞-categories if and only if for every object x of C, the induced functor on fibers $F_x\colon E_x \to E'_x$ is an equivalence.

Proof. Under the equivalence $\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty)$, F corresponds to a natural transformation $\mathrm{Str}^{\mathrm{cc}}(F)\colon \mathrm{Str}^{\mathrm{cc}}(p) \to \mathrm{Str}^{\mathrm{cc}}(p')$, and this natural transformation is a natural equivalence if and only if it is a pointwise equivalence. □

Corollary 2.3.2. *A cocartesian fibration $p\colon E \to C$ is an equivalence if and only if all of its fibers are contractible.*

Proof. Apply the previous theorem to $E' = C$. □

Corollary 2.3.3. *A morphism of animae $p\colon X \to Y$ is an equivalence if and only if all its fibers are contractible.*

Proof. Since X is an anima, every morphism in X is an isomorphism, and in particular p-cocartesian. It follows that p is a left fibration, and thus the claim holds by the previous corollary. □

Lemma 2.3.4. *Let $p\colon E \to C$ be a functor. Then a morphism $\varphi\colon e \to e'$ in E is p-cocartesian if and only if the commutative square*

$$\begin{tikzcd} E_{e'/} \arrow[r, "-\circ\varphi"] \arrow[d, "p"'] & E_{e/} \arrow[d, "p"] \\ C_{pe'/} \arrow[r, "-\circ p\varphi"] & C_{pe/} \end{tikzcd}$$

is a pullback square. A dual criterion holds for p-cartesian morphisms.

Proof. The horizontal maps are left fibrations, hence in particular cocartesian fibrations. It follows from closure under base change that also the map from the pullback to $E_{e/}$ is a cocartesian fibration. By Theorem 2.3.1, the functor $E_{e'/} \to E_{e/} \times_{C_{pe/}} C_{pe'/}$ is an equivalence if and only if this holds fiberwise over $E_{e/}$, which means that for every morphism $\psi\colon e \to e''$ the map

$$\mathrm{Hom}_{E_{e/}}(\varphi, \psi) \to \mathrm{Hom}_{C_{pe/}}(p\varphi, p\psi)$$

is an equivalence. Since this is a morphism of animae, this may be checked fiberwise, where it precisely becomes the definition of cocartesianness for φ. □

Lemma 2.3.5. *Let $q: E \to D$ and $p: D \to C$ be functors and assume that p is a left fibration. Then q is a left fibration if and only if pq is a left fibration.*

Proof. As a consequence of Lemma 2.3.4, we see that a morphism $\varphi: e \to e'$ in E is q-cocartesian if and only if it is pq-cocartesian: in the commutative diagram

$$\begin{tikzcd} E_{e'/} \arrow[r, "-\circ\varphi"] \arrow[d, "q"'] & E_{e/} \arrow[d, "q"] \\ D_{qe'/} \arrow[r, "-\circ q\varphi"] \arrow[d, "p"'] & D_{qe/} \arrow[d, "p"] \\ C_{pqe'/} \arrow[r, "-\circ pq\varphi"] & C_{pqe/}, \end{tikzcd}$$

the bottom square is a pullback square, hence the top square is a pullback square if and only if the bottom one is. It follows that every morphism in E is q-cocartesian if and only if every morphism is pq-cocartesian. The claim follows. □

Proposition 2.3.6. *For a functor $p: E \to C$, the following conditions are equivalent:*

(1) *The functor p is a left fibration;*

(2) *For every object e of E, the induced functor $p: E_{e/} \to C_{pe/}$ is an equivalence;*

(3) *The commutative square*

$$\begin{tikzcd} \mathrm{Fun}([1], E) \arrow[r, "p_*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_0"] \\ E \arrow[r, "p"] & C \end{tikzcd}$$

is a pullback square.

Proof. For (1) $\Rightarrow$ (2), assume that p is a left fibration. Since $E_{e/} \to E$ and $C_{pe/} \to C$ are left fibrations as well, it follows from Lemma 2.3.5 that also $E_{e/} \to C_{pe/}$ is a left fibration, hence it is an equivalence if and only if each of its fibers is contractible. But the fiber over $f \in C_{pe/}$ is the anima $L(e, f)$ of p-cocartesian lifts of f starting in e, hence this is contractible by Lemma 2.1.2.

For (2) $\Rightarrow$ (1), first note that it follows from (2) that every morphism $pe \to c'$ has a lift to some morphism $e \to e'$. It remains to show that every morphism $\varphi: e \to e'$ in E is p-cocartesian. But this is a direct consequence of Lemma 2.3.4: the two vertical maps in the square are equivalences, hence the square is a pullback square.

For (2) $\Rightarrow$ (3), note that the square is a pullback square if and only if the horizontal map in the following diagram is an equivalence:

$$\begin{tikzcd} \mathrm{Fun}([1], E) \arrow[rr, "(\mathrm{ev}_0, p_*)"] \arrow[dr, "\mathrm{ev}_0"'] & & E \times_C \mathrm{Fun}([1], C) \arrow[dl, "\mathrm{pr}"] \\ & E. & \end{tikzcd}$$

By the dual of Example 2.1.7, the two vertical functors are cartesian fibrations. In light of Theorem 2.3.1, the horizontal functor is an equivalence if and only if it induces equivalences on fibers over each object $e \in E$. Unwinding definitions, these functors on fibers are precisely the functors $E_{e/} \to C_{pe/}$ from condition (2). $\square$

2.4 The Hom-functor

Last semester, we had taken it as a black box that every ∞-category C admits a Hom-functor

$$\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}.$$

Using the straightening/unstraightening correspondence, we will now finally construct this functor.

Lemma 2.4.1. *Let C be an ∞-category. Then the functor*

$$\begin{tikzcd} \mathrm{Fun}([1], C) \arrow[rr, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_1"'] & & C \times C \arrow[dl, "\mathrm{pr}_2"] \\ & C & \end{tikzcd}$$

is a cocartesian functor over C.

Proof. Recall from Example 2.1.7 that the target functor $\mathrm{ev}_1 \colon \mathrm{Fun}([1], C) \to C$ is a cocartesian fibration whose cocartesian morphisms are those squares inverted by $\mathrm{ev}_0 \colon \mathrm{Fun}([1], C) \to C$. The projection functor pr_2 is cocartesian by Exercise 2.1.10, and its cocartesian morphisms are those pairs $(g \colon X \to X', f \colon Y \to Y')$ such that g is an isomorphism. It follows immediately from these descriptions that $(\mathrm{ev}_0, \mathrm{ev}_1)$ is a cocartesian functor. $\square$

Construction 2.4.2 (The Hom-functor)**.** Let C be a small ∞-category. We construct a functor $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$.

By Lemma 2.4.1, the functor $(\mathrm{ev}_0, \mathrm{ev}_1) \colon \mathrm{Fun}([1], C) \to C \times C$ defines a morphism in $\mathrm{Cocart}(C)$. Applying the cocartesian straightening functor, we thus obtain a morphism

$$\mathrm{Str}^{\mathrm{cc}}(\mathrm{ev}_1) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$$

in $\mathrm{Fun}(C, \mathrm{Cat}_\infty)$. Now, note that pr_2 is the pullback along $p_C \colon C \to *$ of the map $p_C \colon C \to *$, and it follows from the naturality of straightening that $\mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$ is given by the composite

$$C \xrightarrow{p_C} * \xrightarrow{\mathrm{Str}(p_C)} \mathrm{Cat}_\infty.$$

Since $\mathrm{Str}(p_C)$ is simply the map picking out the object $C \in \mathrm{Cat}_\infty$, this means that $\mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)$ is the constant functor $\mathrm{const}_C \colon C \to \mathrm{Cat}_\infty$. In particular, the map $\mathrm{Str}^{\mathrm{cc}}(\mathrm{ev}_1) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2) \cong \mathrm{const}_C$ gives rise to an object

$$\mathrm{Str}^{\mathrm{cc}}(\mathrm{ev}_1) \quad \in \quad \mathrm{Fun}(C, \mathrm{Cat}_\infty)_{/\mathrm{const}_C} \simeq \mathrm{Fun}(C, (\mathrm{Cat}_\infty)_{/C}),$$

i.e., we may treat it as a functor $C \to (\mathrm{Cat}_\infty)_{/C}$. We now claim that this functor factors through the full subcategory $\mathrm{RFib}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$. We may check this on objects: we have to show that for every object $x \in C$, the functor $\mathrm{Str}^{\mathrm{cc}}(\mathrm{ev}_1)(x) \to \mathrm{Str}^{\mathrm{cc}}(\mathrm{pr}_2)(x)$ is a right fibration. By naturality of $\mathrm{Str}^{\mathrm{cc}}(-)$, this evaluation is computed by forming the induced map on fibers over x:

$$\begin{tikzcd}
 & C \arrow[rr] \arrow[dd] & & C \times C \arrow[dd] \\
C_{/x} \arrow[ur, "s"] \arrow[rr] \arrow[dr] & & \mathrm{Fun}([1], C) \arrow[ur, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[dr] & \\
 & * \arrow[rr, "x"] & & C.
\end{tikzcd}$$

But this functor $s\colon C_{/x} \to C$ is indeed a right fibration by Proposition 2.1.16. We thus obtain a functor

$$C \to \mathrm{RFib}(C) \xrightarrow[\sim]{\mathrm{Str}^{\mathrm{ct}}} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Uncurrying then gives the desired functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$.

Remark 2.4.3. There is an asymmetry in our definition of Hom_C: we could also have considered the *cartesian* functor

$$\begin{tikzcd}
\mathrm{Fun}([1], C) \arrow[rr, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \arrow[dr, "\mathrm{ev}_0"'] & & C \times C \arrow[dl, "\mathrm{pr}_1"] \\
 & C &
\end{tikzcd}$$

to get a functor $C^{\mathrm{op}} \to \mathrm{LFib}(C) \simeq \mathrm{Fun}(C, \mathrm{An})$. The resulting two functors can be shown to agree with each other: more generally, for *bifibrations* $E \to C \times D$ the two ways of constructing a functor $C^{\mathrm{op}} \times D \to \mathrm{An}$ agree, see [Hau+23b, Proposition 6.18].

2.5 Limits and colimits of ∞-categories

As a consequence of straightening/unstraightening, we obtain explicit formulas for limits and colimits in Cat_∞.

Construction 2.5.1. Let $p\colon E \to I$ be a cocartesian fibration.

- We write $E[\mathrm{cc}^{-1}]$ for the localization of E at the collection of p-cocartesian morphisms.
- We write $\Gamma_I(E)$ for the ∞-category of *sections of p*, defined via the following pullback square:

$$\begin{tikzcd}
\Gamma_I(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(I, E) \arrow[d, "p_*"] \\
* \arrow[r, "\mathrm{id}_I"] & \mathrm{Fun}(I, I).
\end{tikzcd}$$

- We write $\Gamma_I^{\mathrm{cc}}(E)$ for the full subcategory of $\Gamma_I(E)$ spanned by the *cocartesian sections*: those sections

$$\begin{tikzcd}
I \arrow[rr, "s"] \arrow[dr, equal] & & E \arrow[dl, "p"] \\
 & I &
\end{tikzcd}$$

  that are cocartesian functors over I (i.e., that send arbitary morphisms in I to p-cocartesian morphisms in E).

Proposition 2.5.2. *Let I be a small ∞-category and let $F\colon I \to \mathrm{Cat}_\infty$ be a functor.*

(1) The functor F admits a colimit in Cat_∞, which can be computed by the formula

$$\operatorname{colim}_I F(i) \simeq \mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}].$$

(2) The functor F admits a limit in Cat_∞, which can be computed by the formula

$$\lim_I F(i) \simeq \Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F)).$$

Proof. (1) We need to show that there exists a natural transformation $\eta_F\colon F \to \mathrm{const}_{\mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}]}$ of functors $I \to \mathrm{Cat}_\infty$ which exhibits $\mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}]$ as a left adjoint object to F under the functor

$$\mathrm{const}\colon \mathrm{Cat}_\infty \to \mathrm{Fun}(I, \mathrm{Cat}_\infty).$$

By straightening/unstraightening, this functor agrees with the functor

$$\mathrm{Cat}_\infty \simeq \mathrm{Cocart}(*) \xrightarrow{p_I^*} \mathrm{Cocart}(I), \qquad C \mapsto (\mathrm{pr}_I\colon C \times I \to I)$$

given by pullback along $p_I\colon I \to *$. A transformation of the form η_F corresponds to a cocartesian functor over I of the form

$$\begin{tikzcd} \mathrm{Un}^{\mathrm{cc}}(F) \arrow[rr, "\overline{\eta}"] \arrow[dr, "p_F"'] & & \mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}] \times I \arrow[dl, "\mathrm{pr}_I"] \\ & I. & \end{tikzcd}$$

We define $\overline{\eta} := (\gamma, p_F)$, where $\gamma\colon \mathrm{Un}^{\mathrm{cc}}(F) \to \mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}]$ is the localization functor. Since the cocartesian morphisms of pr_I are those whose component in $\mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}]$ is an equivalence (see Exercise 2.1.10), $\overline{\eta}$ is indeed a cocartesian functor over I. To show that it exhibits $\mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}]$ as a left adjoint object, we must show that for any other ∞-category C the induced map

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Un}^{\mathrm{cc}}(F)[\mathrm{cc}^{-1}], C) \xrightarrow{(-\circ\overline{\eta})\circ p_I^*} \mathrm{Hom}_{\mathrm{Cocart}(I)}(\mathrm{Un}^{\mathrm{cc}}(F), C \times I) \tag{2.1}$$

is an equivalence. For this, we compare this map with the composite

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Un}^{\mathrm{cc}}(F), C) \xrightarrow{(-\circ(\mathrm{id}, p_F))\circ p_I^*} \mathrm{Hom}_{(\mathrm{Cat}_\infty)_{/I}}(\mathrm{Un}^{\mathrm{cc}}(F), C \times I),$$

which is an equivalence due to the fact that the forgetful functor $(\mathrm{Cat}_\infty)_{/I} \to \mathrm{Cat}_\infty$ is left adjoint to $p_I^*\colon \mathrm{Cat}_\infty \to (\mathrm{Cat}_\infty)_{/I}$. But then also the composite (2.1) is a composite: its source is the full subanima of $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Un}^{\mathrm{cc}}(F), C)$ spanned by those functors $\varphi\colon \mathrm{Un}^{\mathrm{cc}}(F) \to C$ that invert the cocartesian morphisms, which is equivalent to the condition that the functor $(\varphi, p_F)\colon \mathrm{Un}^{cc}(F) \to C \times I$ preserves cocartesian morphisms. This finishes the proof for colimits.

(2) Similar to the case of colimits, we may translate the problem into finding a cocartesian functor

$$\begin{tikzcd} \Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F)) \times I \arrow[rr, "\overline{\varepsilon}"] \arrow[dr, "\mathrm{pr}_I"'] & & \mathrm{Un}^{\mathrm{cc}}(F) \arrow[dl, "p_F"] \\ & I & \end{tikzcd}$$

over I that exhibits $\Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F))$ as a right adjoint to $\mathrm{Un}^{\mathrm{cc}}(F)$ under the pullback functor $p_I^*\colon \mathrm{Cat}_\infty \to \mathrm{Cocart}(I)$. By definition of $\Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F))$, the evaluation functor $\mathrm{Fun}(I, \mathrm{Un}^{\mathrm{cc}}(F)) \times I \to \mathrm{Un}^{\mathrm{cc}}(F)$ restricts to a functor

$$\begin{tikzcd} \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F)) \times I \arrow[rr, "\mathrm{ev}"] \arrow[dr, "\mathrm{pr}_I"'] & & \mathrm{Un}^{\mathrm{cc}}(F) \arrow[dl, "p_F"] \\ & C & \end{tikzcd}$$

over C. We claim that this exhibits $\Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F))$ as a right adjoint object to $\mathrm{Un}^{\mathrm{cc}}(F)$ under the functor $\mathrm{Cat}_\infty \to (\mathrm{Cat}_\infty)_{/I}$: for every other ∞-category C we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cat}_\infty}(C, \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F))) &\simeq \mathrm{fib}(\mathrm{Hom}_{\mathrm{Cat}_\infty}(C, \mathrm{Fun}(I, \mathrm{Un}^{\mathrm{cc}}(F))) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(C, \mathrm{Fun}(I, I))) \\ &\simeq \mathrm{fib}(\mathrm{Hom}_{\mathrm{Cat}_\infty}(C \times I, \mathrm{Un}^{\mathrm{cc}}(F)) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(C \times I, I))) \\ &\simeq \mathrm{Hom}_{(\mathrm{Cat}_\infty)_{/I}}(C \times I, \mathrm{Un}^{\mathrm{cc}}(F)). \end{aligned}$$

By defining $\overline{\varepsilon}$ as the restriction of the evaluation functor, we may immediately deduce that also the functor

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(C, \Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}})(F)) \xrightarrow{(\overline{\varepsilon}\circ -)\circ p_I^*} \mathrm{Hom}_{\mathrm{Cocart}(I)}(C \times I, \mathrm{Un}^{\mathrm{cc}}(F))$$

is an equivalence: a functor $\varphi\colon C \to \Gamma_I(\mathrm{Un}^{\mathrm{cc}})(F)$ lands in the subcategory of cocartesian sections if and only if the associated functor $C \times I \to \mathrm{Un}^{\mathrm{cc}}(F)$ over I is a cocartesian functor. This finishes the proof for limits. □

Corollary 2.5.3. *Let I be a small ∞-category and let $F\colon I \to \mathrm{An}$ be functor.*

(1) The functor F admits a colimit in An*, which can be computed by the formula*

$$\mathrm{colim}_I\, F(i) \simeq \| \mathrm{Un}^{\mathrm{cc}}(F) \|.$$

(2) The functor F admits a limit in An*, which can be computed by the formula*

$$\lim_I F(i) \simeq \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F)).$$

Proof. We compute the limit and colimit in Cat_∞ and observe that it is still in An. For (1) this is clear: all morphisms in $\mathrm{Un}^{\mathrm{cc}}(F)$ are cocartesian, see Proposition 2.3.6, hence inverting all cocartesian morphisms gives an anima. For (2), any section $s\colon I \to \mathrm{Un}^{\mathrm{cc}}(F)$ of p_F is cocartesian, again since all morphisms are cocartesian, so $\Gamma_I^{\mathrm{cc}}(\mathrm{Un}^{\mathrm{cc}}(F)) = \Gamma_I(\mathrm{Un}^{\mathrm{cc}}(F))$. Note that this is already an anima: any morphism of sections is invertible since all fibers of $p_F\colon \mathrm{Un}^{\mathrm{cc}}(F) \to I$ are animae. □

2.6 Hom animae in functor categories

We may use the results from the previous section to compute the Hom animae in a functor category $\mathrm{Fun}(C, D)$. For this, we introduce the notion of an *end* in the ∞-categorical setting.

Definition 2.6.1 (Twisted arrow category)**.** Let C be an ∞-category. We define its *twisted arrow category* $\mathrm{Tw}(C)$ as the source of the left fibration whose cocartesian straightening is the Hom-functor $\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}$:

$$\mathrm{Tw}(C) := \mathrm{Un}^{\mathrm{cc}}(\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}).$$

In particular, it comes equipped with a left fibration $(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ whose fibers are the Hom animae of C:

$$\begin{tikzcd}
\mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(C) \arrow[d, "(s,t)"] \\
* \arrow[r, "(x,y)"] & C^{\mathrm{op}} \times C.
\end{tikzcd}$$

Unwinding the description of the unstraightening construction, we see that $\mathrm{Tw}(C)$ may be described as follows:

- The objects of $\mathrm{Tw}(C)$ are morphisms in C, which we will display as $\left(\begin{smallmatrix} X \\ \downarrow\varphi \\ Y \end{smallmatrix}\right)$. We have $s\left(\begin{smallmatrix} X \\ \downarrow\varphi \\ Y \end{smallmatrix}\right) = X$ and $t\left(\begin{smallmatrix} X \\ \downarrow\varphi \\ Y \end{smallmatrix}\right) = Y$.
- A morphism $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\colon \left(\begin{smallmatrix} X \\ \downarrow\varphi \\ Y \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ \downarrow\varphi' \\ Y' \end{smallmatrix}\right)$ in $\mathrm{Tw}(C)$ is a triple (f,g,α) consisting of a morphism $f\colon X \to X'$ in C^{op} (i.e., $f\colon X' \to X$ in C), a morphism $g\colon Y \to Y'$ in C, and a morphism $\alpha\colon (f,g)_!\left(\begin{smallmatrix} X \\ \downarrow\varphi \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} X' \\ \downarrow g\varphi f \\ Y' \end{smallmatrix}\right) \to \left(\begin{smallmatrix} X' \\ \downarrow\varphi' \\ Y' \end{smallmatrix}\right)$ in the fiber $\mathrm{Tw}(C)_{(X',Y')} = \mathrm{Hom}_C(X',Y')$. We will display this data by means of diagrams of the form

$$\begin{tikzcd}
X \arrow[d, "\varphi"'] & X' \arrow[l, "f"'] \arrow[d, "\varphi'"] \\
Y \arrow[r, "g"] & Y'.
\end{tikzcd}$$

  Note that this is *not* a morphism in the usual arrow category $\mathrm{Fun}([1],C)$, since f points in the opposite direction.

Definition 2.6.2 (End)**.** Let C be an ∞-category and consider a functor $F\colon C^{\mathrm{op}} \times C \to D$. The *end of* F, denoted by $\int_{c\in C} F(c,c)$ if it exists, is the limit of the composite functor

$$\mathrm{Tw}(C) \xrightarrow{(s,t)} C^{\mathrm{op}} \times C \xrightarrow{F} D.$$

Remark 2.6.3. If C is small and D admits small limits, the end of F always exists. Furthermore, if $D \to D'$ is a functor preserving small limits, then it also preserves ends.

Lemma 2.6.4 (Fubini rule for ends)**.** *Let C and C' be ∞-categories and let $F\colon C^{\mathrm{op}} \times C \times C'^{\mathrm{op}} \times C' \to D$ be a functor. Then there is an equivalence*

$$\int_{c\in C}\int_{c'\in C'} F(c,c,c',c') \simeq \int_{(c,c')\in C\times C'} F(c,c,c',c').$$

Proof. Since a limit over a product category may be computed as an iterated limit, it will suffice to produce an equivalence

$$\mathrm{Tw}(C \times C') \simeq \mathrm{Tw}(C) \times \mathrm{Tw}(C')$$

of left fibrations over $C^{\mathrm{op}} \times C \times C'^{\mathrm{op}} \times C'$. By straightening-unstraightening, it will suffice to produce a natural isomorphism between their unstraightenings:

$$(x,y,x',y') \quad\mapsto\quad \mathrm{Hom}_{C\times C'}((x,x'),(y,y')) \simeq \mathrm{Hom}_C(x,y) \times \mathrm{Hom}_{C'}(x',y').$$

Unwinding the definition of the hom functors, this boils down to the canonical equivalence

$$\mathrm{Fun}([1], C \times C') \xrightarrow{\sim} \mathrm{Fun}([1], C) \times \mathrm{Fun}([1], C')$$

over $C \times C \times C' \times C'$, coming from the fact that $\mathrm{Fun}([1], -)$ preserves products. □

Proposition 2.6.5 (Hom animae in functor categories)**.** *Let $F, G\colon C \to D$ be functors. Then we have the following end-formula for the anima of natural transformations from F to G:*

$$\mathrm{Nat}(F, G) \quad \simeq \quad \int_{x \in C} \mathrm{Hom}_D(F(x), G(x)).$$

Here the right-hand side is the end of the composite functor $C^{\mathrm{op}} \times C \xrightarrow{F^{\mathrm{op}} \times G} D^{\mathrm{op}} \times D \xrightarrow{\mathrm{Hom}_D} \mathrm{An}$.

Proof. By definition, the end on the right-hand side is a certain limit in An over the twisted arrow category $\mathrm{Tw}(C)$. As shown in Corollary 2.5.3, this limit is equivalent to the anima of sections of the unstraightening of the composite $\mathrm{Tw}(C) \to \mathrm{An}$. By definition, the unstraightening of Hom_D is the twisted arrow category $\mathrm{Tw}(D) \to D^{\mathrm{op}} \times D$. The unstraightening of the composite $\mathrm{Tw}(C) \to \mathrm{An}$ is thus given by pulling back $\mathrm{Tw}(D)$ along $F^{\mathrm{op}} \times G$ and (s, t):

$$\begin{tikzcd}
P' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathrm{Tw}(D) = \mathrm{Un}(\mathrm{Hom}_D) \arrow[d, "(s,t)"] \\
\mathrm{Tw}(C) \arrow[r, "(s,t)"] & C^{\mathrm{op}} \times C \arrow[rr, "F^{\mathrm{op}} \times G"] & & D^{\mathrm{op}} \times D \arrow[r, "\mathrm{Hom}_D"] & \mathrm{An}.
\end{tikzcd}$$

From the left pullback square it in particular follows that the sections of the map $P \to \mathrm{Tw}(C)$ are equivalent to maps $\mathrm{Tw}(C) \to P$ over $C^{\mathrm{op}} \times C$. Since both sides define left fibrations over $C^{\mathrm{op}} \times C$, the anima of such maps over $C^{\mathrm{op}} \times C$ is equivalent to the anima of natural transformations between their straightenings, which for $\mathrm{Tw}(C)$ is Hom_C and for P is $\mathrm{Hom}_D(F(-), G(-))$. Let us summarize what we have so far:

$$\begin{aligned}
\int_{x \in C} \mathrm{Hom}_D(F(x), G(x)) &= \lim\left(\mathrm{Tw}(C) \xrightarrow{(s,t)} C^{\mathrm{op}} \times C \xrightarrow{F^{\mathrm{op}} \times G} D^{\mathrm{op}} \times D \xrightarrow{\mathrm{Hom}_D} \mathrm{An}\right) \\
&\simeq \Gamma_{\mathrm{Tw}(C)}(P') = \mathrm{Hom}_{\mathrm{LFib}(\mathrm{Tw}(C))}(\mathrm{Tw}(C), P') \\
&\simeq \mathrm{Hom}_{\mathrm{LFib}(C^{\mathrm{op}} \times C)}(\mathrm{Tw}(C), P) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An})}(\mathrm{Hom}_C, \mathrm{Hom}_D \circ (F^{\mathrm{op}} \times G)).
\end{aligned}$$

Now, under the equivalence $\mathrm{Fun}(C^{\mathrm{op}} \times C, \mathrm{An}) \simeq \mathrm{Hom}(C, \mathrm{PSh}(C))$, the functor Hom_C corresponds to the Yoneda embedding $Y^C\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$, while $\mathrm{Hom}_D \circ (F^{\mathrm{op}} \times G)$ corresponds to the composite

$$C \xrightarrow{G} D \xrightarrow{Y^D} \mathrm{PSh}(D) \xrightarrow{F^*} \mathrm{PSh}(C).$$

The Hom anima we need to compute is thus given by

$$\mathrm{Hom}_{\mathrm{Fun}(C, \mathrm{PSh}(C))}(Y^C, F^* \circ Y^D \circ G).$$

By [TO DO], the functor $F^*\colon \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ admits a left adjoint $F_!\colon \mathrm{PSh}(C) \to \mathrm{PSh}(D)$ fitting in a commutative diagram as follows:

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, hook, "Y^C"'] & D \arrow[d, hook, "Y^D"] \\
\mathrm{PSh}(C) \arrow[r, dashed, "F_!"] & \mathrm{PSh}(D).
\end{tikzcd}$$

This provides equivalences

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{PSh}(C))}(Y^C, F^* \circ Y^D \circ G) &\simeq \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{PSh}(D))}(F_! \circ Y^C, Y^D \circ G) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(C,\mathrm{PSh}(D))}(Y^D \circ F, Y^D \circ G) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(C,D)}(F, G) = \mathrm{Nat}(F, G),
\end{aligned}$$

where the last equivalence follows from fully faithfulness of the Yoneda embedding Y^D. □

3 Span categories

In the approach to higher algebra I am taking in this course, a prominent role is played by the notion of a *span category*. In recent years, this notion has gained a lot of prominence among researchers, as it provides a convenient way to encode algebraic structures in a variety of settings, like those of *spectral Mackey functors* [Bar17], *normed ∞-categories* [BH21] or *six-functor formalisms* [Sch23; HM23]. So far, the use of span categories in higher algebra has not yet become mainstream, which is probably due in large part to the fact that the main textbook on higher algebra, [Lur17], does not use them. For this reason, I will dedicate some time in this chapter to a proper introduction of span categories.

Let C be an ∞-category with pullbacks, and let C_L and C_R be collections of morphisms of C closed under composition and pullback. Then the *span category* $\mathrm{Span}_{L,R}(C)$ associated with this triple is an ∞-category whose objects are the objects of C, while the morphisms from X to Y are given by *spans*

$$\begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y, \end{tikzcd}$$

where $f \in C_L$ and $g \in C_R$. The identity morphism is the *identity span* $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$. The composition of a span $X \leftarrow U \rightarrow Y$ with a span $Y \leftarrow V \rightarrow Z$ is given by forming pullbacks:

$$\begin{tikzcd} & & U \times_Y V \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\ & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\ X & & Y & & Z. \end{tikzcd}$$

In this section, we will give a formal definition of $\mathrm{Span}_{L,R}(C)$. We will further discuss the existence of products and coproducts in $\mathrm{Span}_{L,R}(C)$.

3.1 ∞-Categories as complete Segal animae

Constructing ∞-categories is more subtle than constructing ordinary categories: it does not suffice to define composition of morphisms by hand and check unitality and associativity, due to the fact that there are higher coherences that should be encoded. In most cases, the ∞-categories one is interested in can be built up from a few elementary constructions, such as products, pullbacks, and functor categories. However, in certain cases, like that of $\mathrm{Span}_{L,R}(C)$, this is not immediately possible.

In this section, we will introduce an alternative method for constructing ∞-categories, originally due to Rezk [Rez01]: by writing down a simplicial anima $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ satisfying certain *Segal* and *Rezk* conditions. Note that every ∞-category gives rise to such a simplicial anima, called its *nerve*:

Definition 3.1.1. The *nerve* of an ∞-category C is the simplicial anima $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ defined as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\infty}(-,C)} \mathrm{An}.$$

This construction is functorial in C and thus defines a functor $N\colon \mathrm{Cat}_\infty \to \mathrm{sAn}$.

The anima of n-simplices of $N(C)$ is given by

$$N(C)_n = \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C) \simeq \mathrm{Map}([n], C).$$

We may think of points of $\mathrm{Map}([n], C)$ as 'strings of n composable morphisms in C':

- For $n = 0$, $\mathrm{Map}([0], C) = \mathrm{Fun}(*, C)^{\simeq} \simeq C^{\simeq}$ is just the groupoid core of C;
- For $n = 1$, $\mathrm{Map}([1], C)$ is the anima of morphisms in C;
- For $n = 2$, $\mathrm{Map}([2], C)$ is the anima of commutative triangles in C, which is equivalent to $\mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C)$ by the Segal condition (see 213 for a recollection);
- More generally, the 'higher version' of the Segal condition inductively implies that for every $n \geq 0$ we have

  $$\mathrm{Map}([n], C) \xrightarrow{\sim} \mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C) \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \mathrm{Map}([1], C)$$

  for all n.

Note that all the relevant information of C is captured by its nerve: the objects are captured by the anima $N(C)_0 = C^{\simeq}$, the Hom animae $\mathrm{Hom}_C(x, y)$ may be recovered as the fiber over (x, y) of the map $(d_1, d_0)\colon N(C)_1 \to N(C)_0 \times N(C)_0$, and composition in C is captured by $N(C)_2$. But note that $N(C)$ also captures all the higher coherences for composition in C, via the higher simplices in $N(C)$. It thus seems plausible that C may be recovered from $N(C)$. This is the content of the following important theorem:

Theorem 3.1.2 ([JT07, Section 4], [Lur09a, Corollary 4.3.16], [HS23])**.** *The nerve functor*

$$N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$$

is fully faithful. Its image consists of the complete Segal animae (see below). In particular, the nerve functor induces an equivalence

$$N\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{CSeg}(\mathrm{An})$$

between Cat_∞ *and the* ∞*-category of complete Segal animae.*

The power of this theorem is that it allows us to construct ∞-categories by writing down a simplicial anima $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ and checking that it lies in this subcategory of complete Segal animae. This will precisely be how we construct the span category $\mathrm{Span}_{L,R}(C)$ in Section 3.2 below. In the remainder of this section, we will give some more background on Theorem 3.1.2.

Definition 3.1.3 (Segal anima)**.** A simplicial anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is called a *Segal anima* if the Segal condition is satisfied: For every $n \geq 1$, the inclusion maps $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ induce an isomorphism of animae

$$(e_1^*, \dots, e_n^*)\colon X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

We write $\mathrm{Seg}(\mathrm{An}) \subseteq \mathrm{sAn} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An})$ for the full subcategory of Segal animae.

Observe that the nerve $N(C)$ of an ∞-category C is always a Segal anima: this is essentially the content of the Segal Axiom (see the recollections on page 213).

Note that a Segal anima X comes equipped with a composition map

$$- \circ -\colon X_1 \times_{X_0} X_1 \simeq X_2 \xrightarrow{d_1} X_1.$$

One can show that composition is unital and associative.

Convention 3.1.4. Given a Segal anima X, we refer to 0-simplices $x \in X_0$ as *objects* of X, and to 1-simplices $f \in X_1$ as *morphisms* of X. We use the notation $f\colon x \to y$ when $d_1(f) \simeq x$ and $d_0(f) \simeq y$. Given objects $x, y \in X_0$, we define the *Hom anima* in X as the following pullback:

$$\begin{tikzcd}
\mathrm{Hom}_X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d, "{(d_1, d_0)}"] \\
* \arrow[r, "{(x,y)}"] & X_0 \times X_0.
\end{tikzcd}$$

We will sometimes refer to 2-simplices as *homotopies*.

Definition 3.1.5. Let $X \in \mathrm{Seg}(\mathrm{An})$ be a Segal anima. A morphism $f\colon x \to y$ in X is called an *isomorphism* if there exist 2-simplices $\sigma, \tau \in X_2$ satisfying the relations $d_0(\sigma) \simeq f$, $d_1(\sigma) \simeq \mathrm{id}_x$, $d_2(\tau) \simeq f$ and $d_1(\tau) \simeq \mathrm{id}_y$. More informally: f is an isomorphism if there exist morphisms $g, h\colon y \to x$ together with homotopies $gf \simeq \mathrm{id}_x$ and $fg \simeq \mathrm{id}_y$.

We write $X_1^{\simeq} \subseteq X_1$ for the full subanima spanned by the isomorphisms.

Definition 3.1.6 (Complete Segal anima, Rezk [Rez01])**.** A Segal anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is called *complete*, or is said to satisfy the *Rezk condition*, if the map

$$X_0 \to X_1^{\simeq}, \quad x \mapsto \mathrm{id}_x$$

is an equivalence of animae. We denote the full subcategory of complete Segal animae by $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{Seg}(\mathrm{An})$.

The nerve $N(C)$ of an ∞-category is complete by the Rezk Axiom (see the recollections on page 213). In particular, the nerve functor $N\colon \mathrm{Cat}_\infty \to \mathrm{sAn}$ factors through $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$.

The claim that a functor is an equivalence if and only if it is essentially surjective and fully faithful has an analogue for complete Segal animae:

Definition 3.1.7 (Dwyer-Kan equivalences)**.** A morphism $\varphi\colon X \to Y$ of Segal animae is called a *Dwyer-Kan equivalence* if the following two conditions are satisfied:

(1) (Essential surjectivity): For every object $y \in Y_0$ there exists an object $x \in X_0$ satisfying $\varphi(x) \cong y$ in Y_0.

(2) (Fully faithfulness): For objects $x, x' \in X_0$, the induced map $\varphi\colon \operatorname{Hom}_X(x, x') \to \operatorname{Hom}_Y(\varphi x, \varphi x')$ is an equivalence of animae.

Remark 3.1.8. Note that a functor $F\colon C \to D$ between ∞-categories is essentially surjective (resp. fully faithful) precisely if the induced map on nerves $N(F)\colon N(C) \to N(D)$ is essentially surjective (resp. fully faithful). In particular, F is an equivalence if and only if $N(F)$ is a Dwyer-Kan equivalence.

Lemma 3.1.9. *Let $\varphi\colon X \to Y$ be a morphism of complete Segal animae. Then φ is an equivalence if and only if it is a Dwyer-Kan equivalence.*

Proof. It is clear that every equivalence is a Dwyer-Kan equivalence. For the converse, assume φ is a Dwyer-Kan equivalence. We start by showing that φ is *conservative*, meaning that the following square is a pullback square:

$$\begin{tikzcd} X_1^{\sim} \arrow[r, hook] \arrow[d] & X_1 \arrow[d, "\varphi_1"] \\ Y_1^{\sim} \arrow[r, hook] & Y_1. \end{tikzcd}$$

Let $f\colon x \to y$ be a morphism in X_1 and assume that its image $\varphi(f)\colon \varphi(x) \to \varphi(y)$ is an isomorphism in Y. We have to show that already f itself is contained in $X_1^{\sim}$. To produce a left inverse, pick a left inverse $g'\colon \varphi(y) \to \varphi(x)$ of $\varphi(f)$. By assumption on φ, the map $\operatorname{Hom}_X(y, x) \xrightarrow{\sim} \operatorname{Hom}_Y(\varphi y, \varphi x)$ is an equivalence, and in particular there exists a preimage $g\colon y \to x$ of g'. We claim that $gf \simeq \mathrm{id}_x$. But since $\varphi(gf) \simeq \varphi(g)\varphi(f) \simeq g'\varphi(f) \simeq \mathrm{id}_{\varphi(x)} \simeq \varphi(\mathrm{id}_x)$, this follows from the equivalence $\varphi\colon \operatorname{Hom}_X(x, x) \xrightarrow{\sim} \operatorname{Hom}_Y(\varphi x, \varphi x)$. Thus g is a left inverse of f. Since $\varphi(g) = g'$ is invertible in Y, we may apply the argument to g to conclude that also g admits some left inverse h, so that g is invertible in X and hence so is f. This finishes the proof of conservativity of φ.

To show that φ is an equivalence, consider the following commutative square:

$$\begin{tikzcd} X_0 \arrow[r, "s_0", "\sim"'] \arrow[d, "f_0"'] & X_1^{\sim} \arrow[r, hook] \arrow[d, "f_1"] & X_1 \arrow[r, "{(d_1, d_0)}"] \arrow[d, "f_1"] & X_0 \times X_0 \arrow[d, "f_0 \times f_0"] \\ Y_0 \arrow[r, "s_0", "\sim"'] & Y_1^{\sim} \arrow[r, hook] & Y_1 \arrow[r, "{(d_1, d_0)}"] & Y_0 \times Y_0. \end{tikzcd}$$

The horizontal maps in the left-hand square are equivalences by the completeness of X and Y, and so the left square is a pullback square. The middle square is a pullback square by conservativity of φ. The right-hand square is a pullback square, since on horizontal fibers it induces the maps $\operatorname{Hom}_X(x, y) \to \operatorname{Hom}_Y(\varphi x, \varphi y)$, which are equivalences by fully faithfulness of φ. It follows that the outer rectangle is a pullback square, and hence that $\varphi_0\colon X_0 \to Y_0$ is a monomorphism of animae. Since it is also surjective on objects, it follows that φ_0 is an equivalence. The right-hand pullback square then shows that also $\varphi_1\colon X_1 \to Y_1$ is an equivalence and using the Segal condition we conclude that $\varphi_n\colon X_n \to Y_n$ is an equivalence for all n. $\square$

We will now assign an ∞-category to every simplicial anima. For this, let us recall the universal property of the presheaf category $\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An})$ for a small ∞-category C: for every other cocomplete ∞-category E, restriction along the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{PSh}(C)$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C), E) \xrightarrow{\sim} \mathrm{Fun}(C, E) \tag{3.1}$$

between the ∞-category of colimit-preserving functors $\mathrm{PSh}(C) \to E$ and the ∞-category of functors $C \to E$. The inverse to this restriction functor is given by left Kan extension along the Yoneda embedding.

Lemma 3.1.10. *The nerve functor admits a left adjoint* $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}_\infty$.

Proof. Since Cat_∞ admits all colimits, we may form the left Kan extension of $\boldsymbol{\Delta} \hookrightarrow \mathrm{Cat}_\infty$ along the Yoneda embedding $\Delta^\bullet\colon \boldsymbol{\Delta} \hookrightarrow \mathrm{sAn} = \mathrm{PSh}(\boldsymbol{\Delta})$ to obtain a colimit-preserving functor

$$\mathrm{ac}\colon \mathrm{sAn} = \mathrm{PSh}(\boldsymbol{\Delta}) \to \mathrm{Cat}_\infty.$$

To show that this is a left adjoint to the nerve functor, we must produce a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{ac}(X), C) \cong \mathrm{Hom}_{\mathrm{sAn}}(X, N(C))$$

for every simplicial anima X and every ∞-category C. Regarding both sides as functors $\mathrm{sAn}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Cat}_\infty, \mathrm{An})$, it follows from the fact that the Hom-functor preserves limits in its first variable (see e.g. Theorem B.20) that both are limit-preserving, and in particular both are the right Kan extension of their restriction along the Yoneda embedding $\Delta^\bullet\colon \boldsymbol{\Delta} \hookrightarrow \mathrm{sAn}$. It will thus suffice to provide a natural isomorphism between both of these two restrictions. But in this case we may use the composite

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{ac}(\Delta^n), C) \cong \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C) = N(C)_n \cong \mathrm{Hom}_{\mathrm{sAn}}(\Delta^n, N(C)),$$

where the last step uses the Yoneda lemma. ◻

Definition 3.1.11. Given a simplicial anima X, we refer to the ∞-category $\mathrm{ac}(X)$ as the *associated category*.

Since the functor ac is defined in terms of left Kan extensions, the pointwise formula for left Kan extensions allows us to compute it in terms of colimits in Cat_∞, which in turn admit an explicit description by Proposition 2.5.2. The resulting expression can then be simplified somewhat further to obtain a concrete expression for $\mathrm{ac}(X)$ as a certain localization, as explained in [HS23, Corollary 3.8]. This explicit expression can then be used to prove the following important result:

Proposition 3.1.12 (Hebestreit and Steinebrunner [HS23, Corollary 3.12, Lemma 4.1])**.** *For every Segal anima X, the unit map $\eta\colon X \to N(\mathrm{ac}(X))$ is a Dwyer-Kan equivalence.*

We refer to [HS23] for the details of the argument. With this essential ingredient, we may now prove the main result of this section:

Proof of Theorem 3.1.2. Since the nerve functor lands in $\mathrm{CSeg}(\mathrm{An})$, the adjunction $\mathrm{ac} \dashv N$ restricts to an adjunction

$$\mathrm{ac}\colon \mathrm{CSeg}(\mathrm{An}) \rightleftarrows \mathrm{Cat}_\infty : N.$$

We have to show that both the unit and the counit are equivalences. For the unit, let X be a complete Segal space. The unit map $\eta\colon X \to N(\mathrm{ac}(X))$ is a Dwyer-Kan equivalence by Proposition 3.1.12 and thus an equivalence by Lemma 3.1.9, since both sides are complete Segal animae. For the counit, let C be an ∞-category. To show that $\varepsilon\colon \mathrm{ac}(N(C)) \to C$ is an equivalence, it suffices by Remark 3.1.8 to show that $N(\varepsilon)\colon N(\mathrm{ac}(N(C))) \to N(C)$ is an equivalence. But this is a consequence of the triangle identity and the fact that $\eta_{N(C)}\colon N(C) \to N(\mathrm{ac}(N(C)))$ is an equivalence. This finishes the proof. □

3.1.1 Some consequences

For future use, let us record some useful consequences of Theorem 3.1.2.

Lemma 3.1.13. *The subcategory* $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$ *is closed under limits and filtered colimits.*

Proof. Recall that limits and colimits in sAn are computed pointwise. The conditions defining the subcategory $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$ are expressed in terms of finite limits. Since finite limits commute with both limits and filtered colimits[1] in An, the claim follows. □

Corollary 3.1.14. *The nerve functor* $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$ *preserves filtered colimits. In particular, finite limits commute with filtered colimits in* Cat_∞.

Proof. The first claim is immediate from Lemma 3.1.13 and the equivalence $N\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{CSeg}(\mathrm{An})$. The second claim follows since limits commute with filtered colimits in An and hence in sAn. □

Corollary 3.1.15. *For every finite category I, the functor* $\mathrm{Hom}_{\mathrm{Cat}_\infty}(I,-)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ *preserves filtered colimits. In particular the core functor* $(-)^{\simeq}\colon \mathrm{Cat}_\infty \to \mathrm{An}$ *preserves filtered colimits.*

Proof. By definition, the subcategory $\mathrm{Cat}_\infty^{\mathrm{fin}} \subseteq \mathrm{Cat}_\infty$ is the smallest subcategory closed under pushouts which contains $\emptyset$, $[0]$ and $[1]$. Since filtered colimits commute with pullbacks in An, the class I of ∞-categories for which the claim holds is closed under pushouts. It thus remains to show the claim for $I = \emptyset, [0], [1]$. The claim for $\emptyset$ is clear, since $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\emptyset, C)$ is terminal for each C and filtered colimits of terminal objects are terminal. The claim for $[n]$ follows from the previous corollary, since $\mathrm{Hom}_{\mathrm{Cat}_\infty}([n],-)$ agrees with the composite $\mathrm{Cat}_\infty \xrightarrow{N} \mathrm{sAn} \xrightarrow{\mathrm{ev}_n} \mathrm{An}$.
The last claim is the special case $I = [0]$. □

3.2 Constructing span categories

Having established the equivalence between ∞-categories and complete Segal animae, we can return to the question of defining the span category $\mathrm{Span}_{L,R}(C)$. The idea will be to write down a complete Segal anima whose anima of n-simplices consists of n-tuples of spans in C *together with coherent choices for their compositions*. More precisely, an n-simplex in this simplicial anima should be a

[1] See e.g. [Hau25, Section 9.9] for a 'synthetic' proof of the fact that finite limits commute with filtered colimits in An.

diagram in C as follows:

$$\begin{tikzcd}[column sep=tiny]
 & & & & X_{0n} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & & & \\
 & & & \cdots \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & X_{1n} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & & \\
 & & X_{02} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & \cdots \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & \cdots \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & & \\
 & X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11} & & X_{22} & & \cdots & & X_{nn}.
\end{tikzcd}$$

Here all the left-pointing morphisms should be in the class C_L, all the right-pointing morphisms should be in C_R, and all squares should be pullback squares. Let us now make this precise. In the remainder of this section we will closely follow the exposition from [Hau+23b].

Definition 3.2.1. A subcategory[2] $C' \subseteq C$ is called *wide* if it contains all objects of C.

We will usually think of a wide subcategory C' as specifying a collection of morphisms in C closed under composition.

Definition 3.2.2 ([Bar17])**.** An *adequate triple* (C, C_L, C_R) consists of an ∞-category C equipped with two wide subcategories $C_L, C_R \subseteq C$ satisfying the following two conditions:

(1) For morphisms $l\colon X \to Y$ and $r\colon Y' \to Y$ in C with $l \in C_L$ and $r \in C_R$, there exists a pullback square of the form

$$\begin{tikzcd}
X' \arrow[r, "r'"] \arrow[d, "l'"'] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, "l"] \\
Y' \arrow[r, "r"] & Y.
\end{tikzcd} \tag{3.2}$$

(2) In any pullback square in C of the form (3.2) with $l \in C_L$ and $r \in C_R$, we also have $l' \in C_L$ and $r' \in C_R$.

We refer to C_L as the *class of backwards morphisms* and to C_R as the *class of forward morphisms*.

Given another adequate triple (D, D_L, D_R), a *morphism of adequate triples* is a functor $F\colon C \to D$ that sends C_L to D_L and C_R to D_R, and that preserves the pullbacks of the form (3.2). We denote by

$$\mathrm{AdTrip} \subseteq \mathrm{Fun}(\lrcorner, \mathrm{Cat}_\infty)$$

the non-full subcategory spanned by those diagrams $C_L \hookrightarrow C \hookleftarrow C_R$ in Cat_∞ that correspond to adequate triples (C, C_L, C_R) and those morphisms that correspond to morphisms of adequate triples.

Example 3.2.3. For any wide subcategory C' of an ∞-category C, we have the adequate triples $(C, C', C^{\simeq})$ and $(C, C^{\simeq}, C')$. In particular, for any ∞-category C we always have the adequate triples $(C, C, C^{\simeq})$ and $(C, C^{\simeq}, C)$.

[2]Recall that a subcategory in the synthetic setting corresponds to what is sometimes called a 'replete subcategory' in the quasicategorical setting: given an isomorphism $f\colon X \xrightarrow{\sim} Y$ in C, if X is contained in C' then also Y and f are contained in C'.

Example 3.2.4. If C admits all pullbacks, we have the adequate triple (C, C, C) where we take all maps to be both forward and backward maps. This gives rise to a fully faithful embedding

$$\mathrm{Cat}_\infty^{\mathrm{pb}} \hookrightarrow \mathrm{AdTrip},$$

where $\mathrm{Cat}_\infty^{\mathrm{pb}} \subseteq \mathrm{Cat}_\infty$ is the subcategory spanned by those ∞-categories which admit pullbacks, and those functors which preserve pullbacks.

Example 3.2.5. If (C, C_L, C_R) is an adequate triple, then so is (C, C_R, C_L). The resulting endomorphism of AdTrip will be denoted

$$(-)_{\mathrm{rev}} \colon \mathrm{AdTrip} \to \mathrm{AdTrip}\,.$$

We will now introduce the indexing diagrams that classify the functors $[n] \to \mathrm{Span}_{L,R}(C)$.

Definition 3.2.6 (Twisted arrow category)**.** For $n \geq 0$, let $\mathrm{Tw}^r([n])$ be the partially ordered set of tuples (i, j) with $0 \leq i \leq j \leq n$, where the partial order is given by

$$(i, j) \leq (k, l) \qquad \text{if and only if} \qquad i \leq k \quad \text{and} \quad l \leq j\,.$$

We will generally denote (i, j) by $(i \leq j)$. The poset looks as follows:

$$\begin{tikzcd}[column sep=tiny]
&&&& (0 \leq n) \arrow[dl] \arrow[dr] &&&& \\
&&& \ldots \arrow[dl] \arrow[dr] && (1 \leq n) \arrow[dl] \arrow[dr] &&& \\
&& (0 \leq 2) \arrow[dl] \arrow[dr] && \ldots \arrow[dl] \arrow[dr] && \ldots \arrow[dl] \arrow[dr] && \\
& (0 \leq 1) \arrow[dl] \arrow[dr] && (1 \leq 2) \arrow[dl] \arrow[dr] && \ldots \arrow[dl] \arrow[dr] && (n-1 \leq n) \arrow[dl] \arrow[dr] & \\
(0 \leq 0) && (1 \leq 1) && (2 \leq 2) && \ldots && (n \leq n).
\end{tikzcd}$$

Given a morphism of posets $\varphi \colon [n] \to [m]$, we obtain a map $\mathrm{Tw}^r(\varphi) \colon \mathrm{Tw}^r([n]) \to \mathrm{Tw}^r([m])$ by sending $(i \leq j)$ to $(\varphi(i) \leq \varphi(j))$. This assignment is clearly functorial, resulting in a functor

$$\mathrm{Tw}^r \colon \mathbf{\Delta} \to \mathrm{Poset} \subseteq \mathrm{Cat} \subseteq \mathrm{Cat}_\infty.$$

Exercise 3.2.7. Show that the functors $s \colon \mathrm{Tw}^r([n]) \to [n], (i \leq j) \mapsto i$ and $t \colon \mathrm{Tw}^r([n]) \to [n]^{\mathrm{op}}, (i \leq j) \mapsto j$ are cartesian fibrations.

Lemma 3.2.8. *For integers $0 \leq i \leq k \leq l \leq j \leq n$, the commutative square*

$$\begin{tikzcd}
& (i \leq j) \arrow[dl] \arrow[dr] & \\
(i \leq l) \arrow[dr] && (k \leq j) \arrow[dl] \\
& (k \leq l) &
\end{tikzcd}$$

is a pullback square in $\mathrm{Tw}^r([n])$.

Proof. Given $0 \leq i' \leq j' \leq n$ we need to show that we have $(i', j') \leq (i, j)$ if and only if $(i', j') \leq (i, l)$ and $(i', j') \leq (k, j)$. This is clear: in both cases this amounts to asking that $i' \leq i$ and $j \leq j'$. □

Corollary 3.2.9. *The poset* $\mathrm{Tw}^r([n])$ *is an adequate triple, with* $\mathrm{Tw}^r([n])_L$ *consisting of morphisms of the form* $(i \leq j) \leq (i \leq l)$ *and* $\mathrm{Tw}^r([n])_R$ *consisting of morphisms of the form* $(i \leq j) \leq (k \leq j)$. □

From now on, we will equip $\mathrm{Tw}^r([n])$ with the adequate triple structure from Corollary 3.2.9. Note that a morphism of posets $\varphi\colon [n] \to [m]$ induces a morphism of adequate triples $\mathrm{Tw}^r(\varphi)\colon \mathrm{Tw}^r([n]) \to \mathrm{Tw}^r([m])$, resulting in a functor

$$\mathrm{Tw}^r : \mathbf{\Delta} \to \mathrm{AdTrip}\,.$$

Observe that the diagrams in C we drew in our informal description of the functors $[n] \to \mathrm{Span}_{L,R}(C)$ given in the introduction to this section are precisely the morphisms of adequate triples $\mathrm{Tw}^r([n]) \to (C, C_L, C_R)$. This leads to the following definition:

Construction 3.2.10. Let (C, C_L, C_R) be an adequate triple. We define the simplicial anima $\mathrm{NSpan}_{L,R}(C) \in \mathrm{sAn}$ as the following composite:

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Tw}^r} \mathrm{AdTrip}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{AdTrip}}(-,(C,C_L,C_R))} \mathrm{An}.$$

In other words, for $n \geq 0$ we consider the subanima

$$\mathrm{NSpan}_{L,R}(C)_n \subseteq \mathrm{Map}(\mathrm{Tw}^r([n]), C)$$

consisting of the morphisms of adequate triples. This construction is natural in the adequate triple, hence defines a functor

$$\mathrm{NSpan}\colon\ \mathrm{AdTrip} \to \mathrm{sAn}.$$

Proposition 3.2.11 (Barwick [Bar17, Proposition 5.6], Haugseng, Hebestreit, Linskens, and Nuiten [Hau+23b, Theorem 2.1.3])**.** *For every adequate triple* (C, C_L, C_R) *the simplicial anima* $\mathrm{NSpan}_{L,R}(C)$ *is a complete Segal anima.*

Proof. We start by proving the Segal condition. Let us write $X := \mathrm{NSpan}_{L,R}(C)$ for ease of notation. For every $n \geq 0$, consider the subposet $J_n \subseteq \mathrm{Tw}^r([n])$ spanned by the objects $(i \leq j)$ with $j \leq i+1$:

$$\begin{tikzcd}[column sep=tiny]
&&&&& (0\leq n) \arrow[dl] \arrow[dr] \\
&&&& \cdots \arrow[dl] \arrow[dr] && (1\leq n) \arrow[dl] \arrow[dr] \\
&&& (0\leq 2) \arrow[dl] \arrow[dr] && \cdots \arrow[dl] \arrow[dr] && \cdots \arrow[dl] \arrow[dr] \\
&& (0\leq 1) \arrow[dl] \arrow[dr] && (1\leq 2) \arrow[dl] \arrow[dr] && \cdots \arrow[dl] \arrow[dr] && (n-1\leq n) \arrow[dl] \arrow[dr] \\
\mathcal{J}_n & (0\leq 0) && (1\leq 1) && (2\leq 2) && \cdots && (n\leq n)
\end{tikzcd}$$

Note that a functor $\mathrm{Tw}^r([n]) \to C$ preserves the pullback squares from Lemma 3.2.8 if and only if it is right Kan extended along the inclusion $J_n \hookrightarrow \mathrm{Tw}^r([n])$: if we compute this Kan extension row by row, this follows from the pointwise formula for Kan extensions. It follows that restriction to J_n induces a fully faithful functor

$$X_n = \mathrm{NSpan}_{L,R}(C)_n \hookrightarrow \mathrm{Map}(J_n, C)$$

whose image consists of those functors $J_n \to C$ sending the left-pointing morphisms to C_L and the right-pointing morphisms to C_R. We now observe that face maps $e_i\colon [1] \hookrightarrow [n]$ induce inclusions $\mathrm{Tw}^r([1]) \hookrightarrow J_n$ which assemble into an equivalence

$$\mathrm{Tw}^r([1]) \sqcup_{\mathrm{Tw}^r([0])} \mathrm{Tw}^r([1]) \sqcup_{\mathrm{Tw}^r([0])} \cdots \sqcup_{\mathrm{Tw}^r([0])} \mathrm{Tw}^r([1]) \xrightarrow{\sim} J_n,$$

and in particular we get

$$\mathrm{Map}(J_n, C) \xrightarrow{\sim} \mathrm{Map}(\mathrm{Tw}^r([1]), C) \times_{C^{\simeq}} \mathrm{Map}(\mathrm{Tw}^r([1]), C) \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \mathrm{Map}(\mathrm{Tw}^r([1]), C).$$

Under this equivalence, the image of the restriction map $X_n \hookrightarrow \mathrm{Map}(J_n, C)$ corresponds precisely to the target of the Segal map

$$(e_i^*)_{i=1}^n\colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

showing that X satisfies the Segal condition.

For completeness, first note that the functor

$$C^{\simeq} = X_0 \to X_1 \subseteq \mathrm{Map}(\mathrm{Tw}^r([n]), C)$$

is an inclusion of animae. Indeed, since $\mathrm{Tw}^r([1])$ has an initial object we have $\|\mathrm{Tw}^r([1])\| \simeq *$, and thus the constant diagram functor $C \to \mathrm{Fun}(\mathrm{Tw}^r([1]), C)$ may be identified with the fully faithful inclusion $\mathrm{Fun}(\|\mathrm{Tw}^r([n])\|, C) \hookrightarrow \mathrm{Fun}(\mathrm{Tw}^r([1]), C)$. It thus remains to show that a span $X \xleftarrow{l} U \xrightarrow{r} Y$ is an isomorphism in the Segal anima $\mathrm{NSpan}_{L,R}(C)$ if and only if both l and r are isomorphisms in C. Let $Y \xleftarrow{l'} V \xrightarrow{r'} X$ be a left inverse, and let $Y \xleftarrow{l''} W \xrightarrow{r''} X$ be a right inverse. The identities $(l, r) \circ (l', r') = \mathrm{id}$ and $(l'', r'') \circ (l, r) = \mathrm{id}$ are exhibited by commutative diagrams in C of the following form:

$$\begin{tikzcd}
 & & Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] \arrow[ddll, bend right, equal] \arrow[ddrr, bend left, equal] & & \\
 & V \arrow[dl, "l'"'] \arrow[dr, "r'"] & & U \arrow[dl, "l"'] \arrow[dr, "r"] & \\
Y & & X & & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] \arrow[ddll, bend right, equal] \arrow[ddrr, bend left, equal] & & \\
 & U \arrow[dl, "l"'] \arrow[dr, "r"] & & W \arrow[dl, "l''"'] \arrow[dr, "r''"] & \\
X & & Y & & X.
\end{tikzcd}$$

Putting them into one single diagram and forming another pullback, we then obtain:

$$\begin{tikzcd}
 & & & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & & \\
 & & Y \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & X \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\vee"] & & \\
 & V \arrow[dl, "l'"'] \arrow[dr, "r'"] & & U \arrow[dl, "l"'] \arrow[dr, "r"] & & W \arrow[dl, "l''"'] \arrow[dr, "r''"] & \\
Y & & X & & Y & & X.
\end{tikzcd}$$

Since isomorphisms in C are closed under pullbacks, the maps $P \to V$ and $P \to W$ are isomorphisms, so by the 2-out-of-6 property all morphisms along the outer two edges of the diagram are isomorphisms. By the 2-out-of-3 property, then so are the maps $r'\colon V \to X$ and $l''\colon W \to Y$, hence by pullback also $Y \to U$ and $X \to U$. Finally, another instance of the 2-out-of-3 property shows that $l\colon U \to X$ and $r\colon U \to Y$ are isomorphisms, as desired. □

Notation 3.2.12. By Proposition 3.2.11 and Theorem 3.1.2, the functor $\mathrm{NSpan}\colon \mathrm{AdTrip} \to \mathrm{sAn}$ factors uniquely as

$$\begin{tikzcd} \mathrm{AdTrip} \arrow[rr, "\mathrm{NSpan}"] \arrow[dr, dashed, "\mathrm{Span}"'] & & \mathrm{sAn}. \\ & \mathrm{Cat}_\infty \arrow[ur, "N"'] & \end{tikzcd}$$

We refer to the resulting ∞-category $\mathrm{Span}_{L,R}(C)$ as the *span category* of the adequate triple.

Lemma 3.2.13. *There is a natural equivalence* $\mathrm{Span}_{L,R}(C)^{\mathrm{op}} \simeq \mathrm{Span}_{R,L}(C)$*, in the sense that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{AdTrip} \arrow[r, "(-)_{\mathrm{rev}}"] \arrow[d, "\mathrm{Span}"'] & \mathrm{AdTrip} \arrow[d, "\mathrm{Span}"] \\ \mathrm{Cat}_\infty \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{Cat}_\infty. \end{tikzcd}$$

Proof. By definition of $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$, there is a commutative square

$$\begin{tikzcd} \mathrm{Cat}_\infty \arrow[r, "(-)^{\mathrm{op}}"] \arrow[d, hook, "N"'] & \mathrm{Cat}_\infty \arrow[d, hook, "N"] \\ \mathrm{sAn} \arrow[r, "(-)^{\mathrm{op}}"] & \mathrm{sAn}, \end{tikzcd}$$

where the bottom functor is given by precomposition with $(-)^{\mathrm{op}}\colon \mathbf{\Delta} \xrightarrow{\sim} \mathbf{\Delta}$. We thus need to produce a natural equivalence $\mathrm{NSpan}_{L,R}(C)^{\mathrm{op}} \simeq \mathrm{NSpan}(C, C_R, C_L)$ of simplicial animae. Unwinding definitions, this amounts to producing a natural equivalence

$$\mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]^{\mathrm{op}}), (C, C_L, C_R)) \simeq \mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), (C, C_R, C_L)),$$

natural in $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ and (C, C_L, C_R). Equivalently, we must produce a natural equivalence $\mathrm{Tw}^r([n]^{\mathrm{op}}) \simeq \mathrm{Tw}^r([n])_{\mathrm{rev}}$. Such an equivalence is given by sending an object $(i \geq j)$ in $\mathrm{Tw}^r([n]^{\mathrm{op}})$ to the object $(j \leq i)$ in $\mathrm{Tw}^r([n])$. ◻

The span category $\mathrm{Span}_{L,R}(C)$ contains C_R as a wide subcategory on the right-pointing spans $X \xleftarrow{=} X \xrightarrow{r} Y$. Similarly it contains C_L^{op} as a wide subcategory on the left-pointing spans $X \xleftarrow{l} Y \xrightarrow{=} Y$. This is made precise by the following result:

Lemma 3.2.14. *Let $C_L, C_R \subseteq C$ be wide subcategories. Then the span categories of the adequate triples $(C, C^{\simeq}, C_R)$ and $(C, C_L, C^{\simeq})$ from Example 3.2.3 are given by:*

$$\mathrm{Span}_{\simeq,R}(C) \simeq C_R \qquad \textit{and} \qquad \mathrm{Span}_{L,\simeq}(C) \simeq C_L^{\mathrm{op}}.$$

Proof. It suffices to prove the first equivalence; the second one follows by combining the first one with Lemma 3.2.13. For $[n] \in \mathbf{\Delta}^{\mathrm{op}}$, consider the source functor $s\colon \mathrm{Tw}^r([n]) \to [n]$, $(i \leq j) \mapsto i$. Note that this functor admits a fully faithful left adjoint $[n] \to \mathrm{Tw}^r([n])$ sending $i \in [n]$ to $(i \leq n) \in \mathrm{Tw}^r([n])$. Indeed, there is a morphism $(i \leq n) \to (k \leq l)$ in $\mathrm{Tw}^r([n])$ if and only if $i \leq k$. It thus follows from Lemma 3.2.15 that restriction along s induces an inclusion of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C_R) \subseteq \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C) \hookrightarrow \mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Tw}^r([n]), C)$$

whose essential image consists of those functors $\mathrm{Tw}^r[n] \to C$ that send backwards morphisms to isomorphisms and forward morphisms into C_R. Since these are precisely the morphisms of adequate triples into $(C, C^{\simeq}, C_R)$, we obtain a natural equivalence

$$N(C_R)_n = \mathrm{Hom}_{\mathrm{Cat}_\infty}([n], C_R) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), (C, C^{\simeq}, C_R)) = \mathrm{NSpan}_{\simeq,R}(C)_n.$$

This shows that $N(C_R) \simeq \mathrm{NSpan}_{\simeq,R}(C)$, and hence that $C_R \simeq \mathrm{Span}_{\simeq,R}(C)$ by fully faithfulness of the nerve functor. □

Lemma 3.2.15. *Let $p\colon E \to C$ be a functor with a fully faithful left adjoint $i\colon C \hookrightarrow E$ (also known as a Bousfield colocalization). Then p is a localization at the class $W = p^{-1}(C^{\simeq})$ of morphisms in E sent to isomorphisms under p.*

Proof. The adjunction $i \vdash p$ induces for any ∞-category D an adjunction

$$p^*\colon \mathrm{Fun}(C, D) \rightleftarrows \mathrm{Fun}(E, D) : i^*$$

on functor categories. The unit $\mathrm{id} \xrightarrow{\sim} pi$ induces the unit $\mathrm{id} = \mathrm{id}^* \xrightarrow{\sim} (pi)^* = i^*p^*$, hence p^* is fully faithful. To identify the essential image of p^*, note that a functor $F\colon C \to D$ is in the essential image if and only if the counit $F \circ i \circ p = p^*i^*F \to F$ is a natural isomorphism. This happens if and only if F sends the counit $\varepsilon_X\colon i(p(X)) \to X$ in E to an isomorphism in C for every $X \in E$. Since $p(\varepsilon_X)\colon p(i(p(X))) \to p(X)$ is an isomorphism, this is clearly implied by F inverting the maps in W. Conversely, if F inverts all maps ε_X, then for every morphism $f\colon X \to X'$ in $W = p^{-1}(C^{\simeq})$ it inverts three out of the four maps in the diagram

$$\begin{tikzcd} ipX \arrow[r, "ip(f)"] \arrow[d, "\varepsilon_X"'] & ipX' \arrow[d, "\varepsilon_{X'}"] \\ X \arrow[r, "f"] & X', \end{tikzcd}$$

and so by 2-out-of-3 it also inverts f. This shows that the essential image of p^* consists of the functors F that invert W, as desired. □

Corollary 3.2.16. *There are canonical inclusions*

$$C_R \simeq \mathrm{Span}_{\simeq,R}(C) \hookrightarrow \mathrm{Span}_{L,R}(C) \qquad \textit{and} \qquad C_L^{\mathrm{op}} \simeq \mathrm{Span}_{L,\simeq}(C) \hookrightarrow \mathrm{Span}_{L,R}(C).$$

Lemma 3.2.17. *The ∞-category* AdTrip *admits limits and filtered colimits, and the inclusion* $\mathrm{AdTrip} \hookrightarrow \mathrm{Fun}(\lrcorner, \mathrm{Cat}_\infty)$ *preserves limits and filtered colimits.*

Proof. We start with the case for limits. Recall that limits in $\mathrm{Fun}(\lrcorner, \mathrm{Cat}_\infty)$ are computed pointwise. Given a functor $C\colon I \to \mathrm{AdTrip}$, we thus need to show that the triple $(C, C_L, C_R) := (\lim_i C(i), \lim_i C(i)_L, \lim_i C(i)_R)$ is again adequate. Note that subcategory inclusions are closed under limits in Cat_∞: a functor $D \hookrightarrow D'$ is a subcategory inclusion if and only if the commutative square

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, equal] & D \arrow[d] \\ D \arrow[r] & D' \end{tikzcd}$$

is a pullback square, and limits commute with pullback squares. Furthermore, *wide* subcategory inclusions are closed under limits: an inclusion $D \hookrightarrow D'$ is a wide subcategory if and only if the induced map $D^{\simeq} \to D'^{\simeq}$ on groupoid cores is an equivalence of animae, and $(-)^{\simeq}$ preserves limits. Finally, a cospan $X \xrightarrow{l} X'' \xleftarrow{r} X'$ admits a pullback in $C = \lim_i C(i)$ if and only if its image in each $C(i)$ admits a pullback. Since a morphism in C lies in C_L (resp. in C_R) if and only if its image in each $C(i)$ lies in $C(i)_L$ (resp. $C(i)_R$), this shows that (C, C_L, C_R) is an adequate triple. Furthermore, it follows that a functor into C is a morphism of adequate triples if and only if the composite with each projection $C \to C(i)$ is, so that (C, C_L, C_R) is a limit in AdTrip.

For filtered colimits, note that subcategory inclusions are closed under filtered colimits in Cat_∞ since they commute with pullbacks by Lemma 3.1.14. Since every object in a filtered colimit of ∞-categories $D(i)$ comes from one of the $D(i)$, it follows that a filtered colimit of wide subcategory inclusions is again wide. The functors $C(j) \to \operatorname{colim}_i C(i) = C$ preserves pullbacks, since the hom animae in C are defined using a pullback square and hence commute with filtered colimits. Conversely, any cospan in C may be lifted to a cospan in some stage $C(j)$, so C admits all pullbacks of morphisms in C_L along morphisms in C_R. It follows that (C, C_L, C_R) is an adequate triple, and that a functor out of C is a morphism of adequate triples if and only if the composite with each $C(j) \to C$ is, so that (C, C_L, C_R) is a colimit in AdTrip. □

Proposition 3.2.18. *The functor* Span: AdTrip $\to \mathrm{Cat}_\infty$ *preserves limits and filtered colimits.*

Proof. The inclusion $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$ preserves limits and filtered colimits (see Lemma 3.1.14 for the latter claim). It will thus suffice to show that NSpan: AdTrip $\to$ sAn preserves these (co)limits. Since these are computed pointwise, this boils down to showing that the functor

$$\mathrm{Hom}_{\mathrm{AdTrip}}(\mathrm{Tw}^r([n]), -)\colon \mathrm{AdTrip} \to \mathrm{An}$$

preserves them. The case for limits is clear, since the functor is corepresented. For filtered colimits, we use again that filtered colimits commute with pullbacks, so that we may reduce to the cases $n = 0$ and $n = 1$ via the Segal condition. Note that the ∞-categories $\mathrm{Tw}^r([0]) = *$ and $\mathrm{Tw}^r([1]) = \ulcorner$ are finite, so $\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{Tw}^r([n]), -)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits by Corollary 3.1.15. This proves the claim for $n = 0$. For $n = 1$, one observes that the subanimae of morphisms of adequate triples are closed under filtered colimits. □

3.3 Products and coproducts in span categories

We will now provide conditions under which the span category $\mathrm{Span}_{L,R}(C)$ admits finite products or coproducts. This will follow from a more general discussion about adjunctions of span categories, which we will start with.

3.3.1 Adjunctions of span categories

Construction 3.3.1. Consider adequate triples (C, C_L, C_R) and (D, D_L, D_R) and let $F, G\colon C \to D$ be morphisms of adequate triples. Let $\alpha\colon F \Rightarrow G$ be a natural transformation, and assume that the following two conditions are satisfied:

(1) The morphism $\alpha(X)\colon F(X) \to G(X)$ lies in D_R for every object X of C;

(2) For every morphism $l\colon Z \to X$ in C_L the commutative square

$$\begin{tikzcd} F(Z) \arrow[r, "\alpha(X)"] \arrow[d, "F(l)"'] & G(Z) \arrow[d, "G(l)"] \\ F(X) \arrow[r, "\alpha(Y)"] & G(X) \end{tikzcd}$$

is a pullback square in D.

We will construct a natural transformation

$$\mathrm{Span}^R(\alpha)\colon\ \mathrm{Span}(F) \implies \mathrm{Span}(G)$$

of functors $\mathrm{Span}_{L,R}(C) \to \mathrm{Span}_{L,R}(D)$ which is objectwise given by the forward spans

$$F(X) \xleftarrow{=} F(X) \xrightarrow{\alpha(X)} G(X)$$

for $X \in C$. For this, equip $[1]$ with the adequate triple structure $([1], [1]^{\simeq}, [1])$. Then the product $C \times [1]$ is again an adequate triple. Regarding α as a functor $C \times [1] \to D$, we then observe that conditions (1) and (2) are precisely the condition that α is a morphism of adequate triples, hence induces a functor

$$\mathrm{Span}_{L,R}(C \times [1]) \to \mathrm{Span}_{L,R}(D).$$

Using that $\mathrm{Span}(-)$ preserves products by Proposition 3.2.18 and that $\mathrm{Span}_{\mathrm{iso,all}}([1]) \simeq [1]$ by Lemma 3.2.14, this functor takes the form

$$\mathrm{Span}_{L,R}(C) \times [1] \to \mathrm{Span}_{L,R}(D).$$

The restrictions to 0 and 1 are precisely $\mathrm{Span}(F)$ and $\mathrm{Span}(G)$, so this produces the desired natural transformation $\mathrm{Span}^R(\alpha)$.

Remark 3.3.2. The previous construction can be dualized, if we assume instead that the morphism $\alpha(X)\colon F(X) \to G(X)$ is in D_L for every $X \in C$ and that for every morphism $r\colon Z \to Y$ in C_R the commutative square

$$\begin{tikzcd} F(Z) \arrow[r, "\alpha(X)"] \arrow[d, "F(r)"'] & G(Z) \arrow[d, "G(r)"] \\ F(Y) \arrow[r, "\alpha(Y)"] & G(Y) \end{tikzcd}$$

is a pullback square in D. By applying Lemma 3.2.13 to the previous construction, we deduce that α induces a natural transformation

$$\mathrm{Span}^{\mathrm{L}}(\alpha)\colon\ \mathrm{Span}(G) \implies \mathrm{Span}(F)$$

of functors $\mathrm{Span}_{L,R}(C) \to \mathrm{Span}_{L,R}(D)$ given on an object $X \in C$ by the backwards span

$$G(X) \xleftarrow{\alpha(X)} F(X) \xrightarrow{=} F(X)$$

Lemma 3.3.3. *Given a third morphism of adequate triples $H\colon (C, C_L, C_R) \to (D, D_L, D_R)$, let $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ be natural transformations satisfying the conditions (1) and (2) from Construction 3.3.1. Then we have the relations*

$$\mathrm{Span}^R(\mathrm{id}_C) = \mathrm{id}_{\mathrm{Span}_{L,R}(C)} \qquad \textit{and} \qquad \mathrm{Span}^R(\beta \circ \alpha) = \mathrm{Span}^R(\beta) \circ \mathrm{Span}^R(\alpha).$$

The dual claims for $\mathrm{Span}^L(-)$ *also hold.*

Proof. Composition and identities of natural transformations are formed using the posets $[n]$ for $n = 0, 2$. The claim thus follows from the naturality in $[n] \in \mathbf{\Delta}$ of the equivalence $\mathrm{Span}_{L,R}(C \times [n]) \simeq \mathrm{Span}_{L,R}(C) \times [n]$. □

Corollary 3.3.4. *Let (C, C_L, C_R) and (D, D_L, D_R) be adequate triples. Let $F\colon C \rightleftarrows D : G$ be an adjunction of ∞-categories and assume that F and G are morphisms of adequate triples.*

(1) Assume that for every object $X \in C$, the unit map $\eta_X\colon X \to G(F(X))$ is in C_R and the counit map $\varepsilon_X\colon F(G(X)) \to X$ is in D_R. Furthermore, assume that for every morphism $l\colon Z \to X$ in L and every morphism $l'\colon Z' \to X'$ in L', the commutative squares

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "l"'] & G(F(Z)) \arrow[d, "G(F(l))"] \\ X \arrow[r, "\eta_X"] & G(F(X)) \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} F(G(Z')) \arrow[r, "\varepsilon_{Z'}"] \arrow[d, "F(G(l'))"'] & Z' \arrow[d, "l'"] \\ F(G(X')) \arrow[r, "\varepsilon_{X'}"] & X' \end{tikzcd}$$

are pullback squares. Then F and G induce an adjunction

$$\mathrm{Span}(F)\colon \mathrm{Span}(C, L, R) \rightleftarrows \mathrm{Span}(D, L', R') : \mathrm{Span}(G).$$

(2) Dually, assume that for every object $X \in C$, the unit map $\eta_X\colon X \to G(F(X))$ is in C_L and the counit map $\varepsilon_X\colon F(G(X)) \to X$ is in D_L. Furthermore, assume that for every morphism $r\colon Z \to Y$ in C_R and every morphism $r'\colon Z' \to Y'$ in D_R, the commutative squares

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[d, "r"'] & G(F(Z)) \arrow[d, "G(F(r))"] \\ Y \arrow[r, "\eta_Y"] & G(F(Y)) \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} F(G(Z')) \arrow[r, "\varepsilon_{Z'}"] \arrow[d, "F(G(r'))"'] & Z \arrow[d, "r'"] \\ F(G(Y')) \arrow[r, "\varepsilon_{Y'}"] & Y \end{tikzcd}$$

are pullback squares. Then F and G induce an adjunction

$$\mathrm{Span}(G)\colon \mathrm{Span}_{L,R}(D) \rightleftarrows \mathrm{Span}_{L,R}(C) : \mathrm{Span}(F).$$

Proof. We prove (1); as (2) is dual. By Construction 3.3.1 and Lemma 3.3.3 the unit $\eta\colon \mathrm{id} \to GF$ and counit $\varepsilon\colon FG \to \mathrm{id}$ induce transformations $\mathrm{Span}^R(\eta)\colon \mathrm{Span}(\mathrm{id}) \to \mathrm{Span}(G) \circ \mathrm{Span}(F)$ and $\mathrm{Span}^R(\varepsilon)\colon \mathrm{Span}(F) \circ \mathrm{Span}(G) \to \mathrm{id}$. By Lemma 3.3.3 these transformations still satisfy the triangle identities. □

3.3.2 Products and coproducts in span categories

We now apply the previous results on adjunctions of span categories to study products and coproducts in span categories.

Definition 3.3.5. An adequate triple (C, C_L, C_R) is called *weakly extensive* if the following conditions are satisfied:

(1) The ∞-category C admits finite coproducts;

(2) The coproduct functor $C \times C \to C$ is a morphism of adequate triples (i.e., morphisms in C_L and C_R are closed under coproducts, and coproducts of pullback squares of morphisms in C_L along morphisms in C_R are again pullback squares);

(3) For every object $X \in C$, the morphisms $\emptyset \to X$ and $\nabla\colon X \sqcup X \to X$ are in C_L;

(4) For every morphism $r\colon X \to Y$ in C_R, the commutative squares

$$\begin{tikzcd} X \sqcup X \arrow[r, "r \sqcup r"] \arrow[d, "\nabla"'] & Y \sqcup Y \arrow[d, "\nabla"] \\ X \arrow[r, "r"] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[d] \\ X \arrow[r, "r"] & Y \end{tikzcd}$$

are pullback squares.

We say that the triple is *weakly coextensive* if (C, C_R, C_L) is weakly extensive, and we say it is *extensive* if it is both weakly extensive and weakly coextensive. An ∞-category C is called *extensive* if it admits subcategories C_L and C_R such that (C, C_L, C_R) is extensive.

Example 3.3.6. The category Fin of finite sets is extensive. More generally, for a finite group G the category Fin_G of finite G-sets is extensive.

Example 3.3.7. The category Top of topological spaces is extensive.

Example 3.3.8. Any full subcategory $C \subseteq \mathrm{An}$ closed under finite coproducts is extensive.

Lemma 3.3.9. *Let (C, C_L, C_R) be an adequate triple.*

(1) *If the triple is weakly extensive, then $\mathrm{Span}_{L,R}(C)$ admits finite products, and the inclusion $C_L^{\mathrm{op}} \hookrightarrow \mathrm{Span}_{L,R}(C)$ preserves finite products;*

(2) *If the triple is weakly coextensive, then $\mathrm{Span}_{L,R}(C)$ admits finite coproducts, and the inclusion $C_R \hookrightarrow \mathrm{Span}_{L,R}(C)$ preserves finite coproducts;*

(3) *If the triple is extensive, then $\mathrm{Span}_{L,R}(C)$ is semiadditive.*

Proof. For part (1), note that C has binary coproducts if and only if the diagonal functor $\Delta\colon C \to C \times C$ admits a left adjoint $\sqcup\colon C \times C \to C$. The counit of the adjunction is the fold map $\nabla\colon X \sqcup X \to X$ and the unit is the pair of canonical inclusions $(X \hookrightarrow X \sqcup Y, Y \hookrightarrow X \sqcup Y)$. One now observes that the assumptions on (C, C_L, C_R) are precisely those that ensure we may apply part (2) of Corollary 3.3.4, so that this adjunction induces an adjunction

$$\mathrm{Span}_{L,R}(C) \rightleftarrows \mathrm{Span}_{L,R}(C \times C) \simeq \mathrm{Span}_{L,R}(C) \times \mathrm{Span}_{L,R}(C).$$

It follows that $\mathrm{Span}_{L,R}(C)$ admits binary products. For the initial object, we similarly regard the initial object $\emptyset$ of C as a left adjoint to the functor $C \to *$, and use Corollary 3.3.4 to get an adjunction $\mathrm{Span}_{L,R}(C) \rightleftarrows \mathrm{Span}(*) \simeq *$ exhibiting $\emptyset$ as a terminal object of $\mathrm{Span}_{L,R}(C)$. Part (2) is an immediate consequence of (1) using $\mathrm{Span}_{L,R}(C)^{\mathrm{op}} \simeq \mathrm{Span}_{R,L}(C)$. For part (3), note first that the initial object $\emptyset$ in C becomes both initial and terminal in $\mathrm{Span}_{L,R}(C)$, and that the zero morphism between objects X and Y is given by the span

$$X \leftarrow \emptyset \rightarrow Y.$$

Also notice that the coproduct $X \sqcup Y$ in C becomes both the product and the coproduct in $\mathrm{Span}_{L,R}(C)$, via the spans

$$X \sqcup Y \hookleftarrow X \xrightarrow{=} X, \qquad X \sqcup Y \hookleftarrow Y \xrightarrow{=} Y, \qquad X \xleftarrow{=} X \hookrightarrow X \sqcup Y, \qquad Y \xleftarrow{=} Y \hookrightarrow X \sqcup Y.$$

It thus remains to compute the four composites and show that they are id_X, 0, 0 and id_Y. This follows from the following three pullback squares:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ X \arrow[r] & X \sqcup Y, \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\ X \arrow[r] & X \sqcup Y. \end{tikzcd}$$ □

Corollary 3.3.10. *The ∞-category* Span(Fin) *is semiadditive.* □

4 Basics on ∞-operads

Now that we have a solid grasp on cocartesian fibrations and span categories, we return to the topic of ∞-operads. We start with some examples in Section 4.1, where we also prove an equivalent characterization of ∞-operads and discuss its relation to the definition used by Lurie [Lur17]. In Section 4.2 we introduce the *multimorphism operad* $\mathcal{M}_C$ of a symmetric monoidal ∞-category C, allowing us to define the notion of an *O-algebra* in C as an operad map $O \to \mathcal{M}_C$. In Section 4.3 we study the resulting notion of *lax symmetric monoidal functors* $C \to D$, defined as operad maps $\mathcal{M}_C \to \mathcal{M}_D$, and study the (lax) symmetric monoidality of left/right adjoints of (lax) symmetric monoidal functors. In Section 4.4, we provide conditions for a full subcategory D of a symmetric monoidal ∞-category C to inherit a symmetric monoidal structure from C: this may happen if D is closed under tensor products in C, or if D is a Bousfield localization of C which is suitably compatible with the monoidal structure of C. Finally, in Section 4.5 we define the notion of *O-monoids* and *O-monoidal ∞-categories* for an arbitrary ∞-operad O, which for $O = Comm$ specialize to the previous notions of commutative monoids and symmetric monoidal ∞-categories.

4.1 Definitions and examples

Let us recall the definition of an ∞-operad from Definition 1.4.5:

Definition 4.1.1. An *∞-operad* (or: Span(Fin)-*∞-operad*) is a pair $O = (O^{\otimes}, p_O)$ consisting of an ∞-category $O^{\otimes}$ equipped with a functor $p_O \colon O^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ satisfying the following conditions:

(1) The ∞-category $O^{\otimes}$ admits finite products, and the functor p_O preserves finite products;

(2) For every finite set I, the product in $O^{\otimes}$ defines an equivalence
$$\prod_{i \in I} O^{\otimes}_{\{i\}} \xrightarrow{\sim} O^{\otimes}_I.$$
Here we write $O^{\otimes}_I$ for the fiber of p_O over the set I.

(3) For a morphism $f \colon J \to I$ in Fin and objects $X_i \in O^{\otimes}$, the map $\widetilde{f} \colon \prod_{i\in I} X_i \to \prod_{j \in J} X_{f(j)}$ whose j-th component is the projection to $X_{f(j)}$ is *p_O-cocartesian*, meaning that for every $Y \in O^{\otimes}$ the following square is a pullback square:
$$\begin{tikzcd} \mathrm{Hom}_{O^{\otimes}}(\prod_{j\in J} X_{f(j)}, Y) \arrow[r, "- \circ \widetilde{f}"] \arrow[d, "p_O"'] & \mathrm{Hom}_{O^{\otimes}}(\prod_{i \in I} X_i, Y) \arrow[d, "p_O"] \\ \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(p_O(\prod_{j\in J} X_{f(j)}), p_O(Y)) \arrow[r, "- \circ p_O(\widetilde{f})"] & \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(p_O(\prod_{i\in I} X_i), p_O(Y)). \end{tikzcd}$$

If $\mathcal{P} = (\mathcal{P}^{\otimes}, p_{\mathcal{P}})$ is another ∞-operad, then a *morphism of ∞-operads* $f\colon \mathcal{O} \to \mathcal{P}$ is a commutative diagram

$$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f^{\otimes}"] \arrow[dr, "p_{\mathcal{O}}"'] & & \mathcal{P}^{\otimes} \arrow[dl, "p_{\mathcal{P}}"] \\ & \mathrm{Span}(\mathrm{Fin}) & \end{tikzcd}$$

such that the functor $f^{\otimes}$ preserves finite products.

If we write $\mathrm{Cat}_{\infty}^{\times}$ for the (very large) ∞-category of ∞-categories that admit finite products and functors that preserve finite products, we obtain a full subcategory

$$\mathrm{Op}_{\infty} \quad \subseteq \quad (\mathrm{Cat}_{\infty}^{\times})_{/\mathrm{Span}(\mathrm{Fin})}$$

spanned by the ∞-operads. Given two ∞-operads $\mathcal{O}$ and $\mathcal{P}$, we will denote by $\mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P})$ the ∞-category of morphisms of ∞-operads from $\mathcal{O}$ to $\mathcal{P}$, defined as the following fiber:

$$\begin{tikzcd} \mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, \mathcal{P}^{\otimes}) \arrow[d, "p_{\mathcal{P}} \circ -"] \\ \{p_{\mathcal{O}}\} \arrow[r, hook] & \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, \mathrm{Span}(\mathrm{Fin})). \end{tikzcd}$$

In certain contexts, morphisms of ∞-operads $\mathcal{O} \to \mathcal{P}$ are also known as *$\mathcal{O}$-algebras in $\mathcal{P}$*, and a common alternative notation for the ∞-category $\mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{O}, \mathcal{P})$ is $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$.

Warning 4.1.2. As mentioned in Warning 1.4.6, our definition of ∞-operads does not agree with the standard one in the literature, due to Lurie, which we will discuss in Section 4.1.2 below. However, the difference is mostly inconsequential for practical purposes, as most constructions with Lurie's definition go through with our definition. The main difference is that with our definition we can state various conditions on the ∞-category $\mathcal{O}^{\otimes}$ in terms of finite products rather than via a Segal condition.

Terminology 4.1.3. We refer to the fiber $\mathcal{O}_{\langle 1 \rangle} := p_{\mathcal{O}}^{-1}(\langle 1 \rangle)$ over the 1-element set $\langle 1 \rangle := \{1\}$ as the *underlying ∞-category* of the ∞-operad $\mathcal{O}$. We denote its groupoid core by

$$\mathcal{O}^{\simeq} \quad := \quad (\mathcal{O}^{\otimes}_{\langle 1 \rangle})^{\simeq}$$

and refer to it as the *anima of colors of $\mathcal{O}$*. For a finite set I, condition (2) guarantees that every object in $\mathcal{O}^{\otimes}_{I}$ may be uniquely written as a product $\prod_{i \in I} x_i$ for some colors $x_i \in \mathcal{O}^{\simeq}$. In analogy with the classical case, we may also denote such a product as an unordered tuple $\{x_i\}_{i \in I}$. Given another color $y \in \mathcal{O}^{\simeq}$, we define the *anima of multimorphisms in $\mathcal{O}$* from $\{x_i\}_{i \in I}$ to y as the anima of morphisms in $\mathcal{O}^{\otimes}$ that map to the span $I \xleftarrow{=} I \to \langle 1 \rangle$:

$$\begin{tikzcd} \mathcal{O}(\{x_i\}_{i \in I}; y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(\{x_i\}_{i \in I}, y) \arrow[d, "p_{\mathcal{O}}"] \\ * \arrow[r] & \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I, \langle 1 \rangle). \end{tikzcd}$$

A morphism $\varphi\colon X \to Y$ in $\mathcal{O}^{\otimes}$ whose image in Span(Fin) is a forward map $I \xrightarrow{=} I \xrightarrow{g} J$ is called an *active morphism*. We say φ is an *inert morphism* if it is $p_{\mathcal{O}}$-cocartesian and its image in Span(Fin) is a backwards map $I \xleftarrow{f} J \xrightarrow{=} J$. We want to think of the active morphisms as the ones that encode the operad structure, and of the inert morphisms as merely capturing the product structure in $\mathcal{O}^{\otimes}$.

Example 4.1.4. The *commutative ∞-operad $\mathcal{C}omm$* is given by $\mathcal{C}omm^{\otimes} = \mathrm{Span}(\mathrm{Fin})$ and $p_{\mathcal{C}omm} = \mathrm{id}\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathrm{Fin})$. This is the terminal ∞-operad.

Example 4.1.5. The *trivial ∞-operad $\mathcal{T}riv$* is given by $\mathcal{T}riv^{\otimes} = \mathrm{Fin}^{\mathrm{op}}$ and $p_{\mathcal{T}riv} = \mathrm{incl}\colon \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ the canonical inclusion.

Example 4.1.6. The *empty ∞-operad $\mathcal{T}riv_{\emptyset}$* is given by $\mathcal{T}riv_{\emptyset}^{\otimes} = *$ and $p_{\mathcal{T}riv_{\emptyset}}\colon * \xhookrightarrow{\langle 1\rangle} \mathrm{Span}(\mathrm{Fin})$ the inclusion of the one-point set $\langle 1 \rangle$.

Exercise 4.1.7. Show that $\mathcal{T}riv_{\emptyset}$ is the initial ∞-operad.

Example 4.1.8. Let I be an ∞-category, and let $q\colon \mathrm{Fin}(I) \to \mathrm{Fin}$ denote the cartesian unstraightening of $I^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_{\infty}$. We define an ∞-operad $\mathcal{T}riv_I$ by setting

$$\mathcal{T}riv_I^{\otimes} := \mathrm{Fin}(I)^{\mathrm{op}}, \qquad p_{\mathcal{T}riv_I}\colon \mathrm{Fin}(I)^{\mathrm{op}} \xrightarrow{q^{\mathrm{op}}} \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

One can show that $\mathcal{T}riv_I := (\mathrm{Fin}(I)^{\mathrm{op}}, p_{\mathcal{T}riv_I})$ is an ∞-operad, which we refer to as the *trivial ∞-operad generated by I*. Note that the previous two examples are the special cases $I = *$ and $I = \emptyset$. The ∞-operad $\mathcal{T}riv_I$ satisfies the universal property that operad morphisms $\mathcal{T}riv_I \to \mathcal{O}$ are simply functors $I \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ to the underlying ∞-category of $\mathcal{O}$, see also Exercise A.34 below.

Example 4.1.9. The *pointed ∞-operad $\mathcal{E}_0$* is given by

$$\mathcal{E}_0^{\otimes} := \mathrm{Span}_{\mathrm{all,inj}}(\mathrm{Fin}) \qquad \text{and} \qquad p_{\mathcal{E}_0} = \mathrm{incl}\colon\ \mathrm{Span}_{\mathrm{all,inj}}(\mathrm{Fin}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

The active maps in $\mathcal{E}_0$ are the injections $I \hookrightarrow J$ of finite sets.

Example 4.1.10. Other examples of ∞-operads are the associative ∞-operad $\mathcal{A}ssoc$ and the ∞-operads $\mathcal{E}_n$ for $n \geq 0$.

4.1.1 An alternative characterization of ∞-operads

We will next discuss an alternative description of ∞-operads closer to the one found in [Lur17]:

Proposition 4.1.11. *Consider a functor $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$. Then $(\mathcal{O}^{\otimes}, p_{\mathcal{O}})$ is an ∞-operad if and only if the following conditions are satisfied:*

(i) $\mathcal{O}^{\otimes}$ admits all $p_{\mathcal{O}}$-cocartesian lifts of backwards morphisms $I \xleftarrow{f} J \xrightarrow{=} J$ in $\mathrm{Span}(\mathrm{Fin})$*;*

(ii) For every finite set I, the functor

$$\mathcal{O}^{\otimes}_I \to \prod_{i \in I} \mathcal{O}^{\otimes}_{\{i\}}$$

given by cocartesian transport along the maps $\rho_i\colon I \hookleftarrow \{i\} \xrightarrow{=} \{i\}$ in $\mathrm{Span}(\mathrm{Fin})$ *is an equivalence;*

(iii) Consider an object $Y \in \mathcal{O}^{\otimes}$ with image $J := p_{\mathcal{O}}(Y) \in$ $\mathrm{Span}(\mathrm{Fin})$ *and cocartesian lifts $Y \to Y_j$ over the maps ρ_j in* $\mathrm{Span}(\mathrm{Fin})$*. Then the maps $Y \to Y_j$ exhibit Y as a product $\prod_{j\in J} Y_j$ in $\mathcal{O}^{\otimes}$.*

Proof. Throughout the proof we write $p = p_O$ for simplicity. First assume that $O^\otimes$ is an ∞-operad. We show that (i)–(iii) are satisfied:

(i) Consider a morphism $f\colon J \to I$ in Fin and consider $X \in O^\otimes_I$. By condition (2), we may uniquely write X as a product $\prod_{i \in I} X_i$ of objects $X_i \in O^\otimes_{\{i\}}$. We then define $Y := \prod_{j \in J} X_{f(j)}$ and define $\widetilde{f}\colon X \to Y$ to be the map whose j-th component $X = \prod_{i \in I} X_i \to X_{f(j)}$ is the projection map onto $X_{f(j)}$. Then assumption (3) tells us that $\widetilde{f}$ is a p-cocartesian lift of the span $I \xleftarrow{f} J \xrightarrow{=} J$.

(iii) Using (2), we may write the object $Y \in O^\otimes_J$ uniquely as a product $\prod_{j \in J} Y_j$. By assumption (3), the projection maps $Y = \prod_{j \in J} Y_j \to Y_j$ are cocartesian lifts of the maps $\rho_j\colon J \hookleftarrow \{j\} \xrightarrow{=} \{j\}$ in Span(Fin), verifying (iii).

(ii) To show that the map $O^\otimes_I \to \prod_{i \in I} O^\otimes_{\{i\}}$ is an equivalence, it suffices to show it is a left-inverse to the product functor $\prod_{i \in I} O^\otimes_{\{i\}} \xrightarrow{\sim} O^\otimes_I$, as this was assumed to be an equivalence. This is again a consequence of the fact that the projection maps $\prod_{i \in I} X_i \to X_i$ are cocartesian lifts of ρ_i.

We now show that conditions (i)-(iii) imply that $O^\otimes$ is an ∞-operad. We check conditions (1)-(3):

(1) For finite products in $O^\otimes$, consider objects $Y^1, \dots, Y^n$ in $O^\otimes$ and set $T_i := p(Y^i)$ and $T := \bigsqcup_i T_i$. Picking cocartesian lifts $Y^i \to Y^i_t$ over each map ρ_t, condition (iii) tells us that these maps exhibit each Y^i as a product $\prod_{t \in T_i} Y^i_t$. Using (ii), we may pick some object $Y \in O^\otimes_T$ whose cocartesian transport along each ρ_t is Y^i_t, and we similarly see that the maps $Y \to Y^i_t$ exhibit Y as a product $\prod_{i=1}^n \prod_{t \in T} Y^i_t$. It follows that $Y \simeq \prod_{i=1}^n Y^i$, showing that $O^\otimes$ admits finite products. It is also clear from this calculation that $p\colon O^\otimes \to \mathrm{Span(Fin)}$ preserves products.

(2) It will suffice to argue that the product functor $\prod_{s \in S} O^\otimes_s \to O^\otimes_S$ is a right-inverse to the assumed equivalence $O^\otimes_S \xrightarrow{\sim} \prod_{s \in S} O^\otimes_{\{s\}}$ from (ii). But (iii) implies that for $X_s \in O^\otimes_{\{s\}}$ the projection maps $\prod_{s \in S} X_s \to X_s$ are the cocartesian lifts of the ρ_s, which gives the claim.

(3) Consider a morphism $f\colon J \to I$ and let $X_i \in O^\otimes$. We have to show that the map $\widetilde{f}\colon \prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$ is p-cocartesian.

Let us first prove the special case where $p(X_i) = *$ for all $i \in I$. Let $\overline{f}\colon \prod_{i \in I} X_i \to Y$ be a cocartesian lift of f. By the universal property of cocartesian lifts, there is a morphism $Y \to \prod_{j \in J} X_{f(j)}$ which we wish to show is an equivalence. Note that this map lives in the fiber over J, so by condition (2) it suffices to check this after applying cocartesian transport along each of the maps ρ_j. Note that we have $(\rho_j)_! Y \simeq X_{f(j)}$ since the composites of the spans

$$I \xleftarrow{f} J \xrightarrow{=} J \qquad \text{and} \qquad \rho_j\colon J \hookleftarrow \{j\} \to *$$

is the span $\rho_i\colon I \hookleftarrow \{f(j)\} \to *$. But also $(\rho_j)_!(\prod_{j \in J} X_{f(j)}) \simeq X_{f(j)}$. This proves the claim in the special case.

The general case is an immediate consequence by writing each X_i as a product $\prod_{t \in T_i} X_i^t$ and applying the previous argument to $\prod_{i \in I} \prod_{t \in T_i} X_i^t$. ◻

Remark 4.1.12. For later purposes, let us note that condition (iii) in the proposition is equivalent to the condition that for objects $X, Y \in O^{\otimes}$ over $I, J \in \mathrm{Span}(\mathrm{Fin})$ and cocartesian lifts $Y \to Y_j$ over the maps $\rho_j \colon J \hookleftarrow \{j\} \xrightarrow{=} \{j\}$ in Span(Fin), the commutative square

$$\begin{tikzcd} \mathrm{Hom}_{O^{\otimes}}(X,Y) \arrow[r] \arrow[d, "p_O"'] & \prod_{j\in J} \mathrm{Hom}_{O^{\otimes}}(X,Y_j) \arrow[d, "p_O"] \\ \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J) \arrow[r] & \prod_{j\in J} \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,\{j\}) \end{tikzcd}$$

is a pullback square. Indeed, since the bottom map is an equivalence, this square is a pullback square if and only if the top map is also an equivalence, and this holding for all X precisely means that Y is a product of the Y_j.

Corollary 4.1.13. *If O is an ∞-operad, then the inert morphisms in $O^{\otimes}$ are precisely the morphisms of the form $\widetilde{f} \colon \prod_{i\in I} X_i \to \prod_{j\in j} X_{f(j)}$ for maps $f \colon J \to I$ in* Fin *and colors $X_i \in O^{\simeq}$.*

Similarly, if $\mathcal{P}$ is another ∞-operad, and $\varphi^{\otimes} \colon O^{\otimes} \to \mathcal{P}^{\otimes}$ is a functor over Span(Fin)*, then $\varphi^{\otimes}$ preserves finite products (i.e., corresponds to a morphism $\varphi \colon O \to \mathcal{P}$ of ∞-operads) if and only if it preserves the inert morphisms.*

Proof. By assumption $\widetilde{f}$ is inert for every f. Conversely, we saw in the proof of (i) of the previous proposition that for every backwards span $I \xleftarrow{f} J \xrightarrow{=} J$ and every $X \in O_I^{\otimes}$ the map $\widetilde{f} \colon X \simeq \prod_{i\in I} X_i \to \prod_{j\in j} X_{f(j)}$ is a cocartesian lift of f starting in X, so by uniqueness it follows that every cocartesian lift is of this form.

If $\varphi^{\otimes}$ preserves finite products, it in particular preserves the maps $\widetilde{f} \colon \prod_{i\in I} X_i \to \prod_{j\in j} X_{f(j)}$, hence all inert morphisms. Conversely, if $\varphi^{\otimes}$ preserves inert morphisms, it preserves the projection maps $Y \to Y_j$ for $Y \in O_J^{\otimes}$ exhibiting Y as a product of the Y_j, hence it preserves this product. $\square$

Corollary 4.1.14. *Let $O = (O^{\otimes}, p_O)$ be an ∞-operad. Then every morphism $\varphi \colon X \to Z$ in $O^{\otimes}$ factors essentially uniquely as*

$$X \xrightarrow{\widetilde{f}} Y \xrightarrow{\psi} Z,$$

where $\widetilde{f}$ is an inert morphism and ψ is an active morphism.

Proof. Let us denote by $I \xleftarrow{f} K \xrightarrow{g} J$ the image of φ under p. This span may be factored uniquely as a composite of a backwards span $I \xleftarrow{f} K \xrightarrow{=} K$ followed by a forward span $K \xleftarrow{=} K \xrightarrow{g} J$. We then let $\widetilde{f} \colon X \to Y$ be the unique cocartesian lift of f with domain X. It follows that there is a unique morphism $\psi \colon Y \to Z$ over g such that $\psi \circ \widetilde{f} = \varphi$. $\square$

Remark 4.1.15. The argument shows that the factorization is unique up to isomorphism. It is possible to show the stronger statement that the anima of factorizations is contractible. This relies on the fact that the backwards maps and forwards maps form an *orthogonal factorization system* on Span(Fin). We refer the reader to [Hau+23b, Proposition 4.9] for a precise statement of this claim.

4.1.2 Comparison to Lurie's definition

As mentioned in Warning 1.4.6, our definition of ∞-operads does not agree with Lurie's definition, which instead works with ∞-categories over the category Fin_* of finite pointed sets.

Definition 4.1.16. Given a finite set I, we write $I_+ := I \sqcup *$ for the resulting finite pointed set. A morphism $f\colon I_+ \to J_+$ of finite pointed sets is called *inert* if on the preimage of J it restricts to a bijection $f^{-1}(J) \xrightarrow{\cong} J$. We say that f is *active* if $f^{-1}(*) = \{*\}$.

Given a finite set I, the *Segal map* $\rho_i\colon I_+ \to \{i\}_+$ for $i \in I$ is defined as the map that is the identity on $\{i\}$ and sends all other $j \in I \setminus \{i\}$ to the basepoint.

Remark 4.1.17. Recall that there is an equivalence

$$\mathrm{Span}_{\mathrm{inj,all}}(\mathrm{Fin}) \xrightarrow{\sim} \mathrm{Fin}_*,$$

given on objects by sending I to I_+ and on morphisms by sending a span $I \xhookleftarrow{f} K \xrightarrow{g} J$ to the pointed map $h\colon I_+ \to J_+$ given by $h(i) = g(k)$ whenever $i = f(k)$ is in the image of f and $h(i) = *$ otherwise. Under this equivalence, the inert maps in Fin_* correspond to the backwards spans $I \xhookleftarrow{f} J \xrightarrow{=} K$, and the active maps in Fin_* correspond to the forward spans $I \xleftarrow{=} I \xrightarrow{g} J$. The Segal maps ρ_i correspond to the backwards spans

$$I \hookleftarrow \{i\} \xrightarrow{=} \{i\}.$$

We will write $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ for the inclusion of the subcategory $\mathrm{Span}_{\mathrm{inj,all}}(\mathrm{Fin})$.

Definition 4.1.18. Let Fin_* be the category of finite pointed sets. An *∞-operad in the sense of Lurie* is a pair $(\mathcal{O}^\otimes, p_\mathcal{O})$ of an ∞-category $\mathcal{O}^\otimes$ equipped with a functor $p_\mathcal{O}\colon \mathcal{O}^\otimes \to \mathrm{Fin}_*$ satisfying the following conditions:

(1) For any inert morphism $f\colon I_+ \to J_+$ and any object $X \in \mathcal{O}^\otimes_I$ there is a $p_\mathcal{O}$-cocartesian lift $X \to Y$ in $\mathcal{O}^\otimes$.

(2) Let X and X' be objects of $\mathcal{O}^\otimes$ and let $I_+ := p(X)$ and $J_+ := p(Y)$. For every $j \in J$, let $X' \to X'_j$ denote a $p_\mathcal{O}$-cocartesian lift of the Segal map $\rho_j\colon J_+ \to \{j\}_+$. Then the induced commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}^\otimes}(X, X') \arrow[r] \arrow[d, "p_\mathcal{O}"'] & \prod_{j\in J} \mathrm{Hom}_{\mathcal{O}^\otimes}(X, X'_j) \arrow[d, "p_\mathcal{O}"] \\
\mathrm{Hom}_{\mathrm{Fin}_*}(I_+, J_+) \arrow[r] & \prod_{j\in J} \mathrm{Hom}_{\mathrm{Fin}_*}(I_+, \{j\}_+)
\end{tikzcd}$$

is a pullback square.

(3) For every finite collection $\{X_j\}_{j\in J}$ of objects $X_j \in \mathcal{O}^\otimes_{\{j\}}$, there exists an object $X \in \mathcal{O}^\otimes_J$ and a collection of $p_\mathcal{O}$-cocartesian morphisms $X \to X_j$ covering $\rho_j\colon J_+ \to \{j\}_+$.

We say a functor $f\colon \mathcal{O}^\otimes \to \mathcal{P}^\otimes$ over Fin_* is a *morphism of Lurie-∞-operads* if it preserves cocartesian morphisms over inerts. We let $\mathrm{Op}_\infty^{\mathrm{Lurie}} \subseteq (\mathrm{Cat}_\infty)_{/\mathrm{Fin}_*}$ denote the ∞-category of the Lurie ∞-operads and their morphisms.

Lemma 4.1.19. *If $\mathcal{O} = (\mathcal{O}^\otimes, p_\mathcal{O})$ be an ∞-operad in the sense of Definition 4.1.1, then its pullback $p_\mathcal{P}\colon \mathcal{P}^\otimes := \mathcal{O}^\otimes \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Fin}_* \to \mathrm{Fin}_*$ is an ∞-operad in the sense of Lurie. In particular, forming pullbacks along the inclusion* $\mathrm{Fin}_* \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ *induces a 'forgetful functor'*

$$\mathrm{Op}_\infty \to \mathrm{Op}_\infty^{\mathrm{Lurie}}.$$

Proof. Any p_O-cocartesian lift of a backwards span $I \hookleftarrow J \xrightarrow{=} J$ will still be $p_{\mathcal{P}}$-cocartesian, verifying condition (1). Condition (2) is immediate from Remark 4.1.12. Condition (3) is immediate from condition (ii) in Proposition 4.1.11. □

Theorem 4.1.20 ([BHS22, Theorem 5.1.1, Corollary 5.1.15])**.** *The forgetful functor* $\mathrm{Op}_\infty \to \mathrm{Op}_\infty^{\mathrm{Lurie}}$ *is an equivalence.*

The proof of this result is of somewhat technical nature and we will not discuss it, though we will see some of the main ingredients below in Section 7.1, see in particular Theorem 7.1.7.

4.2 Symmetric monoidal ∞-categories and algebras over operads

Let O be a colored operad and let C be a symmetric monoidal category. Recall from Definition 1.2.9 the notion of an *O-algebra in C*, which we defined as an operad map $O \to \mathcal{M}_C$ into the *multimorphism operad* of C; here the multimorphisms of the operad $\mathcal{M}_C$ are taken to be the 'actual' multimorphisms in C, i.e., $\mathcal{M}_C(\{x_i\}_{i\in I}; y) := \mathrm{Hom}_C(\bigotimes_{i\in I} x_i, y)$.

In this section, we will generalize the notion of O-algebras to the ∞-categorical setting. This will in particular involve an appropriate generalization of the construction $C \mapsto \mathcal{M}_C$ for symmetric monoidal ∞-categories C.

Definition 4.2.1. Recall that a *commutative monoid* in an ∞-category D with finite products is a finite-product-preserving functor $\mathrm{Span}(\mathrm{Fin}) \to D$. We write

$$\mathrm{CMon}(D) \subseteq \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), D)$$

for the full subcategory of commutative monoids in D. A *symmetric monoidal ∞-category* is a commutative monoid in Cat_∞. We will often simply call them C, but may for emphasis also write them as triples $(C, \otimes, \mathbb{1})$ or as pairs $(C, \otimes)$. We will also write

$$\mathrm{Cat}_\infty^\otimes \quad := \quad \mathrm{CMon}(\mathrm{Cat}_\infty).$$

Lemma 4.2.2. *Let* $p_O\colon O^\otimes \to \mathrm{Span}(\mathrm{Fin})$ *be a cocartesian fibration. Then* $(O^\otimes, p_O)$ *is an ∞-operad if and only if the cocartesian straightening of* p_O

$$\mathrm{Str}^{\mathrm{cc}}(p_O)\colon\ \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$$

preserves finite products.

Proof. We check the conditions (i)-(iii) from Proposition 4.1.11. Condition (i) is part of the assumption on p_O. Part (ii) is equivalent to the condition that $\mathrm{Str}(p_O)$ preserves finite products, since the spans $\rho_i\colon I \hookleftarrow \{i\} \xrightarrow{=} \{i\}$ exhibit I as a product in $\mathrm{Span}(\mathrm{Fin})$. It thus remains to show that (iii) is automatically satisfied in this case. By Remark 4.1.12, this amounts to showing that for every object $X \in O^\otimes$, the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{O^\otimes}(X,Y) \arrow[r] \arrow[d, "p_O"'] & \prod_{j\in J} \mathrm{Hom}_{O^\otimes}(X,Y_j) \arrow[d, "p_O"] \\
\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,J) \arrow[r] & \prod_{j\in J} \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I,\{j\})
\end{tikzcd}$$

is a pullback square. We will check this by showing that the vertical fibers agree. Consider a span $\alpha\colon I \xleftarrow{f} K \xrightarrow{g} J$. Let $X' := \alpha_! X \in \mathcal{O}^{\otimes}_J$ be the cocartesian transport of X along this span. By definition, precomposition with the map $\varphi\colon X \to X'$ induces a pullback square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}^{\otimes}}(X', Y) \arrow[r, "- \circ \varphi"] \arrow[d, "p_{\mathcal{O}}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X, Y) \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Hom}_{\mathrm{Span(Fin)}}(J, J) \arrow[r, "- \circ (f,g)"] & \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J),
\end{tikzcd}$$

and similarly if we replace the target by Y_j for every $j \in J$. We may thus assume that the span α is the identity span on J. In this case, let $X \to X_j$ denote a cocartesian lift of ρ_j. We then get a commutative diagram as follows:

$$\begin{tikzcd}
\prod_{j \in J} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X_j, Y_j) \arrow[rr] \arrow[dr, "\prod_J"] \arrow[dd, "p_{\mathcal{O}}"'] & & \prod_{j \in J} \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X, Y_j) \arrow[dd, "p_{\mathcal{O}}"] \\
& \mathrm{Hom}_{\mathcal{O}^{\otimes}}(X, Y) \arrow[ur] & \\
\prod_{j \in J} \mathrm{Hom}_{\mathrm{Span(Fin)}}(\{j\}, \{j\}) \arrow[rr] \arrow[dr] & & \prod_{j \in J} \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, \{j\}) \\
& \mathrm{Hom}_{\mathrm{Span(Fin)}}(J, J). \arrow[ur] \arrow[from=uu, crossing over, "p_{\mathcal{O}}"] &
\end{tikzcd}$$

Since the functor $\prod_{j \in J} \mathcal{O}^{\otimes}_{\{j\}} \xrightarrow{\sim} \mathcal{O}^{\otimes}_J$ is an equivalence, the left front square is a pullback square. Also the back square is a pullback square. It follows that the horizontal fibers of the right front square over id_J are equivalent, as desired. $\square$

Definition 4.2.3. Given a symmetric monoidal ∞-category $\mathcal{C}$, we denote its cocartesian unstraightening by

$$\mathcal{C}^{\otimes} \quad := \quad \mathrm{Un}^{\mathrm{cc}}(\mathcal{C}\colon \mathrm{Span(Fin)} \to \mathrm{Cat}_{\infty}).$$

By the previous lemma, the resulting cocartesian fibration $p_{\mathcal{C}}\colon \mathcal{C}^{\otimes} \to \mathrm{Span(Fin)}$ defines an ∞-operad, which we will denote by $\mathcal{M}_{\mathcal{C}} := (\mathcal{C}^{\otimes}, p_{\mathcal{C}})$ and refer to as the *multimorphism operad* associated to $\mathcal{C}$. By functoriality of unstraightening, this defines a (non-full) inclusion

$$\mathcal{M}\colon \mathrm{Cat}_{\infty}^{\otimes} \hookrightarrow \mathrm{Op}_{\infty}.$$

Exercise 4.2.4. Given $\mathcal{C} \in \mathrm{Cat}_{\infty}^{\otimes}$ and objects $x_1, \dots, x_n, y \in \mathcal{C}^{\simeq}$, show that the anima $\mathcal{M}_{\mathcal{C}}((x_1, \dots, x_n); y)$ is naturally equivalent to $\mathrm{Hom}_{\mathcal{C}}(x_1 \otimes \cdots \otimes x_n, y)$.

We have the following useful criterion for checking that an ∞-operad corresponds to a symmetric monoidal ∞-category:

Lemma 4.2.5. *Let $\mathcal{O}$ be an ∞-operad. Assume that the following two conditions are satisfied:*

(1) *For every object $\{x_i\}_{i \in I}$ of $\mathcal{O}^{\otimes}$ the functor*

$$\mathcal{O}(\{x_i\}_{i \in I}; -)\colon \mathcal{O}_{\langle 1 \rangle} \to \mathrm{An}$$

is corepresentable, in the sense that there exists a multimorphism $\varphi\colon \{x_i\}_{i\in I} \to y$ such that for every other color $z \in \mathcal{O}_{\langle 1\rangle}$ precomposition with φ induces an equivalence

$$\mathrm{Hom}_{\mathcal{O}_{\langle 1\rangle}}(y,z) \xrightarrow{\sim} \mathcal{O}(\{x_i\}_{i\in I}; z).$$

This condition determines y up to contractible choice, and we will denote it by $\bigotimes_{i\in I} x_i$.

(2) *For every morphism $f\colon I \to J$ of finite sets, the canonical comparison map*

$$\bigotimes_{i\in I} x_i \to \bigotimes_{j\in J} \bigotimes_{i\in f^{-1}(j)} x_i$$

is an equivalence.

Then $p_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to$ Span(Fin) is a cocartesian fibration, corresponding to a symmetric monoidal structure on the underlying ∞-category $\mathcal{O}^{\otimes}_{\langle 1\rangle}$.

Proof. We need to produce cocartesian lifts of morphisms in Span(Fin). Cocartesian lifts of backwards morphisms exist by the defining property of ∞-operads, so it suffices to produce cocartesian lifts for active spans $\alpha = (I \xleftarrow{=} I \xrightarrow{f} J)$. Let $X = \{x_i\}_{i\in I}$ be an object in $\mathcal{O}^{\otimes}_I$. Applying the assumption on $\mathcal{O}$ to each $X_j := \{x_i\}_{i\in f^{-1}(j)}$, we may find multimorphisms $\varphi_j\colon \{x_i\}_{i\in f^{-1}(j)} \to y_j$ in $\mathcal{O}$ corepresenting the functor $\mathcal{O}(\{x_i\}_{i\in f^{-1}(j)}; -)\colon \mathcal{O}_{\langle 1\rangle} \to \mathrm{An}$. The multimorphisms φ_j define a morphism $\varphi\colon X \to Y := \{y_j\}_{j\in J}$ in $\mathcal{O}^{\otimes}$. We wish to show that φ is a cocartesian lift of α.

To this end, consider any other span $\beta = (J \xleftarrow{g} K \xrightarrow{h} L)$ of finite sets, and consider an object $Z = \{z_l\}_{l\in L}$ of $\mathcal{O}^{\otimes}_L$. We need to show that precomposition with φ induces an equivalence

$$-\circ\varphi\colon\ \mathrm{Hom}^{\beta}_{\mathcal{O}^{\otimes}}(Y,Z) \xrightarrow{\sim} \mathrm{Hom}^{\beta\circ\alpha}_{\mathcal{O}^{\otimes}}(X,Z).$$

Since Z is a product in $\mathcal{O}^{\otimes}$ of the one-element tuples (z_l), we may assume that $L = \langle 1\rangle$, so that there is only a single color $z := z_1$. Using the inert morphisms $\{y_j\} \to \{y_{g(k)}\}_{k\in K}$ and $\{x_i\}_{(i,k)\in I\times_J K}$, we must equivalently show that composition with the map φ induces an equivalence

$$-\circ\varphi\colon \mathcal{O}(\{y_{g(k)}\}_{k\in K}; z) \xrightarrow{\sim} \mathcal{O}(\{x_i\}_{(i,k)\in I\times_J K}; z).$$

Now, applying condition (1) another time to the families $\{y_{g(k)}\}_{k\in K}$ and $\{x_i\}_{(i,k)\in I\times_J K}$, both sides are corepresented by objects $\bigotimes_{k\in K} y_{g(k)}$ and $\bigotimes_{(i,k)\in I\times_J K} x_i$, respectively. But since $y_{g(k)} = \bigotimes_{i\in f^{-1}(g(k))} x_i$, the claim now follows from the equivalence

$$\bigotimes_{(i,k)\in I\times_J K} x_i \xrightarrow{\sim} \bigotimes_{k\in K} \bigotimes_{i\in f^{-1}(g(k))} x_i$$

obtained by applying condition (2) to the projection map $I\times_J K \to K$. □

Definition 4.2.6. Let $\mathcal{O}$ be an ∞-operad and let $\mathcal{C}$ be a symmetric monoidal ∞-category. An *$\mathcal{O}$-algebra in $\mathcal{C}$* is a morphism of operads $\mathcal{O} \to \mathcal{M}_{\mathcal{C}}$. We write

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) := \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{M}_{\mathcal{C}})$$

for the ∞-category of $\mathcal{O}$-algebras in $\mathcal{C}$.

Remark 4.2.7. Lurie refers more generally to an operad map $\mathcal{O} \to \mathcal{P}$ as an *$\mathcal{O}$-algebra in $\mathcal{P}$* and write $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) := \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{P})$.

Example 4.2.8.
- $\mathcal{C}omm$-algebras are called *commutative algebras*, and we write
$$\mathrm{CAlg}(C) := \mathrm{Alg}_{\mathcal{C}omm}(C).$$
- $\mathcal{A}ssoc$-algebras are called *associative algebras*, and we write
$$\mathrm{Alg}(C) := \mathrm{Alg}_{\mathcal{A}ssoc}(C).$$
- The data of a $\mathcal{T}riv$-algebra is a finite-product preserving functor $\mathrm{Fin}^{\mathrm{op}} \to C^{\otimes}$ over $\mathrm{Span}(\mathrm{Fin})$. Since Fin is freely generated under finite coproducts by the point, the evaluation functor induces an equivalence
$$\mathrm{Alg}_{\mathcal{T}riv}(C) \xrightarrow{\sim} C,$$
so that a $\mathcal{T}riv$-algebra is simply an object of C.
- The data of an $\mathcal{T}riv_\emptyset$-algebra is a finite-product preserving functor $* \to C^{\otimes}$ over $\mathrm{Span}(\mathrm{Fin})$. There is only one such functor, given by the terminal object of $C^{\otimes}$. It follows that
$$\mathrm{Alg}_{\mathcal{T}riv_\emptyset}(C) \xrightarrow{\sim} *.$$

4.3 Lax symmetric monoidal functors

The operadic perspective on symmetric monoidal ∞-categories is useful for defining the notion of lax symmetric monoidal functors:

Definition 4.3.1. Let C and D be symmetric monoidal ∞-categories. A *lax symmetric monoidal functor $C \to D$* is a morphism of ∞-operads $\mathcal{M}_C \to \mathcal{M}_D$. An *oplax symmetric monoidal functor* is a morphism of ∞-operads $\mathcal{M}_{C^{\mathrm{op}}} \to \mathcal{M}_{D^{\mathrm{op}}}$. We denote by
$$\mathrm{Cat}_\infty^{\otimes\text{-}\mathrm{lax}} \quad \subseteq \quad \mathrm{Op}_\infty$$
the full subcategory spanned by the (multimorphism operads of) symmetric monoidal ∞-categories.

Remark 4.3.2. Let us unwind the data contained in a lax symmetric monoidal functor $F\colon C \to D$. We have a commutative triangle of the form
$$\begin{tikzcd} C^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr, "p_C"'] & & D^{\otimes} \arrow[dl, "p_D"] \\ & \mathrm{Span}(\mathrm{Fin}). & \end{tikzcd}$$
By passing to fibers over the one-point set $\langle 1 \rangle$, we obtain an underlying functor $F\colon C \to D$. Since $F^{\otimes}$ preserves finite products, it follows that on an arbitrary object $(X_1, \ldots, X_n)$ it is given by
$$F^{\otimes}(X_1, \ldots, X_n) = (F(X_1), \ldots, F(X_n)).$$

The behavior on inert maps is entirely specified as well. For the behavior on active maps, it suffices to understand where $F^{\otimes}$ sends a cocartesian morphism $(x_1, \ldots, x_n) \to x_1 \otimes \cdots \otimes x_n$ lifting the span $\langle n \rangle \overset{=}{\leftarrow} \langle n \rangle \to \langle 1 \rangle$: we obtain a map in $D^{\otimes}$ of the form

$$(F(X_1), \ldots, F(X_n)) \to F(X_1 \otimes \cdots \otimes X_n)$$

lifting the same span. This map factors uniquely through the cocartesian morphism $(F(X_1), \ldots, F(X_n)) \to F(X_1) \otimes \cdots \otimes F(X_n)$, resulting in the expected lax structure map

$$F(X_1) \otimes \cdots \otimes F(X_n) \to F(X_1 \otimes \cdots \otimes X_n).$$

The functor $F^{\otimes}$ may be thought of as a coherent encoding of all these individual structure maps that is fully compatible with composition and tensor products in C and D.

For oplax symmetric monoidal functors the structure maps point the other way.

A general way of constructing lax symmetric monoidal functors is as right adjoints of strong symmetric monoidal ones:

Proposition 4.3.3. *Let $L\colon C \to D$ be a symmetric monoidal functor between symmetric monoidal ∞-categories. Assume that the underlying functor of L admits a right adjoint $R\colon D \to C$. Then the following statements hold:*

(1) *The functor $L^{\otimes}\colon C^{\otimes} \to D^{\otimes}$ admits a right adjoint $R^{\otimes}\colon D^{\otimes} \to C^{\otimes}$.*

(2) *The functor $R^{\otimes}$ is a morphism of ∞-operads.*

(3) *The restriction $R^{\otimes}|_{D^{\otimes}_{\langle 1 \rangle}}\colon D \to C$ is R.*

In particular, the right adjoint of a symmetric monoidal functor is canonically lax symmetric monoidal.

Proof. (1) By the pointwise criterion for right adjoints, it will suffice to show that for every object $X = \{x_i\}_{i \in I} \in D^{\otimes}$ there exists an object $R^{\otimes}X$ together with a counit map $\varepsilon_X\colon L^{\otimes}R^{\otimes}X \to X$ such that for every other $Y \in C^{\otimes}$ the composite

$$\mathrm{Hom}_{C^{\otimes}}(X, R^{\otimes}Y) \xrightarrow{L^{\otimes}} \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}X, L^{\otimes}R^{\otimes}Y) \xrightarrow{\varepsilon_X \circ -} \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}X, Y) \tag{4.1}$$

is an equivalence. We define $R^{\otimes}X := \{Rx_i\}_{i \in I}$. We then have $L^{\otimes}R^{\otimes}X \simeq \{LRx_i\}_{i \in I}$ and so the counit map ε_X may be taken to be the collection of morphisms $\varepsilon_{x_i}\colon LRx_i \to x_i$, resulting in a morphism in $D^{\otimes}_I \simeq \prod_{i \in I} D$.

To show that the composite (4.1) is an equivalence, it suffices to show that it induces an equivalence on the fibers over any span $\alpha\colon I \overset{f}{\leftarrow} K \overset{g}{\to} J$ in $\mathrm{Span}(\mathrm{Fin})$. Let $\widetilde{\alpha}\colon X \to \alpha_! X$ denote a cocartesian lift of the span α. Since $L^{\otimes}$ preserves cocartesian morphisms, also the map $L^{\otimes}(\widetilde{\alpha})\colon L^{\otimes}X \to L^{\otimes}\alpha_! X$ is cocartesian. Precomposition with these maps then induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{C^{\otimes}}(\alpha_! X, R^{\otimes}Y) \arrow[r, "L^{\otimes}"] \arrow[d, "- \circ \widetilde{\alpha}"'] & \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}\alpha_! X, L^{\otimes}R^{\otimes}Y) \arrow[r, "\varepsilon_Y \circ -"] \arrow[d, "- \circ L^{\otimes}\widetilde{\alpha}"] & \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}\alpha_! X, Y) \arrow[d, "- \circ L^{\otimes}\widetilde{\alpha}"] \\
\mathrm{Hom}_{C^{\otimes}}(X, R^{\otimes}Y) \arrow[r, "L^{\otimes}"] & \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}X, L^{\otimes}R^{\otimes}Y) \arrow[r, "\varepsilon_Y \circ -"] & \mathrm{Hom}_{D^{\otimes}}(L^{\otimes}X, Y).
\end{tikzcd}$$

Since the vertical maps induce equivalences on fibers over $\mathrm{Hom}_{\mathrm{Span(Fin)}}(J, J) \xrightarrow{-\circ\alpha} \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, J)$, we have thus reduced to the case of the identity span $\alpha = \mathrm{id}_J$. In this case, the fibers are the Hom animae in the fibers $C_J^\otimes \simeq \prod_{j\in J} C$ and $D_J^\otimes \simeq \prod_{j\in J} D$ respectively. The claim thus follows from the fact that for every $j \in J$ the composite

$$\mathrm{Hom}_C((\alpha_! X)_j, RY_j) \xrightarrow{L} \mathrm{Hom}_D(L(\alpha_! X)_j, LRY_j) \xrightarrow{\varepsilon_{Y_j}\circ -} \mathrm{Hom}_D(L(\alpha_! X)_j, Y_j)$$

is an equivalence due to the adjunction $L \dashv R$.

(2) We will now show that $R^\otimes$ is a functor over Span(Fin), in that it makes the following diagram commute:

$$\begin{tikzcd} D^\otimes \arrow[rr, "R^\otimes"] \arrow[dr, "p_D"'] & & C^\otimes \arrow[dl, "p_C"] \\ & \mathrm{Span(Fin)}. & \end{tikzcd}$$

To this end, we claim that the adjunction counit $\varepsilon\colon L^\otimes R^\otimes \to \mathrm{id}_{D^\otimes}$ induces an equivalence

$$p_C R^\otimes \simeq p_D L^\otimes R^\otimes \xrightarrow{p_D\,\varepsilon} p_D\,\mathrm{id}_{D^\otimes} \simeq p_D.$$

This may be checked objectwise for $X \in D^\otimes$, where it is clear from the construction of ε_X. The condition that $R^\otimes$ preserves finite products may similarly be checked objectwise, showing that $R^\otimes$ is an operad map.

(3) By uniqueness of adjoints, the restriction of $R^\otimes$ to $D = D^\otimes_{\langle 1\rangle}$ is R, and we deduce that $R^\otimes$ is a lax symmetric monoidal refinement of R. □

Warning 4.3.4. In the previous proposition, it does not suffice for L to be a lax symmetric monoidal functor itself. It *would* suffice for L to be an *oplax* symmetric monoidal functor, but producing the lax symmetric monoidal right adjoint is more subtle in this case. Dually, the left adjoint to a lax symmetric monoidal functor is oplax symmetric monoidal. We refer to [Hau+23a, Proposition A] for a detailed discussion.

While the construction of the oplax symmetric monoidal left adjoint of a lax symmetric monoidal functor is subtle in general, this construction simplifies when this left adjoint happens to be actually symmetric monoidal:

Proposition 4.3.5. *Let $R\colon D \to C$ be a lax symmetric monoidal functor between symmetric monoidal ∞-categories. Assume that the underlying functor of R admits a left adjoint $L\colon C \to D$ such that for all $n \geq 0$ and objects $x_1, \ldots, x_n \in C$ the composite*

$$L(\bigotimes_{i=1}^n x_i) \xrightarrow{L(\bigotimes_{i=1}^n \eta_{x_i})} L(\bigotimes_{i=1}^n RLx_i) \xrightarrow{\mathrm{lax}_R} LR(\bigotimes_{i=1}^n Lx_i) \xrightarrow{\varepsilon} \bigotimes_{i=1}^n L(x_i)$$

is an isomorphism in D. Then the following statements hold:

1. *The functor $R^\otimes\colon D^\otimes \to C^\otimes$ admits a left adjoint $L^\otimes\colon C^\otimes \to D^\otimes$.*
2. *The functor $L^\otimes$ is a morphism of ∞-operads.*
3. *The restriction of $L^\otimes$ to $C = C^\otimes_{\langle 1\rangle}$ is L.*

In particular, R admits a strong symmetric monoidal left adjoint $L\colon (C, \otimes_C) \to (D, \otimes_D)$.

Note that it suffices to check the condition on L for $n = 0$ and $n = 2$, i.e. it suffices to show that the two canonical maps

$$L(\mathbb{1}_C) \to \mathbb{1}_D \qquad \text{and} \qquad L(x \otimes y) \to L(x) \otimes L(y)$$

are isomorphisms.

Proof. (1) As in the previous proof, we will use the pointwise criterion for left adjoints. We define $L^\otimes$ objectwise as $L^\otimes(\{x_i\}_{i\in I}) := \{L(x_i)\}_{i\in I}$; the unit maps $\eta_{x_i}\colon x_i \to RLx_i$ induce a candidate unit map $\eta_X\colon X \to R^\otimes L^\otimes X$. We must show that for every object $Y \in C^\otimes_J$ the composite

$$\mathrm{Hom}_{D^\otimes}(L^\otimes X, Y) \xrightarrow{R^\otimes} \mathrm{Hom}_{C^\otimes}(R^\otimes L^\otimes X, R^\otimes Y) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_{C^\otimes}(X, R^\otimes Y)$$

is an equivalence. This may be tested fiberwise over any span $\alpha\colon I \xleftarrow{f} K \xrightarrow{g} J$. Let

$$X \to \alpha_! X, \qquad L^\otimes X \to \alpha_! L^\otimes X, \qquad \text{and} \qquad R^\otimes L^\otimes X \to \alpha_! R^\otimes L^\otimes X$$

denote cocartesian lifts of α in $C^\otimes$ and $D^\otimes$, and let $\mathrm{lax}_R\colon \alpha_! R^\otimes L^\otimes X \to R^\otimes \alpha_! L^\otimes X$ denote the lax structure map of R associated to α. We then get a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{id}_J}_{D^\otimes}(\alpha_! L^\otimes X, Y) \arrow[rr, "\simeq"] \arrow[d, "R^\otimes"'] & & \mathrm{Hom}^{\alpha}_{D^\otimes}(L^\otimes X, Y) \arrow[dd, "R^\otimes"] \\
\mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(R^\otimes \alpha_! L^\otimes X, R^\otimes Y) \arrow[d, "-\circ \mathrm{lax}_R"'] & & \\
\mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(\alpha_! R^\otimes L^\otimes X, R^\otimes Y) \arrow[rr, "\simeq"] \arrow[d, "-\circ\alpha_!\eta_X"'] & & \mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(R^\otimes L^\otimes X, R^\otimes Y) \arrow[d, "-\circ\eta_X"] \\
\mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(\alpha_! X, R^\otimes Y) \arrow[rr, "\simeq"] & & \mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(X, R^\otimes Y).
\end{tikzcd}$$

It will thus suffice to show that the left vertical composite is an equivalence. Each of these hom animae are hom animae of the fibers of $C^\otimes$ and $D^\otimes$ over J, so by working fiberwise we may reduce to the case $J = \langle 1 \rangle$. Composing the vertical composite with the adjunction equivalence $\mathrm{Hom}_C(\alpha_! X, RY) \simeq \mathrm{Hom}_D(L\alpha_! X, Y)$, we obtain a map

$$\mathrm{Hom}_D(\alpha_! L^\otimes X, Y) \to \mathrm{Hom}_D(L\alpha_! X, Y)$$

which we must show is an equivalence. But unwinding definitions shows that this map is given by precomposition with the composite

$$L\alpha_! X \xrightarrow{L\alpha_!\eta_X} L\alpha_! R^\otimes L^\otimes X \xrightarrow{\mathrm{lax}_R} LR\alpha_! L^\otimes X \xrightarrow{\varepsilon_{\alpha_! L^\otimes X}} \alpha_! L^\otimes X.$$

But the assumption on L guarantees by induction that this composite is an isomorphism. This shows that the left adjoint $L^\otimes$ exists. The standard argument as in previous proofs shows it is an operad map, and essentially by construction it preserves cocartesian morphisms so that it is in fact a symmetric monoidal functor. □

4.4 Monoidal subcategories and monoidal localizations

In this section we will introduce conditions for subcategories and localizations of symmetric monoidal ∞-categories to again be symmetric monoidal.

Construction 4.4.1. Let $\mathcal{O}$ be an ∞-operad. Given a subanima $\mathcal{P}^{\simeq} \subseteq \mathcal{O}^{\simeq}$, let $\mathcal{P}^{\otimes} \subseteq \mathcal{O}^{\otimes}$ be the full subcategory spanned by objects $\{x_i\}_{i\in I}$ with $x_i \in \mathcal{P}^{\simeq}$ for all i. Observe that the composite $p_{\mathcal{P}}\colon \mathcal{P}^{\otimes} \hookrightarrow \mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathrm{Span}(\mathrm{Fin})$ exhibits $\mathcal{P}^{\otimes}$ as an ∞-operad and the inclusion $\mathcal{P}^{\otimes} \hookrightarrow \mathcal{O}^{\otimes}$ is a morphism of ∞-operads:

(1) $\mathcal{P}^{\otimes}$ admits products inherited from $\mathcal{O}^{\otimes}$;

(2) By definition the equivalence $\prod_{i\in I} \mathcal{O}^{\otimes}_{\{i\}} \xrightarrow{\sim} \mathcal{O}^{\otimes}_I$ restricts to $\prod_{i\in I} \mathcal{P}^{\otimes}_{\{i\}} \xrightarrow{\sim} \mathcal{P}^{\otimes}_I$;

(3) By fully faithfulness, $p_{\mathcal{O}}$-cocartesian morphisms $\widetilde{f}\colon \prod_{i\in I} X_i \to \prod_{j\in J} X_{f(j)}$ in $\mathcal{O}^{\otimes}$ are still $p_{\mathcal{P}}$-cocartesian as morphisms in $\mathcal{P}^{\otimes}$.

We refer to the pair $\mathcal{P} = (\mathcal{P}^{\otimes}, p_{\mathcal{P}})$ as the *full suboperad of $\mathcal{O}$ spanned by the colors $\mathcal{P}^{\simeq}$*.

If $\mathcal{O} = \mathcal{M}_C$ comes from a symmetric monoidal category C, then the choice of a subanima of colors corresponds to a choice of a full subcategory $D \subseteq C$. We will now discuss various conditions that imply that D inherits a symmetric monoidal structure from C, either by restricting the monoidal structure or by localizing it.

Lemma 4.4.2. *Let $(C, \otimes_C, \mathbb{1}_C)$ be a symmetric monoidal ∞-category. If $D \subseteq C$ is a full subcategory whose collection of objects $D^{\simeq} \subseteq C^{\simeq}$ contains $\mathbb{1}_C$ and is closed under $\otimes_C$, then D inherits a symmetric monoidal structure $(D, \otimes_D, \mathbb{1}_D)$ from C such that the inclusion $i\colon D \hookrightarrow C$ is a (strong) symmetric monoidal functor.*

Proof. The previous construction provides a suboperad $D^{\otimes} \subseteq C^{\otimes}$ and it remains to show that the functor $D^{\otimes} \hookrightarrow C^{\otimes} \xrightarrow{p} \mathrm{Span}(\mathrm{Fin})$ is still a cocartesian fibration. By fully faithfulness of the inclusion, it will suffice to show that the p-cocartesian morphisms in $C^{\otimes}$ starting in an object $(x_1, \dots, x_n) \in D^{\otimes}_{\langle n\rangle}$ are contained in $D^{\otimes}$. By the Segal condition we may immediately reduce this to the cocartesian morphisms of the form $(x_1, \dots, x_n) \to x_1 \otimes \cdots \otimes x_n$. This is clear for $n = 1$. For $n = 0$ this is the condition that $\mathbb{1}_C \in D$. For $n = 2$ this is the condition that $x \otimes y \in D$ whenever $x, y \in D$. The case for $n \geq 3$ may be reduced to that for $n = 2$ by induction, using that composites of cocartesian morphisms are again cocartesian. □

Recall that a *Bousfield localization* is a functor $L\colon C \to D$ that admits a fully faithful right adjoint $i\colon D \hookrightarrow C$. In this case a morphism $x \to y$ is called an *L-equivalence* if the map $L(x) \to L(y)$ in D is an isomorphism.

Proposition 4.4.3. *Let $(C, \otimes_C, \mathbb{1}_C)$ be a symmetric monoidal ∞-category, and let $i\colon D \hookrightarrow C$ be the inclusion of a full subcategory such that i admits a left adjoint $L\colon C \to D$. Assume that for every L-equivalence $f\colon x \to y$ in C and for every $z \in C$, the morphism $f \otimes_C \mathrm{id}_z\colon x \otimes z \to y \otimes z$ is also an L-equivalence. Then the following hold:*

(1) *The suboperad $D^{\otimes}$ of $C^{\otimes}$ is a symmetric monoidal structure on D;*

(2) *The inclusion $i^\otimes\colon D^\otimes \hookrightarrow C^\otimes$ admits a left adjoint $L^\otimes\colon C^\otimes \to D^\otimes$ which is a strong symmetric monoidal refinement of the functor L.*

Proof. Step 1: Let us start by observing that the assumption on L implies that for every span $\alpha\colon K \xleftarrow{f} S \xrightarrow{g} J$ of finite sets the induced functor

$$\alpha_!\colon C^I \xrightarrow{f^*} C^K \xrightarrow{g_\otimes} C^J, \qquad (x_i)_{i\in I} \mapsto \big(\bigotimes_{k\in g^{-1}(j)} x_{f(k)}\big)_{j\in J}$$

preserves componentwise L-equivalences. This is clear for the restriction functor f^*. For the functor $g_\otimes$, it remains to show that if $x_k \to y_k$ are L-equivalences in C for $k = 1, \dots, n$, then their tensor product $x_1 \otimes_C \cdots \otimes_C x_n \to y_1 \otimes_C \cdots \otimes_C y_n$ is also an L-equivalence. For $n = 2$ this follows by writing this map as a composite of $x_1 \otimes y_1 \to x_2 \otimes y_1$ and $x_2 \otimes y_1 \to x_2 \otimes y_2$, both of which are L-equivalences by assumption. The case for general n follows by induction.

Step 2: We will now show that $i^\otimes$ has a left adjoint $L^\otimes$. To simplify notation, we will treat objects of D as objects in C without writing the inclusion functor i. As in the proof of the previous proposition, we may construct $L^\otimes$ objectwise, and we set $L^\otimes(\{x_k\}_{k\in K}) := \{L(x_k)\}_{k\in K}$. The unit map $\eta_X\colon X \to L^\otimes X$ is given by the collection of maps $\{\eta_{x_k}\colon x_k \to L(x_k)_{k\in K}\}$. We must show that for any other $Y \in D^\otimes$ the precomposition map

$$\mathrm{Hom}_{D^\otimes}(L^\otimes X, Y) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_{C^\otimes}(X, Y)$$

is an equivalence. It suffices to check this on fibers $\mathrm{Hom}^\alpha_{C^\otimes}(-,-)$ over any span $\alpha\colon K \xleftarrow{f} S \xrightarrow{g} J$. As in the proof of Proposition 4.3.3, we will reduce this to the case of the identity span where it will follow from the adjunction $L \dashv i$. Let $X \to \alpha_! X$ and $L^\otimes X \to \alpha_! L^\otimes X$ denote p_C-cocartesian lifts of α starting in X and $L^\otimes X$, respectively, so that precomposition with these maps induces two vertical equivalences as follows:

$$\begin{tikzcd}
\mathrm{Hom}^{\mathrm{id}_J}_{D^\otimes}(\alpha_! L^\otimes X, Y) \arrow[r, "-\circ\alpha_!(\eta_X)"] \arrow[d, "\simeq"'] & \mathrm{Hom}^{\mathrm{id}_J}_{C^\otimes}(\alpha_! X, Y) \arrow[d, "\simeq"] \\
\mathrm{Hom}^{\alpha}_{D^\otimes}(L^\otimes X, Y) \arrow[r, "-\circ\eta_X"] & \mathrm{Hom}^{\alpha}_{C^\otimes}(X, Y).
\end{tikzcd}$$

It thus suffices to show that the top map is an equivalence. Using the identification $C^\otimes_J \simeq C^J$, we may identify this map with the product over $j \in J$ of the maps

$$\mathrm{Hom}_C((\alpha_! L^\otimes X)_j, Y_j) \xrightarrow{-\circ\alpha_!(\eta_X)_j} \mathrm{Hom}_C((\alpha_! X)_j, Y_j),$$

so we may show that each of these is an equivalence. Applying the adjunction $L \dashv i$, this map is in turn equivalent to the map

$$\mathrm{Hom}_D(L(\alpha_! L^\otimes X)_j, Y_j) \xrightarrow{-\circ L\alpha_!(\eta_X)_j} \mathrm{Hom}_D(L(\alpha_! X)_j, Y_j).$$

But now note that the map $\eta_{x_k}\colon x_k \to L(x_k)$ is an L-equivalence for each k due to fully faithfulness of $i\colon D \hookrightarrow C$, and so by step 1 the maps $\alpha_!(\eta_X)_j\colon \alpha_!(X)_j \to \alpha_!(L^\otimes X)_j$ are also L-equivalences for all j, i.e., each $L\alpha_!(\eta_X)_j$ is an isomorphism. This shows the existence of the left adjoint $L^\otimes$. The same argument as in Proposition 4.3.3 shows that $L^\otimes$ is a morphism of ∞-operads.

Step 3: We will now show that the suboperad $D^\otimes \subseteq C^\otimes$ is a symmetric monoidal ∞-category. Given an object $X \in D^\otimes$ and a span α, we need to show that there exists a p_D-cocartesian lift of α in $D^\otimes$ starting in X. We construct this as the composite

$$X \to \alpha_! X \xrightarrow{\eta} L^\otimes \alpha_! X,$$

where the first map is a p_C-cocartesian lift of α in $C^\otimes$, and the second map is the unit for the adjunction $i^\otimes \vdash L^\otimes$. To show that this map is p_D-cocartesian, we have to show that for every other object $Y \in D^\otimes$ the following commutative square is a pullback square:

$$\begin{tikzcd}
\mathrm{Hom}_{D^\otimes}(L^\otimes \alpha_! X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{D^\otimes}(X, Y) \arrow[d, "p"] \\
\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(J, K) \arrow[r, "-\circ\alpha"] & \mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(I, K).
\end{tikzcd}$$

But by the adjunction equivalence $\mathrm{Hom}_{D^\otimes}(L^\otimes \alpha_! X, Y) \simeq \mathrm{Hom}_{C^\otimes}(\alpha_! X, Y)$ this immediately follows from p_D-cocartesianness of the map $X \to \alpha_! X$. This shows part (1).

Step 4: For (2), it remains to show that $L^\otimes$ is strong symmetric monoidal, i.e., that it sends p_C-cocartesian morphisms to p_D-cocartesian morphisms. Given a p_C-cocartesian morphism $X \to \alpha_! X$, applying $L^\otimes$ gives $L^\otimes X \to L^\otimes \alpha_! X$, and we need to show that this agrees with the composite $L^\otimes X \to \alpha_! L^\otimes X \xrightarrow{\eta} L^\otimes \alpha_! L^\otimes X$. But this again follows from the assumption on L: since $\eta_X : X \to L^\otimes X$ is an L-equivalence, so is the map $\alpha_! X \to \alpha_! L^\otimes X$, hence applying $L^\otimes$ gives an isomorphism. □

We record a special case for which the conditions of Proposition 4.4.3 are satisfied:

Lemma 4.4.4. *Let C be a symmetric monoidal ∞-category which admits internal homs in the sense that the functor $- \otimes x : C \to C$ admits a right adjoint $\underline{\mathrm{Hom}}_C(x, -) : C \to C$ for every object x of C. Let D be a full subcategory of C such that the inclusion $D \hookrightarrow C$ admits a left adjoint, and assume that for objects $x \in D$ and $z \in C$ the internal hom $\underline{\mathrm{Hom}}_C(z, x)$ is again in D. Then D obtains a symmetric monoidal structure such that the localization functor $L : C \to D$ is symmetric monoidal.*

Proof. We must prove the condition from Proposition 4.4.3. Let $f : x \to y$ be an L-equivalence and let $z \in C$ be an arbitrary object. We must show that the morphism $L(f \otimes \mathrm{id}_z) : L(x \otimes z) \to L(y \otimes z)$ is invertible in D. By the Yoneda lemma, this amounts to showing that for every object $w \in D$ the map

$$(f \otimes \mathrm{id}_z)^* : \mathrm{Hom}_C(y \otimes z, w) \to \mathrm{Hom}_C(x \otimes z, w)$$

is an equivalence of animae. Applying the adjunction $- \otimes z \dashv \underline{\mathrm{Hom}}_C(z, -)$, this map can be identified with the map

$$f^* : \mathrm{Hom}_C(y, \underline{\mathrm{Hom}}_C(z, w)) \to \mathrm{Hom}_C(x, \underline{\mathrm{Hom}}_C(z, w)).$$

Since $\underline{\mathrm{Hom}}_C(z, w) \in D$ by assumption, this map is an equivalence due to the fact that f is an L-equivalence. □

4.5 $\mathcal{O}$-monoidal ∞-categories

In the previous two sections, we have discussed various results for symmetric monoidal ∞-categories, i.e., commutative monoids in Cat_∞. All of this can directly be generalized to monoids over an arbitrary ∞-operad $\mathcal{O}$:

Definition 4.5.1. Let D be an ∞-category with finite products. An *$\mathcal{O}$-monoid in D* is a finite-product-preserving functor $\mathcal{O}^{\otimes} \to D$. We write

$$\mathrm{Mon}_{\mathcal{O}}(D) := \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, D) \quad \subseteq \quad \mathrm{Fun}(\mathcal{O}^{\otimes}, D)$$

for the full subcategory of $\mathcal{O}$-monoids in D.

Definition 4.5.2. An *$\mathcal{O}$-monoidal ∞-category* is an $\mathcal{O}$-monoid in Cat_{∞}. When $\mathcal{O} = \mathcal{A}ssoc$, we refer to $\mathcal{O}$-monoidal ∞-categories simply as *monoidal ∞-categories*.

Example 4.5.3. When $\mathcal{O} = \mathcal{T}riv$, an $\mathcal{O}$-monoidal ∞-category is just an ∞-category.

The multimorphism operad functor $\mathcal{M}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Op}_{\infty}$ admits an analogue $\mathcal{M}_{-/\mathcal{O}}\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_{\infty}) \to (\mathrm{Op}_{\infty})_{/\mathcal{O}}$ for $\mathcal{O}$-monoidal ∞-categories. This relies on the following lemma, which is completely analogous to Lemma 4.2.2 and is left to the reader:

Lemma 4.5.4. *Let $\mathcal{O}$ be an ∞-operad and let $q\colon C^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration. Then the composite*

$$C^{\otimes} \xrightarrow{q} \mathcal{O}^{\otimes} \xrightarrow{p_{\mathcal{O}}} \mathrm{Span}(\mathrm{Fin})$$

exhibits $C^{\otimes}$ as an ∞-operad if and only if the functor $\mathrm{Str}^{\mathrm{cc}}(q)\colon \mathcal{O}^{\otimes} \to \mathrm{Cat}_{\infty}$ preserves finite products. In this case, the morphism $q\colon C^{\otimes} \to \mathcal{O}^{\otimes}$ is automatically a morphism of ∞-operads.

Definition 4.5.5. Given an $\mathcal{O}$-monoidal ∞-category C, we let $p_C\colon C^{\otimes} \to \mathcal{O}^{\otimes}$ denote its cocartesian unstraightening, which by the lemma defines an ∞-operad $\mathcal{M}_{C/\mathcal{O}} = (C^{\otimes}, p_{\mathcal{O}} \circ p_C)$. Since p_C is a morphism of operads, it turns $\mathcal{M}_{C/\mathcal{O}}$ into an object of the slice category $(\mathrm{Op}_{\infty})_{/\mathcal{O}}$, producing a functor

$$\mathcal{M}_{-/\mathcal{O}}\colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_{\infty}) \to (\mathrm{Op}_{\infty})_{/\mathcal{O}}.$$

If $\mathcal{P} \to \mathcal{O}$ and $\mathcal{Q} \to \mathcal{O}$ are two ∞-operads over $\mathcal{O}$, we denote by $\mathrm{Fun}_{(\mathrm{Op}_{\infty})_{/\mathcal{O}}}(\mathcal{P}, \mathcal{Q})$ the ∞-category of operad maps $\mathcal{P} \to \mathcal{Q}$ over $\mathcal{O}$, defined as the following fiber:

$$\begin{tikzcd}
\mathrm{Fun}_{(\mathrm{Op}_{\infty})_{/\mathcal{O}}}(\mathcal{P}, \mathcal{Q}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{P}, \mathcal{Q}) \arrow[d] \\
* \arrow[r] & \mathrm{Fun}_{\mathrm{Op}_{\infty}}(\mathcal{P}, \mathcal{O}).
\end{tikzcd}$$

As a special case, we define the ∞-category *$\mathcal{P}/\mathcal{O}$-algebras in C* as

$$\mathrm{Alg}_{\mathcal{P}/\mathcal{O}}(C) := \mathrm{Fun}_{(\mathrm{Op}_{\infty})_{/\mathcal{O}}}(\mathcal{P}, \mathcal{M}_{C/\mathcal{O}}).$$

Definition 4.5.6. Given $\mathcal{O}$-monoidal ∞-categories C and D, a *lax $\mathcal{O}$-monoidal functor $C \to D$* is an operad morphism $\mathcal{M}_{C/\mathcal{O}} \to \mathcal{M}_{D/\mathcal{O}}$ over $\mathcal{O}^{\otimes}$.

Proposition 4.5.7 (cf. [Lur17, Corollary 7.3.2.7])**.** *Let $L\colon C \to D$ be an $\mathcal{O}$-monoidal functor between $\mathcal{O}$-monoidal ∞-categories. Assume that for every color $o \in \mathcal{O}^{\simeq}$, the functor $L_o\colon C_o \to D_o$ admits a right adjoint $R_o\colon D_o \to C_o$. Then also the functor $L^{\otimes}\colon C^{\otimes} \to D^{\otimes}$ admits a right adjoint $R^{\otimes}\colon D^{\otimes} \to C^{\otimes}$ which is a morphism of operads over $\mathcal{O}$. In particular, L admits a lax $\mathcal{O}$-monoidal right adjoint R.*

Proof. The proof is entirely analogous to that of Proposition 4.3.3 and we will leave it to the reader. □

Proposition 4.5.8 (cf. [Lur17, Corollary 7.3.2.12]). *Let $R\colon \mathcal{D} \to \mathcal{C}$ be an $\mathcal{O}$-monoidal functor between $\mathcal{O}$-monoidal ∞-categories. Assume that the following conditions hold:*

(1) *For every color $x \in \mathcal{O}^{\simeq}$, the functor $R_x\colon \mathcal{D}_x \to \mathcal{C}_x$ admits a left adjoint $L_x\colon \mathcal{C}_x \to \mathcal{D}_x$.*

(2) *For every multimorphism $\alpha\colon X = (x_1, \dots, x_n) \to y$ in $\mathcal{O}^{\otimes}$, the composite transformation*

$$L_y\alpha_!(-) \xrightarrow{\eta} L_y\alpha_!(R_X L_X(-)) \xrightarrow{\mathrm{lax}_R^{\alpha}} L_y R_y \alpha_! L_X(-) \xrightarrow{\varepsilon} \alpha_! L_X(-)$$

of functors $\mathcal{C}_X^{\otimes} \to \mathcal{D}_y^{\otimes}$ is an isomorphism.

Then the functor $R^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ admits a left adjoint $L^{\otimes}\colon \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ which is a morphism of cocartesian fibrations over $\mathcal{O}^{\otimes}$. In particular, R admits a strong $\mathcal{O}$-monoidal left adjoint L.

Proof. The proof is entirely analogous to that of Proposition 4.3.5 and we will leave it to the reader. □

5 Cartesian and cocartesian monoidal structures

Let C be an ∞-category. If C admits finite coproducts, one would expect there to be a symmetric monoidal structure on C given by the coproducts in C: the necessary coherences should all exist uniquely due to the universal property of coproducts. Dually, if C admits finite products, there ought to be a symmetric monoidal structure on C given by products. The goal of this chapter is to construct these two symmetric monoidal structures.

Before diving into the abstract definition, let us briefly describe the situation in classical category theory. If C is a category, we may construct a colored operad $\mathcal{O}_C^{\amalg}$, called the *cocartesian operad of* C, whose sets of multimorphisms are given by

$$\mathcal{O}_C^{\amalg}((x_1, \ldots, x_n); y) \quad := \quad \prod_{i=1}^{n} \mathrm{Hom}_C(x_i, y),$$

and whose composition is given by composition in C. In other words, a multimorphism from $(x_1, \ldots, x_n)$ to y in $\mathcal{O}_C^{\amalg}$ consists of a collection of morphisms $f_i \colon x_i \to y$. This operad comes from a symmetric monoidal structure on C if and only if C admits finite coproducts: we are asking for the existence of some object x together with a natural bijection

$$\mathrm{Hom}_C(x, y) \cong \prod_{i=1}^{n} \mathrm{Hom}_C(x_i, y),$$

which is the same as saying that x is a coproduct $x_1 \sqcup \cdots \sqcup x_n$.

The goal of this chapter is to repeat this process in the setting of ∞-categories: we will construct an ∞-operad $\mathcal{O}_C^{\amalg}$ for an arbitrary ∞-category C, and show that it corresponds to a symmetric monoidal structure on C if and only if C admits finite coproducts. To motivate the ∞-categorical definition of $\mathcal{O}_C^{\amalg}$ we will give, let us stay in the classical setting for a moment and compute the fibration $C^{\amalg} := (\mathcal{O}_C^{\amalg})^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ associated with the colored operad $\mathcal{O}_C^{\amalg}$ via Construction 1.4.1:

- The objects of $C^{\amalg}$ are unordered tuples $\{x_i\}_{i \in I}$ of objects of C;
- A morphism $\{x_i\}_{i \in I} \to \{y_j\}_{j \in J}$ in $C^{\amalg}$ consists of a span

  $$I \xleftarrow{f} K \xrightarrow{g} J$$

  of finite sets together with morphisms $\varphi_k \colon x_{f(k)} \to y_{g(k)}$ for every $k \in K$.

- Identities and compositions are computed in the 'obvious way'.

The crucial observation for extending this to the ∞-categorical setting is that $C^{\amalg}$ is itself a span category. To see this, let $\mathrm{Fin}(C)$ denote the category whose objects are the unordered tuples $\{x_i\}_{i\in I}$ and whose objects $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ are given by a morphism $g\colon I \to J$ of finite sets and a collection of morphisms $\varphi_i\colon x_i \to y_{g(i)}$ for all $i \in I$. There is a forgetful functor

$$q_C\colon \mathrm{Fin}(C) \to \mathrm{Fin}, \qquad \{x_i\}_{i\in I} \mapsto I, \qquad (g,(\varphi_i)) \mapsto g$$

and this is a cartesian fibration whose cartesian morphisms are those maps $(g,(\varphi_i))$ such that each φ_i is an isomorphism; i.e., they are maps of the form

$$\{(y_{g(i)})_{i\in I}\} \xrightarrow{(g,(\mathrm{id}_{y_{g(i)}})_i)} \{y_j\}_{j\in J}.$$

The category $C^{\amalg}$ described above is then precisely the span category $\mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$. We will take this to be the *definition* of $C^{\amalg}$ in the setting of ∞-categories; the canonical functor $C^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$ then exhibits this as an ∞-operad that we will denote by $\mathcal{O}_C^{\amalg}$ and refer to as the *cocartesian operad of* C. If C admits finite coproducts, this ∞-operad corresponds to a symmetric monoidal structure $(C,\sqcup)$ on C, called the *cocartesian monoidal structure*.

The dual situation for finite *products* is a bit more tricky: since the multimorphism operad of a category encodes morphisms *out of* the tensor product, it is not immediate how to encode the cartesian symmetric monoidal structure on C directly. We will solve this by equipping C^{op} with the *co*cartesian monoidal structure and then passing to opposites again, resulting in a symmetric monoidal ∞-category $(C,\times)$. The resulting cocartesian fibration $C^{\times} \to \mathrm{Span}(\mathrm{Fin})$ can still be described in terms of span categories, but the description is a bit more complicated than the one for $C^{\amalg}$.

5.1 Cocartesian morphisms in span categories

As discussed in the introduction of this chapter, we will construct the cocartesian operad as a span category itself. This requires us to have a solid understanding of cocartesian morphisms in span categories. This is the content of the following theorem:

Theorem 5.1.1 (Barwick, [Hau+23b, Theorem 3.1])**.** *Let* $p\colon (E,E_L,E_R) \to (C,C_L,C_R)$ *be a morphism of adequate triples, and consider a span*

$$X \xleftarrow{\varphi} U \xrightarrow{\psi} Y,$$

with $\varphi \in E_L$ *and* $\psi \in E_R$*. Then this morphism is cocartesian with respect to* $\mathrm{Span}(p)\colon \mathrm{Span}_{L,R}(E) \to \mathrm{Span}_{L,R}(C)$ *whenever the following conditions are satisfied:*

(1) *The morphism* φ *is* p*-cartesian and the morphism* ψ *is* p*-cocartesian;*

(2) *(Left cancellation for* E_L*) Given a commutative triangle*

$$\begin{tikzcd} & U \arrow[dr, "\varphi"] & \\ T \arrow[ur, "\xi"] \arrow[rr, "\varphi\xi"'] & & X \end{tikzcd}$$

such that $p(\xi) \in C_L$ and $\varphi\xi \in E_L$, we have $\xi \in E_L$;

(3) *(Right cancellation for E_R) Given a commutative triangle*

$$\begin{tikzcd} & Y \arrow[dr, "\xi"] & \\ U \arrow[ur, "\psi"] \arrow[rr, "\xi\psi"] & & T \end{tikzcd}$$

such that $p(\xi) \in C_R$ and $\xi\psi \in E_R$, we have $\xi \in E_R$;

(4) *(Enough p-cocartesian lifts) For any pullback square in C of the form*

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, "l'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & J \arrow[d, "l"] \\ pU \arrow[r, "p\psi"] & pY, \end{tikzcd}$$

with $l \in C_L$, and for any $W \in E_K$, the morphism g admits a p-cocartesian lift $\psi' \colon W \to V$ which also satisfies the condition from (3).

(5) *(Beck-Chevalley) For any commutative square in E of the form*

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\xi'"'] & V \arrow[d, "\xi"] \\ U \arrow[r, "\psi"] & Y \end{tikzcd}$$

such that its image in C is a pullback square, $\xi' \in E_L$, $\psi' \in E_R$, and $p(\xi) \in C_L$. Then ψ' is p-cocartesian if and only if $\xi \in E_L$ and the square is a pullback square.

Proof. We consider any other span $X \leftarrow W \to Z$ in E. We have to show that we may uniquely fill the following dashed diagram

$$\begin{tikzcd} & & W \arrow[dl, dashed, "(i)"] \arrow[dr, dashed, "(ii)"] \arrow[dd, phantom, very near start, "\vee"] \arrow[dll, bend right] \arrow[drr, bend left] & & \\ & U \arrow[dl, "\varphi"] \arrow[dr, "\psi"] & & \bullet \arrow[dl, dashed, "(ii)"] \arrow[dr, dashed, "(iii)"] & \\ X & & Y & & Z \end{tikzcd}$$

whenever we are given a similar filling for the resulting diagram in C; here all left-pointing maps are in E_L and all right-pointing maps are in E_R.

(i) Since φ is p-cartesian, there exists a unique lift $W \to U$ making the left triangle commute, which lies in E_L by condition (2).

(ii) We claim that this lift $W \to U$ may uniquely be written as the pullback along ψ of some morphism $V \to Y$ in E_L compatible with the given pullback square in C; more precisely, we claim that the following commutative square is a pullback square:

$$\begin{tikzcd}
(E_{/Y})^L \arrow[r, "\psi^*"] \arrow[d, "p"'] & (E_{/U})^L \arrow[d, "p"] \\
(C_{/J})^L \arrow[r, "(p\psi)^*"] & (C_{/K})^L.
\end{tikzcd}$$

Here the superscript $(-)^L$ means that we take the full subcategories spanned by the morphisms in E_L and C_L, respectively.

(a) For essential surjectivity of $(E_{/Y})^L \to (E_{/U})^L \times_{(C_{/K})^L} (C_{/J})^L$, we consider a morphism $\xi' \colon W \to U$ and assume we are given a pullback square in C of the form

$$\begin{tikzcd}
pW \arrow[r, "g"] \arrow[d, "p\xi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & J \arrow[d, "l"] \\
pU \arrow[r, "p\psi"] & pY
\end{tikzcd}$$

with $l \in C_L$. By assumption (4) we can find a p-cocartesian lift $\xi' \colon W \to V$ of g satisfying condition (3). The composite $\psi \circ \xi' \colon W \to Y$ then factors through a map $\xi \colon V \to Y$, giving a commutative square in E of the form considered in (5). By assumption (5), the map ξ then lies in E_L and the resulting square is a pullback square. This shows that the pair (w, l) lies in the essential image, as desired.

(b) For fully faithfulness, consider two objects $\xi_1 \colon V_1 \to Y$ and $\xi_2 \colon V_2 \to Y$ of $(E_{/Y})^L$. We must show that the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{E_{/Y}}(V_1, V_2) \arrow[r, "\psi^*"] \arrow[d, "p"] & \mathrm{Hom}_{E_{/U}}(U \times_Y V_1, U \times_Y V_2) \arrow[d, "p"] \\
\mathrm{Hom}_{C_{/K}}(pV_1, pV_2) \arrow[r, "(p\psi)^*"] & \mathrm{Hom}_{C_{/J}}(K \times_J pV_1, K \times_J pV_2)
\end{tikzcd}$$

is a pullback square. By the universal property of pullbacks, the right vertical map is equivalent to the map

$$p \colon \mathrm{Hom}_{E_{/Y}}(U \times_Y V_1, V_2) \to \mathrm{Hom}_{C_{/K}}(K \times_J pV_1, pV_2),$$

hence the claim follows from the fact that the projection map $U \times_Y V_1 \to V_1$ is p-cocartesian by assumption (4).

(iii) Finally, given a p-cocartesian morphism $W \to V$, the given map $W \to Z$ uniquely factors through a map $V \to Z$, which lies in E_R by condition (3). □

In practice, we will often be interested in the situation where the subcategories E_L and E_R of E are determined by C_L, C_R and the functor $p \colon E \to C$. To set this up, we start with the following auxiliary lemma:

Lemma 5.1.2. *Let $p\colon E \to C$ be a functor, and consider a commutative square*

$$\begin{tikzcd} X' \arrow[r, "r'"] \arrow[d, "l'"'] & X \arrow[d, "l"] \\ Y' \arrow[r, "r"] & Y \end{tikzcd}$$

in E where l is a p-cartesian morphism. Assume that the image of this square in C is a pullback square. Then this square is a pullback square if and only if the morphism l' is p-cartesian.

Proof. Consider for any object $Z \in E$ the following two diagrams:

$$\begin{tikzcd} \mathrm{Hom}_E(Z, X') \arrow[r, "r'\circ-"] \arrow[d, "l'\circ-"'] & \mathrm{Hom}_E(Z, X) \arrow[r, "p"] \arrow[d, "l\circ-"] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Hom}_C(pZ, pX) \arrow[d, "pl\circ-"] \\ \mathrm{Hom}_E(Z, Y') \arrow[r, "r\circ-"] & \mathrm{Hom}_E(Z, Y) \arrow[r, "p"] & \mathrm{Hom}_C(pZ, pY), \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}_E(Z, X') \arrow[r, "p"] \arrow[d, "l'\circ-"'] & \mathrm{Hom}_C(pZ, pX') \arrow[r, "pr\circ-"] \arrow[d, "pl'\circ-"] \arrow[dr, phantom, "\lrcorner" very near start] & \mathrm{Hom}_C(pZ, pX) \arrow[d, "pl\circ-"] \\ \mathrm{Hom}_E(Z, Y') \arrow[r, "p"] & \mathrm{Hom}_C(pZ, pY') \arrow[r, "pr\circ-"] & \mathrm{Hom}_C(pZ, pY). \end{tikzcd}$$

The right-hand squares of both diagrams are pullback squares: for the first one since l is p-cartesian, for the second one since the image of the given square in C is a pullback square. As the functor p preserves composition, the outer rectangles of both diagrams agree, so by the pasting law the left-hand square in the first diagram is a pullback square if and only if the left-hand square in the second diagram is a pullback square. Since the former holds for all Z if and only if the given diagram is a pullback square and the latter holds for all Z if and only if l' is p-cartesian, this finishes the proof. □

Proposition 5.1.3. *Let (C, C_L, C_R) be an adequate triple and let $p\colon E \to C$ be a functor such that every morphism in C_L admits p-cartesian lifts with given target (for example: p is a cartesian fibration). Then:*

(1) *The triple $(E, E_L^{p\text{-cart}}, E_R)$ is adequate, where $E_R := p^{-1}(C_R)$ consists of morphisms over C_R and $E_L^{p\text{-cart}} := p^{-1}(C_L) \cap E^{p\text{-cart}}$ consists of p-cartesian morphisms over C_L.*

(2) *The functor $p\colon E \to C$ is a morphism of adequate triples.*

Proof. We will first construct the pullback in E of a morphism $l \in E_L^{p\text{-cart}}$ along some $r \in E_R$. Since $pl \in C_L$ and $pr \in C_R$, we may form the pullback

$$\begin{tikzcd} pX \times_{pY} pY' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & pX \arrow[d, "pl"] \\ pY' \arrow[r, "pr"] & pY \end{tikzcd}$$

in C. By assumption on p, there exists a p-cartesian lift $l'\colon X' \to Y'$ of the morphism $pX \times_{pY} pY' \to pY'$. The composite $rl'\colon X' \to Y$ then uniquely factorizes through some $r'\colon X' \to X$ lifting the projection $pX \times_{pY} pY' \to pX$, using that l is p-cartesian. All in all, we have lifted the pullback square in C to a commutative square in E of the form

$$\begin{tikzcd} X' \arrow[r, "r'"] \arrow[d, "l'"'] & X \arrow[d, "l"] \\ Y' \arrow[r, "r"] & Y. \end{tikzcd} \tag{5.1}$$

Since l and l' are p-cartesian, it follows from Lemma 5.1.2 that this square is a pullback square in E, showing that the required pullbacks exist. Conversely, for any such pullback square we have $l \in E_L^{p\text{-cart}}$ and $r \in E_R$. This shows part (1).

For part (2), consider a pullback square in E of the form (5.1). We need to show that applying p to this square gives a pullback square in C. Proceeding as before, we may always form the pullback of pl and pr in C and lift this to a pullback square in E. But by uniqueness of pullback squares of l and r, this new square must then agree with the given one. In particular their images in C agree, hence this image is a pullback square. □

Remark 5.1.4. For the morphism of adequate triples $p\colon (E, E_L^{p\text{-cart}}, E_R) \to (C, C_L, C_R)$, parts (2) and (3) of Theorem 5.1.1 are vacuous. For a p-cocartesian morphism $\psi\colon U \to Y$ satisfying the conditions from part (4), we have that part (5) is satisfied if and only if for every pullback square as in (4) the canonical 'Beck-Chevalley' transformation $g'_! l'^* \to l^* g_!$ is a natural isomorphism of functors $E_{pU} \to E_J$.

5.2 Cocartesian monoidal structures

With the characterization of cocartesian morphisms in span categories at hand, we will now return to our discussion of cocartesian monoidal structures. Throughout this section we fix an ∞-category C.

5.2.1 Cocartesian operads

We start by constructing the ∞-operad $O_C^{\amalg}$.

Construction 5.2.1. We define an ∞-category $\mathrm{Fin}(C)$ as follows:

- Consider the unique functor $C^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that sends the one-point set to C and preserves finite products. (It is necessarily right Kan extended from the point.)
- Let $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ denote the cartesian straightening of $C^{(-)}$.

Note that objects of $\mathrm{Fin}(C)$ are (finite) unordered tuples $\{x_i\}_{i\in I}$, while morphisms $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ are pairs $(f, (\varphi_i)_{i\in I})$ where $f\colon I \to J$ is a morphism of finite sets and $\varphi_i\colon x_i \to y_{f(i)}$ is a morphism in C for all $i \in I$.

We have the following observations about $\mathrm{Fin}(C)$:

- The fiber of q over $\langle 1\rangle$ is equivalent to C, and the resulting inclusion $j\colon C \hookrightarrow \mathrm{Fin}(C)$ is fully faithful;

- A morphism $(f, (\varphi_i)_{i\in I})\colon \{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ is q-cartesian if and only if each φ_i is an isomorphism in C. We denote by
$$\mathrm{Fin}(C)_{\mathrm{ct}} \subseteq \mathrm{Fin}(C)$$
the wide subcategory spanned by the q-cartesian morphisms.

- The ∞-category $\mathrm{Fin}(C)$ admits finite coproducts, given by unordered concatenation of unordered tuples, and the functor $q\colon \mathrm{Fin}(C) \to \mathrm{Fin}$ preserves finite coproducts. More precisely, given a finite collection of objects $X_i \in \mathrm{Fin}(C)$ for $i \in I$, we may uniquely write each X_i as $\{x_i^j\}_{j\in J_i}$ for some finite set J_i, and then the coproduct is given by $X = \{x_j^i\}_{(i,j)\in \bigsqcup_{i\in I} J_i}$, with the maps $X_i \to X$ given by the q-cartesian morphisms over the inclusions $J_i \hookrightarrow \bigsqcup_{i\in I} J_i$. The proof of this fact is entirely analogous to the claim in Lemma 4.2.2 that unordered concatenation is the product in $C^{\otimes}$ for a symmetric monoidal ∞-category C, and we will leave it to the reader.

We will now show that the ∞-category $\mathrm{Fin}(C)$ is in fact *freely generated* by C under finite coproducts:

Lemma 5.2.2. *For any other ∞-category D with finite coproducts, restriction along the fully faithful inclusion $i\colon C \hookrightarrow \mathrm{Fin}(C)$ induces an equivalence*
$$i^*\colon\ \mathrm{Fun}^{\amalg}(\mathrm{Fin}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D).$$

Proof. We claim that an inverse is given by left Kan extension.

Step 1: Given a functor $F\colon C \to D$, we show that its left Kan extension $i_!F$ along i exists. By the pointwise formula for left Kan extensions, it suffices to show that each of the colimits
$$(i_!F)(x_1, \dots, x_n) = \mathrm{colim}_{y\in C_{/(x_1,\dots,x_n)}} F(y)$$
exists. Observe that the canonical map
$$\bigsqcup_{i=1}^{n} C_{/x_i} \to C_{/(x_1,\dots,x_n)} := C \times_{\mathrm{Fin}(C)} \mathrm{Fin}(C)_{/(x_1,\dots,x_n)}$$
sending $(\varphi\colon y \to x_i)$ to the map $(\{i\} \hookrightarrow \langle n\rangle, \varphi)$ is an equivalence. Since D has finite colimits, we see that colimits indexed by the slices $C_{/(x_1,\dots,x_n)}$ exist in D if and only if colimits indexed by each $C_{/x_i}$ exist. But the latter is clear: as $C_{/x_i}$ admits a terminal object, colimits always exist and are given by evaluation at that terminal object. This shows that $i_!F$ exists and that it is given on objects by
$$(i_!F)(x_1, \dots, x_n) = F(x_1) \sqcup \cdots \sqcup F(x_n).$$

Step 2: It follows from the pointwise description of $i_!F$ that it preserves finite coproducts. In particular, the restriction functor thus admits a left adjoint
$$i_!\colon\ \mathrm{Fun}(C, D) \to \mathrm{Fun}^{\amalg}(\mathrm{Fin}(C), D).$$

Step 3: We now show that the unit $\mathrm{id} \to i^*i_!$ and counit $i_!i^* \to \mathrm{id}$ are equivalences. For the unit this is immediate: by fully faithfulness of i also $i_!$ is fully faithful. The claim for $i_!i^* \to \mathrm{id}$ also follows: since $\mathrm{Fin}(C)$ is generated by C under colimits, restriction to C is conservative, and $i^*i_!i^* \to i^*$ is an isomorphism by the triangle identity. □

Lemma 5.2.3. *Let C be an ∞-category. Then the triple* $(\mathrm{Fin}(C), \mathrm{Fin}(C)_{\mathrm{ct}}, \mathrm{Fin}(C))$ *is an adequate triple. Furthermore, it is extensive, in the sense of Definition 3.3.5.*

Proof. The adequate triple is obtained by applying Proposition 5.1.3 to the cartesian fibration $q \colon \mathrm{Fin}(C) \to \mathrm{Fin}$ from the previous construction. For extensiveness, we check the conditions from Definition 3.3.5:

(1) We have already seen that $\mathrm{Fin}(C)$ admits finite coproducts, given by unordered concatenation.

(2) The q-cartesian morphisms are closed under coproducts. It readily follows from Lemma 5.1.2 that coproducts of pullback squares along q-cartesian morphisms are again pullback squares.

(3) The morphisms $\emptyset \to X$ and $\nabla \colon X \sqcup X \to X$ in $\mathrm{Fin}(C)$ are q-cartesian.

(4) The fact that the commutative squares

$$\begin{tikzcd} X \sqcup X \arrow[r, "f \sqcup f"] \arrow[d, "\nabla"'] & Y \sqcup Y \arrow[d, "\nabla"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \emptyset \arrow[r, equal] \arrow[d] & \emptyset \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

are pullback squares again follows from Lemma 5.1.2, since their images in Fin are pullback squares and the vertical maps are q-cartesian. □

Construction 5.2.4 (Cocartesian operad)**.** Let C be an ∞-category. We define the ∞-category $C^{\amalg}$ as

$$C^{\amalg} := \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)).$$

Combining Lemma 5.2.3 with Lemma 3.3.9, we see that this is a semiadditive ∞-category, with finite products and finite coproducts both induced by the unordered concatenation of tuples in C. We denote by

$$p_C^{\amalg} := \mathrm{Span}(q) \colon C^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$$

the induced functor to Span(Fin), which preserves finite products.

Proposition 5.2.5. *Let C be an ∞-category. Then the pair $(C^{\amalg}, p_C^{\amalg})$ is an ∞-operad.*

Proof. We have already argued that $C^{\amalg} = \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$ has finite products and that $p_C^{\amalg} = \mathrm{Span}(q)$ preserves them, giving condition (1) of the definition. By construction, the fiber of $C^{\amalg}$ over a finite set I is the I-fold product C^I, giving condition (2). For condition (3), consider objects $X_i \in \mathrm{Fin}(C)$ and a morphism $f \colon I \to J$ of finite sets. Then the lift $\widetilde{f} \colon \prod_{i \in I} X_i \to \prod_{j \in J} X_{f(j)}$ in $\mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$ is a backwards span of the form

$$\coprod_{i \in I} X_i \xleftarrow{(f, (\mathrm{id})_j)} \coprod_{j \in J} X_j \xrightarrow{(\mathrm{id}, (\mathrm{id}))} \coprod_{j \in J} X_j.$$

Since the left-pointing morphism is q-cartesian and the right-pointing morphism is an identity, the conditions from Theorem 5.1.1 are trivially satisfied, showing that this span is $\mathrm{Span}(q)$-cocartesian as desired. □

Definition 5.2.6 (Cocartesian operad)**.** For an ∞-category C, we refer to the ∞-operad $\mathcal{O}_C^{\amalg} := (C^{\amalg}, p_C^{\amalg})$ from Proposition 5.2.5 as the *cocartesian operad of* C.

Warning 5.2.7. In various sources, notably in Lurie [Lur17], no distinction is being made between the ∞-operad $\mathcal{O}_C^{\amalg}$ and its total ∞-category $C^{\amalg}$: both are denoted by $C^{\amalg}$. Although the notation $\mathcal{O}_C^{\amalg}$ is slightly more verbose, I think it is useful to distinguish between the operad and its total category.

The operad $\mathcal{O}_C^{\amalg}$ does precisely what we want it to do: its multimorphisms $(x_1, \ldots, x_n) \to y$ are simply the morphisms $(x_1, \ldots, x_n) \to y$ in $\mathrm{Fin}(C)$, which indeed correspond to tuples $(\varphi_i \colon x_i \to y)$ of morphisms in C.

Remark 5.2.8. Note that all the steps in the construction of $\mathcal{O}_C^{\amalg} = (C^{\amalg}, p_C^{\amalg})$ are functorial in C: we may write it as the composite

$$\mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathrm{Fin}^{\mathrm{op}}, \mathrm{Cat}_\infty) \xrightarrow{\mathrm{Un}^{\mathrm{ct}}} \mathrm{Cart}(\mathrm{Fin}) \xrightarrow[5.1.3]{E \mapsto (E, E^{p\text{-cart}}, E)} \mathrm{AdTrip}_{/\mathrm{Fin}} \xrightarrow{\mathrm{Span}} (\mathrm{Cat}_\infty)_{/\mathrm{Span}(\mathrm{Fin})}.$$

In particular, we obtain a functor

$$\mathcal{O}^{\amalg} \colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty .$$

5.2.2 Algebras over cocartesian operads

Algebras over a cocartesian operad $\mathcal{O}_C^{\amalg}$ are simple: as we will show next, they correspond to C-indexed families of commutative algebras.

Construction 5.2.9. Let C be an ∞-category. We construct a functor

$$\iota \colon \mathrm{Span}(\mathrm{Fin}) \times C \to C^{\amalg} = \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)),$$

informally given by sending a pair (I, x) to the 'constant' I-tuple $(x)_{i \in I}$, and by sending a pair $(I \xleftarrow{f} J \xrightarrow{g} K, \varphi \colon x \to y)$ of a span in Fin and a morphism in C to the span

$$(x)_{i \in I} \xleftarrow{(f, (\mathrm{id}_x)_{j \in J})} (x)_{j \in J} \xrightarrow{(g, (\varphi)_{j \in J})} (y)_{k \in K}.$$

(Note that the left-pointing morphism is indeed q-cartesian.) Formally, this map is constructed as follows:

- Consider the natural transformation $\Delta \colon \mathrm{const}_C \to C^{(-)}$ of functors $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which on the one-point set is given by the identity $\mathrm{id}_C \colon C \to C$; this property determines the entire transformation since $C^{(-)}$ is by definition right Kan extended from the point. For a finite set I, it is given by the diagonal functor

$$\Delta \colon C \to C^I, \quad x \mapsto (x)_{i \in I}.$$

- By cartesian unstraightening, we obtain a morphism of cartesian fibrations over Fin:

$$\begin{tikzcd} \mathrm{Fin} \times C \arrow[rr, "\mathrm{Un}^{\mathrm{ct}}(\Delta)"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{Fin}(C) \arrow[dl, "q"] \\ & \mathrm{Fin}. & \end{tikzcd}$$

- By using Proposition 5.1.3, both sides become adequate triples with left class given by the cartesian morphisms. The functor $\mathrm{Un}^{\mathrm{ct}}(\Delta)$ preserves cartesian morphisms and becomes a morphism of adequate triples. Since a morphism on the left-hand side is cartesian if and only if the C-component is an isomorphism, we may use the equivalence $\mathrm{Span}_{\simeq,\mathrm{all}}(C) \simeq C$ to get a map on spans:

$$\begin{tikzcd} \mathrm{Span}(\mathrm{Fin}) \times C \arrow[rr, "\iota"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C)) = C^{\amalg} \arrow[dl, "\mathrm{Span}(q)"] \\ & \mathrm{Span}(\mathrm{Fin}). & \end{tikzcd}$$

Notation 5.2.10. Let C, D and E be ∞-categories and assume D and E admit finite coproducts. We denote by

$$\mathrm{Fun}^{\amalg,-}(E \times C, D) \quad \subseteq \quad \mathrm{Fun}(E \times C, D)$$

the full subcategory consisting of functors $F\colon E \times C \to D$ that preserve finite coproducts in the first variable, in the sense that $F(-,x)\colon E \to D$ preserves finite coproducts for all $x \in C$.

Lemma 5.2.11 ([BH21, Lemma C.4], cf. [CLL25, Proof of Proposition 3.30])**.** *Let C and D be ∞-categories and assume D admits finite coproducts. Then restriction along ι induces an equivalence*

$$\iota^*\colon \mathrm{Fun}^{\amalg}(\mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C)), D) \xrightarrow{\sim} \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D),$$

with inverse given by left Kan extension along ι.

Proof. Step 1: We start with some auxiliary diagrams. Consider the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, hook, "m"] \arrow[dr, hook, "i"'] & \mathrm{Fin} \times C \arrow[r, hook, "l"] \arrow[d, "\iota'"] & \mathrm{Span}(\mathrm{Fin}) \times C \arrow[d, "\iota"] \\ & \mathrm{Fin}(C) \arrow[r, hook, "k"] & \mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C)). \end{tikzcd}$$

In particular, passing to restriction functors induces a commutative diagram as follows:

$$\begin{tikzcd} \mathrm{Fun}(C, D) & \mathrm{Fun}^{\amalg,-}(\mathrm{Fin} \times C, D) \arrow[l, "m^*"', "\simeq"] & \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D) \arrow[l, "l^*"'] \\ & \mathrm{Fun}^{\amalg}(\mathrm{Fin}(C), D) \arrow[ul, "\simeq", "i^*"'] \arrow[u, "\simeq", "\iota'^*"'] & \mathrm{Fun}^{\amalg}(\mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C)), D). \arrow[l, "k^*"'] \arrow[u, "\iota^*"] \end{tikzcd}$$

The functor m^* is an equivalence, since Fin is the free ∞-category with finite coproducts. The functor i^* was shown to be an equivalence in Lemma 5.2.2. It follows by 2-out-of-3 that ι'^* is an equivalence as well. Since the inverses to m^* and i^* are given by left Kan extension along i^* and m^*, the inverse to ι'^* is given by left Kan extension along ι'.

Step 2: We now prove the following auxiliary statement: for every object $X = \{x_i\}_{i\in I}$ of $\mathrm{Fin}(C)$, the relative slice category of ι' over X embeds fully faithfully into the relative slice category of ι over X, and this inclusion admits a left adjoint. To see this, let us denote these two relative slice categories as follows:

$$\mathcal{P} := \mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C))_{/X} \times_{\mathrm{Span}_{\mathrm{ct},\mathrm{all}}(\mathrm{Fin}(C))} (\mathrm{Span}(\mathrm{Fin}) \times C);$$

$$Q := \mathrm{Fin}(C)_{/X} \times_{\mathrm{Fin}(C)} (\mathrm{Fin} \times C).$$

The inclusions l and k induce a fully faithful inclusion $Q \hookrightarrow \mathcal{P}$. We claim that this inclusion admits a left adjoint $\mathcal{P} \to Q$. To this end, note that an object of $\mathcal{P}$ consists of a pair (J, y) and a span

$$\{y\}_{j\in J} \xleftarrow{(f,(\mathrm{id}_y))} \{y\}_{k\in K} \xrightarrow{(g,(\varphi_k\colon y\to x_i))} \{x_i\}_{i\in I} = X$$

in $\mathrm{Fin}(C)$, where the left-pointing map is q-cartesian. We will send this to the object of Q given by the morphism

$$\{y\}_{k\in K} \xrightarrow{(g,(\varphi_k\colon y\to x_i))} \{x_i\}_{i\in I} = X.$$

If we include this back into $\mathcal{P}$ by taking the left-pointing map to be the identity, the resulting object receives a morphism in $\mathcal{P}$ from the original object. An explicit computation shows that this morphism is a unit for an adjunction. The pointwise criterion for adjunctions thus provides a left adjoint $\mathcal{P} \to Q$, as desired.

Step 3: We now return to the question of interest by showing that ι^* admits a left adjoint given by left Kan extension. Consider a functor $F\colon \mathrm{Span}(\mathrm{Fin}) \times C \to D$ which preserves finite coproducts in the first variable. We need to show that the left Kan extension $\iota_! F\colon \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)) \to D$ exists and preserves finite coproducts. For the former, we may use the pointwise criterion for left Kan extensions and show that for each object $\{x_i\}_{i\in I}$ of $\mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$ the colimit

$$(\iota_! F)(\{x_i\}_{i\in I}) = \mathrm{colim}_{(J,y)\in\mathcal{P}} F(J, y)$$

exists in D, where $\mathcal{P}$ denotes the relative slice category of ι considered in Step 2. Since the inclusion $Q \hookrightarrow \mathcal{P}$ is a right adjoint, it is in particular final, hence the above colimit exists in D if and only if so does $\mathrm{colim}_{(J,y)\in Q} F(J, y)$. But this is indeed the case, since Q is the relative slice category of ι' and the pointwise left Kan extension of $l^*F\colon \mathrm{Fin} \times C \to D$ along ι' exists by Step 1.

We conclude that $\iota_! F$ exists, and that the canonical map $\iota'_! l^* F \to k^* \iota_! F$ is a natural isomorphism. It remains to show that $\iota_! F$ preserves finite coproducts, but this can be checked after restricting along the inclusion $k\colon \mathrm{Fin}(C) \hookrightarrow \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$, and thus follows from the fact that $\iota'_! l^* F$ preserves finite coproducts.

Step 4: Finally, we show that the resulting adjunction

$$\iota_!\colon \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D) \rightleftarrows \mathrm{Fun}^{\amalg}(\mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)), D) :\iota^*$$

is an adjoint equivalence. Since $\mathrm{Fin}(C)$ is generated under coproducts by the image of $\mathrm{Fin} \times C$, the functor ι^* is conservative, hence it suffices to show that for every $F \in \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin}) \times C, D)$ the unit $F \to \iota^*\iota_! F$ is an isomorphism. By essential surjectivity of l, this in turn reduces to $l^*F \xrightarrow{\sim} l^*\iota^*\iota_! F$. Under the isomorphisms $l^*\iota^*\iota_! F = \iota'^* k^* \iota_! F \simeq \iota'^* \iota'_! l^* F$, this corresponds to the unit of the adjunction $\iota'_! \dashv \iota'^*$, hence is an equivalence by Step 1. □

We may now deduce the characterization of algebras over cocartesian operads we are after:

Proposition 5.2.12. *Let C be an ∞-category.*

(1) *If E is an ∞-category with finite products, restriction along ι induces an equivalence*

$$\mathrm{Fun}^{\times}(C^{\amalg}, E) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{CMon}(E)).$$

(2) *If $\mathcal{O}$ is an ∞-operad, we obtain an equivalence*

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}_{C}, \mathcal{O}) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{O})).$$

(3) *If D is a symmetric monoidal ∞-category, we obtain an equivalence*

$$\mathrm{Alg}_{\mathcal{O}^{\amalg}_{C}}(D) \simeq \mathrm{Fun}(C, \mathrm{CAlg}(D)).$$

Proof. Part (3) is an instance of (2) by taking $\mathcal{O} = \mathcal{M}_D$ to be the multimorphism operad of D. Part (2) follows by applying Part (1) to $E = \mathcal{O}^{\otimes}$ and $E = \mathrm{Span}(\mathrm{Fin})$, and passing to horizontal fibers:

$$\begin{tikzcd}
\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}^{\amalg}_{C}, \mathcal{O}) \arrow[r] \arrow[d, dashed, "\simeq"'] & \mathrm{Fun}^{\times}(C^{\amalg}, \mathcal{O}^{\otimes}) \arrow[r, "p_{\mathcal{O}}\circ -"] \arrow[d, "\simeq"] & \mathrm{Fun}^{\times}(C^{\amalg}, \mathrm{Span}(\mathrm{Fin})) \arrow[d, "\simeq"] \\
\mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{O})) \arrow[r] & \mathrm{Fun}(C, \mathrm{CMon}(\mathcal{O}^{\otimes})) \arrow[r, "p_{\mathcal{O}}\circ"] & \mathrm{Fun}(C, \mathrm{CMon}(\mathrm{Span}(\mathrm{Fin}))).
\end{tikzcd}$$

It thus remains to prove part (1). For this, recall from Construction 5.2.4 that $C^{\amalg}$ is semiadditive. It thus follows from results proved last semester that the forgetful functor $\mathrm{CMon}(E) \to E$ induces two vertical equivalences in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}^{\times}(C^{\amalg}, \mathrm{CMon}(E)) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Fun}(C, \mathrm{CMon}(\mathrm{CMon}(E))) \arrow[d, "\simeq"] \\
\mathrm{Fun}^{\times}(C^{\amalg}, E) \arrow[r] & \mathrm{Fun}(C, \mathrm{CMon}(E)).
\end{tikzcd}$$

Showing that the bottom functor in this square is an equivalence this reduces to showing that the top functor is an equivalence. Since a functor between semiadditive ∞-categories preserves finite products if and only if it preserves finite coproducts, this top functor may be rewritten as the restriction functor

$$\iota^*\colon \mathrm{Fun}^{\amalg}(C^{\amalg}, \mathrm{CMon}(E)) \to \mathrm{Fun}^{\amalg,-}(\mathrm{Span}(\mathrm{Fin})\times C, \mathrm{CMon}(E)) \simeq \mathrm{Fun}(C, \mathrm{Fun}^{\amalg}(\mathrm{Span}(\mathrm{Fin}), \mathrm{CMon}(E))).$$

This is an equivalence by Lemma 5.2.11. □

5.2.3 Classification of cocartesian operads

As a consequence of the results from the previous subsection, we obtain an explicit classification of ∞-operads equivalent to $\mathcal{O}^{\amalg}_{C}$ for some ∞-category C.

Definition 5.2.13. An ∞-operad $\mathcal{O}$ is called *cocartesian* if there exists an equivalence $\mathcal{O} \simeq \mathcal{O}^{\amalg}_{C}$ for some ∞-category C. We denote by $\mathrm{Op}^{\amalg}_{\infty} \subseteq \mathrm{Op}_\infty$ the full subcategory spanned by the cocartesian ∞-operads.

Example 5.2.14. The commutative operad $\mathcal{C}omm$ is cocartesian: it is equivalent to $\mathcal{O}^{\amalg}_{\mathrm{pt}}$.

Proposition 5.2.15. *Let $\mathcal{O}$ be a cocartesian ∞-operad and let $\mathcal{P}$ be an ∞-operad such that the ∞-category $\mathcal{P}^{\otimes}$ is semiadditive. Then the underlying category functor induces an equivalence*

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle}, \mathcal{P}_{\langle 1\rangle})$$

between operad maps $\mathcal{O} \to \mathcal{P}$ and functors $\mathcal{O}_{\langle 1\rangle} \to \mathcal{P}_{\langle 1\rangle}$.

Proof. Picking an equivalence $\mathcal{O} \simeq \mathcal{O}_C^{\amalg}$ for some C, we may identify $\mathcal{O}_{\langle 1\rangle}$ with C. The inclusion $\mathcal{O}_{\langle 1\rangle} \hookrightarrow \mathcal{O}^{\otimes}$ then corresponds to the composite inclusion $C \hookrightarrow \mathrm{Span}(\mathrm{Fin}) \times C \overset{\iota}{\hookrightarrow} C^{\amalg}$, where ι is the inclusion from Construction 5.2.9. The functor in question then factors as

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}_C^{\amalg}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{CAlg}(\mathcal{P})) \to \mathrm{Fun}(C, \mathcal{P}_{\langle 1\rangle}),$$

where the first functor is the equivalence from Proposition 5.2.12(2) and the second functor is induced by $\mathrm{CAlg}(\mathcal{P}) \to \mathcal{P}_{\langle 1\rangle}$ given by evaluation at $\langle 1\rangle \in \mathrm{Span}(\mathrm{Fin})$. It thus remains to show that the latter is an equivalence. Unwinding definitions, we see that this map sits as the left vertical map in a diagram of fiber sequences in Cat_∞ as follows:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{P}) \arrow[r, hook] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}"] & \mathrm{CMon}(\mathcal{P}^{\otimes}) \arrow[r, "(p_{\mathcal{P}})_*"] \arrow[d, "\mathrm{ev}_{\langle 1\rangle}", "\simeq"'] & \mathrm{CMon}(\mathrm{Span}(\mathrm{Fin})) \arrow[d, "\mathrm{ev}_{\langle 1\rangle}", "\simeq"'] \\
\mathcal{P} \arrow[r, hook] & \mathcal{P}^{\otimes} \arrow[r, "p_{\mathcal{P}}"] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

Since $\mathcal{P}^{\otimes}$ and $\mathrm{Span}(\mathrm{Fin})$ are semiadditive, the middle and right vertical maps are equivalences by results from last semester, and the claim follows. $\square$

Corollary 5.2.16. *The functor $\mathcal{O}^{\amalg}\colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ is fully faithful and induces an equivalence*

$$\mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Op}_\infty^{\amalg}.$$

Proof. Observe that $\mathcal{O}^{\amalg}\colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ admits a retraction, given by $(-)_{\langle 1\rangle}\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty$. It follows that for two ∞-categories C and D, the composite

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(C, D) \to \mathrm{Hom}_{\mathrm{Op}_\infty}(\mathcal{O}_C^{\amalg}, \mathcal{O}_D^{\amalg}) \to \mathrm{Hom}_{\mathrm{Cat}_\infty}(C, D)$$

is the identity. Since the second map is an equivalence by Proposition 5.2.15, the first map is an equivalence as well, showing fully faithfulness. Since $\mathrm{Op}_\infty^{\amalg}$ is by definition the image of $\mathcal{O}^{\amalg}$, the second claim follows immediately. $\square$

Corollary 5.2.17. *The following conditions are equivalent for an ∞-operad $\mathcal{O}$:*

(1) *The ∞-operad $\mathcal{O}$ is cocartesian;*

(2) *The ∞-category $\mathcal{O}^{\otimes}$ is semiadditive;*

(3) *The following two conditions are satisfied:*

   (a) *For every color $x \in \mathcal{O}^{\simeq}$, the anima $\mathcal{O}(\emptyset; x)$ of nullary operations is contractible;*

   (b) *For objects $\{x_i\}_{i\in I}, \{y_j\}_{j\in J}$ of $\mathcal{O}^{\otimes}$ and a color $z \in \mathcal{O}^{\simeq}$, composition with the multimorphisms $\{x_i\}_{i\in I} \to \{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}$ and $\{y_j\}_{j\in J} \to \{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}$ obtained from condition (a) provides an equivalence*

$$\mathcal{O}(\{x_i\}_{i\in I} \sqcup \{y_j\}_{j\in J}; z) \xrightarrow{\sim} \mathcal{O}(\{x_i\}_{i\in I}; z) \times \mathcal{O}(\{y_j\}_{j\in J}; z).$$

Proof. The equivalence between (2) and (3) is a simple unwinding of definitions: condition (a) says that the terminal object of $O^{\otimes}$ is also initial, and condition (b) says that the maps $X \simeq X \times * \to X \times Y$ and $Y \simeq * \times Y \to X \times Y$ in $O^{\otimes}$ exhibit the product also as a coproduct.

Since $C^{\sqcup} = \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$ is semiadditive, it is clear that (1) implies (2). To see that (2) implies (1), let O be such an ∞-operad, and write $C := O^{\otimes}_{\langle 1\rangle}$ for its underlying ∞-category. By Proposition 5.2.15, the identity $\mathrm{id}_C \colon C \to O_{\langle 1\rangle}$ extends to a morphism of ∞-operads $O^{\amalg}_C \to O$. It is clear that it induces an equivalence on colors, so it remains to show that it induces equivalences on all multimorphism animae:

$$O^{\amalg}_C(\{x_i\}_{i\in I}; y) \xrightarrow{\sim} O(\{x_i\}_{i\in I}; y).$$

But using condition (3), we may inductively reduce this to the case of the one-point set $I = \langle 1\rangle$, where it is simply the equality $C = O^{\otimes}_{\langle 1\rangle}$. □

5.2.4 Cocartesian monoidal structures

We will now characterize when the cocartesian operad $O^{\amalg}_C$ defines a symmetric monoidal structure on C.

Proposition 5.2.18. *Let C be an ∞-category. Then the following conditions are equivalent:*

(1) *The ∞-category C admits finite coproducts;*

(2) *The cartesian fibration $q \colon \mathrm{Fin}(C) \to \mathrm{Fin}$ is a cocartesian fibration;*

(3) *The functor $\mathrm{Span}(q) \colon C^{\amalg} = \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)) \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration.*

Proof. Recall from Exercise 2.1.12 that every q-cartesian morphism in $\mathrm{Fin}(C)$ whose image in Fin is an isomorphism is itself already an isomorphism. This implies that the following square is a pullback square:

$$\begin{tikzcd}
\mathrm{Fin}(C) \arrow[r, hook, "\simeq"] \arrow[d, "q"'] & \mathrm{Span}_{\simeq,\mathrm{all}}(\mathrm{Fin}(C)) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)) \arrow[d, "\mathrm{Span}(q)"] \\
\mathrm{Fin} \arrow[r, hook, "\simeq"] & \mathrm{Span}_{\simeq,\mathrm{all}}(\mathrm{Fin}) \arrow[r, hook] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

In particular, (3) implies (2) since cocartesian fibrations are closed under pullback.

If q is a cocartesian fibration, then the covariant pushforward functor associated with the map $\nabla \colon * \sqcup * \to *$ in Fin provides a functor $\sqcup \colon C \times C \to C$ which is left adjoint to the diagonal functor $C \to C \times C$, $X \mapsto (X, X)$, which means that X has finite coproducts. Conversely, assume that C has finite coproducts, and consider an object $\{X_i\}_{i\in I} \in C^I$ together with a morphism $f \colon I \to J$ of finite sets. Define $Y_j := \bigsqcup_{i\in f^{-1}(j)} X_i$. The canonical inclusions $\varphi_i \colon X_i \hookrightarrow Y_{f(i)}$ determine a map $(f, (\varphi_i)) \colon \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ which lifts f. This morphism is q-cocartesian: unwinding the definitions this will boil down to showing that for every $Z \in C$ precomposition with the inclusions $X_i \to Y_{f(i)}$ induce equivalences $\mathrm{Hom}_C(Y_j, Z) \xrightarrow{\sim} \prod_{i\in f^{-1}(j)} \mathrm{Hom}_C(X_i, Z)$, which is the defining property of the coproduct. This shows that (1) and (2) are equivalent.

It remains to show that (2) implies (3). Consider an object $X \in \mathrm{Fin}(C)$ with $I := q(X)$, and consider a span $I \xleftarrow{f} K \xrightarrow{g} J$ of finite sets. We need to lift this to a $\mathrm{Span}(q)$-cocartesian morphism in $\mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C))$. Let $\varphi \colon U \to X$ be a q-cartesian lift of f, and let $\psi \colon U \to Y$ be a q-cocartesian lift of g. It will now suffice to prove the following:

Claim: The resulting span $X \xleftarrow{\varphi} U \xrightarrow{\psi} Y$ is a $\mathrm{Span}(q)$-cocartesian lift of the given span (f, g).

To show the claim, we check all conditions from Theorem 5.1.1. Condition (1) holds by construction, and conditions (2), (3) and (4) are vacuous. It remains to check (5). Consider a commutative square in $\mathrm{Fin}(C)$ of the form

$$\begin{tikzcd} W \arrow[r, "\psi'"] \arrow[d, "\xi'"'] & V \arrow[d, "\xi"] & & K \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "f"] \\ U \arrow[r, "\psi"] & Y & \overset{q}{\rightsquigarrow} & I \arrow[r, "g"] & J \end{tikzcd}$$

and assume that its image in Fin is a pullback square and that ξ' is a q-cartesian morphism. We must show that then the morphism ψ' is q-cocartesian if and only if ξ is q-cartesian (since then the square is automatically a pullback square by Lemma 5.1.2). Now, if we write $U = \{u_i\}_{i \in I}$ for $I := qU$, then by q-cartesianness of ξ' and q-cocartesianness of ψ, we get isomorphisms $W \cong \{u_{f'(k)}\}_{k \in K}$ and $Y \cong \left\{\bigsqcup_{i \in g^{-1}(j)} u_i\right\}_{j \in J}$. If we similarly write $V = \{v_l\}_{l \in L}$, then the given diagram in $\mathrm{Fin}(C)$ takes the form

$$\begin{tikzcd} \{u_{f'(k)}\}_{k \in K} \arrow[r, "\psi'"] \arrow[d, "\xi'"'] & \{v_l\}_{l \in L} \arrow[d, "\xi"] \\ \{u_i\}_{i \in I} \arrow[r, "\psi"] & \{\bigsqcup_{i \in g^{-1}(j)} u_i\}_{j \in J}. \end{tikzcd}$$

With this description, we find that:

- The morphism ψ' is q-cocartesian if and only if for each $l \in L$ the maps $\psi'_k \colon u_{f'(k)} \to v_l$ for $k \in g'^{-1}(l)$ induce an isomorphism

$$(\psi'_k) \colon \bigsqcup_{k \in g'^{-1}(l)} u_{f'(k)} \xrightarrow{\sim} v_l.$$

- The morphism ξ is q-cartesian if and only if for each $l \in L$ the map

$$\xi_l \colon v_l \to \bigsqcup_{i \in g^{-1}(f(l))} u_i$$

  is an isomorphism.

But since the diagram in Fin is a pullback, we have that $f' \colon K \to I$ induces for each $l \in L$ a bijection $g'^{-1}(l) \xrightarrow{\cong} g^{-1}(f(l))$, and it follows that the composite

$$\bigsqcup_{k \in g'^{-1}(l)} u_{f'(k)} \xrightarrow{(\psi'_k)} v_l \xrightarrow{\xi_l} \bigsqcup_{i \in g^{-1}(f(l))} u_i$$

of these two maps is an isomorphism. The claim thus follows by 2-out-of-3. $\square$

Definition 5.2.19 (Cocartesian monoidal structure)**.** Let C be an ∞-category with finite coproducts. Then the cocartesian fibration $\mathrm{Span}(q)\colon C^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$ from Proposition 5.2.18 corresponds to a symmetric monoidal ∞-category $\mathrm{Str}^{\mathrm{cc}}(\mathrm{Span}(q))\colon \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$ which we will denote by $(C, \amalg)$ and refer to as the *cocartesian monoidal structure on C*.

Lemma 5.2.20. *Let C and D be ∞-categories with finite coproducts and let $F\colon C \to D$ be a functor. Then F preserves finite coproducts if and only if the induced map* $\mathrm{Span}(F)\colon \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C)) \to \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(D))$ *preserves cocartesian morphisms over* $\mathrm{Span}(\mathrm{Fin})$.

Proof. It is always an operad map, so the condition is about preserving cocartesian fibrations over the forward spans. This may be checked after pullback along the inclusion $\mathrm{Fin} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$, so it remains to show that F preserves finite coproducts if and only if the functor $\mathrm{Fin}(F)\colon \mathrm{Fin}(C) \to \mathrm{Fin}(D)$ preserves cocartesian morphisms. This readily follows from the coproduct description of the cocartesian morphisms in $\mathrm{Fin}(C)$ and $\mathrm{Fin}(D)$ given in the previous proof. □

Definition 5.2.21. Let $\mathrm{Cat}_\infty^{\otimes,\amalg} \subseteq \mathrm{Cat}_\infty^{\otimes}$ denote the full subcategory spanned by the symmetric monoidal ∞-categories of the form $(C, \amalg)$ for some ∞-category C. We say that a symmetric monoidal ∞-category $(D, \otimes)$ is *cocartesian monoidal* if it is equivalent to $(C, \amalg)$ for some ∞-category C with finite coproducts.

Corollary 5.2.22. *The equivalence $O^{\amalg}\colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Op}_\infty^{\amalg}$ from Corollary 5.2.16 restricts to an equivalence*

$$\mathrm{Cat}_\infty^{\amalg} \xrightarrow{\sim} \mathrm{Cat}_\infty^{\otimes,\amalg}.$$

Proof. We need to check that the equivalence from Corollary 5.2.16 restricts to objects and morphisms. On objects this is the content of Proposition 5.2.18. On morphisms this is the content of Lemma 5.2.20. □

Lemma 5.2.23. *A symmetric monoidal ∞-category $(D, \otimes)$ is cocartesian monoidal if and only if the following two conditions are satisfied:*

(1) *The monoidal unit $\mathbb{1} \in D$ is an initial object;*

(2) *For any two objects X and Y of D, the two maps $X \simeq X \otimes \mathbb{1} \to X \otimes Y$ and $Y \simeq \mathbb{1} \otimes Y \to X \otimes Y$ induced by the maps $\mathbb{1} \to X$ and $\mathbb{1} \to Y$ from (a) exhibit the tensor product $X \otimes Y$ as a coproduct of X and Y.*

Proof. In light of the equivalence

$$\mathcal{M}_D(\{x_i\}_{i\in I}; y) \simeq \mathrm{Hom}_D(\bigotimes_{i\in I} x_i; y),$$

this is an immediate consequence of Corollary 5.2.17. □

5.3 Cartesian monoidal structures

The definition of cocartesian monoidal structures may be dualized:

Definition 5.3.1 (Cartesian monoidal structure)**.** Let C be an ∞-category with finite products. Then we may equip C^{op} with the cocartesian monoidal structure $(C^{\mathrm{op}}, \amalg)$. We define the *cartesian monoidal structure on* C as the composite

$$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{(C^{\mathrm{op}},\amalg)} \mathrm{Cat}_\infty \xrightarrow[\sim]{(-)^{\mathrm{op}}} \mathrm{Cat}_\infty.$$

Since $(-)^{\mathrm{op}}$ is an equivalence and hence preserves finite products, this is indeed a commutative monoid in Cat_∞, which we will denote by $(C, \times)$. Its underlying ∞-category is $(C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, and its tensor product is given by the cartesian product in C. This assignment $C \mapsto (C, \times)$ determines a functor $\mathrm{Cat}_\infty^\times \to \mathrm{Cat}_\infty^\otimes$, whose essential image we denote by

$$\mathrm{Cat}_\infty^{\otimes,\times} \subseteq \mathrm{Cat}_\infty^\otimes.$$

We also denote by $\mathcal{O}_C^\times := \mathcal{M}_{(C,\times)}$ the multimorphism operad of $(C, \times)$, and let $C^\times := (\mathcal{O}_C^\times)^\otimes$ be the total ∞-category of $\mathcal{O}_C^\times$.

Corollary 5.3.2. *The functor* $\mathrm{Cat}_\infty^\times \to \mathrm{Cat}_\infty^{\otimes,\times}, C \mapsto (C, \times)$ *is an equivalence.*

Proof. The equivalence $(-)^{\mathrm{op}} \colon \mathrm{Cat}_\infty \xrightarrow{\sim} \mathrm{Cat}_\infty$ induces vertical equivalences in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Cat}_\infty^\times \arrow[r, "C\mapsto(C,\times)"] \arrow[d, "(-)^{\mathrm{op}}"', "\sim"] & \mathrm{Cat}_\infty^{\otimes,\times} \arrow[d, "\sim"', "(-)^{\mathrm{op}}"] \\
\mathrm{Cat}_\infty^\amalg \arrow[r, "D\mapsto(D,\amalg)"] & \mathrm{Cat}_\infty^{\otimes,\amalg}.
\end{tikzcd}$$

Since the bottom functor is an equivalence by Corollary 5.2.22, so is the top functor. □

Lemma 5.3.3. *A symmetric monoidal ∞-category $(D, \otimes)$ is cartesian monoidal if and only if the following two conditions are satisfied:*

(1) *The monoidal unit $\mathbb{1} \in D$ is a terminal object;*

(2) *For any two objects X and Y of D, the two maps $X \otimes Y \to X \otimes \mathbb{1} \cong X$ and $X \otimes Y \to \mathbb{1} \otimes Y \cong Y$ induced by the maps $X \to \mathbb{1}$ and $Y \to \mathbb{1}$ from (a) exhibit the tensor product $X \otimes Y$ as a product of X and Y.*

Proof. This is an instance of *Lemma* 5.2.23 applied to D^{op}. □

5.3.1 Cartesian operads via iterated spans

Note that our definition of the cartesian monoidal structure on C is somewhat indirect: rather than directly writing down the ∞-operad $\mathcal{O}_C^\times$ encoding this structure, we first pass to opposites and consider the *cocartesian* operad of C^{op}. The main reason for this is that the cartesian monoidal structure is not as easy to describe in terms of operads as the cocartesian one: operads are good for describing maps *out of* tensor products, while the universal property of products is in terms of *maps into* them. That being said, we will now give a description of the total category $C^\times$ in terms of the span category of $(C^{\mathrm{op}})^\amalg$. I thank Sil Linskens for useful discussions about this approach.

The process we will apply is sometimes called *dualization of cocartesian fibrations*. Let $p\colon E \to B$ be a cocartesian fibration of ∞-categories, corresponding to a functor $F = \mathrm{Str}^{\mathrm{cc}}(p)\colon B \to \mathrm{Cat}_\infty$. We may then compose with $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ and unstraighten to get back another cocartesian fibration $\overline{p}\colon \overline{E} \to B$, i.e., $\overline{p} = \mathrm{Un}^{\mathrm{cc}}((-)^{\mathrm{op}} \circ F)$. We may think of $\overline{E}$ as obtained from E by taking fiberwise opposites. It turns out that $\overline{E}$ admits an explicit description as a span category of E^{op}:

Proposition 5.3.4 (Barwick, Glasman, and Nardin [BGN18])**.** *Let $p\colon E \to B$ be a cocartesian fibration classifying $F\colon B \to \mathrm{Cat}_\infty$. Then the cocartesian fibration classified by $(-)^{\mathrm{op}} \circ F\colon B \to \mathrm{Cat}_\infty$ is given by*

$$\mathrm{Span}(p^{\mathrm{op}})\colon \mathrm{Span}_{\mathrm{cart,fw}}(E^{\mathrm{op}}) \to \mathrm{Span}_{\mathrm{all},\simeq}(B^{\mathrm{op}}) \simeq B.$$

The construction of $\overline{E}$ happens in two steps. As a first step, we turn the cocartesian fibration $p\colon E \to B$ into a cartesian fibration $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to B^{\mathrm{op}}$; this is the cartesian fibration classifying $(-)^{\mathrm{op}} \circ F$. As a second step, we then need to pass from the *cartesian* unstraightening of $(-)^{\mathrm{op}} \circ F$ to the *cocartesian* unstraightening; this process is called *dualization* and is given by $(-)^\vee = \mathrm{Span}_{\mathrm{ct,fw}}(-)$. If C is an ∞-category with finite products, the cocartesian fibration $C^\times \to \mathrm{Span}(\mathrm{Fin})$ is by construction the dualization of the cocartesian fibration $(C^{\mathrm{op}})^{\amalg} \to \mathrm{Span}(\mathrm{Fin})$. The proposition thus allows us to identify the total category $C^\times$ as a span category of $((C^{\mathrm{op}})^{\amalg})^{\mathrm{op}}$. We will now provide an independent proof that does not rely on Proposition 5.3.4.

Notation 5.3.5. Let C be an ∞-category with finite products. Since C^{op} then admits finite coproducts, its cocartesian ∞-operad $(C^{\mathrm{op}})^{\amalg} := \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(C^{\mathrm{op}})) \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration. We will use the following notation:

- We write $\overline{C} := ((C^{\mathrm{op}})^{\amalg})^{\mathrm{op}} \simeq \mathrm{Span}_{\mathrm{all,ct}}(\mathrm{Fin}(C^{\mathrm{op}}))$. We write $\overline{q} := \mathrm{Span}(q)^{\mathrm{op}}\colon \overline{C} \to \mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}$ for the resulting cartesian fibration.
- We denote by
$$\overline{C}_{\mathrm{ct}} \subseteq \overline{C} \qquad \text{and} \qquad \overline{C}_{\mathrm{fw}} \subseteq \overline{C}$$
the two wide subcategories spanned by the $\overline{q}$-cartesian morphisms and the morphisms inverted by $\overline{q}$, respectively.
- Applying Proposition 5.1.3 to $\overline{q}$, we see that the triple $(\overline{C}, \overline{C}_{\mathrm{ct}}, \overline{C}_{\mathrm{fw}})$ is an adequate triple, and $\overline{q}$ is a morphism of adequate triples to $(\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}, \mathrm{all}, \mathrm{iso})$.

Proposition 5.3.6. *Let C be an ∞-category with finite products. Then the functor*

$$\mathrm{Span}_{\mathrm{ct,fw}}(\overline{C}) \xrightarrow{\mathrm{Span}(\overline{q})} \mathrm{Span}_{\mathrm{all},\simeq}(\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}) \simeq \mathrm{Span}(\mathrm{Fin})$$

is a cocartesian fibration which exhibits $\mathrm{Span}_{\mathrm{ct,fw}}(\overline{C})$ as an ∞-operad. Furthermore, there is an equivalence

$$C^\times \simeq \mathrm{Span}_{\mathrm{ct,fw}}(\overline{C})$$

of ∞-categories over $\mathrm{Span}(\mathrm{Fin})$.

Proof. Step 1: We first show that $\mathrm{Span}(\overline{q})$ is a cocartesian fibration. For this, we apply Theorem 5.1.1 to the map $\overline{q}\colon (\overline{C}, \overline{C}_{\mathrm{ct}}, \overline{C}_{\mathrm{fw}}) \to (\mathrm{Span}(\mathrm{Fin})^{\mathrm{op}}, \mathrm{all}, \mathrm{iso})$. Consider a span $I \xleftarrow{f} K \xrightarrow{g} J$ of finite sets, and let $X \in \overline{C}_I$ live in the fiber over I. We may then form the following data:

- First, we may find a q-cartesian morphism $\varphi\colon Y \to X$ in $\mathrm{Fin}(C^{\mathrm{op}})$;
- Second, we pick a q-*co*cartesian morphism $\psi\colon Y \to Z$ in $\mathrm{Fin}(C^{\mathrm{op}})$.

By the claim in the proof of Proposition 5.2.18, the resulting span

$$X \xleftarrow{\varphi} Y \xrightarrow{\psi} Z$$

is a $\mathrm{Span}(q)$-cocartesian morphism in $(C^{\mathrm{op}})^{\amalg} = \mathrm{Span}_{\mathrm{all},\mathrm{ct}}(\mathrm{Fin}(C^{\mathrm{op}}))$. By passing to opposite categories, this means that the span

$$(\psi,\varphi)\colon Z \xleftarrow{\psi} Y \xrightarrow{\varphi} X$$

is a $\overline{q}$-cartesian morphism in $\overline{C}$. By Theorem 5.1.1, the span

$$X \xleftarrow{(\psi,\varphi)} Z \xrightarrow{=} Z$$

in $\overline{C}$ is then a $\mathrm{Span}(\overline{q})$-cocartesian morphism in $\mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\overline{C})$ lifting the given span (f,g), as desired.

Step 2: We now show that $\mathrm{Span}(\overline{q})\colon \mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\overline{C}) \to \mathrm{Span}(\mathrm{Fin})$ defines an ∞-operad. By Lemma 4.2.2, it suffices to show that the cocartesian straightening

$$\mathrm{Str}^{\mathrm{cc}}(\mathrm{Span}(\overline{q}))\colon\ \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_{\infty}$$

preserves finite products. Since the inclusion $\mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ is essentially surjective and preserves finite products, this may be checked after precomposition with this inclusion. We now observe that the pullback of $\mathrm{Span}(\overline{q})$ along this inclusion is given by

$$\begin{tikzcd}
\mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}}) \arrow[r, hook] \arrow[d, "\mathrm{Span}(q)"'] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\overline{C}) \arrow[d, "\mathrm{Span}(\overline{q})"] \\
\mathrm{Fin}^{\mathrm{op}} = \mathrm{Span}_{\mathrm{all},\simeq}(\mathrm{Fin}) \arrow[r, hook] & \mathrm{Span}(\mathrm{Fin}),
\end{tikzcd}$$

By functoriality of unstraightening, we must therefore show that the fiber of $\mathrm{Span}(q)$ over a finite set I is the I-fold product of the fiber over $\langle 1\rangle$. But indeed, the fiber of $\mathrm{Span}(q)$ over I is the fiber of $\mathrm{Fin}(C^{\mathrm{op}})^{\mathrm{op}} \to \mathrm{Fin}^{\mathrm{op}}$ over I, which is $((C^{\mathrm{op}})^I)^{\mathrm{op}} \simeq C^I$.

Step 3: By steps 1 and 2, we now know that $\mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\overline{C}) \to \mathrm{Span}(\mathrm{Fin})$ defines a symmetric monoidal structure on $(C^{\mathrm{op}})^{\mathrm{op}} = C$. For the equivalence $C^{\times} \simeq \mathrm{Span}_{\mathrm{ct},\mathrm{fw}}(\overline{C})$, it thus remains to show that this symmetric monoidal structure is a cartesian monoidal structure, i.e., that it satisfies the conditions from Lemma 5.3.3. But unwinding definitions reveals that the monoidal unit is the initial object of C^{op}, hence is terminal in C, and similarly that the maps $X \otimes Y \to X$ and $X \otimes Y \to Y$ in C correspond to the canonical inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ for the coproduct in C^{op}, hence exhibit $X \otimes Y$ as a product of X and Y in C. □

All in all, we see that we may think of morphisms in $C^{\times}$ as consisting of diagrams in $\mathrm{Fin}(C^{\mathrm{op}})$ of the form

$$\{x_i\}_{i\in I} \xleftarrow{\varphi} \{x_{f(k)}\}_{k\in K} \xrightarrow{\psi} \{\prod_{k\in g^{-1}(j)} x_{f(k)}\}_{j\in J} \xleftarrow{(\varphi_j)} \{y_j\}_{j\in J},$$

where φ is a q-cartesian morphism (corresponding to an inert morphism in $C^{\times}$), ψ is a q-cocartesian morphism (corresponding to a cocartesian active morphism in $C^{\times}$) and the maps $\varphi_j\colon \prod_{k\in g^{-1}(j)} x_{f(k)} \to y_j$ are morphisms in C (corresponding to morphisms $y_j \to \prod_{k\in g^{-1}(j)} x_{f(k)}$ in C^{op}).

5.3.2 Algebras in cartesian monoidal structures

Let C be an ∞-category with finite products, and let $\mathcal{O}$ be an ∞-operad. We have previously introduced two a priori different algebraic structures associated to $\mathcal{O}$:

- We may consider $\mathcal{O}$-monoids in C: finite-product-preserving functors $M\colon \mathcal{O}^{\otimes} \to C$;
- We may also consider $\mathcal{O}$-algebras in C with respect to the cartesian monoidal structure $(C, \times)$: operad maps $M\colon \mathcal{O} \to \mathcal{M}_{(C,\times)}$.

At an informal level, these two notions encode precisely the same structure: an object M_x for every color $x \in \mathcal{O}^{\simeq}$ and a multimorphism

$$P_M\colon M_{x_1} \times M_{x_2} \times \cdots \times M_{x_n} \to M_y$$

for every multimorphism $P \in \mathcal{O}((x_1, \ldots, x_n); y)$. This makes it plausible that $\mathcal{O}$-monoids and $\mathcal{O}$-algebras define equivalent ∞-categories. This heuristic is made precise by the following result:

Proposition 5.3.7 ([Lur17, Proposition 2.4.1.7])**.** *For an ∞-category C with finite products, there exists a finite-product-preserving functor $\Pi_C\colon C^{\times} \to C$ such that for every ∞-operad $\mathcal{O}$ composition with Π_C induces an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathcal{O}}((C,\times)) = \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O}, \mathcal{O}_C^{\times}) \xrightarrow{\sim} \mathrm{Fun}^{\times}(\mathcal{O}^{\otimes}, C) = \mathrm{Mon}_{\mathcal{O}}(C).$$

In particular, the functor $\mathrm{Op}_\infty \to \mathrm{Cat}_\infty^{\times}, \mathcal{O} \mapsto \mathcal{O}^{\otimes}$ is left adjoint to the cartesian operad functor $\mathcal{O}^{\times}\colon \mathrm{Cat}_\infty^{\times} \to \mathrm{Op}_\infty$.

Since I did not manage to find a simple proof of this result, I will treat it as a black box and refer to Lurie's treatment (which unfortunately is of rather technical nature). Informally, the functor $\Pi_C\colon C^{\times} \to C$ is given on objects by

$$\Pi_C(\{x_i\}_{i\in I}) := \prod_{i\in I} x_i.$$

6 The tensor product of spectra

Last semester, we introduced the *stabilization* $\mathrm{Sp}(C)$ of an ∞-category C with finite limits, which is the universal stable ∞-category with a left exact functor to C, or equivalently is the *cofree* stable ∞-category generated by C. The question we will address in this chapter is the following: if C admits a symmetric monoidal structure, does $\mathrm{Sp}(C)$ naturally inherit a symmetric monoidal structure as well?

Our discussion will follow that of Nikolaus [Nik16], who not only addresses this question for the stabilization $\mathrm{Sp}(C)$, but also for the ∞-category C_* of pointed objects, the ∞-category $\mathrm{CMon}(C)$ of commutative monoids, and the ∞-category $\mathrm{CGrp}(C)$ of commutative groups. As a first step, we will define operadic analogues of these constructions: For an ∞-operad $\mathcal{O}$ with sufficient (co)limits, we will construct:

- The stable ∞-operad $\mathcal{S}p(\mathcal{O})$ of spectrum objects in $\mathcal{O}$;
- The additive ∞-operad $\mathcal{CG}rp(\mathcal{O})$ of commutative group objects in $\mathcal{O}$;
- The semiadditive ∞-operad $\mathcal{CM}on(\mathcal{O})$ of commutative monoid objects in $\mathcal{O}$;
- The pointed ∞-operad $\mathcal{O}_*$ of pointed objects in $\mathcal{O}$.

Each of these constructions comes with a universal property, exhibiting it as the *cofree* operad with the corresponding algebraic structure obtained from $\mathcal{O}$. If $\mathcal{O} = \mathcal{M}_C$ is the multimorphism operad of a symmetric monoidal ∞-category and the required (co)limits in C are compatible with the tensor product, then the resulting four ∞-operads are again symmetric monoidal ∞-categories.

As a key application, applying these four constructions to the cartesian symmetric monoidal ∞-category $(\mathrm{An}, \times)$ (or more precisely, its multimorphism operad $\mathcal{O}^{\times}_{\mathrm{An}}$) will equip the ∞-categories An_*, $\mathrm{CMon}(\mathrm{An})$, $\mathrm{CGrp}(\mathrm{An})$ and $\mathrm{Sp} = \mathrm{Sp}(\mathrm{An})$ with symmetric monoidal structures.

The chapter is structured as follows. We start in Section 6.1 by introducing the notion of an *operadic (co)limit*, which will allow us to formulate the universal properties the four constructions listed above should have. The trick for obtaining monoidal structures on these four algebraic constructions is by writing them in terms of functor categories. For spectrum objects, this will use the description of $\mathrm{Sp}(C)$ in terms of *reduced excisive functors*, which we explain in Section 6.2. We then discuss the *Day convolution monoidal structure* on functor categories in Section 6.3. We then deduce the symmetric monoidal structure on Sp in Section 6.4, and bootstrap to the case for arbitrary $\mathcal{S}p(\mathcal{O})$ in Section 6.5.

6.1 Operadic (co)limits

For our upcoming discussion of stable ∞-operads, we will need a suitable notion of limits and colimits that respects the operadic structure. It will not suffice for the underlying ∞-category $\mathcal{O}_{\langle 1\rangle}$ of an ∞-operad $\mathcal{O}$ to admit finite limits; we also need these limits to interact suitably with the operad's composition rules. This leads to the notion of *operadic limits*, which are defined by requiring that the multimorphism animae $\mathcal{O}((x_1,\dots,x_n);-)$ preserve these limits. A similar, dual notion exists for operadic colimits.

Definition 6.1.1. Let $\mathcal{O}$ be an ∞-operad and let $F\colon I\to\mathcal{O}_{\langle 1\rangle}$ be a functor.

(1) A cone $\eta\colon\ \mathrm{const}_y\to F$ in $\mathcal{O}_{\langle 1\rangle}$ is called an *operadic limit* if for all colors $x_1,\dots,x_n\in\mathcal{O}^{\simeq}$ the induced map
$$\mathcal{O}((x_1,\dots,x_n);y)\to\lim_{i\in I}\mathcal{O}((x_1,\dots,x_n);F(i))$$
is an equivalence of animae.

(2) A cocone $\varepsilon\colon F\to\mathrm{const}_x$ in $\mathcal{O}_{\langle 1\rangle}$ is called an *operadic colimit* if for all colors $x_2,\dots,x_n,y\in\mathcal{O}^{\simeq}$ the induced map
$$\mathcal{O}((x,x_2,\dots,x_n);y)\to\lim_{i\in I}\mathcal{O}((F(i),x_2,\dots,x_n);y)$$
is an equivalence of animae.

Remark 6.1.2. By taking $n=1$ in the definition of an operadic limit, we see that an operadic limit is, in particular, a limit in the underlying ∞-category $\mathcal{O}_{\langle 1\rangle}$. Similarly, by taking $n=1$ in the definition of an operadic colimit, we find that an operadic colimit is a colimit in $\mathcal{O}_{\langle 1\rangle}$. For this reason, we will generally assume that certain (co)limits exist in $\mathcal{O}_{\langle 1\rangle}$ and then ask whether they are in fact operadic.

The envelope construction allows for an alternative formulation: a cone in $\mathcal{O}_{\langle 1\rangle}$ is an operadic limit if and only if, after applying the inclusion $\mathcal{O}_{\langle 1\rangle}\hookrightarrow\mathrm{Env}(\mathcal{O})$, it becomes a limit cone in the usual sense. Similarly, a cocone in $\mathcal{O}_{\langle 1\rangle}$ is an operadic colimit if and only if applying the functor $(-)\otimes\{x_2\}\otimes\cdots\otimes\{x_n\}\colon\mathcal{O}_{\langle 1\rangle}\to\mathrm{Env}(\mathcal{O})$ results in a colimit cone in $\mathrm{Env}(\mathcal{O})$ for all choices of colors $x_2,\dots,x_n$.

Example 6.1.3. If $\mathcal{O}=\mathcal{M}_C$ for a symmetric monoidal ∞-category C, then any limit in C is automatically operadic. This is because the tensor product provides a natural equivalence
$$\mathcal{M}_C((x_1,\dots,x_n);-)\simeq\mathrm{Hom}_C(x_1\otimes\cdots\otimes x_n,-),$$
and the functor $\mathrm{Hom}_C(X,-)$ preserves limits for any object X.

Example 6.1.4. The claim from the previous example does not hold for general operads. For instance, consider the trivial operad $\mathcal{Triv}$. Its underlying ∞-category is the terminal category $*$, which has all limits, and in particular a terminal object. However, for $n\neq 1$, we have $\mathcal{Triv}((x_1,\dots,x_n);-)=\emptyset$, so this terminal object is not operadic terminal.

Example 6.1.5. If $\mathcal{O}=\mathcal{M}_C$ for a symmetric monoidal ∞-category C, then a colimit in C is operadic if and only if it is preserved by the functor $x\otimes -\colon C\to C$ for every object $x\in C$; the argument is similar to Example 6.1.3.

This brings us to the key definitions for this chapter, which are direct analogues of the corresponding notions for ∞-categories:

Definition 6.1.6. An ∞-operad $\mathcal{O}$ is called:

(1) *Pointed* if $\mathcal{O}_{\langle 1\rangle}$ has a zero object which is both operadic initial and operadic terminal.

(2) *Semiadditive* if $\mathcal{O}_{\langle 1\rangle}$ is semiadditive and all finite products and coproducts are operadic.

(3) *Additive* if $\mathcal{O}_{\langle 1\rangle}$ is additive and all finite products and coproducts are operadic.

(4) *Stable* if $\mathcal{O}_{\langle 1\rangle}$ is stable and all finite limits and colimits are operadic.

This allows us to define various subcategories of the ∞-category Op_∞ of ∞-operads:

- We define $\mathrm{Op}_\infty^* \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with an operadic terminal object and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{pt}} \subseteq \mathrm{Op}_\infty^*$ the full subcategory spanned by pointed ∞-operads.
- We define $\mathrm{Op}_\infty^\times \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with finite operadic products and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{sadd}}, \mathrm{Op}_\infty^{\mathrm{add}} \subseteq \mathrm{Op}_\infty^\times$ the full subcategories spanned by semiadditive and additive ∞-operads, respectively.
- We define $\mathrm{Op}_\infty^{\mathrm{lex}} \subseteq \mathrm{Op}_\infty$ as the (non-full) subcategory spanned by ∞-operads with finite operadic limits and maps preserving them. We denote by $\mathrm{Op}_\infty^{\mathrm{st}} \subseteq \mathrm{Op}_\infty^{\mathrm{lex}}$ the full subcategory spanned by stable ∞-operads.

The goal of the remainder of this chapter is to show that the four fully faithful inclusion functors

$$\begin{aligned}
\mathrm{Op}_\infty^{\mathrm{pt}} &\hookrightarrow \mathrm{Op}_\infty^* \\
\mathrm{Op}_\infty^{\mathrm{sadd}} &\hookrightarrow \mathrm{Op}_\infty^\times \\
\mathrm{Op}_\infty^{\mathrm{add}} &\hookrightarrow \mathrm{Op}_\infty^\times \\
\mathrm{Op}_\infty^{\mathrm{st}} &\hookrightarrow \mathrm{Op}_\infty^{\mathrm{lex}}
\end{aligned}$$

all admit right adjoints, which will correspond to 'cofreely adding' the respective algebraic structure to an ∞-operad.

6.2 Stabilization via excisive functors

Last semester, we defined the stabilization $\mathrm{Sp}(\mathcal{C})$ of an ∞-category $\mathcal{C}$ with finite limits as the inverse limit of a tower of loop spaces. We will now explore an alternative, equivalent characterization due to Lurie, which defines spectrum objects as 'homology-like' functors. This perspective will be the key to constructing the tensor product of spectra, and more generally the stabilization $\mathrm{Sp}(\mathcal{O})$ of an ∞-operad $\mathcal{O}$. Our main goal in this section is to establish the equivalence of these two viewpoints. The material in this section is largely based on the presentation in [Lur17, Section 1.4.2].

We begin by recalling the ∞-category of pointed finite animae: the smallest full subcategory $\mathrm{An}_*^{\mathrm{fin}} \subseteq \mathrm{An}_*$ that contains the 0-sphere $S^0 := * \sqcup *$ and is closed under finite colimits. It has the following universal property: for any pointed ∞-category D with finite colimits, evaluation at S^0 induces an equivalence

$$\mathrm{Fun}_*^{\mathrm{rex}}(\mathrm{An}_*^{\mathrm{fin}}, D) \xrightarrow{\sim} D,$$

where $\mathrm{Fun}_*^{\mathrm{rex}}$ denotes the full subcategory of functors that are pointed and preserve finite colimits.

Definition 6.2.1. Let $F\colon C \to D$ be a functor between ∞-categories.

(1) If C admits pushouts, we say that F is *excisive* if it sends pushout squares in C to pullback squares in D.

(2) If C admits a terminal object, we say that F is *reduced* if it sends the terminal object of C to a terminal object of D.

We write $\mathrm{Exc}(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory of excisive functors, $\mathrm{Fun}_*(C,D) \subseteq \mathrm{Fun}(C,D)$ for the full subcategory of reduced functors, and $\mathrm{Exc}_*(C,D)$ for their intersection.

Exercise 6.2.2. Show that for any spectrum E the functor $\mathrm{Hom}_{\mathrm{Sp}}(E, \mathbb{S}[-])\colon \mathrm{An} \to \mathrm{An}$ is reduced and excisive.

With these definitions, we can state the alternative construction of stabilization.

Definition 6.2.3. Let C be an ∞-category with finite limits. The *∞-category of spectrum objects in C*, denoted $\mathrm{Sp}_L(C)$, is the full subcategory of $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$ spanned by the reduced, excisive functors:

$$\mathrm{Sp}_L(C) := \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C).$$

Our goal is to show that this definition is equivalent to the tower-based definition of $\mathrm{Sp}(C)$ from last semester. We will do this by establishing the universal property of $\mathrm{Sp}_L(C)$:

Theorem 6.2.4. *Let D be a stable ∞-category and let C be an ∞-category with finite limits. Composition with the evaluation functor $\Omega^\infty\colon \mathrm{Sp}_L(C) \to C$, given by $F \mapsto F(S^0)$, induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{\mathrm{lex}}(D, \mathrm{Sp}_L(C)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(D, C).$$

Before proving this, let us deduce the desired comparision:

Corollary 6.2.5. *For an ∞-category C with finite limits, there is a natural equivalence* $\mathrm{Sp}_L(C) \xrightarrow{\sim} \mathrm{Sp}(C)$.

Proof. In light of the natural equivalences

$$\mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(D, \mathrm{Sp}_L(C)) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{lex}}}(D, C) \simeq \mathrm{Hom}_{\mathrm{Cat}_\infty^{\mathrm{st}}}(D, \mathrm{Sp}(C)),$$

this follows directly from the Yoneda lemma. □

The proof of the theorem will require some preparation. We will break down the argument into a series of lemmas.

Lemma 6.2.6. *Let C be a pointed ∞-category and let D be arbitrary. Then the ∞-category* $\mathrm{Fun}_*(C, D)$ *is pointed.*

Proof. Let $*_C$ and $*_D$ denote the zero objects of C and D, respectively. Let $F_0 \colon C \to D$ be the constant functor with value $*_D$. Since F_0 is clearly reduced, it defines an object in $\mathrm{Fun}_*(C, D)$. The functor F_0 is a terminal object in $\mathrm{Fun}(C, D)$, and thus also in the full subcategory $\mathrm{Fun}_*(C, D)$. To show that it is a zero object, it remains to prove that it is also an initial object. For any other object $G \in \mathrm{Fun}_*(C, D)$, we need to show that the mapping anima $\mathrm{Hom}_{\mathrm{Fun}_*(C,D)}(X_0, Y)$ is contractible. Since the inclusion $\mathrm{Fun}_*(C, D) \hookrightarrow \mathrm{Fun}(C, D)$ is fully faithful, this is equivalent to showing that $\mathrm{Hom}_{\mathrm{Fun}(C,D)}(X_0, Y)$ is contractible.

Evaluation at the zero object $*_C \in C$ induces a functor $\mathrm{ev}_{*_C} \colon \mathrm{Fun}(C, D) \to D$. We have $\mathrm{ev}_{*_C}(F_0) = *_D$ and, since G is a reduced functor, $\mathrm{ev}_{*_C}(G) = *_D$. Thus, the mapping anima in D between these evaluated objects is contractible. The proof is concluded by showing that the restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(C,D)}(F_0, G) \to \mathrm{Hom}_D(F_0(*_C), G(*_C))$$

is an equivalence. This follows from the observation that the constant functor F_0 is a left Kan extension of its restriction along the fully faithful inclusion $\{*_C\} \hookrightarrow C$. □

Corollary 6.2.7. *Let C be a pointed ∞-category with finite colimits, and let D be an ∞-category with finite limits. Then the ∞-category* $\mathrm{Exc}_*(C, D)$ *is pointed and admits finite limits.*

Proof. The zero object $F_0 \in \mathrm{Fun}_*(C, D)$ from Lemma 6.2.6 is clearly excisive, so is also contained in $\mathrm{Exc}_*(C, D)$, showing that the latter is pointed. The existence of finite limits in $\mathrm{Exc}_*(C, D)$ follows from the fact that limits in functor categories are computed pointwise, and the properties of being reduced and excisive are preserved under such limits. □

Corollary 6.2.8. *Let C be a pointed ∞-category with finite colimits, and let D be an ∞-category with finite limits. Then the forgetful functors*

$$\mathrm{Fun}_*(C, D_*) \to \mathrm{Fun}_*(C, D) \qquad \text{and} \qquad \mathrm{Exc}_*(C, D_*) \to \mathrm{Exc}_*(C, D)$$

induced by $D_* \to D$ are equivalences.

Proof. Recall from last semester that the functor category $\mathrm{Fun}(C, D_*)$ is equivalent to $\mathrm{Fun}(C, D)_*$. By restricting to reduced (excisive) functors, this identifies the forgetful functor in the statement to the forgetful functors $\mathrm{Fun}_*(C, D)_* \to \mathrm{Fun}_*(C, D)$ and $\mathrm{Exc}_*(C, D)_* \to \mathrm{Exc}_*(C, D)$. But these are equivalences since $\mathrm{Fun}_*(C, D)$ and $\mathrm{Exc}_*(C, D)$ are pointed. □

Remark 6.2.9. For the equivalence $\mathrm{Fun}_*(C, D_*) \xrightarrow{\sim} \mathrm{Fun}_*(C, D)$ we in fact only need C to be pointed and D to have a terminal object.

We will now prove an alternative criterion for excisive functors. This criterion is closely related to the criterion that a pointed ∞-category is stable if and only if it has an invertible loop functor.

Construction 6.2.10. Let C be a pointed ∞-category with finite colimits, let D be a pointed ∞-category with finite limits and let $F \colon C \to D$ be a reduced functor. Given an object X, applying F

to the pushout square defining $\Sigma_C X$ gives a commutative diagram in D of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma_C X \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & F(\Sigma_C X), \end{tikzcd}$$

using that $F(0) \simeq 0$. This induces a map $\eta_X\colon F(X) \to \Omega_D F(\Sigma_C X)$ by the universal property of Ω_D.

Proposition 6.2.11 (Excisiveness criterion)**.** *Let C be a pointed ∞-category with finite colimits and let D be a pointed ∞-category with finite limits. A reduced functor $F\colon C \to D$ is excisive if and only if for every object $X \in C$, the canonical map $\eta_X\colon F(X) \to \Omega_D F(\Sigma_C X)$ is an isomorphism.*

Proof. The implication ($\Rightarrow$) is straightforward: if F is excisive, it sends the pushout square defining $\Sigma_C X$ to a pullback square in D, which exhibits $F(X)$ as the loop space $\Omega_D F(\Sigma_C X)$.

For the converse ($\Leftarrow$), assume that $\eta_X\colon F(X) \to \Omega_D F(\Sigma_C X)$ is an equivalence for all $X \in C$. Let

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "u"] \\ Y \arrow[r, "v"'] & W \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

be a pushout square in C. We want to show that its image under F is a pullback square in D, or equivalently that the induced map $F(Z) \to F(X) \times_{F(W)} F(Y)$ is an isomorphism. To this end, we extend the pushout diagram in C as follows:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[d] & \\ Y \arrow[r, "v"'] \arrow[d] & W \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & \mathrm{cofib}(g) \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \arrow[r] \arrow[d] & \Sigma_C Z \arrow[r] \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] & \Sigma_C Y \arrow[d] \arrow[ul, phantom, "\ulcorner" very near start] \\ & 0 \arrow[r] & \Sigma_C X \arrow[r] \arrow[ul, phantom, "\ulcorner" very near start] & \Sigma_C W. \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

Applying F, this results in the analogous diagram in D, and by composing some of the maps we obtain:

$$\begin{tikzcd} F(Z) \arrow[r, "F(f)"] \arrow[d, "="'] & F(X) \arrow[r] \arrow[d, "F(u)"] & 0 \arrow[r, equal] \arrow[d] & 0 \times_0 0 \arrow[d] \\ F(Z) \arrow[r] & F(W) \arrow[r] & F(\Sigma_C Z) \arrow[r] & F(\Sigma_C X) \times_{F(\Sigma_C W)} F(\Sigma_C Y) \\ F(Z) \arrow[r, "F(g)"] \arrow[u, "="] & F(Y) \arrow[r] \arrow[u, "F(v)"'] & 0 \arrow[r, equal] \arrow[u] & 0 \times_0 0. \arrow[u] \end{tikzcd}$$

Passing to pullbacks along the vertical direction, we thus obtain the following commutative diagram

$$\begin{tikzcd} F(Z) \arrow[r] \arrow[dr, "\eta_Z"'] & F(X)\times_{F(W)} F(Y) \arrow[d] \arrow[drr, "\eta_X\times_{\eta_W}\eta_Y"] & & \\ & \Omega_D F(\Sigma_C Z) \arrow[rr] & & \Omega_D F(\Sigma_C X)\times_{\Omega_D F(\Sigma_C W)} \Omega_D F(\Sigma_C Y). \end{tikzcd}$$

By inspection, the two diagonal composites are the maps η_Z and $\eta_X \times_{\eta_W} \eta_Y$, which by assumption are isomorphisms in D. The claim thus follows by 2-out-of-6. □

Lemma 6.2.12. *Let C be a pointed ∞-category with finite colimits and let D be an ∞-category with finite limits. Then the ∞-category $\mathrm{Exc}_*(C, D)$ is stable.*

Proof. We saw in Corollary 6.2.7 that $\mathrm{Exc}_*(C, D)$ is pointed and has finite limits, computed pointwise. By the characterization of stable ∞-categories from last semester, it suffices to show that its loop functor $\Omega\colon \mathrm{Exc}_*(C, D) \to \mathrm{Exc}_*(C, D)$ is an equivalence. The loop functor on a functor category is given by post-composition, so for a functor $F \in \mathrm{Exc}_*(C, D)$, its loop is $\Omega(F) = \Omega_D \circ F$.

Let us define a 'shift' endofunctor on $\mathrm{Exc}_*(C, D)$ by pre-composition with suspension: $\mathrm{Shift}(F) := F \circ \Sigma_C$. Since Σ_C preserves pushout squares, we see that $\mathrm{Shift}(F)$ is excisive whenever F is, resulting in a functor $\mathrm{Shift}\colon \mathrm{Exc}_*(C, D) \to \mathrm{Exc}_*(C, D)$.

The excisiveness criterion now provides us with a natural isomorphism

$$\eta_F\colon F \xrightarrow{\sim} \Omega_D \circ (F \circ \Sigma_C) = \Omega(\mathrm{Shift}(F)) = \mathrm{Shift}(\Omega(F))$$

for $F \in \mathrm{Exc}_*(C, D)$. This defines natural isomorphism of functors $\mathrm{id} \xrightarrow{\sim} \Omega \circ \mathrm{Shift}$ and $\mathrm{id} \xrightarrow{\sim} \mathrm{Shift} \circ \Omega$, showing that Ω is an equivalence. This finishes the proof. □

Lemma 6.2.13. *If C is a stable ∞-category, the evaluation functor $\Omega^\infty\colon \mathrm{Sp}_L(C) \to C$ is an equivalence.*

Proof. An object of $\mathrm{Sp}_L(C)$ is a reduced, excisive functor $F\colon \mathrm{An}_*^{\mathrm{fin}} \to C$. Since C is a stable ∞-category, a functor into it is excisive if and only if it preserves finite colimits. Therefore, the subcategory of reduced, excisive functors coincides with the subcategory of reduced, finite-colimit-preserving functors:

$$\mathrm{Sp}_L(C) = \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \simeq \mathrm{Fun}_*^{\mathrm{rex}}(\mathrm{An}_*^{\mathrm{fin}}, C).$$

By the universal property of $\mathrm{An}_*^{\mathrm{fin}}$, the evaluation functor at the 0-sphere, ev_{S^0}, induces an equivalence

$$\mathrm{ev}_{S^0}\colon \mathrm{Fun}_*^{\mathrm{rex}}(\mathrm{An}_*^{\mathrm{fin}}, C) \xrightarrow{\sim} C.$$

The functor Ω^∞ is, by definition, this evaluation functor. The result follows. □

We are now equipped to prove the main result of this section:

Proof of Theorem 6.2.4. Since the ∞-category D is stable, a functor with domain D is left-exact if and only if it is reduced and excisive. We therefore need to prove that composing with $\mathrm{ev}_{S^0}\colon \mathrm{Sp}_L(C) \to C$ induces an equivalence

$$\mathrm{Exc}_*(D, \mathrm{Sp}_L(C)) \simeq \mathrm{Exc}_*(D, C).$$

Plussing in the definition of $\mathrm{Sp}_L(C)$, we get

$$\mathrm{Exc}_*(D, \mathrm{Sp}_L(C)) = \mathrm{Exc}_*(D, \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C)) \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{Exc}_*(D, C)) = \mathrm{Sp}_L(\mathrm{Exc}_*(D, C)),$$

where in the middle equivalence we use currying/uncurrying and observe that both sides correspond to functors $D \times \mathrm{An}_*^{\mathrm{fin}} \to C$ that are excisive in both variables separately. We thus need to show that evaluation at S^0 induces an equivalence

$$\mathrm{Sp}_L(\mathrm{Exc}_*(D, C)) \xrightarrow{\sim} \mathrm{Exc}_*(D, C).$$

But this is an instance of the previous lemma, since $\mathrm{Exc}_*(D, C)$ is stable by Lemma 6.2.12. $\square$

6.3 Day convolution

Let C and D be symmetric monoidal ∞-categories. It is natural to wonder whether the ∞-category $\mathrm{Fun}(C, D)$ of functors from C to D inherits a symmetric monoidal structure. The *Day convolution*, first introduced by Brian Day for ordinary symmetric monoidal categories, provides an affirmative answer to this question under suitable assumptions on C and D.

Let us first recall the classical construction. Let $(C, \otimes_C, \mathbb{1}_C)$ and $(D, \otimes_D, \mathbb{1}_D)$ be symmetric monoidal categories, and assume that C is small and that D admits all small colimits. For any two functors $F, G : C \to D$, their *Day convolution* $F \otimes_{\mathrm{Day}} G$ is a functor $C \to D$ defined as the left Kan extension of the composite functor

$$C \times C \xrightarrow{F \times G} D \times D \xrightarrow{\otimes_D} D$$

along the tensor product functor $\otimes_C : C \times C \to C$. More explicitly, for an object $X \in C$, the value of $F \otimes_{\mathrm{Day}} G$ is given by the colimit

$$(F \otimes_{\mathrm{Day}} G)(X) = \mathrm{colim}_{X_0 \otimes_C X_1 \to X} F(X_0) \otimes_D G(X_1),$$

where the colimit is taken over the category of morphisms into X whose source is of the form $X_0 \otimes_C X_1$ for $X_0, X_1 \in C$. This construction has two key features:

(a) If the tensor product $\otimes_D : D \times D \to D$ preserves small colimits in each variable separately, then the Day convolution equips the functor category $\mathrm{Fun}(C, D)$ with a symmetric monoidal structure.

(b) Under this monoidal structure, the category of commutative algebra objects $\mathrm{CAlg}(\mathrm{Fun}(C, D))$ is equivalent to the category of lax symmetric monoidal functors from C to D.

The goal of this section is to discuss an ∞-categorical analogue of Day convolution. A first version of such a construction was proved by Glasman [Gla16], which was subsequently generalized by Lurie [Lur17, Section 2.2.6].

6.3.1 Day convolution operads

Lurie's construction of the Day convolution monoidal structure on $\mathrm{Fun}(C, D)$ has a nice feature, characterizing it as an internal hom in the ∞-category of ∞-operads. We will take this universal property as the defining property of Day convolution operads:

Definition 6.3.1. Let $\mathcal{O}$ and $\mathcal{P}$ be ∞-operads. An ∞-operad $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ is called a *Day convolution operad for $\mathcal{O}$ and $\mathcal{P}$* if it comes equipped with a morphism of ∞-operads

$$\mathrm{ev}\colon \mathcal{D}ay(\mathcal{O}, \mathcal{P}) \times \mathcal{O} \to \mathcal{P}$$

satisfying the following universal property: if Q is another ∞-operad, the composite

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(Q, \mathcal{D}ay(\mathcal{O}, \mathcal{P})) \xrightarrow{-\times \mathcal{O}} \mathrm{Fun}_{\mathrm{Op}_\infty}(Q \times \mathcal{O}, \mathcal{D}ay(\mathcal{O}, \mathcal{P}) \times \mathcal{O}) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Fun}_{\mathrm{Op}_\infty}(Q \times \mathcal{O}, \mathcal{P})$$

is an equivalence. Informally, this means we have the following classification of operad morphisms into $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$:

$$\{\text{Operad morphisms } Q \to \mathcal{D}ay(\mathcal{O}, \mathcal{P})\} \quad \leftrightsquigarrow \quad \{\text{Operad morphisms } Q \times \mathcal{O} \to \mathcal{P}\}.$$

Note that this property determines $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ up to contractible choice, if it exists.

Remark 6.3.2. Let $\mathcal{O}$ be an ∞-operad for which $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ exists for all $\mathcal{P}$. Then the functor $- \times \mathcal{O}\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty$ admits a right adjoint

$$\mathcal{D}ay(\mathcal{O}, -)\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty\,.$$

The universal property of $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ allows us to deduce most of its basic structure. Let us first describe its underlying ∞-category.

Lemma 6.3.3. *Assume $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ exists. Then the underlying ∞-category of $\mathcal{D}ay(\mathcal{O}, \mathcal{P})$ is the ∞-category of functors $\mathcal{O}_{\langle 1\rangle} \to \mathcal{P}_{\langle 1\rangle}$ between the underlying ∞-categories of $\mathcal{O}$ and $\mathcal{P}$:*

$$\mathcal{D}ay(\mathcal{O}, \mathcal{P})_{\langle 1\rangle} \quad \simeq \quad \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle}, \mathcal{P}_{\langle 1\rangle}).$$

Proof. Recall from Exercise A.34 that the underlying ∞-category $\mathcal{O}_{\langle 1\rangle}$ of an ∞-operad $\mathcal{O}$ is equivalent to the ∞-category $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv, \mathcal{O})$ of operad maps $\mathcal{T}riv \to \mathcal{O}$. By adjunction, we thus get

$$\mathcal{D}ay(\mathcal{O}, \mathcal{P})_{\langle 1\rangle} \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv, \mathcal{D}ay(\mathcal{O}, \mathcal{P})) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv \times \mathcal{O}, \mathcal{P}).$$

Recall that $\mathcal{T}riv^{\otimes} = \mathrm{Fin}^{\mathrm{op}}$ and that $p_{\mathcal{T}riv}\colon \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ is the inclusion. The product operad $\mathcal{T}riv \times \mathcal{O}$ is thus given by the diagonal functor in the following pullback square:

$$\begin{tikzcd}
\mathrm{Fin}^{\mathrm{op}} \times_{\mathrm{Span}(\mathrm{Fin})} \mathcal{O}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Fin}^{\mathrm{op}} \arrow[r, hook] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

Essentially by definition of $\mathcal{O}$ being an ∞-operad, the left vertical map in this square is a cocartesian fibration, and its cocartesian straightening $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ preserves finite products. In particular, this cocartesian straightening is equivalent to

$$\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \qquad I \mapsto \mathcal{O}^{I}_{\langle 1\rangle},$$

and it follows that the left vertical map is of the form $q^{\mathrm{op}}\colon \mathrm{Fin}(\mathcal{O}_{\langle 1\rangle})^{\mathrm{op}} \to \mathrm{Fin}^{\mathrm{op}}$. In other words, we have an identification of ∞-operads $\mathcal{T}riv \times \mathcal{O} \simeq \mathcal{T}riv_{\mathcal{O}_{\langle 1\rangle}}$. We thus conclude

$$\mathcal{D}ay(\mathcal{O},\mathcal{P})_{\langle 1\rangle} \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv \times \mathcal{O},\mathcal{P}) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv_{\mathcal{O}_{\langle 1\rangle}},\mathcal{P}) \simeq \mathrm{Fun}(\mathcal{O}_{\langle 1\rangle},\mathcal{P}_{\langle 1\rangle}),$$

where the last equivalence is another instance of Exercise A.34. □

As another consequence of the universal property, we may describe the commutative algebras in Day convolution operads:

Lemma 6.3.4. *Assume $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ exists. Then commutative algebras in $\mathcal{D}ay(\mathcal{O},\mathcal{P})$ correspond to operad morphisms $\mathcal{O} \to \mathcal{P}$:*

$$\mathrm{CAlg}(\mathcal{D}ay(\mathcal{O},\mathcal{P})) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O},\mathcal{P}).$$

Proof. Since $\mathcal{C}omm$ is the terminal ∞-operad, we have $\mathcal{O} \simeq \mathcal{C}omm \times \mathcal{O}$, and so

$$\begin{aligned}\mathrm{CAlg}(\mathcal{D}ay(\mathcal{O},\mathcal{P})) &= \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{C}omm, \mathcal{D}ay(\mathcal{O},\mathcal{P})) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{C}omm \times \mathcal{O},\mathcal{P}) \\ &\simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{O},\mathcal{P}).\end{aligned}$$

□

A result by Lurie, which we leave as a black box, shows that Day convolution operads always exist if $\mathcal{O} = \mathcal{M}_C$ is a multimorphism operad of a symmetric monoidal ∞-category C:

Theorem 6.3.5 (Lurie [Lur17, Theorem 2.2.6.2])**.** *Let C be a symmetric monoidal ∞-category. For every ∞-operad $\mathcal{P}$, there exists a Day convolution operad $\mathcal{D}ay(\mathcal{M}_C,\mathcal{P})$. In particular, the functor $-\times \mathcal{M}_C\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty$ admits a right adjoint*

$$\mathcal{D}ay(\mathcal{M}_C,-)\colon \mathrm{Op}_\infty \to \mathrm{Op}_\infty.$$

□

Remark 6.3.6. In fact, Lurie proves a much more general result: if $f\colon \mathcal{O}' \to \mathcal{O}$ is a cocartesian fibration of ∞-operads, then the pullback functor $f^*\colon (\mathrm{Op}_\infty)_{/\mathcal{O}} \to (\mathrm{Op}_\infty)_{/\mathcal{O}'}$ admits a right adjoint

$$\mathrm{Nm}_{\mathcal{O}'/\mathcal{O}}\colon (\mathrm{Op}_\infty)_{/\mathcal{O}'} \to (\mathrm{Op}_\infty)_{/\mathcal{O}}.$$

By precomposing this functor with f^*, it then follows that the product functor $\mathcal{O}' \times_{\mathcal{O}} -\colon (\mathrm{Op}_\infty)_{/\mathcal{O}} \to (\mathrm{Op}_\infty)_{/\mathcal{O}}$ admits a right adjoint

$$\mathcal{D}ay_{/\mathcal{O}}(\mathcal{O}',-)\colon (\mathrm{Op}_\infty)_{/\mathcal{O}} \to (\mathrm{Op}_\infty)_{/\mathcal{O}}.$$

If $\mathcal{O} = \mathcal{C}omm$ is the terminal ∞-operad and $\mathcal{O}' \simeq \mathcal{M}_C$ for some C, this specializes to Theorem 6.3.5.

As a consequence of the theorem, if C and D are symmetric monoidal ∞-categories, we may form the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C,\mathcal{M}_D)$. This ∞-operad has the following properties:

- By Lemma 6.3.3, the underlying ∞-category of $\mathcal{D}ay(\mathcal{M}_C,\mathcal{M}_D)$ is the ∞-category of functors from C to D:
  $$\mathcal{D}ay(\mathcal{M}_C,\mathcal{M}_D)_{\langle 1\rangle} \quad\simeq\quad \mathrm{Fun}(C,D).$$
  In particular, the colors of $\mathcal{D}ay(\mathcal{M}_C,\mathcal{M}_D)_{\langle 1\rangle}$ may be identified with functors $C \to D$.

- By Lemma 6.3.4, commutative algebras in $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ are lax symmetric monoidal functors from C to D:

$$\mathrm{CAlg}(\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)) \quad \simeq \quad \mathrm{Fun}^{\otimes\text{-lax}}(C, D).$$

Our goal will be to show that, under suitable conditions on C and D, the ∞-operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal structure on $\mathrm{Fun}(C, D)$. For this, we need a short digression.

6.3.2 Hinich's model for Day convolution operads

While the universal property of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ gives us access to some of its abstract properties, it does not provide an explicit description of it. In this subsection, we will make this operad concrete by discussing a model for it given by Hinich [Hin20]. We will follow the recent 'synthetic' exposition of it by Winges [Win25].

The main ingredient for this is an explicit description of the unstraightening of the functor

$$(\mathrm{Cat}_\infty^\otimes)^{\mathrm{op}} \times \mathrm{Cat}_\infty^\otimes \to \mathrm{Cat}, \qquad (C, D) \mapsto \mathrm{Fun}^{\otimes\text{-lax}}(C, D).$$

Definition 6.3.7. We denote by

$$\mathrm{Ar}^{\mathrm{oplax}} \subseteq (\mathrm{Cat}_\infty)_{/[1]}$$

the full subcategory spanned by the cocartesian fibrations $E \to [1]$.

Note that the objects of $\mathrm{Ar}^{\mathrm{oplax}}$ correspond under straightening to functors $[1] \to \mathrm{Cat}_\infty$, i.e., to functors $F \colon C \to D$ between small ∞-categories. Since the morphisms in $\mathrm{Ar}^{\mathrm{oplax}}$ do not need to preserve the cocartesian edges, they correspond to *laxly* commutative squares of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d, "F"'] & C' \arrow[d, "F'"] \arrow[dl, Rightarrow] \\ D \arrow[r] & D'. \end{tikzcd}$$

Observe that $\mathrm{Ar}^{\mathrm{oplax}}$ is closed under finite products in $(\mathrm{Cat}_\infty)_{/[1]}$, and that pulling back along the inclusions $\{0\} \hookrightarrow [1]$ and $\{1\} \hookrightarrow [1]$ provides a finite-product-preserving functor

$$(s, t) \colon \mathrm{Ar}^{\mathrm{oplax}} \to \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty.$$

Theorem 6.3.8 (Winges [Win25, Theorem 2.7])**.** *Let $\mathcal{O}$ be an ∞-operad, and consider the induced functor*

$$(s, t) \colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Ar}^{\mathrm{oplax}}) \to \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \times \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty).$$

Then:

(1) *The fiber over a pair (C, D) of $\mathcal{O}$-monoidal ∞-categories is the ∞-category* $\mathrm{Fun}^{\mathcal{O}\text{-lax}}(C, D)$ *of lax $\mathcal{O}$-monoidal functors $C \to D$.*

(2) *For fixed C, the restriction of (s, t) to $\{C\} \times \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)$ is a cocartesian fibration classifying*

$$\mathrm{Fun}^{\mathcal{O}\text{-lax}}(C, -) \colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \to \mathrm{Cat}_\infty.$$

(3) *For fixed D, the restriction of (s,t) to* $\mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty) \times \{D\}$ *is a cartesian fibration classifying*

$$\mathrm{Fun}^{\mathcal{O}\text{-lax}}(-, D) \colon \mathrm{Mon}_{\mathcal{O}}(\mathrm{Cat}_\infty)^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

Remark 6.3.9. Winges in fact shows the stronger claim that this identification of the fibers is natural in both C and D simultaneously. Formulating this precisely requires the notion of *orthofibrations*, which are functors of the form $E \to C \times D$ that correspond under a generalized form of straightening to functors $C^{\mathrm{op}} \times D \to \mathrm{Cat}_\infty$. What Winges shows is that the functor (s,t) from Theorem 6.3.8 is the orthofibration classifying the assignment $(C, D) \mapsto \mathrm{Fun}^{\mathcal{O}\text{-lax}}(C, D)$.

With this result at hand, we may now prove the explicit description of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$:

Theorem 6.3.10 (Hinich [Hin20, Proposition 2.8.9], Winges [Win25, Theorem 1.1])**.** *Let C and D be symmetric monoidal ∞-categories, which we identify with commutative algebras in* Cat_∞ *with the cartesian monoidal structure. Then the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ exists, and is given by the following pullback square:*

$$\begin{tikzcd}
\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\times}_{\mathrm{Ar}^{\mathrm{oplax}}} \arrow[d, "(s,t)"] \\
\mathit{Comm} \arrow[r, "(C,D)"] & \mathcal{O}^{\times}_{\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty}.
\end{tikzcd}$$

Proof. Let us write $\mathcal{D}$ for this pullback. We will show directly that $\mathcal{D}$ satisfies the defining universal property of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$, in particular proving its existence without relying on Lurie's Theorem 6.3.5. It will suffice to produce for any ∞-operad $\mathcal{P}$ a natural equivalence

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P} \times \mathcal{M}_C, \mathcal{M}_D).$$

For the left-hand side, note that $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, -)$ commutes with pullbacks. Since Comm is the terminal ∞-operad, we have $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathit{Comm}) \simeq *$. Since operad morphisms from $\mathcal{P}$ into a cartesian operad correspond to $\mathcal{P}$-monoids by Proposition 5.3.7, we obtain equivalences

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}^{\times}_{\mathrm{Ar}^{\mathrm{oplax}}}) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathrm{Ar}^{\mathrm{oplax}}), \qquad \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}^{\times}_{\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty}) \simeq \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty).$$

It follows that the ∞-category $\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{D})$ is equivalent to the fiber of the map

$$\mathrm{Mon}_{\mathcal{P}}(\mathrm{Ar}^{\mathrm{oplax}}) \to \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty) \times \mathrm{Mon}_{\mathcal{P}}(\mathrm{Cat}_\infty)$$

over the pair $(C_{\mathcal{P}}, D_{\mathcal{P}})$, where $C_{\mathcal{P}}$ and $D_{\mathcal{P}}$ are obtained from C and D by precomposition with the operad map $p_{\mathcal{P}} \colon \mathcal{P} \to \mathit{Comm}$. By Theorem 6.3.8, this fiber is the ∞-category $\mathrm{Fun}^{\mathcal{P}\text{-lax}}(C_{\mathcal{P}}, D_{\mathcal{P}}) = \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}}, \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}})$ of lax $\mathcal{P}$-monoidal functors $C_{\mathcal{P}} \to D_{\mathcal{P}}$. By the definition of $C_{\mathcal{P}}$ and $D_{\mathcal{P}}$ we have

$$\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}} \simeq \mathcal{P} \times_{\mathit{Comm}} \mathcal{M}_C \qquad \text{and} \qquad \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}} \simeq \mathcal{P} \times_{\mathit{Comm}} \mathcal{M}_D,$$

hence

$$\mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{M}_{C_{\mathcal{P}}/\mathcal{P}}, \mathcal{M}_{D_{\mathcal{P}}/\mathcal{P}}) \simeq \mathrm{Fun}_{(\mathrm{Op}_\infty)_{/\mathcal{P}}}(\mathcal{P} \times_{\mathit{Comm}} \mathcal{M}_C, \mathcal{P} \times_{\mathit{Comm}} \mathcal{M}_D) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P} \times_{\mathit{Comm}} \mathcal{M}_C, \mathcal{M}_D),$$

where the second equivalence uses the universal property of pullbacks. This was precisely what we wanted to show. $\square$

Remark 6.3.11. The above description of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ immediately generalizes to a description of the O-monoidal Day convolution operad $\mathcal{D}ay_{/O}(\mathcal{M}_{C/O}, \mathcal{M}_{D/O}) \in (\mathrm{Op}_\infty)_{/O}$ for O-monoidal ∞-categories C and D (cf. Remark 6.3.6). The proof is essentially identical, up to replacing Op_∞ with $(\mathrm{Op}_\infty)_{/O}$ everywhere. It is this generality in which Hinich and Winges state and prove the result.

As a consequence of this result, we may explicitly compute the multimorphism anime of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$:

Proposition 6.3.12. *Let C and D be symmetric monoidal ∞-categories, let I be a finite set, and let $\{F_i : C \to D\}_{i\in I}$ and $G : C \to D$ be functors. Then there is an equivalence*

$$O(\{F_i\}_{i\in I}; \{G\}) \simeq \mathrm{Nat}\left(\bigotimes_D^I \circ \prod_{i\in I} F_i, G \circ \bigotimes_C^I\right)$$

between the anima of multimorphisms $\{F_i\}_{i\in I} \to G$ in the ∞-operad $O := \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ and the anima of natural transformations α fitting in the following diagram:

$$\begin{tikzcd} C^I \arrow[r, "\prod_{i\in I} F_i"] \arrow[d, "\otimes_C^I"'] & D^I \arrow[d, "\otimes_D^I"] \arrow[dl, phantom, "\swArrow"] \\ C \arrow[r, "G"'] & D. \end{tikzcd}$$

Proof. Let us write $O := \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ for simplicity. By definition, the anima of multimorphisms $O(\{F_i\}_{i\in I}; G)$ is the fiber of the projection map

$$p_O : \mathrm{Hom}_{O^\otimes}(\{F_i\}_{i\in I}, G) \to \mathrm{Hom}_{\mathrm{Span(Fin)}}(I, \langle 1\rangle)$$

over the active span $I \xleftarrow{=} I \to \langle 1\rangle$. By Theorem 6.3.10, the operad O sits in the following pullback square of ∞-operads:

$$\begin{tikzcd} O \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & O^\times_{\mathrm{Ar}^{\mathrm{oplax}}} \arrow[d, "(s,t)"] \\ Comm \arrow[r, "(C,D)"] & O^\times_{\mathrm{Cat}_\infty \times \mathrm{Cat}_\infty}. \end{tikzcd}$$

Because limits in functor categories are computed pointwise, the anima $\mathrm{Hom}_{O^\otimes}(\{F_i\}_{i\in I}, G)$ is itself the fiber in a pullback square of mapping animae. Taking the fibers of these mapping animae over the active span $I \to \langle 1\rangle$, we find that $O(\{F_i\}_{i\in I}; G)$ is the fiber in the following pullback square of animae:

$$\begin{tikzcd} O(\{F_i\}_{i\in I}; G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & O^\times_{\mathrm{Ar}^{\mathrm{oplax}}}(\{\mathrm{Un}(F_i)\}_{i\in I}; \mathrm{Un}(G)) \arrow[d, "(s_*,t_*)"] \\ * \arrow[r, "(\otimes_C^I, \otimes_D^I)"] & O^\times_{\mathrm{Cat}_\infty}(\{C\}_{i\in I}; C) \times O^\times_{\mathrm{Cat}_\infty}(\{D\}_{i\in I}; D). \end{tikzcd}$$

Here the object $\mathrm{Un}(F_i) \in \mathrm{Ar}^{\mathrm{oplax}}$ is the unstraightening of F_i regarded as a functor $F_i : [1] \to \mathrm{Cat}_\infty$, and similarly for $\mathrm{Un}(G)$. Since Cat_∞ and $\mathrm{Ar}^{\mathrm{oplax}}$ have finite products, we may write the three multimorphism animae in the right vertical map as Hom animae in $\mathrm{Ar}^{\mathrm{oplax}}$ and Cat_∞ out of the corresponding products, which results in a pullback square as follows:

$$\begin{tikzcd} O(\{F_i\}_{i\in I}; G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathrm{Ar}^{\mathrm{oplax}}}(\mathrm{Un}(\prod_{i\in I} F_i), \mathrm{Un}(G)) \arrow[d, "(s_*,t_*)"] \\ * \arrow[r, "(\otimes_C^I, \otimes_D^I)"] & \mathrm{Hom}_{\mathrm{Cat}_\infty}(C^I, C) \times \mathrm{Hom}_{\mathrm{Cat}_\infty}(D^I, D). \end{tikzcd} \tag{6.1}$$

By applying Theorem 6.3.8 to the trivial ∞-operad, we get that the functor $(s,t)\colon \mathrm{Ar}^{\mathrm{oplax}} \to \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty$ is a cartesian fibration over $\mathrm{Cat}_\infty \times \{D\}$ and a cocartesian fibration over $\{C\} \times \mathrm{Cat}_\infty$. The fibers of (s,t) are given by the functor categories $\mathrm{Fun}(C,D)$, with functoriality in C given by precomposition and functoriality in D given by postcomposition. In particular, the pullback (6.1) is given by

$$O(\{F_i\}_{i\in I}; G) \simeq \mathrm{Hom}_{\mathrm{Fun}(C^I,D)}(\otimes_D^I \circ \prod_{i\in I} F_i, G \circ \otimes_C^I),$$

as claimed. $\square$

6.3.3 The Day convolution monoidal structure

With the explicit description of the multimorphism animae of $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ at hand, we may now return to the question of when this ∞-operad corresponds to a symmetric monoidal structure on the functor category $\mathrm{Fun}(C,D)$:

Proposition 6.3.13 (Lurie [Lur17, Proposition 2.2.6.16], Winges [Win25, Proposition 4.1])**.** *Let $(C, \otimes_C, \mathbb{1}_C)$ and $(D, \otimes_D, \mathbb{1}_D)$ be symmetric monoidal ∞-categories. Let $\mathcal{K}$ be the collection of all relative slice categories $(\otimes_C^n)_{/c}$ defined by the pullback squares*

$$\begin{tikzcd} (\otimes_C^n)_{/c} \arrow[r] \arrow[d] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ C^n \arrow[r, "(\otimes_C^n, c)"] & C \times C \end{tikzcd}$$

for every natural number $n \geq 0$ and every object $c \in C$. Assume that the following two conditions are satisfied:

(1) The ∞-category D admits all $\mathcal{K}$-indexed colimits.

(2) The tensor product functor $\otimes_D\colon D \times D \to D$ preserves $\mathcal{K}$-indexed colimits in each variable separately.

Then the Day convolution operad $\mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal structure $\otimes_{\mathrm{Day}}$ on $\mathrm{Fun}(C,D)$.

Remark 6.3.14. The conditions in Proposition 6.3.13 are automatically satisfied if C is a small ∞-category, D is a cocomplete ∞-category (i.e., admits all small colimits), and if the tensor product of D preserves small colimits in each variable.

Proof of Proposition 6.3.13. We wish to show that the Day convolution operad $O := \mathcal{D}ay(\mathcal{M}_C, \mathcal{M}_D)$ defines a symmetric monoidal ∞-category, i.e., that the functor $p_O\colon O^\otimes \to \mathrm{Span}(\mathrm{Fin})$ is a cocartesian fibration. Recall from Lemma 4.2.5 that it is enough to show that the following two conditions are satisfied:

(a) For every finite collection of functors $\{F_i\}_{i\in I}$, there exists a functor $G = \bigotimes_{i\in I}^{\mathrm{Day}} F_i$ and a multimorphism $\{F_i\}_{i\in I} \to G$ in O inducing for every other functor $H\colon C \to D$ an equivalence

$$\mathrm{Hom}_{O_{\langle 1\rangle}}(G,H) \xrightarrow{\sim} O(\{F_i\}; H).$$

(b) For a map $f\colon I \to J$ of finite sets, if we let $G_j := \bigotimes^{\mathrm{Day}}_{i\in f^{-1}(j)} F_i$ for all $j \in J$, then the canonical map
$$\bigotimes^{\mathrm{Day}}_{i\in I} F_i \to \bigotimes^{\mathrm{Day}}_{j\in J} G_j$$
is a natural isomorphism of functors $C \to D$.

For part (a), we define G as the left Kan extension of the composite functor
$$\prod_{i\in I} C \xrightarrow{\prod_{i\in I} F_i} \prod_{i\in I} D \xrightarrow{\otimes_D} D$$
along the tensor product functor $\otimes_C\colon C^I \to C$. Explicitly, for an object $c \in C$, its value is given by the colimit:
$$G(c) := \operatorname{colim}_{\{c'_i\}_{i\in I}\in(\otimes^I_C)_{/c}} \left(\bigotimes_{i\in I} F_i(c'_i)\right).$$
The existence of this left Kan extension is guaranteed by condition (1) of our hypothesis, as the indexing category $(\otimes^I_C)_{/c}$ is in our collection $\mathcal{K}$. The universal property of this left Kan extension provides a natural transformation
$$\eta\colon \bigotimes^{D}_{i\in I} \circ \prod_{i\in I} F_i \Longrightarrow G \circ \bigotimes^{C}_{i\in I},$$
which by Proposition 6.3.12 corresponds to a multimorphism $\eta\colon \{F_i\}_{i\in I} \to G$ in $\mathcal{O}$. The universal property of left Kan extension precisely says that for any other functor $H\colon C \to D$, composition with η induces an equivalence
$$\mathrm{Nat}(G,H) \xrightarrow{\sim} \mathrm{Nat}\left(\bigotimes^{I}_{D} \circ \prod_{i\in I} F_i, H \circ \bigotimes^{I}_{C}\right) \simeq \mathrm{Hom}_{\mathcal{O}_{\langle 1\rangle}}(G,H),$$
as desired.

We now show that also (b) holds. Let $f\colon I \to J$ be a map in Fin, and write $G_j := \bigotimes^{\mathrm{Day}}_{i\in f^{-1}(j)} F_i$ for all $j \in J$. We need to show that the canonical transformation
$$\bigotimes^{\mathrm{Day}}_{i\in I} F_i \to \bigotimes^{\mathrm{Day}}_{j\in J} G_j$$
is a natural isomorphism of functors $C \to D$. To help visualize this, consider the following diagram:
$$\begin{tikzcd}
C^I \arrow[r, "\prod_{i\in I} F_i"] \arrow[d, "(\bigotimes^f_C)"'] \arrow[dd, bend right=60, "\otimes^I_C"'] & D^I \arrow[d, "(\bigotimes^f_D)"] \arrow[dd, bend left=60, "\otimes^I_D"] \\
C^J \arrow[r, dashed, "\prod_{j\in J} G_j"] \arrow[d, "\otimes^J_C"'] & D^J \arrow[d, "\otimes^J_D"] \\
C \arrow[r, dashed] & D.
\end{tikzcd}$$

Since left Kan extensions compose, it will suffice to show that the composite $\bigotimes^J_D \circ \prod_{j\in J} G_j\colon C^J \to D$ is the left Kan extension of the composite $\bigotimes^I_D \circ \prod_{i\in I} F_i\colon C^I \to D$ along $(\bigotimes^f_C)\colon C^I \to C^J$. By the pointwise formula for Kan extensions, this amounts to showing that for every tuple $(c_j)_{j\in J} \in C^J$, the map

$$\operatorname{colim}_{(c'_i)\in(\otimes^f_C)_{/c}} \bigotimes_{i\in I}^{D} F_i(c'_i) \to \otimes_{j\in J} G_j(c_j)$$

is a natural isomorphism. Since the functor $\otimes^f_C\colon C^I \to C^J$ is a product of the individual functors $\otimes^{I_j}_C\colon C^{I_j} \to C$, the indexing category $(\otimes^f_C)_{/c}$ on the left is a product over $j \in J$ of the relative slice categories $(\otimes^{I_j}_C)_{/c_j}$. But we also have

$$G_j(c_j) = \operatorname{colim}_{(c''_i)\in(\otimes^{I_j}_C)_{/c_j}} \bigotimes_{i\in I_j}^{D} F_i(c''_i),$$

and since the functor $\bigotimes^D_{J\in J}\colon D^J \to D$ preserves the relevant colimits in each variable separately it follows that the map is a natural isomorphism. This finishes the proof. □

Remark 6.3.15. The proof of the proposition provides an explicit description of the tensor product functor

$$\bigotimes_{\mathrm{Day}}^{I}\colon\ \mathrm{Fun}(C,D)^I \to \mathrm{Fun}(C,D)$$

for every finite set I. For $I = \langle 0\rangle$ and $I = \langle 2\rangle$, this specializes as follows:

- The monoidal unit $\mathbb{1}_{\mathrm{Day}}\colon C \to D$ of the Day convolution monoidal structure is the left Kan extension of $\mathbb{1}_D\colon * \to D$ along $\mathbb{1}_C\colon * \to C$.
- The Day convolution $F \otimes_{\mathrm{Day}} G$ of two functors $F, G\colon C \to D$ is the left Kan extension of $C \times C \xrightarrow{F\times G} D \times D \xrightarrow{\otimes_D} D$ along $\otimes_C\colon C \times C \to C$.

We have the following special case of the proposition:

Corollary 6.3.16. *Let C be a small symmetric monoidal ∞-category. Then functor category* $\mathrm{Fun}(C, \mathrm{An})$ *admits a symmetric monoidal structure via Day convolution.*

Proof. The proposition applies since An admits small colimits and the functor $-\times-\colon \mathrm{An}\times\mathrm{An} \to \mathrm{An}$ preserves colimits in both variables. □

For later use, we will also record the following description of the internal hom of $\mathrm{Fun}(C, \mathrm{An})$.

Proposition 6.3.17. *Let C be a small symmetric monoidal ∞-category. Then the Day convolution monoidal structure on* $\mathrm{Fun}(C,\mathrm{An})$ *is closed: for every functor $F\colon C \to \mathrm{An}$, tensoring with F admits a right adjoint $\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,\mathrm{An})}(F,-)\colon \mathrm{Fun}(C,\mathrm{An}) \to \mathrm{Fun}(C,\mathrm{An})$. Furthermore, this right adjoint is pointwise given by the following end expression:*

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(C,\mathrm{An})}(F,G)(X) \simeq \int_{Y\in C} \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X \otimes_C Y)).$$

Here we use the notion of an *end* from Definition 2.6.2.

Proof. It will suffice to show that for every third functor $H\colon C \to \mathrm{An}$ there exists a natural equivalence

$$\mathrm{Nat}(H \otimes^{\mathrm{Day}} F, G) \simeq \mathrm{Nat}(H, \int_{Y\in C} \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X \otimes_C Y))).$$

Since the Day convolution is a left Kan extension, the left-hand side is equivalent to the anima

$$\mathrm{Nat}(H(-) \times F(-), G(- \otimes_C -))$$

of natural transformations of functors $C \times C \to \mathrm{An}$. By Proposition 2.6.5, this anima of natural transformations may be computed as the following end:

$$\int_{(X,Y)\in C\times C} \mathrm{Hom}_{\mathrm{An}}(H(X) \times F(Y), G(X \otimes Y)).$$

Using the Fubini rule for ends, Lemma 2.6.4, we may write this as an iterated end $\int_{X\in C}\int_{Y\in C}$. Using the adjunction $- \times F(Y) \dashv \mathrm{Hom}_{\mathrm{An}}(F(Y), -)$ and using that $\mathrm{Hom}_{\mathrm{An}}(H(X), -)$ preserves limits, hence ends, we may transform this expression as follows: may rewrite this end as

$$\begin{aligned}\int_{X\in C}\int_{Y\in C} &\mathrm{Hom}_{\mathrm{An}}(H(X), \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X\otimes Y)))\\ &\simeq \int_{X\in C} \mathrm{Hom}_{\mathrm{An}}(H(X), \int_{Y\in C} \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X \otimes Y)).\end{aligned}$$

Using the end-description of animae of natural transformations from Proposition 2.6.5 another time, this is precisely the anima $\mathrm{Nat}(H, \int_{Y\in C} \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X \otimes_C Y)))$ we are after. $\square$

6.4 The tensor product of spectra

We will now combine the results from the previous two sections to upgrade the tensor product $X \otimes Y$ of spectra introduced last semester to a symmetric monoidal strucure $(\mathrm{Sp}, \otimes, \mathbb{S})$ on the ∞-category of spectra.

Let us start by outlining the strategy. In Section 6.2, we established an equivalence $\mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ between the ∞-category of spectra and the ∞-category of reduced excisive functors from finite pointed animae to animae. In particular, we may regard Sp as a full subcategory of the functor category $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$. The results from Section 6.3 allow us to equip this functor category with the Day convolution monoidal structure. We will then show that Sp can be exhibited as a symmetric monoidal Bousfield localization of $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$.

As the first step in this plan, we need to equip $\mathrm{An}_*^{\mathrm{fin}}$ with a monoidal structure. More generally, if C is a symmetric monoidal ∞-category, then under mild conditions the ∞-category C_* of pointed objects in C inherits a symmetric monoidal structure:

Lemma 6.4.1. *Let C be a symmetric monoidal ∞-category that admits finite colimits and assume that its tensor product $\otimes\colon C \times C \to C$ preserves finite colimits in both variables separately.*

(1) *The functor category $\mathrm{Fun}([1], C)$ admits a symmetric monoidal structure via Day convolution, whose unit is the canonical map $\emptyset \to \mathbb{1}$ from the initial object of C to its monoidal unit, and whose tensor product is given by the pushout product of morphisms in C:*

$$(f\colon X \to Y) \quad \square \quad (f'\colon X' \to Y') \quad := \quad f \square f'\colon (X \otimes Y') \sqcup_{X\otimes X'} (Y \otimes X') \to (Y \otimes Y').$$

(2) *If C admits a terminal object, then the subcategory $C_* \subseteq \mathrm{Fun}([1], C)$ is symmetric monoidal, and the localization functor*

$$\mathrm{cofib}\colon \mathrm{Fun}([1], C) \to C_*, \qquad (f\colon X \to Y) \quad \mapsto \quad (* \to \mathrm{cofib}(f))$$

is symmetric monoidal. In particular, the monoidal structure on C_* may be described as

$$(x\colon * \to X) \quad \otimes \quad (y\colon * \to Y) \quad = \quad \mathrm{cofib}(X \otimes * \sqcup_{*\otimes *} * \otimes Y \to X \otimes Y).$$

(3) *The zero object of C_* is operadic terminal and operadic initial, making $\mathcal{M}_{C_*}$ a pointed ∞-operad.*

Definition 6.4.2. Let C be an ∞-category with finite products and pushouts such that the functor $X \times -\colon C \to C$ preserves pushouts for all $X \in C$. Then we may apply part (2) of the lemma to the cartesian monoidal structure on C. We refer to the resulting symmetric monoidal structure $(C_*, \wedge, S^0)$ on the pointed objects in C as the *smash product*. The monoidal unit is $S^0 := * \sqcup *$, while the smash product $X \wedge Y$ is given by

$$X \wedge Y \quad := \quad \mathrm{cofib}(X \vee Y \to X \times Y),$$

where $X \vee Y := X \sqcup_* Y$.

In particular, this provides smash product monoidal structures on the ∞-categories An_* and $\mathrm{An}_*^{\mathrm{fin}}$ of pointed animae and finite pointed animae, respectively.

Proof of Lemma 6.4.1. For part (1), we equip $[1] = \{0 \le 1\}$ with the cartesian monoidal structure, which may more intuitively be thought of as the monoidal structure given by taking the *minimum*. To show that $\mathrm{Fun}([1], C)$ admits a Day convolution monoidal structure, we must check the two conditions of Proposition 6.3.13. This follows from the assumption on C, since the relative slices of the functors $\min\colon [1]^n \to [1]$ are finite ∞-categories. By Remark 6.3.15, the monoidal unit is the left Kan extension of the functor $\mathbb{1}\colon * \to C$ along the inclusion $1\colon * \hookrightarrow [1]$, which is $\emptyset \to \mathbb{1}$ by the pointwise formula. Similarly the tensor product is given by left Kan extending $[1] \times [1] \to C \times C \xrightarrow{\otimes} C$ along $\min\colon [1] \times [1] \to [1]$, and the pointwise formula results in the pushout product in this case.

For part (2), observe that the inclusion $C_* \hookrightarrow \mathrm{Fun}([1], C)$ admits a left adjoint given by taking cofibers. By Proposition 4.4.3, it thus remains to show that tensoring in $\mathrm{Fun}([1], C)$ with an object preserves those morphisms (= commutative squares in C) which induce isomorphisms on cofibers. To see this, consider such a morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y'. \end{tikzcd}$$

Let us write $Z := \mathrm{cofib}(f\colon X \to Y)$ and $Z' := \mathrm{cofib}(f'\colon X' \to Y')$, so that the condition on the square is that the induced map $Z \to Z'$ is an isomorphism. Our task is to show that for every other morphism $(g\colon A \to B)$ in C the induced commutative square

$$\begin{tikzcd} (X \otimes B) \sqcup_{X \otimes A} (Y \otimes A) \arrow[r] \arrow[d, "f \square g"'] & (X' \otimes B) \sqcup_{X' \otimes A} (Y' \otimes A) \arrow[d, "f' \square g"] \\ Y \otimes B \arrow[r] & Y' \otimes B \end{tikzcd}$$

again induces an isomorphism on (vertical) cofibers. Since $Z \xrightarrow{\sim} Z'$, it will suffice to express the two vertical cofibers purely in terms of g, Z and Z'. We will do so by showing that the bottom right square in the following commutative diagram is a pushout square:

$$\begin{tikzcd}
X \otimes A \arrow[rr] \arrow[dr] \arrow[dd] & & * \otimes A \arrow[dr] \arrow[dd] & \\
& X \otimes B \arrow[rr] \arrow[dd] & & * \otimes B \arrow[dd] \\
Y \otimes A \arrow[rr] \arrow[dr] & & Z \otimes A \arrow[dr] & \\
& (X \otimes B) \sqcup_{X \otimes A} (Y \otimes A) \arrow[rr] \arrow[d, "f \square g"'] & & (* \otimes B) \sqcup_{* \otimes A} (Z \otimes A) \arrow[d] \\
& Y \otimes B \arrow[rr] & & Z \otimes B.
\end{tikzcd}$$

But this follows from an iterated application law for pushout squares: the left and right faces of the diagram are pushouts by definition, while the back square and front rectangle are pushout squares since $- \otimes A$ and $- \otimes B$ preserve the pushout square defining Z as a cofiber of $X \to Y$.

For part (3), the condition for operadic terminality is vacuous by Example 6.1.3, and the condition for operadic initiality is saying that taking the tensor product with $*$ results in $*$, which is clear. □

As the next ingredient, we will discuss *excisive approximations*:

Proposition 6.4.3. *Let C be an ∞-category that admits finite limits, cofibers and sequential colimits, and assume that the loop functor $\Omega\colon C_* \to C_*$ preserves sequential colimits. Then the inclusion functor $\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$ admits a left adjoint*

$$P_1\colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C) \to \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \simeq \mathrm{Sp}(C).$$

Proof. We can write the inclusion as a composite of two fully faithful functors

$$\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C),$$

where the first functor is the inclusion of reduced excisive functors into all reduced functors, and the second is the inclusion of reduced functors into all functors. It is sufficient to show that both of these inclusions admit left adjoints.

We begin with the second inclusion. Let $F\colon \mathrm{An}_*^{\mathrm{fin}} \to C$ be a functor. Since the one-point anima $*$ is an initial object in $\mathrm{An}_*^{\mathrm{fin}}$, there is a unique natural transformation $\eta\colon \mathrm{const}_* \to \mathrm{id}$ of endofunctors of $\mathrm{An}_*^{\mathrm{fin}}$. Applying F yields a natural transformation $F(\eta)\colon \mathrm{const}_{F(*)} \to F$. We define a new functor $F^{\mathrm{red}}\colon \mathrm{An}_*^{\mathrm{fin}} \to C$ by taking the pointwise cofiber of this transformation:

$$F^{\mathrm{red}}(X) := \mathrm{cofib}(F(\eta)_X\colon F(*) \to F(X)).$$

The quotient map provides a natural transformation $\eta_F\colon F \to F^{\mathrm{red}}$. Since plugging in $X = *$ results in the cofiber of the identity map of $F(*)$, we see that F^{red} is reduced. If F is already reduced, then

$F(*) \xrightarrow{\sim} *$, so η_F is an isomorphism. Furthermore, one can check that applying the construction $(-)^{\mathrm{red}}$ to the natural transformation η_F yields a natural isomorphism $(F)^{\mathrm{red}} \xrightarrow{\sim} (F^{\mathrm{red}})^{\mathrm{red}}$. By the recognition principle for Bousfield localizations (Proposition A.2.54 in the notes from last semester), we conclude that $(-)^{\mathrm{red}}$ defines a left adjoint to the inclusion $\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$.

Next, we construct a left adjoint to the first inclusion, $\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$. First, observe that the forgetful functors

$$\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C_*) \to \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \qquad \text{and} \qquad \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C_*) \to \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$$

are equivalences of ∞-categories (Corollary 6.2.8), so we may assume without loss of generality that C is a pointed ∞-category. Let $F\colon \mathrm{An}_*^{\mathrm{fin}} \to C$ be a reduced functor. Recall from Construction 6.2.10 that for any reduced functor G, there is a canonical natural transformation $G \to \Omega \circ G \circ \Sigma$. By iterating this, we obtain a sequence of natural transformations

$$F \to \Omega \circ F \circ \Sigma \to \Omega^2 \circ F \circ \Sigma^2 \to \dots$$

in the functor category $\mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$. Since C admits sequential colimits, so does the functor category, and we can define a new functor F^{exc} as the colimit of this sequence:

$$F^{\mathrm{exc}} := \mathrm{colim}_{n\in\mathbb{N}}(\Omega^n \circ F \circ \Sigma^n).$$

We claim that F^{exc} is excisive. By the excisiveness criterion (Proposition 6.2.11), it suffices to show that the canonical transformation $F^{\mathrm{exc}} \to \Omega \circ F^{\mathrm{exc}} \circ \Sigma$ is an isomorphism. This follows by construction, using our assumption that the loop functor Ω commutes with sequential colimits:

$$\Omega \circ F^{\mathrm{exc}} \circ \Sigma = \Omega\left(\mathrm{colim}_{n\in\mathbb{N}}(\Omega^n \circ F \circ \Sigma^n)\right) \circ \Sigma \xrightarrow{\sim} \mathrm{colim}_{n\in\mathbb{N}}(\Omega^{n+1} \circ F \circ \Sigma^{n+1}) = F^{\mathrm{exc}}.$$

The colimit construction provides a canonical natural transformation $\eta_F\colon F \to F^{\mathrm{exc}}$. If F is already excisive, then each map in the sequence $F \to \Omega F \Sigma \to \dots$ is an isomorphism, which implies that $\eta_F\colon F \to F^{\mathrm{exc}}$ is an isomorphism as well. Moreover, applying the $(-)^{\mathrm{exc}}$ construction to η_F results in a natural isomorphism. Another application of the recognition criterion for Bousfield localizations (Proposition A.2.54) shows that $(-)^{\mathrm{exc}}$ is left adjoint to the inclusion $\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$. This completes the proof. □

Combining everything, we can now construct the tensor product on Sp:

Theorem 6.4.4. *The ∞-category* Sp *admits a unique symmetric monoidal structure such that the reduced excisive approximation functor*

$$P_1\colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \to \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \simeq \mathrm{Sp}$$

is symmetric monoidal with respect to Day convolution on its source.

Proof. Equip $\mathrm{An}_*^{\mathrm{fin}}$ with the smash product monoidal structure from Definition 6.4.2, and equip An with the cartesian monoidal structure. Since An admits small colimits and $- \times -\colon \mathrm{An} \times \mathrm{An} \to \mathrm{An}$ preserves colimits in both variables, Proposition 6.3.13 guarantees that the Day convolution monoidal structure on $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ exists.

We saw in Proposition 6.4.3 that the inclusion $\mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ admits a left adjoint P_1. We will show that this Bousfield localization satisfies the criterion from Lemma 4.4.4 in terms of internal homs. Recall from Proposition 6.3.17 that the Day convolution monoidal structure on $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ admits an internal Hom given by

$$\underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}},\mathrm{An})}(F, G)(X) = \int_{Y \in \mathrm{An}_*^{\mathrm{fin}}} \mathrm{Hom}_{\mathrm{An}}(F(Y), G(X \wedge Y)).$$

By Lemma 4.4.4, it now remains to show that this internal hom is reduced and excisive whenever G is.

We first check that it is reduced. If $X = *$ is the terminal anima, then so is $X \wedge Y$, hence also $G(X \wedge Y)$ by reducedness of G. Since $\mathrm{Hom}_{\mathrm{An}}(F(Y), -)$ preserves terminal animae and an end of terminal animae is terminal (as it is given by a limit), this shows that the internal hom is reduced.

We now check that it is also excisive. Note that $- \wedge Y$ preserves pushouts, and so the functor $G(- \wedge Y) \colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ sends pushouts to pullbacks for all Y. The same then holds true for $\mathrm{Hom}_{\mathrm{An}}(F(Y), G(- \wedge Y))$, and thus also for the end-expression since its defining limit commutes with pullbacks. We conclude that $\underline{\mathrm{Hom}}_{\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}},\mathrm{An})}(F, G)$ is excisive, finishing the proof. $\square$

Proposition 6.4.5. *The unreduced suspension spectrum functor* $\mathbb{S}[-] \colon \mathrm{An} \to \mathrm{Sp}$ *lifts to a strongly monoidal functor* $\mathbb{S}[-] \colon (\mathrm{An}, \times) \to (\mathrm{Sp}, \otimes)$. *In particular, the monoidal unit of* Sp *is the sphere spectrum* $\mathbb{S}$.

Proof. We will show that $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ is lax symmetric monoidal and that it has a strong monoidal left adjoint.

Under the equivalence $\mathrm{Sp} \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$, we may identify Ω^∞ with the composite

$$\mathrm{Sp} \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An}) \xrightarrow{\mathrm{ev}_{S^0}} \mathrm{An}.$$

The first functor is lax symmetric monoidal by construction of the tensor product of Sp. The second functor is lax symmetric monoidal by the functoriality of Day convolution: any lax symmetric monoidal functor $C' \to C$ induces a lax symmetric monoidal functor $\mathrm{Fun}(C, D) \to \mathrm{Fun}(C', D)$ by precomposition.

We now claim that the left adjoint of this composite is strong monoidal. Again, for the first functor this holds true by definition of the tensor product on Sp. For the second functor, let us write

$$F \colon \mathrm{An} \to \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$$

for the left Kan extension functor along $S^0 \colon * \to \mathrm{An}_*^{\mathrm{fin}}$. To show this is symmetric monoidal, we use the criterion from Proposition 4.3.5, which tells us it is enough to show that the 'canonical' maps

$$F(*) \to \mathbb{1}^{\mathrm{Day}} \qquad \text{and} \qquad F(X \times Y) \to F(X) \otimes^{\mathrm{Day}} F(Y)$$

are isomorphisms. For the monoital unit $\mathbb{1}^{\mathrm{Day}}$ this is clear from the description of Remark 6.3.15: it is indeed the left Kan extension of the unit $*$ of An along the unit S^0 of $\mathrm{An}_*^{\mathrm{fin}}$. The tensor product $F(X) \otimes^{\mathrm{Day}} F(Y)$ is the left Kan extension of the composite $\mathrm{An}_*^{\mathrm{fin}} \times \mathrm{An}_*^{\mathrm{fin}} \xrightarrow{F(X) \times F(Y)} \mathrm{An} \times \mathrm{An} \xrightarrow{- \times -} \mathrm{An}$ along $- \wedge - \colon \mathrm{An}_*^{\mathrm{fin}} \times \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}_*^{\mathrm{fin}}$. But by definition of $F(X)$ and $F(Y)$, it follows that the first functor is the left Kan extension of $(X, Y) \colon * \to \mathrm{An} \times \mathrm{An}$ along the inclusion $(S^0, S^0) \colon * \to \mathrm{An}_*^{\mathrm{fin}} \times \mathrm{An}_*^{\mathrm{fin}}$ of the monoidal unit. We conclude that $F(X) \otimes^{\mathrm{Day}} F(Y)$ is the left Kan extension of $(X \times Y) \colon * \to \mathrm{An}$ along the composite

$$* \xrightarrow{(S^0, S^0)} \mathrm{An}_*^{\mathrm{fin}} \times \mathrm{An}_*^{\mathrm{fin}} \xrightarrow{- \wedge -} \mathrm{An}_*^{\mathrm{fin}}.$$

But since $S^0 \wedge S^0 = S^0$, this is simply the unit of $\mathrm{An}_*^{\mathrm{fin}}$, resulting in $F(X \times Y)$, as desired. $\square$

Proposition 6.4.6. *The tensor product on* Sp *commutes with colimits in each variable separately.*

Proof. Since Sp is a symmetric monoidal Bousfield localization of $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$, it suffices to prove the claim for the latter. To this end, let I be a small ∞-category and let $F\colon I \to \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$ be an I-indexed diagram of functors. We need to show that for any other $G \in \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{An})$, the comparison map

$$\mathrm{colim}_{i\in I}(F_i \otimes^{\mathrm{Day}} G) \to (\mathrm{colim}_{i\in I} F_i) \otimes^{\mathrm{Day}} G$$

is an equivalence. This follows from the Yoneda lemma: for every functor $H\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathrm{An}$ we have natural equivalences

$$\begin{aligned}
\mathrm{Nat}(\mathrm{colim}_{i\in I}(F_i \otimes^{\mathrm{Day}} G), H) &\simeq \lim_{i\in I} \mathrm{Nat}(F_i \otimes^{\mathrm{Day}} G, H)\\
&\simeq \lim_{i\in I} \mathrm{Nat}(F_i(-) \times G(-), H(- \wedge -))\\
&\simeq \mathrm{Nat}(\mathrm{colim}_{i\in I}(F_i(-) \times G(-)), H(- \wedge -))\\
&\simeq \mathrm{Nat}((\mathrm{colim}_{i\in I} F_i(-)) \times G(-), H(- \wedge -))\\
&\simeq \mathrm{Nat}((\mathrm{colim}_{i\in I} F_i) \otimes^{\mathrm{Day}} G, H).
\end{aligned}$$

This finishes the proof. □

Corollary 6.4.7. *For spectra X and Y, the tensor product $X \otimes Y$ agrees with the tensor product constructed last semester.*

Proof. This follows from the previous two results: the functor $-\otimes -\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ is completely characterized by the condition that it commutes with colimits in both variables and that $\mathbb{S}[X]\otimes\mathbb{S}[Y] \cong \mathbb{S}[X \times Y]$. □

6.5 Stabilization of ∞-operads

In Section 6.1, we introduced various classes of ∞-operads with additional algebraic structure: pointed, semiadditive, additive, and stable. The goal of this section is to show that each of the four fully faithful inclusions

$$\mathrm{Op}_\infty^{\mathrm{pt}} \hookrightarrow \mathrm{Op}_\infty^{*}, \qquad \mathrm{Op}_\infty^{\mathrm{sadd}} \hookrightarrow \mathrm{Op}_\infty^{\times}, \qquad \mathrm{Op}_\infty^{\mathrm{add}} \hookrightarrow \mathrm{Op}_\infty^{\times} \qquad \text{and} \qquad \mathrm{Op}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Op}_\infty^{\mathrm{lex}}$$

admits a right adjoint. These right adjoints can be thought of as *cofreely* adding the respective algebraic structure to an ∞-operad. The constructions will rely on the Day convolution machinery developed in Section 6.3.

Let us focus on the case for spectrum objects; the other three cases are similar in spirit.

Construction 6.5.1. Let $\mathcal{O}$ be an ∞-operad that admits finite operadic limits. Consider the ∞-category $\mathrm{An}_*^{\mathrm{fin}}$ equipped with the smash product monoidal structure from Definition 6.4.2. We denote by

$$\mathcal{S}p(\mathcal{O}) \quad\subseteq\quad \mathcal{D}ay(\mathcal{M}_{\mathrm{An}_*^{\mathrm{fin}}}, \mathcal{O})$$

the full suboperad whose colors are those functors $F\colon \mathrm{An}_*^{\mathrm{fin}} \to \mathcal{O}_{\langle 1\rangle}$ that are reduced and excisive. Note that $\mathcal{S}p(\mathcal{O})_{\langle 1\rangle} \simeq \mathrm{Sp}(\mathcal{O}_{\langle 1\rangle})$ by Corollary 6.2.5, justifying the notation.

Theorem 6.5.2 (Nikolaus [Nik16, Theorem 4.11, Corollary 4.12]). *Let $\mathcal{O}$ be an ∞-operad that admits finite operadic limits. Then the ∞-operad $\mathcal{S}p(\mathcal{O})$ is stable, and the resulting functor*

$$\mathcal{S}p\colon \mathrm{Op}_\infty^{\mathrm{lex}} \to \mathrm{Op}_\infty^{\mathrm{st}}$$

is right adjoint to the inclusion functor.

Proof sketch. The most subtle part of the proof is verifying that $\mathrm{Sp}(\mathcal{O})$ is indeed stable. This requires showing that the relevant (co)limits are operadic. We do this in various steps:

(1) **The presheaf case:** First, one proves the theorem for an operad of the form $\mathcal{O} = \mathcal{M}_C$ where $C = \mathrm{PSh}(D) = \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{An})$ is a presheaf category of some symmetric monoidal ∞-category D equipped with the Day convolution monoidal structure. In this case, the tensor product on $\mathrm{PSh}(D)$ preserves colimits in both variables. It thus follows from Proposition 6.3.13 that the functor category $\mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{PSh}(D))$ admits a symmetric monoidal structure given by Day convolution. The inclusion

$$\mathrm{Sp}(\mathrm{PSh}(D)) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, \mathrm{PSh}(D))$$

admits a left adjoint. Note that we may identify this inclusion with the functor

$$\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Sp}) \hookrightarrow \mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, An)).$$

By the case for spectra, Theorem 6.4.4, we thus know that this inclusion has a symmetric monoidal left adjoint. We also conclude that the tensor product on $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Sp})$ commutes with colimits in both variables, so that $\mathrm{Sp}(\mathrm{PSh}(D))$ is a stable ∞-operad.

(2) **The general case:** For an arbitrary operad $\mathcal{O}$, consider its envelope $D := \mathrm{Env}(\mathcal{O})$. The unit of the adjunction $\mathrm{Env} \dashv \mathcal{M}$ provides an operad map $\mathcal{O} \hookrightarrow \mathcal{M}_{\mathrm{Env}(\mathcal{O})}$ which is fully faithful, in the sense of inducing equivalences on multimorphism animae. Moreover, by Remark 6.1.2 $\mathcal{O}$ admits finite operadic limits if and only if $\mathrm{Env}(\mathcal{O})$ admits finite limits (which in that case are automatically operadic).

We now consider the Yoneda embedding $Y\colon \mathrm{Env}(\mathcal{O}) = D \hookrightarrow \mathrm{PSh}(D)$. This also preserves finite limits. Moreover, it can be equipped with a symmetric monoidal structure. Indeed, by the universal property of the Day convolution monoidal structure on $\mathrm{PSh}(D)$, equipping Y with a *lax* symmetric monoidal structure is the same as providing a lax symmetric monoidal functor $D^{\mathrm{op}} \times D \to \mathrm{An}$. Equivalently, by the universal property of cartesian operads, this is in turn the same as a finite-product-preserving functor $(D^{\mathrm{op}} \times D)^{\otimes} \to \mathrm{An}$. To this end, we use that the twisted arrow category functor $\mathrm{Tw}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ preserves finite products, hence induces a functor

$$\mathrm{Tw}\colon \mathrm{Cat}_\infty^{\otimes} \to \mathrm{Cat}_\infty^{\otimes}.$$

Moreover, the two projections $s\colon \mathrm{Tw}(D) \to D^{\mathrm{op}}$ and $t\colon \mathrm{Tw}(D) \to D$ are natural in D and thus also refine to symmetric monoidal functors. We thus obtain a functor $\mathrm{Tw}(D)^{\otimes} \to (D^{\mathrm{op}} \times D)^{\otimes}$, which one can check is a left fibration. Its straightening is the desired functor $(D^{\mathrm{op}} \times D)^{\otimes} \to \mathrm{An}$.

All in all, we get fully faithful inclusions of ∞-operads

$$\mathcal{S}p(\mathcal{O}) \hookrightarrow \mathcal{S}p(\mathrm{Env}(\mathcal{O})) \hookrightarrow \mathcal{S}p(\mathrm{PSh}(\mathrm{Env}(\mathcal{O}))).$$

Since each of these inclusions preserves finite operadic limits, this reduces the general case to the presheaf case.

Next, we argue that the assignment $\mathcal{O} \mapsto \mathcal{S}p(\mathcal{O})$ is a right adjoint. As usual, we apply the recognition principle for Bousfield colocalizations: the construction of $\mathcal{S}p(\mathcal{O})$ immediately provides an operad map $\Omega^\infty\colon \mathcal{S}p(\mathcal{O}) \to \mathcal{O}$ given by evaluation at S^0, and it will suffice to show the following two conditions:

- $\Omega^\infty\colon \mathcal{S}p(\mathcal{O}) \to \mathcal{O}$ is an equivalence whenever $\mathcal{O}$ is a stable ∞-operad;
- The map $\mathcal{S}p(\Omega^\infty)\colon \mathcal{S}p(\mathcal{S}p(\mathcal{O})) \to \mathcal{S}p(\mathcal{O})$ is an equivalence for arbitrary $\mathcal{O}$.

But the former is clear from Lemma 6.2.13, and the latter can be reduced to the former by using the self-equivalence $\mathcal{S}p(\mathcal{S}p(\mathcal{O})) \simeq \mathcal{S}p(\mathcal{S}p(\mathcal{O}))$ given by swapping the two copies of $\mathrm{An}_*^{\mathrm{fin}}$. $\square$

Definition 6.5.3. We say a symmetric monoidal ∞-category C is *stably symmetric monoidal* if C is stable and $-\otimes-\colon C \times C \to C$ preserves finite colimits in both variables, or equivalently if $\mathcal{M}_C$ is a stable ∞-operad.

Corollary 6.5.4. *Let C be a symmetric monoidal ∞-category with finite limits. Then $\Omega^\infty\colon \mathrm{Sp}(C) \to C$ admits a canonical lax symmetric monoidal structure. Furthermore, for every stably symmetric monoidal ∞-category D, composition with Ω^∞ induces an equivalence of ∞-categories*

$$- \circ \Omega^\infty\colon\ \mathrm{Fun}^{\otimes\text{-lax}}(D, \mathrm{Sp}(C)) \xrightarrow{\sim} \mathrm{Fun}^{\otimes\text{-lax}}(D, C).$$

The pointed, semiadditive and additive cases work similarly.

Definition 6.5.5. If $\mathcal{O}$ is an ∞-operad with an operadic terminal object, we denote by

$$\mathcal{O}_* \quad \subseteq \quad \mathcal{D}ay(\mathcal{M}_{[1]}, \mathcal{O})$$

the full suboperad whose colors are those functors $F\colon [1] \to \mathcal{O}_{\langle 1\rangle}$ satisfying $F(0) = *$. If in addition $\mathcal{O}$ has finite operadic products, then we denote by

$$\mathcal{CG}rp(\mathcal{O}) \quad \subseteq \quad \mathcal{CM}on(\mathcal{O}) \quad \subseteq \quad \mathcal{D}ay(\mathcal{M}_{\mathrm{Span}(\mathrm{Fin})}, \mathcal{O})$$

the full suboperads whose colors are those functors $F\colon \mathrm{Span}(\mathrm{Fin}) \to \mathcal{O}_{\langle 1\rangle}$ that are commutative groups or commutative monoids in $\mathcal{O}_{\langle 1\rangle}$, respectively.

It is immediate from the definitions that the underlying ∞-categories of these ∞-operads are given by

$$(\mathcal{O}_*)_{\langle 1\rangle} = (\mathcal{O}_{\langle 1\rangle})_*, \qquad \mathcal{CM}on(\mathcal{O})_{\langle 1\rangle} \simeq \mathrm{CMon}(\mathcal{O}_{\langle 1\rangle}) \qquad \text{and} \qquad \mathcal{CG}rp(\mathcal{O})_{\langle 1\rangle} \simeq \mathrm{CGrp}(\mathcal{O}_{\langle 1\rangle}),$$

justifying the notation.

Theorem 6.5.6 (Nikolaus [Nik16, Theorems 4.11 and 5.7])**.** *Let $\mathcal{O}$ be an object in Op_∞^* or $\mathrm{Op}_\infty^\times$ respectively. Then the operads $\mathcal{O}_*$, $\mathcal{CM}on(\mathcal{O})$ and $\mathcal{CG}rp(\mathcal{O})$ are pointed, semiadditive, and additive, respectively. Moreover, the resulting functors*

$$(-)_*\colon\ \mathrm{Op}_\infty^* \to \mathrm{Op}_\infty^{\mathrm{pt}}, \qquad \mathcal{CM}on\colon\ \mathrm{Op}_\infty^\times \to \mathrm{Op}_\infty^{\mathrm{sadd}} \qquad \textit{and} \qquad \mathcal{CG}rp\colon\ \mathrm{Op}_\infty^\times \to \mathrm{Op}_\infty^{\mathrm{add}}$$

are right adjoint to the respective inclusion functors.

Proof. To show $\mathcal{O}_*$ is pointed, we use the embeddings $\mathcal{O}_* \hookrightarrow (\mathcal{M}_{\mathrm{Env}(\mathcal{O})})_* \hookrightarrow (\mathcal{M}_{\mathrm{PSh}(\mathrm{Env}(\mathcal{O}))})_*$ to reduce to the presheaf case, where the claim follows from part (3) of Lemma 6.4.1. The operad map $\mathcal{O}_* \to \mathcal{O}$ then satisfies the criterion for Bousfield colocalizations, showing that $\mathcal{O} \mapsto \mathcal{O}_*$ is a right adjoint to the inclusion.

The proofs for $\mathcal{CMon}(\mathcal{O})$ and $\mathcal{CGrp}(\mathcal{O})$ are similar. The only difference is that in the presheaf case the left adjoint to the inclusions $\mathrm{CMon}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathcal{C})$ and $\mathrm{CGrp}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), \mathcal{C})$ are no longer easy to write down explicitly; their existence is guaranteed by the *adjoint functor theorem*. □

Proposition 6.5.7. *The lax symmetric monoidal functors*

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{CMon}(\mathrm{An}) \xrightarrow{\mathrm{fgt}} \mathrm{An}_* \xrightarrow{\mathrm{fgt}} \mathrm{An}$$

admit strongly monoidal left adjoints:

$$(\mathrm{An}, \times) \xrightarrow{(-)_+} (\mathrm{An}_*, \wedge) \xrightarrow{F^{\mathrm{CMon}}} (\mathrm{CMon}(\mathrm{An}), \otimes) \xrightarrow{(-)^{\mathrm{grp}}} (\mathrm{CGrp}(\mathrm{An}), \otimes) \xhookrightarrow{\mathbf{B}^\infty} (\mathrm{Sp}, \otimes).$$

Here, $(-)_+$ is the functor that adjoins a disjoint basepoint, F^{CMon} is the free commutative monoid functor, $(-)^{\mathrm{grp}}$ is the group completion functor, and $\mathbf{B}^\infty$ is the Eilenberg-MacLane spectrum functor, identifying $\mathrm{CGrp}(\mathrm{An})$ *with the full subcategory* $\mathrm{Sp}^{\geq 0}$ *of connective spectra.*

Proof. The argument for each of these functors is analogous to that of Proposition 6.4.5. □

7 Other topics in operad theory

In this chapter, we give a quick overview of some other topics in operad theory. We will not provide all the proofs.

This chapter is not part of the exam material.

7.1 The envelope construction

Recall from Section 1.3 that every colored operad $\mathcal{O}$ has an *envelope* $\mathrm{Env}(\mathcal{O})$, which is a symmetric monoidal category with a symmetric monoidal functor to Fin which captures all the operad structure of $\mathcal{O}$. Also recall from Exercise 1.3.7 that the construction $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ is left adjoint to the assignment $C \mapsto \mathcal{M}_C$ of the multimorphism operad to a symmetric monoidal category C. In particular, $\mathcal{O}$-algebras in C correspond to symmetric monoidal functors $A\colon \mathrm{Env}(\mathcal{O}) \to C$.

In this section, we will introduce the analogous envelope construction for ∞-operads, and show that there exists a similar adjunction

$$\mathrm{Env}\colon \mathrm{Op}_\infty \rightleftarrows \mathrm{Cat}_\infty^\otimes : \mathcal{M}.$$

We will further show that Env induces a fully faithful functor $\mathrm{Env}^{/\mathrm{Fin}}\colon \mathrm{Op}_\infty \hookrightarrow (\mathrm{Cat}_\infty^\otimes)_{/(\mathrm{Fin},\amalg)}$ and characterize its image.

Definition 7.1.1 (Envelope of an ∞-operad)**.** Let $\mathcal{O} = (\mathcal{O}^\otimes, p_\mathcal{O})$ be an ∞-operad. We define its *envelope*, denoted $\mathrm{Env}(\mathcal{O})$, as the following pullback in $\mathrm{Cat}_\infty^\otimes$:

$$\begin{tikzcd}
(\mathrm{Env}(\mathcal{O}), \otimes) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{O}^\otimes, \times) \arrow[d, "p_\mathcal{O}"] \\
(\mathrm{Fin}, \amalg) \arrow[r, "i"] & (\mathrm{Span}(\mathrm{Fin}), \oplus).
\end{tikzcd}$$

Here we equip $\mathcal{O}^\otimes$ and Span(Fin) with the cartesian monoidal structure, and we equip Fin with the cocartesian monoidal structure. Since Span(Fin) is semiadditive, its monoidal structure is also cocartesian monoidal. Since $p_\mathcal{O}$ preserves finite products, it induces a symmetric monoidal functor with respect to the cartesian monoidal structures. Similarly, since the inclusion $i\colon \mathrm{Fin} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ preserves finite coproducts, it induces a symmetric monoidal functor with respect to the cocartesian monoidal structures.

The assignment $\mathcal{O}^\otimes \mapsto \mathrm{Env}(\mathcal{O})$ is clearly functorial in $\mathcal{O}$ and thus defines a functor

$$\mathrm{Env}\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty^\otimes.$$

Since pullbacks in $\mathrm{Cat}_\infty^\otimes$ are computed as the pullback of their underlying ∞-categories, the ∞-category $\mathrm{Env}(\mathcal{O})$ admits an explicit description: its objects are unordered tuples $\{x_i\}_{i\in I}$ of colors $x_i \in \mathcal{O}^\simeq$, while the morphisms $\{x_i\}_{i\in I} \to \{y_j\}_{j\in J}$ are pairs $(f, (\varphi_j))$ consisting of a map $f\colon I \to J$ of finite sets together with a J-indexed collection of multimorphisms $\varphi_j \in \mathcal{O}(\{x_i\}_{i\in J_i}; y_j)$. The symmetric monoidal structure is given by unordered concatenation of tuples. Note that this is entirely analogous to the definition of the envelope of a colored operad in Definition 1.3.3.

Our definition of the envelope is different from the one usually given in the literature. This makes sense: our definition only works because we work with a model for ∞-operads in which the total categories $\mathcal{O}^\otimes$ admit finite products, and thus can be equipped with the cartesian monoidal structure. The usual definition from [Lur17] instead directly writes down the cocartesian fibration $\mathrm{Env}(\mathcal{O})^\otimes \to \mathrm{Span}(\mathrm{Fin})$ associated to $\mathrm{Env}(\mathcal{O})$. We will now give such a description in our setting as well.

As a first step, we will explicitly compute the ∞-category $\mathrm{Fin}^\amalg$.

Lemma 7.1.2. *There is an equivalence*

$$\begin{tikzcd} \mathrm{Fin}(\mathrm{Fin}) \arrow[rr, "\simeq"] \arrow[dr, "q"'] & & \mathrm{Ar}(\mathrm{Fin}) \arrow[dl, "t"] \\ & \mathrm{Fin} & \end{tikzcd}$$

of cartesian fibrations over Fin.

Proof. It suffices to provide a natural isomorphism $\mathrm{Fin}^{(-)} \xrightarrow{\sim} \mathrm{Fin}_{/(-)}$ of functors $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ after straightening. This is implemented via the coproduct functor: $\mathrm{Fin}^I \xrightarrow{\sim} \mathrm{Fin}_{/I}$, $(S_i)_{i\in I} \mapsto S := \bigsqcup_{i\in I} S_i \to \bigsqcup_{i\in I}\{i\} = I$. ◻

Construction 7.1.3. We consider the full subcategory

$$\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \subseteq \mathrm{Ar}(\mathrm{Span}(\mathrm{Fin})) := \mathrm{Fun}([1], \mathrm{Span}(\mathrm{Fin}))$$

spanned by the 'active' spans $I \xleftarrow{=} I \xrightarrow{f} J$. Sending such span to its target J gives a functor

$$t\colon \ \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \to \mathrm{Span}(\mathrm{Fin}).$$

Corollary 7.1.4. *There is an equivalence*

$$\begin{tikzcd} \mathrm{Fin}^\amalg \arrow[rr, "\simeq"] \arrow[dr, "p^\amalg_{\mathrm{Fin}}"'] & & \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[dl, "t"] \\ & \mathrm{Span}(\mathrm{Fin}) & \end{tikzcd}$$

over Span(Fin).

Proof. By the previous lemma, we have $\mathrm{Fin}^\amalg = \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Fin}(\mathrm{Fin})) \simeq \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Ar}(\mathrm{Fin}))$. To construct a functor to $\mathrm{Ar}(\mathrm{Span}(\mathrm{Fin}))$, consider the following composite:

$$\mathrm{Fin}^\amalg \times [1] \simeq \mathrm{Span}_{\mathrm{ct,all}}(\mathrm{Ar}(\mathrm{Fin})) \times \mathrm{Span}_{\simeq,\mathrm{all}}([1]) \simeq \mathrm{Span}_{\mathrm{ct}\times\simeq,\mathrm{all}}(\mathrm{Ar}(\mathrm{Fin}) \times [1]) \xrightarrow{\mathrm{ev}} \mathrm{Span}(\mathrm{Fin}).$$

This results in a functor $\mathrm{Fin}^\amalg \to \mathrm{Ar}(\mathrm{Span}(\mathrm{Fin}))$, which one can check restricts to the required equivalence. ◻

Corollary 7.1.5. *Let $\mathcal{O}$ be an ∞-operad. Then the cocartesian fibration* $\mathrm{Env}(\mathcal{O})^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$ *encoding the symmetric monoidal ∞-category* $\mathrm{Env}(\mathcal{O})$ *sits in a pullback diagram as follows:*

$$\begin{tikzcd}
\mathrm{Env}(\mathcal{O})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[r, "s"] \arrow[d, "t"'] & \mathrm{Span}(\mathrm{Fin}) \\
\mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

Proof. By definition of $\mathrm{Env}(\mathcal{O})$, we have the following pullback square of ∞-categories over $\mathrm{Span}(\mathrm{Fin})$:

$$\begin{tikzcd}
\mathrm{Env}(\mathcal{O})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{O}^{\otimes})^{\times} \arrow[d, "(p_{\mathcal{O}})^{\times}"] \\
\mathrm{Fin}^{\sqcup} \arrow[r, "i^{\sqcup}"] & \mathrm{Span}(\mathrm{Fin})^{\oplus}.
\end{tikzcd}$$

By the corollay we have $\mathrm{Fin}^{\amalg} \simeq \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin}))$. Finally, we observe that there is a pullback square

$$\begin{tikzcd}
(\mathcal{O}^{\otimes})^{\times} \arrow[r] \arrow[d, "(p_{\mathcal{O}})^{\times}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Span}(\mathrm{Fin})^{\times} \arrow[r] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

TO DO: Is this really true? Perhaps one should argue differently... □

Proposition 7.1.6 (Envelope adjunction)**.** *The envelope functor* $\mathrm{Env}\colon \mathrm{Op}_{\infty} \to \mathrm{Cat}_{\infty}^{\otimes}$ *is left adjoint to the multimorphism operad functor* $\mathcal{M}\colon \mathrm{Cat}_{\infty}^{\otimes} \to \mathrm{Op}_{\infty}$ *from Definition 4.2.3:*

$$\mathrm{Env}\colon \mathrm{Op}_{\infty} \rightleftarrows \mathrm{Cat}_{\infty}^{\otimes} : \mathcal{M}.$$

Proof. Step 1: We start by constructing a unit map $\eta_{\mathcal{O}}\colon \mathcal{O} \to \mathcal{M}_{\mathrm{Env}(\mathcal{O})}$ for every ∞-operad $\mathcal{O}$. Informally, the colors of $\mathcal{M}_{\mathrm{Env}(\mathcal{O})}$ are given by unordered tuples $\{x_i\}_{i\in I}$ of colors of $\mathcal{O}$, and the map $\eta_{\mathcal{O}}$ will include $\mathcal{O}$ as a full suboperad of $\mathcal{M}_{\mathrm{Env}(\mathcal{O})}$ spanned by the one-element-tuples $\{x\}$. More formally, we define $\eta_{\mathcal{O}}\colon \mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ as the unique map fitting in the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \arrow[r, dashed, "\eta_{\mathcal{O}}"] \arrow[d, "p_{\mathcal{O}}"'] \arrow[rr, bend left, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Env}(\mathcal{O})^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes} \arrow[d, "p_{\mathcal{O}}"] \\
\mathrm{Span}(\mathrm{Fin}) \arrow[r, "S\mapsto \mathrm{id}_S"] & \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[r] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

Step 2: We now construct the counit map $\varepsilon_{\mathcal{C}}\colon \mathrm{Env}(\mathcal{M}_{\mathcal{C}}) \to \mathcal{C}$ for every symmetric monoidal ∞-category $\mathcal{C}$. Informally, $\varepsilon_{\mathcal{C}}$ will send an unordered tuple $\{x_i\}_{i\in I}$ of objects of $\mathcal{C}$ to their tensor product $\bigotimes_{i\in I} x_i$ in $\mathcal{C}$. More formally, consider the operad map

$$\eta_{\mathcal{M}_{\mathcal{C}}}\colon \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathrm{Env}(\mathcal{M}_{\mathcal{C}})}$$

constructed in Step 1. We may regard this as a lax symmetric monoidal functor from C to $\mathrm{Env}(\mathcal{M}_C)$. We argue that it admits a strong symmetric monoidal left adjoint. For this, we apply the criterion from Proposition 4.3.5. First we show that the underlying functor $C \to \mathrm{Env}(\mathcal{M}_C)$ admits a left adjoint. By definition, it sits in a pullback square as follows:

$$\begin{tikzcd} C \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Env}(\mathcal{M}_C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\otimes} \arrow[d, "p_C"] \\ * \arrow[r, "\langle 1\rangle"] & \mathrm{Fin} \arrow[r, hook, "i"] & \mathrm{Span}(\mathrm{Fin}). \end{tikzcd}$$

Since p_C is a cocartesian fibration, so is $\mathrm{Env}(\mathcal{M}_C) \to \mathrm{Fin}$. We now claim that a left adjoint $\varepsilon_C \colon \mathrm{Env}(\mathcal{M}_C) \to C$ is given by cocartesian transport along the unique map $I \to *$ in Fin. More precisely: if $\{x_i\}_{i\in I}$ is an arbitrary object of $\mathrm{Env}(\mathcal{M}_C)$, we may find a cocartesian lift $\{x_i\}_{i\in I} \to \otimes_{i\in I} x_i$ in $C^{\otimes}$ over the span $I \xleftarrow{=} I \to \langle 1\rangle$. The universal property for this cocartesian lift then says that $\bigotimes_{i\in I} x_i$ is a left adjoint object to $\{x_i\}_{i\in I}$. By the pointwise criterion for adjoints, this assembles into a functor $\varepsilon_C \colon \mathrm{Env}(\mathcal{M}_C) \to C$. It is clear from its description that ε_C sends concatenation of tuples in $\mathrm{Env}(\mathcal{M}_C)$ to tensor products in C, so the condition from Proposition 4.3.5 is satisfied, turning ε_C into a strong symmetric monoidal functor.

Step 3: We now check that the triangle identities are satisfied. The identity

$$\begin{tikzcd} & \mathcal{M}_{\mathrm{Env}(\mathcal{M}_C)} \arrow[dr, "\mathcal{M}_{\varepsilon_C}"] & \\ \mathcal{M}_C \arrow[ur, "\eta_{\mathcal{M}_C}"] \arrow[rr, equal] & & \mathcal{M}_C \end{tikzcd}$$

commutes via the counit $\varepsilon\eta \xrightarrow{\sim} \mathrm{id}$ of the adjunction $\varepsilon \dashv \eta$, using fully faithfulness of $\eta_{\mathcal{M}_C} \colon C \hookrightarrow \mathrm{Env}(\mathcal{M}_C)$. For the identity

$$\begin{tikzcd} & \mathrm{Env}(\mathcal{M}_{\mathrm{Env}(O)}) \arrow[dr, "\varepsilon_{\mathrm{Env}(O)}"] & \\ \mathrm{Env}(O) \arrow[ur, "\mathrm{Env}(\eta_O)"] \arrow[rr, equal] & & \mathrm{Env}(O), \end{tikzcd}$$

this can be checked after passing to operads: we need to show that we have the following commutative diagram over Span(Fin):

$$\begin{tikzcd} & \mathrm{Env}(\mathcal{M}_{\mathrm{Env}(O)})^{\otimes} \arrow[dr, "\varepsilon_{\mathrm{Env}(O)}"] & \\ \mathrm{Env}(O)^{\otimes} \arrow[ur, "\mathrm{Env}(\eta_O)^{\otimes}"] \arrow[rr, equal] & & \mathrm{Env}(O)^{\otimes}. \end{tikzcd}$$

Using the explicit description of the total category $\mathrm{Env}(O)^{\otimes}$ from Corollary 7.1.5, we claim that we may explicitly describe the functor $\varepsilon_{\mathrm{Env}(O)}$ as the following map induced on pullbacks:

$$\begin{tikzcd}
\mathrm{Env}(\mathcal{M}_{\mathrm{Env}(O)})^{\otimes} \arrow[rr, dashed, "\varepsilon_{\mathrm{Env}(O)}"] \arrow[d] & & \mathrm{Env}(O)^{\otimes} \arrow[d] \\
\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[rr, "- \circ -"] \arrow[d, "\mathrm{pr}_1"'] & & \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[d, "t"] \\
\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[d, "t"'] & & \mathrm{Span}(\mathrm{Fin}) \\
\mathrm{Span}(\mathrm{Fin}), & &
\end{tikzcd}$$

where $- \circ -$ denote the composition functor. Indeed, $\varepsilon_{\mathrm{Env}(O)}$ is a left adjoint to $\eta_{\mathcal{M}_{\mathrm{Env}(O)}}$, which is similarly induced by the inclusion functor $\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \to \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin}))$ sending $(I \to J)$ to $(I \to J \xrightarrow{=} J)$, and $- \circ -$ is a left adjoint to this inclusion functor. The relation $\varepsilon_{\mathrm{Env}(O)} \circ \mathrm{Env}(\eta_O)^{\otimes} = \mathrm{id}$ thus follows from the fact that the bottom map in the following diagram is the identity:

$$\begin{tikzcd}
\mathrm{Env}(O)^{\otimes} \arrow[r, dashed, "\mathrm{Env}(\eta_O)^{\otimes}"] \arrow[d, "\mathrm{Env}(\eta_O)^{\otimes}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Env}(\mathcal{M}_{\mathrm{Env}(O)})^{\otimes} \arrow[r, dashed, "\varepsilon_{\mathrm{Env}(O)}"] \arrow[d] & \mathrm{Env}(O)^{\otimes} \arrow[d] \\
\mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[r, "f \mapsto (\mathrm{id}, f)"] \arrow[rr, bend right, equal] & \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})) \arrow[r, "- \circ -"] & \mathrm{Ar}^{\mathrm{act}}(\mathrm{Span}(\mathrm{Fin})).
\end{tikzcd}$$

This finishes the proof. The exposition in this proof should probably be improved at some point... $\square$

The following result by Barkan, Haugseng and Steinebrunner implies that this functor is an equivalence of ∞-categories.

Theorem 7.1.7 ([BHS22, Proposition 4.2.1, Theorem 4.2.6]). *There exists a functor* $\mathrm{Env}^{\mathrm{Lurie}} \colon \mathrm{Op}_{\infty}^{\mathrm{Lurie}} \to \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, \mathrm{Cat}_{\infty})$ *that makes the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Op}_{\infty} \arrow[r, "\mathrm{Env}"] \arrow[d, "- \times_{\mathrm{Span}(\mathrm{Fin})} \mathrm{Fin}_*"'] & \mathrm{Cat}_{\infty}^{\otimes} = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathrm{Cat}_{\infty}) \arrow[d, "\simeq"] \\
\mathrm{Op}_{\infty}^{\mathrm{Lurie}} \arrow[r, "\mathrm{Env}^{\mathrm{Lurie}}"] & \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, \mathrm{Cat}_{\infty}).
\end{tikzcd}$$

Moreover, taking slices over the termain ∞-operad Comm and using that $\mathrm{Env}(\mathit{Comm}) = (\mathrm{Fin}, \amalg)$, *the both horizontal functors induce fully faithful functors*

$$\mathrm{Op}_{\infty} \hookrightarrow (\mathrm{Cat}_{\infty}^{\otimes})_{/(\mathrm{Fin}, \amalg)}, \qquad \mathrm{Op}_{\infty}^{\mathrm{Lurie}} \hookrightarrow (\mathrm{Cat}_{\infty}^{\otimes})_{/(\mathrm{Fin}, \amalg)}$$

whose essential images consist precisely of those symmetric monoidal functors $\pi \colon (C, \otimes) \to (\mathrm{Fin}, \amalg)$ *that are equifibered, meaning that for every pair of finite sets* $S, T \in \mathrm{Fin}$*, the tensor product in* C *induces an equivalence*

$$C_S \times C_T \xrightarrow{\sim} C_{S \sqcup T}, \quad (X, Y) \mapsto X \otimes Y$$

on fibers. In particular, the functor $\mathrm{Op}_{\infty} \to \mathrm{Op}_{\infty}^{\mathrm{Lurie}}$ *is an equivalence.*

7.2 More operadic constructions

In this section, we briefly introduce several other ways to build new operads from old ones. These constructions will not play a big role in the remainder of these notes, and are mainly provided for completeness. We will state the main definitions and universal properties, referring the interested reader to the literature for detailed proofs.

Monoidal structures on slice categories

Let C be a symmetric monoidal category and let A be a commutative algebra object of C. Then the overcategory $C_{/A}$ inherits the structure of a symmetric monoidal category: the tensor product of a map $X \to A$ with a map $Y \to A$ is given by the composition

$$X \otimes Y \to A \otimes A \xrightarrow{m} A,$$

where m denotes the multiplication on A. This has the following ∞-categorical analogue:

Definition 7.2.1. Let $\mathcal{O}$ be an ∞-operad, and let $A\colon \mathcal{C}omm \to \mathcal{O}$ be a commutative algebra in $\mathcal{O}$. We define the *slice operad* $\mathcal{O}_{/A}$ as the following pullback:

$$\begin{tikzcd} (\mathcal{O}_{/A})^{\otimes} \arrow[r] \arrow[d, "p_{\mathcal{O}_{/A}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1], \mathcal{O}^{\otimes}) \arrow[d, "\mathrm{ev}_1"] \\ \mathcal{C}omm^{\otimes} \arrow[r, "A"] & \mathcal{O}^{\otimes}. \end{tikzcd}$$

Theorem 7.2.2 (cf. Lurie [Lur17, Theorem 2.2.2.4])**.** *Let A be a commutative algebra in an ∞-operad $\mathcal{O}$. Then the pair $\mathcal{O}_{/A} := ((\mathcal{O}_{/A})^{\otimes}, p_{\mathcal{O}_{/A}})$ is an ∞-operad. Moreover, if $\mathcal{O} = \mathcal{M}_C$ is the multimorphism operad of a symmetric monoidal ∞-category, then $\mathcal{O}_{/A}$ corresponds to a symmetric monoidal structure on the slice category $C_{/A}$.*

Observation 7.2.3. It follows straight from the definition of $\mathcal{O}_{/A}$ that it admits the following universal property: for every other ∞-operad, there is a natural equivalence

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O}_{/A}) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, \mathcal{O})_{/A_{\mathcal{P}}},$$

where $A_{\mathcal{P}}$ is the composite $\mathcal{P} \xrightarrow{p_{\mathcal{P}}} \mathcal{C}omm \xrightarrow{A} \mathcal{O}$. In particular for $\mathcal{O} = \mathcal{M}_C$ and $\mathcal{P} = \mathcal{C}omm$ we get

$$\mathrm{CAlg}(C_{/A}) \simeq \mathrm{CAlg}(C)_{/A}.$$

Coproducts of ∞-operads

The ∞-category Op_∞ admits finite coproducts that admit the following explicit description:

Construction 7.2.4. Let $\mathcal{O}$ and $\mathcal{P}$ be ∞-operads. We define a functor $p_{\mathcal{O} \boxplus \mathcal{P}}$ as the composite

$$\mathcal{O}^{\otimes} \times \mathcal{P}^{\otimes} \xrightarrow{p_{\mathcal{O}} \times p_{\mathcal{P}}} \mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin}) \xrightarrow{- \sqcup -} \mathrm{Span}(\mathrm{Fin}),$$

where the second functor is the coproduct in Span(Fin) (given by disjoint unions of finite sets). Since the ∞-categories $\mathcal{O}^{\otimes}_{\langle 0\rangle}$ and $\mathcal{P}^{\otimes}_{\langle 0\rangle}$ are contractible, we obtain pullback squares

$$\begin{tikzcd}
\mathcal{O}^{\otimes} \arrow[r, "i_{\mathcal{O}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes} \arrow[d, "p_{\mathcal{O}}\times p_{\mathcal{P}}"] \\
\mathrm{Span}(\mathrm{Fin}) \arrow[r, "I\mapsto(I,\emptyset)"] & \mathrm{Span}(\mathrm{Fin})\times\mathrm{Span}(\mathrm{Fin})
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathcal{P}^{\otimes} \arrow[r, "i_{\mathcal{P}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes} \arrow[d, "p_{\mathcal{O}}\times p_{\mathcal{P}}"] \\
\mathrm{Span}(\mathrm{Fin}) \arrow[r, "I\mapsto(\emptyset,I)"] & \mathrm{Span}(\mathrm{Fin})\times\mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

Theorem 7.2.5 (cf. Lurie [Lur17, Theorem 2.2.3.6])**.** *The pair $\mathcal{O}\boxplus\mathcal{P} := (\mathcal{O}^{\otimes}\times\mathcal{P}^{\otimes}, p_{\mathcal{O}\boxplus\mathcal{P}})$ is an ∞-operad. Moreover, the maps $i_{\mathcal{O}}$ and $i_{\mathcal{P}}$ define morphisms of ∞-operads $i_{\mathcal{O}}\colon \mathcal{O}\to\mathcal{O}\boxplus\mathcal{P}$ and $i_{\mathcal{P}}\colon\mathcal{P}\to\mathcal{O}\boxplus\mathcal{P}$ that exhibit $\mathcal{O}\boxplus\mathcal{P}$ as a coproduct of $\mathcal{O}$ and $\mathcal{P}$ in* Op_{∞}.

Note that the anima of colors of $\mathcal{O}\boxplus\mathcal{P}$ is the disjoint union $\mathcal{O}^{\simeq}\sqcup\mathcal{P}^{\simeq}$ of the colors of $\mathcal{O}$ and the colors of $\mathcal{P}$. The multimorphism animae are given by

$$(\mathcal{O}\boxplus\mathcal{P})(\{x_i\}_{i\in I};y) = \begin{cases}\mathcal{O}(\{x_i\}_{i\in I};y) & y\in\mathcal{O}^{\simeq}\text{ and }x_i\in\mathcal{O}^{\simeq}\text{ for all }i\in I,\\ \mathcal{P}(\{x_i\}_{i\in I};y) & y\in\mathcal{P}^{\simeq}\text{ and }x_i\in\mathcal{P}^{\simeq}\text{ for all }i\in I,\\ \emptyset & \text{otherwise.}\end{cases}$$

Tensor products of ∞-operads

Given ∞-operads $\mathcal{O}$ and $\mathcal{P}$, there exists an ∞-operad $\mathcal{O}\otimes_{\mathrm{BV}}\mathcal{P}$ which encodes objects that admit compatible $\mathcal{O}$-algebra and $\mathcal{P}$-algebra structures.

Construction 7.2.6. We define the *tensor product functor* on Span(Fin) as the composite

$$\mathrm{Span}(\mathrm{Fin})\times\mathrm{Span}(\mathrm{Fin})\simeq\mathrm{Span}(\mathrm{Fin}\times\mathrm{Fin})\xrightarrow{\mathrm{Span}(\times)}\mathrm{Span}(\mathrm{Fin}),$$

where the second functor is induced by the cartesian product functor $\times\colon\mathrm{Fin}\times\mathrm{Fin}\to\mathrm{Fin}$.

Definition 7.2.7. Let $\mathcal{O}_1,\mathcal{O}_2$ and $\mathcal{P}$ be ∞-operads. A *bifunctor of ∞-operads* from $(\mathcal{O}_1,\mathcal{O}_2)$ to $\mathcal{P}$ is a product-preserving functor $f\colon\mathcal{O}_1^{\otimes}\times\mathcal{O}_2^{\otimes}\to\mathcal{P}^{\otimes}$ fitting in a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{O}_1^{\otimes}\times\mathcal{O}_2^{\otimes} \arrow[r, "f"] \arrow[d, "p_{\mathcal{O}_1}\times p_{\mathcal{O}_2}"'] & \mathcal{P}^{\otimes} \arrow[d, "p_{\mathcal{P}}"] \\
\mathrm{Span}(\mathrm{Fin})\times\mathrm{Span}(\mathrm{Fin}) \arrow[r, "\otimes"] & \mathrm{Span}(\mathrm{Fin}).
\end{tikzcd}$$

We denote by

$$\mathrm{BiFun}((\mathcal{O}_1,\mathcal{O}_2);\mathcal{P})\subseteq\mathrm{Fun}_{/\mathrm{Span}(\mathrm{Fin})}(\mathcal{O}_1^{\otimes}\times\mathcal{O}_2^{\otimes},\mathcal{P}^{\otimes})$$

the full subcategory spanned by the bifunctors of ∞-operads.

Informally speaking, a bifunctor $f\colon(\mathcal{O}_1,\mathcal{O}_2)\to\mathcal{P}$ consists of the following data:

- For every pair (x,y) of colors $x\in\mathcal{O}_1^{\simeq}$ and $y\in\mathcal{O}_2^{\simeq}$ we get a color $f(x,y)\in\mathcal{P}^{\simeq}$;

- For every pair (φ, ψ) of multimorphisms $\varphi \in O_1(\{x_i\}_{i\in I}; z)$ and $\psi \in O_2(\{y_j\}_{j\in J}; w)$, we get a multimorphism
$$f(\varphi, \psi) \in \mathcal{P}(\{f(x_i, y_j)\}_{(i,j)\in I\times J}; f(z, w)).$$
- The assignment $(\varphi, \psi) \mapsto f(\varphi, \psi)$ preserves composition of multimorphisms in φ as well as composition of multimorphisms in ψ.

Definition 7.2.8. We say that a bifunctor $f\colon (O_1, O_2) \to \mathcal{P}$ of ∞-operads *exhibits $\mathcal{P}$ as a tensor product of O_1 and O_2* if for every other ∞-operad Q, precomposition with f induces an equivalence of ∞-categories
$$f^*\colon \mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{P}, Q) \xrightarrow{\sim} \mathrm{BiFun}((O_1, O_2); Q).$$

Remark 7.2.9. More generally, if I is a finite set and if $\{O_i\}_{i\in I}$ is an I-indexed family of ∞-operads, we may define a notion of a *multimorphism of ∞-operads* $f\colon \{O_i\}_{i\in I} \to \mathcal{P}$. For $I = \langle 1\rangle$, this simply reduces to the notion of a morphism of ∞-operads. It is not difficult to see that such multimorphisms compose, and that they equip the ∞-category of ∞-operads with the structure of an ∞-operad.

Theorem 7.2.10 ([Lur17, Proposition 2.2.5.6])**.** *Let O_1 and O_2 be ∞-operads. Then there exists a tensor product $O_1 \otimes_{\mathrm{BV}} O_2$ of O_1 and O_2.*

In fact, the ∞-operad of multimorphisms of ∞-operads corresponds to a closed symmetric monoidal structure $(\mathrm{Op}_\infty, \otimes_{\mathrm{BV}})$.

This tensor product of operads is classically known as the *Boardman-Vogt tensor product* and was first written down by Boardman and Vogt [BV73]. Informally speaking, $O_1 \otimes_{\mathrm{BV}} O_2$ is the ∞-operad whose colors are pairs (x, y) of colors of O_1 and O_2, and whose multimorphisms are 'freely generated' by multimorphisms of the form $\varphi \otimes y\colon \{(x_i, y)\}_{i\in I} \to (z, y)$ for $\varphi \in O_1(\{x_i\}_{i\in I}; z)$ and $x \otimes \psi\colon \{(x, y_j)\}_{j\in J} \to (x, w)$ for $\psi \in O_2(\{y_j\}_{j\in J}; w)$.

The fact that $(\mathrm{Op}_\infty, \otimes_{\mathrm{BV}})$ is closed symmetric monoidal means that for ∞-operads O and $\mathcal{P}$ there exists an internal hom $\mathcal{F}un_{\mathrm{Op}_\infty}(O, \mathcal{P})$ which comes with an equivalence
$$\mathrm{Fun}_{\mathrm{Op}_\infty}(Q \otimes_{\mathrm{BV}} O, \mathcal{P}) \simeq \mathrm{Fun}_{\mathrm{Op}_\infty}(Q, \mathcal{F}un_{\mathrm{Op}_\infty}(O, \mathcal{P}))$$
for every ∞-operad Q. If $\mathcal{P} = \mathcal{M}_C$ is a multimorphism operad of a symmetric monoidal ∞-category, we set
$$\mathcal{A}lg_O(C) := \mathcal{F}un_{\mathrm{Op}_\infty}(O, \mathcal{M}_C),$$
so that we get an equivalence
$$\mathrm{Alg}_{\mathcal{P}}(\mathcal{A}lg_O(C)) \simeq \mathrm{Alg}_{\mathcal{P}\times_{\mathrm{BV}} O}(C).$$

The tensor product on Op_∞ admits the following interaction with the product on Cat_∞ and the tensor product on $\mathrm{Cat}_\infty^\otimes$:

Proposition 7.2.11 ([Lur17, Proposition 2.2.5.15])**.** *The trivial operad functor $\mathcal{T}riv\colon \mathrm{Cat}_\infty \to \mathrm{Op}_\infty$ can be refined to a symmetric monoidal functor*
$$\mathcal{T}riv\colon (\mathrm{Cat}_\infty, \times) \to (\mathrm{Op}_\infty, \otimes_{\mathrm{BV}}).$$
In particular, the tensor product of two trivial ∞-operads $\mathcal{T}riv_C$ and $\mathcal{T}riv_D$ is $\mathcal{T}riv_{C\times D}$.

Proposition 7.2.12 ([BS23, Theorem E])**.** *The operadic envelope functor* $\mathrm{Env}\colon \mathrm{Op}_\infty \to \mathrm{Cat}_\infty^\otimes$ *can be refined to a symmetric monoidal functor*

$$\mathrm{Env}\colon (\mathrm{Op}_\infty, \otimes_{\mathrm{BV}}) \to (\mathrm{Cat}_\infty^\otimes, \otimes).$$

Finally, we record the famous *Dunn additivity* theorem, proved in the classical setting by Dunn [Dun88] and Fiedorowicz and Vogt [FV15]:

Theorem 7.2.13 (Dunn additivity, Lurie [Lur17, Theorem 5.1.2.2])**.** *For* $n, m \geq 0$*, there exists a bifunctor* $(\mathcal{E}_n, \mathcal{E}_m) \to \mathcal{E}_{n+m}$ *which exhibits* $\mathcal{E}_{n+m}$ *as a tensor product of* $\mathcal{E}_n$ *and* $\mathcal{E}_m$*.*

In particular, since $\mathcal{E}_1 \simeq \mathcal{A}ssoc$, we see that $\mathcal{E}_n$ is the n-fold tensor power of the associative operad $\mathcal{A}ssoc$ for all $n \geq 1$. Consequently, we may think of $\mathcal{E}_n$-algebras as objects that admit n compatible associative algebra structures.

Part II

Higher algebra

8 Algebras and modules

Having established the symmetric monoidal structure on the ∞-category of spectra, the remainder of this course will be devoted to the study of the resulting 'higher algebra' of spectra. In this chapter, we lay the abstract groundwork, developing the general theory of algebras and modules in an arbitrary symmetric monoidal ∞-category C that will be specialized to the spectral setting in Chapter 9. We begin by formally defining left modules over an associative algebra in C using the left module operad, $\mathcal{LM}od$. We then establish some of the most fundamental constructions from classical algebra in this general context, like the free-forgetful adjunction between modules and their underlying objects. A central part of the chapter is dedicated to the relative tensor product, $M \otimes_A N$, which is key to defining the restriction and extension of scalars functors associated with a morphism of algebras.

The theory developed in this chapter is deep and often technically demanding. For reasons of time and space, we will frequently refer to Jacob Lurie's foundational text, Higher Algebra, where these topics are treated in exhaustive detail.

8.1 Basic definitions and properties

In order to talk about modules over associative algebras, recall from Example 1.2.8 the 'left module operad' $\mathcal{LM}od$. It comes equipped with the following morphisms of ∞-operads:

$$\begin{tikzcd} \mathcal{A}ssoc \arrow[r, hook, "\mathfrak{a}"] \arrow[dr, equal] & \mathcal{LM}od \arrow[d] & \mathcal{T}riv \arrow[l, hook', "\mathfrak{m}"'] \\ & \mathcal{A}ssoc. & \end{tikzcd}$$

We may describe their total ∞-categories $\mathcal{A}ssoc^{\otimes}$ and $\mathcal{LM}od^{\otimes}$ as follows:

- The objects of $\mathcal{A}ssoc^{\otimes}$ are finite sets I. A morphism from I to J consists of a span
$$I \xleftarrow{f} K \xrightarrow{g} J$$
of finite sets together with a collection of linear orders $(\preceq_j)_{j\in J}$ on the fibers $g^{-1}(j)$. Composition is given by composition of spans and the canonical ordering.

- The objects of $\mathcal{LM}od^{\otimes}$ are pairs (I_A, I_M) of finite sets. A morphism from (I_A, I_M) to (J_A, J_M) is a span
$$I_A \sqcup I_M \xleftarrow{f} K \xrightarrow{g} J_A \sqcup J_M$$

together with a collection of linear orders $(\preceq_j)_{j\in J_A}$ on the fibers $g^{-1}(j)$, such that g induces a bijection between $f^{-1}(I_M)$ and J_M.

The inclusion $\mathfrak{a}\colon \mathcal{A}ssoc^{\otimes} \hookrightarrow \mathcal{LM}od^{\otimes}$ sends I to $(I, \emptyset)$, while the inclusion $\mathfrak{m}\colon \mathcal{T}riv^{\otimes} \hookrightarrow \mathcal{LM}od^{\otimes}$ sends I to $(\emptyset, I)$. The map $\mathcal{LM}od^{\otimes} \to \mathcal{A}ssoc^{\otimes}$ sends (I_A, I_M) to $I_A \sqcup I_M$.

Definition 8.1.1. Let C be a symmetric monoidal ∞-category. We denote by

$$\mathrm{LMod}(C) := \mathrm{Alg}_{\mathcal{LM}od}(C)$$

the ∞-category of $\mathcal{LM}od$-algebras in C. Restriction along $\mathfrak{a}$ defines a functor $\mathfrak{a}^*\colon \mathrm{LMod}(C) \to \mathrm{Alg}(C) := \mathrm{Alg}_{\mathcal{A}ssoc}(C)$. For an associative algebra $A \in \mathrm{Alg}(C)$ we define the ∞-category $\mathrm{LMod}_A(C)$ of *left modules over A* as the fiber

$$\begin{tikzcd}
\mathrm{LMod}_A(C) \arrow[r] \arrow[d] & \mathrm{LMod}(C) \arrow[d, "\mathfrak{a}^*"] \\
* \arrow[r, "A"] & \mathrm{Alg}(C).
\end{tikzcd}$$

Definition 8.1.2. An *associative ring spectrum* is an associative algebra of the symmetric monoidal ∞-category Sp. A *commutative ring spectrum* is a commutative algebra of Sp. We write

$$\mathrm{Alg} := \mathrm{Alg}(\mathrm{Sp}) \qquad \text{and} \qquad \mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$$

for the resulting ∞-categories. For an associative ring spectrum A, we write

$$\mathrm{LMod}_A := \mathrm{LMod}_A(\mathrm{Sp})$$

for its ∞-category of left modules.

Remark 8.1.3. We will often drop the word ‘spectrum’ and simply say ‘algebras’ and ‘commutative algebras’ when it is clear from context that we are working with spectra. This is especially convenient in the relative setting when talking about A-algebras for a ring spectrum A.

Remark 8.1.4. Because of the adjunction $\mathrm{Env}\colon \mathrm{Op}_\infty \rightleftarrows \mathrm{Cat}_\infty^{\otimes}$, an associative algebra contains the same data as that of a symmetric monoidal functor $\mathrm{Env}(\mathcal{A}ssoc) \to C$. This operadic envelope is equivalent to the category whose objects are finite sets I and whose morphisms $I \to J$ are pairs $(f, \preceq_j)$ of a map $f\colon I \to J$ and a collection of linear orders $\preceq_j$ on each of the preimages $f^{-1}(j)$ of f. The symmetric monoidal structure is disjoint union. In particular, if A is an associative algebra in C, then the maps $\langle 0\rangle \to \langle 1\rangle$ and $\langle 2\rangle \to \langle 1\rangle$ corresponding to the canonical ordering $\{1 \le 2\}$ provide structure maps

$$e\colon \mathbb{1} \to A \qquad \text{and} \qquad m\colon A\otimes A \to A.$$

Furthermore, the following two diagrams commute:

$$\begin{tikzcd}
A\otimes \mathbb{1} \arrow[r, "\mathrm{id}_A\otimes e"] \arrow[dr, "\simeq"'] & A\otimes A \arrow[d, "m"] & A \arrow[l, "e\otimes \mathrm{id}_A"'] \arrow[dl, "\simeq"] \\
& A, &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes m"'] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A.
\end{tikzcd}$$

Similarly, a module over A is given by a symmetric monoidal functor $\mathrm{Env}(\mathcal{LM}od) \to C$ which on $\mathrm{Env}(\mathcal{A}ssoc)$ restricts to A. The map $(\langle 1\rangle, \langle 1\rangle) \to (\emptyset, \langle 1\rangle)$ in $\mathrm{Env}(\mathcal{LM}od)$ coming from the fold map $\langle 1\rangle \sqcup \langle 1\rangle \to \langle 1\rangle$ gives rise to a map

$$\mathrm{act}\colon A \otimes M \to M,$$

and the relations in $\mathrm{Env}(\mathcal{LM}od)$ ensure that the following two diagrams commute:

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "1\otimes \mathrm{id}_M"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\mathrm{act}"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{id}_A \otimes \mathrm{act}"] \arrow[d, "m\otimes \mathrm{id}_M"'] & A \otimes M \arrow[d, "\mathrm{act}"] \\ A \otimes M \arrow[r, "\mathrm{act}"] & M. \end{tikzcd}$$

For future purposes, it will be convenient to introduce left modules the more general setting of ∞-categories that are 'left-tensored' over C.

Definition 8.1.5. Let C be a monoidal ∞-category. We say that an ∞-category D is *left-tensored over* C if there exists an $\mathcal{LM}od$-monoidal ∞-category, given by a cocartesian fibration of ∞-operads $p_D\colon D^{\otimes} \to \mathcal{LM}od^{\otimes}$, satisfying the following two properties:

- The underlying monoidal ∞-category $\mathfrak{a}^*(p_D)\colon \mathfrak{a}^*(D^{\otimes}) \to \mathcal{A}ssoc^{\otimes}$ is equivalent to C;
- The underlying $\mathcal{T}riv$-monoidal ∞-category $\mathfrak{m}^*(p_D)\colon \mathfrak{m}^*(D^{\otimes}) \to \mathcal{T}riv^{\otimes}$ corresponds to D under the equivalence $\mathrm{Alg}_{\mathcal{T}riv}(\mathrm{Cat}_\infty) \simeq \mathrm{Cat}_\infty$.

Definition 8.1.6. Let D be left-tensored over C. Pullback along $\mathfrak{a}$ defines a forgetful functor $\mathfrak{a}^*\colon \mathrm{Alg}_{\mathrm{LMod}}(D^{\otimes}) \to \mathrm{Alg}_{\mathcal{A}ssoc}(C) = \mathrm{Alg}(C)$. Given an associative algebra $A \in \mathrm{Alg}(C)$, we define the ∞-category $\mathrm{LMod}_A(D)$ of A-modules in D as the fiber of this functor over A.

Remark 8.1.7. By pulling back along the operad map $\mathcal{LM}od \to \mathcal{A}ssoc$, every monoidal ∞-category C may be canonically regarded as left-tensored over itself. In this case, Definition 8.1.6 reduces to Definition 8.1.1.

Remark 8.1.8. All the definitions we just made also make sense for modules over *commutative* algebras, where we would replace the ∞-operad $\mathcal{LM}od$ by its commutative version $\mathcal{M}od$ from Example 1.2.7. It sits in a commutative diagram of ∞-operads as follows:

$$\begin{tikzcd} & \mathcal{A}ssoc \arrow[r] \arrow[d, hook] & \mathcal{C}omm \arrow[d, hook] \\ \mathcal{T}riv \arrow[r, hook] & \mathcal{LM}od \arrow[r] \arrow[d] & \mathcal{M}od \arrow[d] \\ & \mathcal{A}ssoc \arrow[r] & \mathcal{C}omm. \end{tikzcd}$$

In particular, restriction along the inclusion $\mathcal{C}omm \hookrightarrow \mathcal{M}od$ defines a forgetful functor $\mathrm{Mod}(C) := \mathrm{Alg}_{\mathcal{M}od}(C) \to \mathrm{CAlg}(C)$, allowing us to define an ∞-category $\mathrm{Mod}_A(C)$ for every commutative algebra A in C. It turns out this ∞-category only depends on the underlying associative algebra of A:

Proposition 8.1.9. *Let C be a symmetric monoidal ∞-category. Then the following commutative square is a pullback square:*

$$\begin{tikzcd} \mathrm{Mod}(C) \arrow[r, "-|_{\mathcal{LM}od}"] \arrow[d, "-|_{\mathcal{C}omm}"'] & \mathrm{LMod}(C) \arrow[d, "-|_{\mathcal{A}ssoc}"] \\ \mathrm{CAlg}(C) \arrow[r, "-|_{\mathcal{A}ssoc}"] & \mathrm{Alg}(C). \end{tikzcd}$$

In particular, passing to vertical fibers over $R \in \mathrm{CAlg}(C)$ induces an equivalence

$$\mathrm{Mod}_R(C) \xrightarrow{\sim} \mathrm{LMod}_{R'}(C),$$

where we denote the underlying associative algebra of R by R'.

Proof. This follows by combining Glasman [Gla14, Proposition 7] with [Lur17, Proposition 4.5.1.4]. □

8.1.1 Trivial algebras

We will now show that the monoidal unit $\mathbb{1} \in C$ is always a commutative algebra, and modules over it are simply objects of C. In fact, we may do this more generally for *unital ∞-operads*:

Definition 8.1.10. An ∞-operad $\mathcal{O}$ is called *unital* if its total ∞-category $\mathcal{O}^{\otimes}$ is pointed, i.e., if the terminal object (given by the empty tuple $\emptyset_{\mathcal{O}} := \{\}$) is also an initial object. Equivalently, $\mathcal{O}$ is unital if we have

$$\mathcal{O}(\{\}; x) \simeq *$$

for every color $x \in \mathcal{O}^{\simeq}$.

Example 8.1.11. The ∞-operads $\mathcal{C}omm$, $\mathcal{A}ssoc$ and $\mathcal{E}_k$ are unital. The ∞-operads $\mathcal{T}riv$, $\mathcal{M}od$ and $\mathcal{LM}od$ are not unital.

Example 8.1.12. If $\mathcal{O}$ and $\mathcal{P}$ are unital ∞-operads, then so are their product $\mathcal{O} \times \mathcal{P}$ and tensor product $\mathcal{O} \otimes_{\mathrm{BV}} \mathcal{P}$.

Definition 8.1.13. Let $\mathcal{O}$ be a unital ∞-operad and let $C \colon \mathcal{O}^{\otimes} \to \mathrm{Cat}_{\infty}$ be an $\mathcal{O}$-monoidal ∞-category. For every color $x \in \mathcal{O}^{\simeq}$, the unique multimorphism $e_x \in \mathcal{O}(\{\}; x)$ defines a functor

$$\mathbb{1}_x \colon * \simeq C(\{\}) \to C(\{x\}) = C_x.$$

We refer to the corresponding object $\mathbb{1}_x \in C_x$ as the *monoidal unit of C_x*.

Note that every $\mathcal{O}$-algebra A comes equipped with 'unit maps' $e_x \colon \mathbb{1}_x \to A_x$ in C_x for every color $x \in \mathcal{O}^{\simeq}$: this map is the unique lift of the map $\{\} \to \{A_x\}$ along the cocartesian morphism $\{\} \to \{\mathbb{1}_x\}$. The following result by Lurie shows that, convesely, the monoidal units $\mathbb{1}_x \in C_x$ can always be assembled into an $\mathcal{O}$-algebra $\mathbb{1}_C$ for which the maps $\mathbb{1}_x \to (\mathbb{1}_C)_x$ are isomorphisms:

Proposition 8.1.14 ([Lur17, Proposition 3.2.1.8])**.** *Let $\mathcal{O}$ be a unital ∞-operad and let C be an $\mathcal{O}$-monoidal ∞-category. Then the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(C)$ of $\mathcal{O}$-algebras in C admits an initial object $\mathbb{1}_C \in \mathrm{Alg}_{/\mathcal{O}}(C)$. Furthermore, an $\mathcal{O}$-algebra is initial if and only if the unit maps $e_x \colon \mathbb{1}_x \to A_x$ are isomorphisms for all $x \in \mathcal{O}^{\simeq}$.*

The proof for this uses the theory of operadic left Kan extensions, which I do not want to introduce at this point as that would take too much time. But let me at least give the construction of $\mathbb{1}_C$ as an O-algebra of C, following an argument I learned from Ferdinand Wagner [HW21, Lemma II.45a]. It will suffice to define a cocartesian functor

$$\begin{tikzcd} O^{\otimes} \arrow[r, dashed, "\mathbb{1}_C"] \arrow[dr, equal] & C^{\otimes} \arrow[d, "p_C"] \\ & O^{\otimes} \end{tikzcd}$$

of cocartesian fibrations over $O^{\otimes}$. By straightening-unstraightening, we may equivalently define a natural transformation of functors $\mathrm{const}_* \to \mathrm{Str}^{\mathrm{cc}}(p_C)$ of functors $O^{\otimes} \to \mathrm{Cat}_\infty$. Since $O^{\otimes}$ admits an initial object $\emptyset_O$, there is a natural transformation $\mathrm{const}_{\emptyset_O} \to \mathrm{id}_{O^{\otimes}}$ of functors $O^{\otimes} \to O^{\otimes}$. As the fiber of p_C over $\emptyset_O$ is the terminal ∞-category, composing this transformation with $\mathrm{Str}^{\mathrm{cc}}(p_C)$ gives the desired map $\mathrm{const}_* \to \mathrm{Str}^{\mathrm{cc}}(p_C)$. It is clear from this construction that this O-algebra $\mathbb{1}_C$ satisfies the condition that the maps $\mathbb{1}_x \to (\mathbb{1}_C)_x$ are isomorphisms.

Corollary 8.1.15 ([Lur17, Corollary 3.2.1.9])**.** *Let C be a symmetric monoidal ∞-category. Then a commutative algebra R is initial in $\mathrm{CAlg}(C)$ if and only if the unit map $e\colon \mathbb{1} \to R$ is an equivalence in C. In particular, the underlying object of the commutative algebra $\mathbb{1}_C \in \mathrm{CAlg}(C)$ from Proposition 8.1.14 is the monoidal unit of C.*

8.1.2 Free modules

Let A be an associative ring and let M_0 be an abelian group. The tensor product $M := A \otimes M_0$ admits the structure of a left A-module, given by

$$A \otimes M = A \otimes (A \otimes M_0) \simeq (A \otimes A) \otimes M_0 \xrightarrow{m \otimes M_0} A \otimes M_0 = M.$$

Moreover, M comes equipped with a morphism $\varphi\colon M_0 \simeq \mathbb{1} \otimes M_0 \xrightarrow{e \otimes M_0} A \otimes M_0 = M$ that exhibits M as the *free* A-module on M_0: for every other left A-module N, composition with φ induces a bijection $\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ab}}(M_0, N)$, where $\mathrm{Hom}_A(M, N)$ is the set of A-module homomorphisms from M to N.

This admits the following ∞-categorical refinement:

Proposition 8.1.16 ([Lur17, Proposition 4.2.4.2, Corollary 4.2.4.4])**.** *Let C be a monoidal ∞-category and let D be an ∞-category left tensored over C.*

(1) *The forgetful functor* $\mathrm{LMod}(D) \xrightarrow{(\mathfrak{a}^*, \mathfrak{m}^*)} \mathrm{Alg}(C) \times D$ *admits a left adjoint*

$$F\colon \mathrm{Alg}(D) \times C \to \mathrm{LMod}(D), \qquad (A, M_+) \mapsto F_A(M_0)$$

satisfying $\mathfrak{a}^* F_A(M_0) \xrightarrow{\sim} A$ *and* $\mathfrak{m}^* F_A(M_0) \xrightarrow{\sim} A \otimes M_0$.

(2) *For an associative algebra $A \in \mathrm{Alg}(C)$, the forgetful functor $\mathfrak{m}^*\colon \mathrm{LMod}_A(D) \to D$ admits a left adjoint*

$$F_A\colon D \to \mathrm{LMod}_A(D),$$

where the underlying object of $F_A(M_0)$ is $A \otimes M_0$.

Remark 8.1.17. Let $C = D = \mathrm{Sp}$, and take M_0 to be the sphere spectrum. Then we see that for a ring spectrum R, its underlying spectrum can be equipped with a canonical structure of a left R-module which satisfies the property that $\mathrm{Hom}_{\mathrm{LMod}_R}(R, M) \simeq \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, M)$.

As a consequence, we may deduce that modules over a trivial algebra are simply objects of D:

Corollary 8.1.18 ([Lur17, Proposition 4.2.4.9])**.** *Let C be a monoidal ∞-category and let $A \in \mathrm{Alg}(C)$ be an algebra object such that the unit map $e\colon \mathbb{1} \to A$ is an isomorphism in C. Then for every left C-tensored ∞-category D the forgetful functor $\mathrm{LMod}_A(D) \to D$ is an equivalence of ∞-categories.*

Proof. By the previous proposition, the forgetful functor admits a left adjoint $F_A\colon D \to \mathrm{LMod}_A(D)$. Since the forgetful functor is conservative, it suffices to show that the unit map $M_0 \to A \otimes M_0$ is an isomorphism. Since this map is given by tensoring the map $e\colon \mathbb{1} \to A$ by M_0, this follows from the assumption on A. □

8.2 A simplicial model for algebras and modules

While the operadic definition of algebras and modules is conceptually powerful and places them within a unified framework, they admit an alternative description in terms of simplicial objects that is more concrete and computationally convenient. This alternative perspective will allow us to directly analyze how limits and colimits behave in module categories by relating them to limits and colimits in the underlying category. We will now explain this in more detail, referring to Lurie's work for proofs.

8.2.1 Associative algebras as simplicial objects

Recall from last semester the functor

$$\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Span}(\mathrm{Fin}) = \mathcal{C}omm^{\otimes}, \qquad P \mapsto \mathrm{Cut}(P),$$

where for a non-empty finite linear order P we write $\mathrm{Cut}(P)$ for the set of *Dedekind cuts of P*, defined as non-trivial partitions (P_0, P_1) of P satisfying $p_0 \le p_1$ for all $p_0 \in P_0$ and $p_1 \in P_1$. Given a morphism $\varphi\colon P \to Q$ of linear orders, the span $\mathrm{Cut}(\varphi)$ of finite sets is given as:

$$\mathrm{Cut}(Q) \hookleftarrow \{(Q_0, Q_1) \mid \varphi^{-1}(Q_0) \neq \emptyset, \varphi^{-1}(Q_1) \neq \emptyset\} \xrightarrow{(Q_0,Q_1)\mapsto(\varphi^{-1}(Q_0),\varphi^{-1}(Q_1))} \mathrm{Cut}(P).$$

Note that $\mathrm{Cut}(P)$ admits a canonical ordering by saying that $(P_0, P_1) \le (P_0', P_1')$ if and only if $P_0 \subseteq P_0'$. In particular, we obtain canonical orderings on each of the fibers of the right-pointing map in the span $\mathrm{Cut}(\varphi)$, so that Cut canonically refines to a functor

$$\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{A}ssoc^{\otimes}.$$

Definition 8.2.1. Let C be a monoidal ∞-category. We define an *$\mathbb{A}_\infty$-algebra object of C* to be a commutative diagram of the form

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "A"] \arrow[dr, "\mathrm{Cut}"'] & C^{\otimes} \arrow[d, "p_C"] \\ & \mathcal{A}ssoc^{\otimes} \end{tikzcd}$$

such that A sends (the dual in $\mathbf{\Delta}^{\mathrm{op}}$ of) each map $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ in $\mathbf{\Delta}$ to an inert morphism $A_n \to A_1$ in $C^{\otimes}$. We denote by

$$\mathrm{Alg}_{\mathbb{A}_\infty}(C) \subseteq \mathrm{Fun}_{/\mathcal{A}ssoc^{\otimes}}(\mathbf{\Delta}^{\mathrm{op}}, C^{\otimes})$$

the full subcategory spanned by the $\mathbb{A}_\infty$-algebras in C.

Note that Cut sends e_i to the span $\langle n\rangle \hookleftarrow \{i\} \xrightarrow{=} \{i\}$, so in essence the condition is demanding that $A_n \in C^{\otimes}_n \simeq C^n$ is the n-tuple all of whose components are identified with A_1.

Theorem 8.2.2 ([Lur17, Proposition 4.1.3.19])**.** *Let C be a monoidal ∞-category. Then restriction along* $\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{A}ssoc^{\otimes}$ *induces an equivalence of ∞-categories*

$$\mathrm{Cut}^*\colon \mathrm{Alg}(C) = \mathrm{Alg}_{\mathcal{A}ssoc}(C) \xrightarrow{\sim} \mathrm{Alg}_{\mathbb{A}_\infty}(C).$$

While we will not prove this theorem, this description of associative algebras is perhaps not surprising: for cartesian monoidal structures this was precisely how we defined monoid objects in C last semester!

8.2.2 Left modules as simplicial objects

The simplicial model extends nicely to describe left modules over an associative algebra. For this, we need the following analogue of Cut:

Construction 8.2.3. We define a functor $\mathrm{LCut}\colon \mathbf{\Delta}^{\mathrm{op}} \times [1] \to \mathcal{LM}od^{\otimes}$ as the following composite:

$$\mathbf{\Delta}^{\mathrm{op}} \times [1] \xrightarrow{\mathrm{Cut}\times[1]} \mathcal{A}ssoc^{\otimes} \times [1] \to \mathcal{LM}od^{\otimes},$$

where the second functor sends $(I,1)$ to $(I,\emptyset)$ and $(I,0)$ to $(I,\{m\})$, with the maps $(I,\{m\}) \to (I,\emptyset)$ given by the span $(I,\{m\}) \hookleftarrow (I,\emptyset) \xrightarrow{=} (I,\emptyset)$.

Definition 8.2.4. Let C be a monoidal ∞-category. We define an $\mathbb{A}_\infty$*-module in C* to be a commutative diagram of the form

$$\begin{tikzcd} \mathbf{\Delta}^{\mathrm{op}} \times [1] \arrow[r, "F"] \arrow[d, "\mathrm{LCut}"'] & C^{\otimes} \arrow[d, "p_C"] \\ \mathcal{LM}od^{\otimes} \arrow[r] & \mathcal{A}ssoc^{\otimes} \end{tikzcd}$$

such that the functor F satisfies the following conditions:

(1) The restriction $F|_{\mathbf{\Delta}^{\mathrm{op}}\times\{1\}}\colon \mathbf{\Delta}^{\mathrm{op}} \to C^{\otimes}$ is an $\mathbb{A}_\infty$-algebra in C;

(2) The morphism $F([n],0) \to F([n],1)$ is an inert morphism in $C^{\otimes}$ for all $[n] \in \mathbf{\Delta}^{\mathrm{op}}$;

(3) If $\alpha\colon [n] \to [m]$ is an inert[1] morphism in $\mathbf{\Delta}$ satisfying $\alpha(n) = m$, then the induced morphism $F(\alpha^{\mathrm{op}}, \mathrm{id}_0)\colon F([m],0) \to F([n],0)$ is inert in $C^{\otimes}$.

We write $\mathrm{LMod}^{\mathbb{A}_\infty}(C) \subseteq \mathrm{Fun}_{/\mathcal{A}ssoc^{\otimes}}(\mathbf{\Delta}^{\mathrm{op}} \times [1], C^{\otimes})$ for the resulting full subcategory.

[1]An inert morphism in $\mathbf{\Delta}$ is the inclusion of an interval, i.e. an injective order-preserving map $\alpha\colon [n] \hookrightarrow [m]$ satisfying $\alpha(i+1) = \alpha(i)+1$.

Theorem 8.2.5 (Lurie [Lur17, Proposition 4.2.2.12])**.** *Let C be a monoidal ∞-category. Then restriction along* LCut *defines an equivalence of ∞-categories*

$$\mathrm{LCut}^*\colon \mathrm{LMod}(C) = \mathrm{Alg}_{\mathcal{LM}od}(C) \xrightarrow{\sim} \mathrm{LMod}^{\mathbb{A}_\infty}(C).$$

Again, we will not prove this theorem, but we will give some intuitions. Given such a functor $F\colon \mathbf{\Delta}^{\mathrm{op}} \times [1] \to C^{\otimes}$, we obtain objects $A := F([1], 1)$ and $M := F([0], 0)$. Condition (1) guarantees that the restriction of F to $\mathbf{\Delta}^{\mathrm{op}} \times \{1\}$ corresponds to an algebra structure on A. Conditions (2) and (3) are saying that the object $F([n], 0) \in C^{\otimes}_{n+1} \simeq C^{n+1}$ is of the form $(A, \dots, A, M)$. The restriction $F|_{\mathbf{\Delta}^{\mathrm{op}} \times \{0\}}\colon \mathbf{\Delta}^{\mathrm{op}} \to C^{\otimes}$ thus encodes the 'action maps' $A^{\otimes n} \otimes M \to A^{\otimes m} \otimes M$.

8.2.3 Limits and colimits of modules

This simplicial model for modules has a powerful consequence: it allows us to understand how limits and colimits behave in module categories.

Corollary 8.2.6 ([Lur17, Corollary 4.2.3.3])**.** *Let C be a symmetric monoidal ∞-category and $A \in \mathrm{Alg}(C)$ an associative algebra. Let $\mathfrak{m}^*\colon \mathrm{LMod}_A(C) \to C$ be the forgetful functor, which sends a module to its underlying object in C. Let I be a small ∞-category.*

1. *If C admits I-indexed limits, then the forgetful functor $\mathfrak{m}^*$ creates and preserves I-indexed limits.*
2. *If C admits I-indexed colimits and the tensor product functor $- \otimes_C X\colon C \to C$ preserves I-indexed colimits for all $X \in C$, then the forgetful functor $\mathfrak{m}^*$ creates and preserves small colimits.*

Lurie's proof for this uses relative Kan extensions, and I will not explain it. Heuristically, what is happening is that we are given a diagram $M_\bullet\colon I \to \mathrm{LMod}_A(C)$ of which we wish to construct a limit or colimit. By Theorem 8.2.5, this corresponds to a diagram of functors $M_\bullet\colon I \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}} \times [1], C^{\otimes})$ with compatible identifications $M_i|_{\mathbf{\Delta}^{\mathrm{op}} \times \{1\}} \cong A$. Informally, this means M_i is given by

$$\begin{aligned} M_i\colon \mathbf{\Delta}^{\mathrm{op}} \times [1] &\to C^{\otimes} \\ ([n], 0) &\mapsto (A, \dots, A, M_i) \\ ([n], 1) &\mapsto (A, \dots, A). \end{aligned}$$

The limit and colimit of $M_\bullet$ should then be given by the similar functors, but where M_i is replaced by $\lim_i M_i$ or $\mathrm{colim}_i M_i$, respectively.

Corollary 8.2.7. *For any associative ring spectrum A, the ∞-category* LMod_A *of left A-modules is stable.*

Proof. The ∞-category of spectra Sp is stable, so it admits all finite limits and colimits. The tensor product functor $- \otimes X\colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits in both variables. By Corollary 8.2.6, the forgetful functor $\mathrm{LMod}_A \to \mathrm{Sp}$ creates and preserves finite limits and colimits. Furthermore, since commutative square in LMod_A is a pullback or pushout square if and only if its image in Sp is, the stability of Sp implies stability of LMod_A. $\square$

8.3 Relative tensor products and change of algebras

If R is a classical commutative ring, then the category $\mathrm{Mod}_R(\mathrm{Ab})$ of R-modules comes equipped with a symmetric monoidal structure given by the *relative tensor product* $M \otimes_R N$ of two R-modules M and N. Moreover, if $f: R \to S$ is a morphism of commutative rings, then the relative tensor product $S \otimes_R M$ admits the structure of an S-module for every R-module M, resulting in a symmetric monoidal functor $f^*: \mathrm{Mod}_R(\mathrm{Ab}) \to \mathrm{Mod}_S(\mathrm{Ab})$ which is left adjoint to the forgetful functor $\mathrm{Mod}_S(\mathrm{Ab}) \to \mathrm{Mod}_R(\mathrm{Ab})$.

The goal of this section is to introduce this structure in an arbitrary symmetric monoidal ∞-category that admits operadic geometric realizations.

8.3.1 Bimodules

Let A and B be ordinary associative algebras. Then we may speak of (A, B)-*bimodules*: objects M that are simultaneously a left A-module and a right B-module in a compatible way. This notion extends to the ∞-categorical setting.

Definition 8.3.1. We define a colored operad $\mathcal{BM}od$. It has three colors: $\mathfrak{a}_-$, $\mathfrak{a}_+$ and $\mathfrak{m}$. The multimorphism sets $\mathcal{BM}od(\{X_i\}_{i\in I}; Y)$ are then given as follows:

- If $Y = \mathfrak{a}_-$, then this set is empty unless $X_i = \mathfrak{a}_-$ for all $i \in I$, in which case it is the set of linear orders on I;
- If $Y = \mathfrak{a}_+$, then this set is empty unless $X_i = \mathfrak{a}_+$ for all $i \in I$, in which case it is the set of linear orders on I;
- If $Y = \mathfrak{m}$, then this set is the set of linear orders $\{i_1 < i_2 < \cdots < i_n\}$ on I with the following property: there exists precisely one index $i_k \in I$ such that $X_{i_k} = \mathfrak{m}$, $X_{i_j} = \mathfrak{a}_-$ for $j < k$ and $X_{i_j} = \mathfrak{a}_+$ for $j > k$.

Composition in $\mathcal{BM}od$ is given by composition of linear orders.

Observation 8.3.2. Note that we may identify the operad $\mathcal{LM}od$ with the full suboperad of $\mathcal{BM}od$ spanned by the colors $\mathfrak{a}_-$ and $\mathfrak{m}$. We let $\mathcal{RM}od \subseteq \mathcal{BM}od$ denote the full suboperad spanned by the colors $\mathfrak{m}$ and $\mathfrak{a}_+$, giving rise to a commutative diagram of operad maps:

$$
\begin{tikzcd}
\mathcal{A}ssoc \arrow[dr, hook, "\mathfrak{a}_-"] & & \mathcal{T}riv \arrow[dl, hook', "\mathfrak{m}"'] \arrow[dr, hook, "\mathfrak{m}"] & & \mathcal{A}ssoc \arrow[dl, hook', "\mathfrak{a}_+"'] \\
& \mathcal{LM}od \arrow[dd, equal] \arrow[dr, "\substack{\mathfrak{m}\mapsto\mathfrak{m} \\ \mathfrak{a}_+\mapsto\mathfrak{a}_+}"] & & \mathcal{RM}od \arrow[dd, equal] \arrow[dl, "\substack{\mathfrak{a}_-\mapsto\mathfrak{a}_- \\ \mathfrak{m}\mapsto\mathfrak{m}}"'] & \\
& & \mathcal{BM}od \arrow[dl, "\substack{\mathfrak{a}_-\mapsto\mathfrak{a}_- \\ \mathfrak{m},\mathfrak{a}_+\mapsto\mathfrak{m}}"'] \arrow[dr, "\substack{\mathfrak{a}_-,\mathfrak{m}\mapsto\mathfrak{m} \\ \mathfrak{a}_+\mapsto\mathfrak{a}_+}"] & & \\
& \mathcal{LM}od \arrow[dr, "\mathfrak{a}_-{,}\mathfrak{m}\mapsto\mathfrak{a}"'] & & \mathcal{RM}od \arrow[dl, "\mathfrak{m}{,}\mathfrak{a}_+\mapsto\mathfrak{a}"] & \\
& & \mathcal{A}ssoc. & &
\end{tikzcd}
$$

Notation 8.3.3. For a monoidal ∞-category we write

$$\mathrm{RMod}(C) := \mathrm{Alg}_{\mathcal{RMod}}(C) \qquad \text{and} \qquad \mathrm{BMod}(C) := \mathrm{Alg}_{\mathcal{BMod}}(C).$$

Similarly, if A and B are associative algebras in C, we write

$$\mathrm{RMod}_B(C) \qquad \text{and} \qquad {}_A\mathrm{BMod}_B(C)$$

for the fibers over B and (A, B) of the two forgetful functors $\mathfrak{a}_+^*\colon \mathrm{RMod}(C) \to \mathrm{Alg}(C)$ and $(\mathfrak{a}_-^*, \mathfrak{a}_+^*)\colon \mathrm{BMod}(C) \to \mathrm{Alg}(C) \times \mathrm{Alg}(C)$, respectively.

Observation 8.3.4. There is an operad map $\mathcal{BMod} \to \mathcal{Mod}$ which sends $\mathfrak{m}$ to $\mathfrak{m}$ and sends both $\mathfrak{a}_-$ and $\mathfrak{a}_+$ to $\mathfrak{a}$. Composing it with the inclusion $\mathcal{LMod} \hookrightarrow \mathcal{BMod}$ gives the canonical map $\mathcal{LMod} \to \mathcal{Mod}$, and similarly for $\mathcal{RMod}$. It follows that for a commutative algebra A in a symmetric monoidal ∞-category C, restriction along these operad maps define forgetful functors

$$\begin{tikzcd}
 & \mathrm{Mod}_A(C) \arrow[dl, "\sim"'] \arrow[d] \arrow[dr, "\sim"] & \\
\mathrm{LMod}_A(C) & {}_A\mathrm{BMod}_A(C) \arrow[l] \arrow[r] & \mathrm{RMod}_A(C).
\end{tikzcd}$$

Here the left diagonal equivalence is the one from Proposition 8.1.9, and the right one is dual.

Observation 8.3.5. There are equivalences of ∞-operads

$$\mathrm{rev}\colon \mathcal{Assoc} \xrightarrow{\sim} \mathcal{Assoc}, \qquad \mathrm{rev}\colon \mathcal{LMod} \xrightarrow{\sim} \mathcal{RMod} \qquad \text{and} \qquad \mathrm{rev}\colon \mathcal{BMod} \xrightarrow{\sim} \mathcal{BMod}$$

given on objects by $\mathfrak{a} \mapsto \mathfrak{a}$, $\mathfrak{a}_- \mapsto \mathfrak{a}_+$, $\mathfrak{a}_+ \mapsto \mathfrak{a}_-$ and $\mathfrak{m} \mapsto \mathfrak{m}$, and on multimorphisms by reversing the order on the finite set. For a symmetric monoidal ∞-category C, the functor $\mathrm{rev}^*\colon \mathrm{Alg}(C) \to \mathrm{Alg}(C)$ sends an associative algebra A to its *opposite algebra*, denoted A^{rev}, which has the same underlying object but whose multiplication is given by

$$A \otimes A \xrightarrow{\sim} A \otimes A \xrightarrow{m_A} A,$$

where the first map is the swap map. For associative algebras A and B we thus get equivalences

$$\mathrm{LMod}_A(C) \xrightarrow{\sim} \mathrm{RMod}_{A^{\mathrm{rev}}}(C) \qquad \text{and} \qquad {}_A\mathrm{BMod}_B(C) \xrightarrow{\sim} {}_{B^{\mathrm{rev}}}\mathrm{BMod}_{A^{\mathrm{rev}}}(C).$$

Informally speaking, an (A, B)-bimodule M comes equipped with structure maps

$$A^{\otimes n} \otimes M \otimes B^{\otimes m} \to M$$

for all $n, m \geq 0$ that are compatible with the multiplication maps of A and B. By setting $m = 0$ or $n = 0$ we obtain a left A-module structure and a right B-module structure on M, respectively. These structures are compatible with each other in a very strong sense: the A-module structure on M can be upgraded to an A-module structure internal to the ∞-category $\mathrm{RMod}_B(C)$ and vice versa:

Theorem 8.3.6 ([Lur17, Theorem 4.3.2.7])**.** *Let A and B be associative algebras in a monoidal ∞-category C.*

(1) *The ∞-category* $\mathrm{RMod}_B(C)$ *is left tensored over* C*, and there is an equivalence*

$$ {}_A\mathrm{BMod}_B(C) \simeq \mathrm{LMod}_A(\mathrm{RMod}_B(C)); $$

(2) *The ∞-category* $\mathrm{LMod}_B(C)$ *is right tensored over* C*, and there is an equivalence*

$$ {}_A\mathrm{BMod}_B(C) \simeq \mathrm{RMod}_B(\mathrm{LMod}_A(C)). $$

Corollary 8.3.7. *For associative algebras* A *and* B *in* C*, there are equivalences*

$$ {}_A\mathrm{BMod}_{\mathbb{1}}(C) \xrightarrow{\sim} \mathrm{LMod}_A(C) \qquad \textit{and} \qquad {}_{\mathbb{1}}\mathrm{BMod}_B(C) \xrightarrow{\sim} \mathrm{RMod}_B(C). $$

Proof. In light of the equivalences $\mathrm{RMod}_{\mathbb{1}}(C) \xrightarrow{\sim} C$ and $\mathrm{LMod}_{\mathbb{1}}(C) \xrightarrow{\sim} C$ from Corollary 8.1.18, this follows directly from the previous theorem. □

8.3.2 The relative tensor product

Let B be a discrete associative ring. A central construction in the theory of modules is the *relative tensor product*: Given a right B-module M and a left B-module N, their relative tensor product $M \otimes_B N$ is obtained as the quotient of $M \otimes N$ by identifying $mb \otimes n$ with $m \otimes bn$ for all $m \in M$, $n \in N$ and $b \in B$. In other words, $M \otimes_B N$ is the coequalizer of the following diagram:

$$ M \otimes B \otimes N \rightrightarrows M \otimes N. $$

In the ∞-categorical setting, this coequalizer needs to be replaced by the geometric realization of a simplicial diagram, called the *two-sided bar construction*. The construction is somewhat subtle, and is handled by Lurie in [Lur17, Construction 4.4.2.7]. We will first give the non-coherent description, and then indicate its coherent version in Remark 8.3.10 below.

Construction 8.3.8 (Bar construction)**.** Let B be an associative algebra in a monoidal ∞-category C, let M be a right B-module, and let N be a left B-module. The *two-sided bar construction* for M and N over B, denoted $\mathrm{Bar}_B(M,N)_\bullet$, is a simplicial object in C described as follows:

- In degree $[n] \in \Delta^{\mathrm{op}}$, it is given by $\mathrm{Bar}_B(M,N)_n := M \otimes B^{\otimes n} \otimes N$;
- The face maps $d_i\colon \mathrm{Bar}_B(M,N)_n \to \mathrm{Bar}_B(M,N)_{n-1}$ are given by:
  - $d_0 = \mathrm{act}_M \otimes \mathrm{id}_{B^{\otimes(n-1)}} \otimes \mathrm{id}_N\colon M \otimes B \otimes B^{\otimes(n-1)} \otimes N \to M \otimes B^{\otimes(n-1)} \otimes N.$
  - For $0 < i < n$, d_i is induced by the multiplication map $m\colon A \otimes A \to A$ on the i-th and $(i+1)$-th factors of $M^{\otimes n}$.
  - $d_n = \mathrm{id}_M \otimes \mathrm{id}_{B^{\otimes(n-1)}} \otimes \mathrm{act}_N\colon M \otimes B^{\otimes(n-1)} \otimes B \otimes N \to M \otimes B^{\otimes(n-1)} \otimes N.$
- The degeneracy maps $s_i\colon \mathrm{Bar}_B(M,N)_n \to \mathrm{Bar}_B(M,N)_{n+1}$ are induced by inserting the unit $e\colon \mathbb{1} \to B$ at the $(i+1)$-th position.

Definition 8.3.9 (Relative tensor product)**.** Let C be a monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $-\otimes-\colon C \times C \to C$ preserves geometric realizations in both variables. For an associative algebra B, a right B-module M and a left B-module N, their *relative tensor product* is the object of C given by the geometric realization of the bar construction:

$$M \otimes_B N := |\mathrm{Bar}_B(M, N)_\bullet| = \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \mathrm{Bar}_B(M, N)_n.$$

If M is in fact an (A, B)-bimodule and N is a (B, C)-bimodule, then $\mathrm{Bar}_B(M, N)_\bullet$ in fact turns out to be a simplicial object in ${}_A\mathrm{BMod}_C(C)$, so that also $M \otimes_B N$ is an object of ${}_A\mathrm{BMod}_C(C)$.

Remark 8.3.10. To formally construct $\mathrm{Bar}_B(M, N)_\bullet$, Lurie defines in [Lur17, Definition 4.4.1.1] an ∞-operad $\mathcal{T}ens_2$ and shows in [Lur17, Proposition 4.4.1.11] that it sits in a pushout square of ∞-operads as follows:

$$\begin{tikzcd}
\mathcal{A}ssoc \arrow[r, hook, "\mathfrak{a}_+"] \arrow[d, hook, "\mathfrak{a}_-"'] \arrow[dr, phantom, "\ulcorner", very near end] & \mathcal{B}\mathcal{M}od \arrow[d, hook] \\
\mathcal{B}\mathcal{M}od \arrow[r, hook] & \mathcal{T}ens_2.
\end{tikzcd}$$

In particular, $\mathcal{T}ens_2$ encodes the data of a pair (M, N) consisting of an (A, B)-bimodule M and a (B, C)-bimodule N: for every monoidal ∞-category C there is a pullback square

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{T}ens_2}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{BMod}(C) \arrow[d, "\mathfrak{a}_-^*"] \\
\mathrm{BMod}(C) \arrow[r, "\mathfrak{a}_+^*"] & \mathrm{Alg}(C).
\end{tikzcd}$$

For this reason, I will denote the five colors of $\mathcal{T}ens_2$ by $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{c}$.

We may construct a functor $\mathrm{BCut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}ens_2^{\otimes}$ which sends $[n]$ to the tuple $(\mathfrak{m}, \mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}, \mathfrak{n})$ in $\mathcal{T}ens_2$, i.e., the object that encodes the tensor product $M \otimes B^n \otimes N$. To do this, consider two copies of the functor $\mathrm{LCut}\colon \mathbf{\Delta}^{\mathrm{op}} \times [1] \to \mathcal{B}\mathcal{M}od^{\otimes}$ from Construction 8.2.3. Using the two inclusions $\mathcal{B}\mathcal{M}od \hookrightarrow \mathcal{T}ens_2$, this gives two functors $\mathbf{\Delta}^{\mathrm{op}} \times [1] \to \mathcal{T}ens_2^{\otimes}$. By construction, their restrictions to $\mathbf{\Delta}^{\mathrm{op}} \times \{1\}$ agree with the composite of $\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{A}ssoc^{\otimes} \hookrightarrow \mathcal{T}ens_2^{\otimes}$, so we obtain a cospan of functors $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}ens_2^{\otimes}$. Unwinding definitions, this is given on $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ by the following diagram:

$$\begin{tikzcd}
& (\mathfrak{m}, \mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}) \arrow[d] \\
(\mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}, \mathfrak{n}) \arrow[r] & (\mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}).
\end{tikzcd}$$

Since both maps are inert morphisms, it follows that the pullback of this diagram exists in $\mathcal{T}ens_2^{\otimes}$ and is given by the tuple $(\mathfrak{m}, \mathfrak{b}, \mathfrak{b}, \ldots, \mathfrak{b}, \mathfrak{n})$, providing the desired functor $\mathrm{BCut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{T}ens_2^{\otimes}$.

Now, given an (A, B)-bimodule M and a (B, C)-bimodule N, we may think of this data as defining a $\mathcal{T}ens_2$-algebra in C, which in particular defines a functor $\mathcal{T}ens_2^{\otimes} \to C^{\otimes}$. Precomposing this with BCut gives a functor $\mathbf{\Delta}^{\mathrm{op}} \to C^{\otimes}$. One can check that this actually factors through the active morphisms. Using cocartesian transport, we can then turn this into the desired functor $\mathrm{Bar}_B(M, N)_\bullet\colon \mathbf{\Delta}^{\mathrm{op}} \to C$.

The relative tensor product assembles into a functor

$$\mathrm{BMod}(C) \times_{\mathrm{Alg}(C)} \mathrm{BMod}(C) \to \mathrm{BMod}(C).$$

Furthermore, it satisfies the following properties:

- *Associativity*: for bimodules ${}_AM_B$, ${}_BN_C$ and ${}_CP_D$ there is a natural isomorphism

$$(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P) \quad \in \quad {}_A\mathrm{BMod}_D(C).$$

  This is proved in [Lur17, Proposition 4.4.3.14].

- *Unitality*: for a bimodule ${}_AM_B$ there are natural isomorphisms

$$A \otimes_A M \simeq M \qquad \text{and} \qquad M \otimes_B B \simeq M \quad \in \quad {}_A\mathrm{BMod}_B(C).$$

  This is proved in [Lur17, Proposition 4.4.3.16]

If we restrict attention to the case $A = B = C$, we see that the relative tensor product over A takes the form of a functor

$$- \otimes_A -\colon {}_A\mathrm{BMod}_A(C) \times {}_A\mathrm{BMod}_A(C) \to {}_A\mathrm{BMod}_A(C)$$

which satisfies associativity and unitality. Lurie shows that this assembles into a monoidal structure on ${}_A\mathrm{BMod}_A(C)$:

Proposition 8.3.11 ([Lur17, Proposition 4.4.3.12])**.** *Let C be a monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $- \otimes -\colon C \times C \to C$ preserves geometric realizations in both variables. For an associative algebra A, there exists a monoidal structure on ${}_A\mathrm{BMod}_A(C)$ given by the relative tensor product over A.*

We will now specialize to the case $C = \mathrm{Sp}$, equipped with its tensor product. Since the tensor product of spectra preserves colimits in both variables (Proposition 6.4.6) the same follows for the relative tensor product. Restricting to the case $A = \mathbb{S}$, and $B = C = R$ an associative ring spectrum, the relative tensor product takes the form of a functor

$$- \otimes_R -\colon \mathrm{RMod}_R \times {}_R\mathrm{BMod}_R \to \mathrm{RMod}_R$$

which preserves colimits in both variables. Hence its curried functor becomes a colimit-preserving functor of the form

$${}_R\mathrm{BMod}_R \to \mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_R).$$

Theorem 8.3.12 ([Lur17, Proposition 7.1.2.4])**.** *For every associative ring spectrum R, the functor ${}_R\mathrm{BMod}_R \to \mathrm{Fun}^{\mathrm{L}}(\mathrm{RMod}_R, \mathrm{RMod}_R)$ is an equivalence.*

8.3.3 Modules over commutative algebras

Let R be a discrete commutative ring. Given R-modules M and N, the relative tensor product $M \otimes_R N$ is again an R-module, with the scalar multiplication given by $r(m \otimes n) = rm \otimes n = m \otimes rn$. In fact, the construction $(M, N) \mapsto M \otimes_R N$ equips the category $\mathrm{Mod}_R(\mathrm{Ab})$ of R-modules with a symmetric monoidal structure. The analogous statement also holds in the ∞-categorical setting:

Theorem 8.3.13 ([Lur17, Theorem 4.5.2.1])**.** *Let C be a symmetric monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product of C preserves geometric realizations in both variables. Let R be a commutative algebra in C. Then:*

(1) *The ∞-category $\mathrm{Mod}_R(C)$ admits a symmetric monoidal structure.*

(2) *The functor $\mathrm{Mod}_R(C) \to {}_R\mathrm{BMod}_R(C)$ from Observation 8.3.4 refines to a monoidal functor, where the target has the monoidal structure from Proposition 8.3.11.*

In particular, the tensor product on $\mathrm{Mod}_R(C)$ is given as the following composite:

$$\mathrm{Mod}_R(C)\times\mathrm{Mod}_R(C) \to {}_R\mathrm{BMod}_R(C)\times{}_R\mathrm{BMod}_R(C) \xrightarrow{-\otimes_R-} {}_R\mathrm{BMod}_R(C) \to \mathrm{LMod}_R(C) \xrightarrow{\sim} \mathrm{Mod}_R(C).$$

Proposition 8.3.14 ([Lur17, Proposition 3.4.1.3, Corollary 7.1.3.4])**.** *Let C be a symmetric monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product of C preserves geometric realizations in both variables. Let R be a commutative algebra in C.*

(1) *The forgetful functors $\mathrm{Alg}(\mathrm{Mod}_R(C)) \to \mathrm{Alg}(C)$ and $\mathrm{CAlg}(\mathrm{Mod}_R(C)) \to \mathrm{CAlg}(C)$ lift to equivalence of ∞-categories*

$$\mathrm{Alg}(\mathrm{Mod}_R(C)) \xrightarrow{\sim} \mathrm{Alg}(C)_{R/} \qquad \textit{and} \qquad \mathrm{CAlg}(\mathrm{Mod}_R(C)) \to \mathrm{CAlg}(C)_{R/}.$$

(2) *Let A be an associative algebra in $\mathrm{Mod}_R(C)$, corresponding under (1) to a morphism of algebras $R \to A$ in C. The forgetful functor $\mathrm{Mod}_R(C) \to C$ induces an equivalence of ∞-categories*

$$\mathrm{LMod}_A(\mathrm{Mod}_R(C)) \simeq \mathrm{LMod}_A(C).$$

Proposition 8.3.15. *Let R be a commutative ring spectrum. There exists a functor*

$$\underline{\mathrm{Hom}}_R(-,-)\colon \mathrm{Mod}_R^{\mathrm{op}} \times \mathrm{Mod}_R \to \mathrm{Mod}_R$$

which comes equipped with a natural equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M, \underline{\mathrm{Hom}}_R(N,P)) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(M \otimes_R N, P)$$

for $M, N, P \in \mathrm{Mod}_R$. In particular, $\underline{\mathrm{Hom}}_R(N,-)$ is a right adjoint of $-\otimes_R N$.

Proof. By the Yoneda lemma, it suffices to show that for all fixed N and P, the assignment $M \mapsto \mathrm{Hom}_{\mathrm{Mod}_R}(M \otimes_R N, P)$ is representable. Since the functor $-\otimes_R N$ preserves colimits, this assignment defines a limit-preserving functor $\mathrm{Mod}_R^{\mathrm{op}} \to \mathrm{An}$. The claim is now an instance of the adjoint functor theorem, which says that if C is a presentable ∞-category, then any limit-preserving functor $C^{\mathrm{op}} \to \mathrm{An}$ is representable; see [Lur09b, Proposition 5.5.2.2]. □

Remark 8.3.16. It follows directly from the Yoneda lemma and the associativity of $-\otimes_R-$ that the adjunction equivalence can be internalized to a natural equivalence

$$\underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_R(N,P)) \simeq \underline{\mathrm{Hom}}_R(M \otimes_R N, P).$$

8.3.4 Restriction and extension of scalars

Let $f\colon A \to B$ be a morphism of classical commutative rings. Every B-module M may be regarded as an A-module via the scalar multiplication $am := f(a)m$, resulting in a functor

$$f_*\colon \mathrm{Mod}_B(\mathrm{Ab}) \to \mathrm{Mod}_A(\mathrm{Ab})$$

called *restriction of scalars*[2]. Furthermore, this functor admits a left adjoint

$$f^*\colon \mathrm{Mod}_A(\mathrm{Ab}) \to \mathrm{Mod}_B(\mathrm{Ab})$$

sending an A-module to the relative tensor product $B \otimes_A M$ of B and M over A, where we regard B as a (B, A)-bimodule. Furthermore, this left adjoint is symmetric monoidal: there are natural isomorphisms of B-modules

$$B \otimes_A (M \otimes_A N) \xrightarrow{\sim} (B \otimes_A M) \otimes_B (B \otimes_A N)$$

for all A-modules M and N.

It turns out that this structure exists for commutative algebras in an arbitrary symmetric monoidal ∞-category C.

Theorem 8.3.17 ([Lur17, Theorem 4.5.3.1])**.** *Let C be a symmetric monoidal ∞-category. Assume that C admits geometric realizations and that the tensor product $\otimes\colon C \times C \to C$ preserves geometric realizations in both variables separately. Then there exists a cocartesian fibration*

$$p\colon \mathrm{Mod}(C)^{\otimes} \to \mathrm{CAlg}(C) \times \mathrm{Span}(\mathrm{Fin})$$

such that for every commutative algebra R the pullback along $\{R\} \hookrightarrow \mathrm{CAlg}(C) \times \mathrm{Span}(\mathrm{Fin})$ *is the cocartesian fibration*

$$p_R\colon \mathrm{Mod}_R(C)^{\otimes} \to \mathrm{Span}(\mathrm{Fin})$$

encoding the symmetric monoidal structure on $\mathrm{Mod}_R(C)$ *from Theorem 8.3.13.*

By straightening-unstraightening, the cocartesian fibration p corresponds to a functor

$$\mathrm{Str}^{\mathrm{cc}}\colon \mathrm{CAlg}(C) \times \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Cat}_\infty$$

which preserves finite products in the second variable. By currying, this may equivalently be encoded as a functor

$$\mathrm{CAlg}(C) \to \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), \mathrm{Cat}_\infty) = \mathrm{Cat}_\infty^{\otimes}$$

from commutative algebras in C to symmetric monoidal ∞-category. In particular, every morphism $f\colon R \to S$ in $\mathrm{CAlg}(C)$ induces a symmetric monoidal functor

$$f^*\colon \mathrm{Mod}_R(C) \to \mathrm{Mod}_S(C).$$

When $R = \mathbb{1}$ is the monoidal unit, the resulting symmetric monoidal functor

$$C \simeq \mathrm{Mod}_{\mathbb{1}}(C) \to \mathrm{Mod}_S(C)$$

is the free module functor $F_S\colon C \to \mathrm{Mod}_S(C)$ from Proposition 8.1.16.

We will now specialize to $C = \mathrm{Sp}$:

[2]While this assignment is *contravariant* in the ring homomorphism, the covariant notation f_* comes from the algebo-geometric perspective: f_* is geometrically thought of as the pushforward on quasi-coherent sheaves along the map $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Proposition 8.3.18. *Let $f\colon R \to S$ be a morphism of commutative ring spectra.*

(1) *The functor $f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ admits a right adjoint*

$$f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R.$$

For every S-module M, the underlying object of $f_*(M)$ is naturally equivalent to M.

(2) *The functor $f_*\colon \mathrm{Mod}_S \to \mathrm{Mod}_R$ admits a further right adjoint*

$$\underline{\hom}_R(S,-)\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$$

which satisfies the property that $f_*\underline{\hom}_R(S,-) \simeq \underline{\mathrm{Hom}}_R(S,-)$ as functors $\mathrm{Mod}_R \to \mathrm{Mod}_R$.

Proof. (1) To check that f^* admits a right adjoint, we use the adjoint functor theorem, which says that it suffices to show that f^* preserves small colimits. This may be tested after postcomposing with the forgetful functor $\mathrm{Mod}_S \to \mathrm{Sp}$. But this composite is given by the relative tensor product $S \otimes_R -$, which preserves colimits.

To see that the underlying object of $f_*(M)$ is naturally equivalent to M, we may equivalently pass to left adjoints and show that for every object $M_0 \in C$ we have a natural equivalence $f^*(F_S(M_0)) \simeq F_R(M_0)$. But this is immediate from the functoriality of f^* and the fact that F_S and F_R are instances of this functoriality applied to the unit maps $\eta_S\colon \mathbb{S} \to S$ and $\eta_R\colon \mathbb{S} \to R$, which satisfy $\eta_R = f \circ \eta_R$ by initiality of $\mathbb{S} \in \mathrm{CAlg}$.

(2) To check that f_* admits a further adjoint, it again suffices to see it preserves colimits. This may again be checked on underlying spectra, where it is clear. The fact that $f_*\underline{\mathrm{Hom}}_R(S,-) \simeq \underline{\mathrm{Hom}}_R(S,-)$ may be checked after left adjoints, where it amounts to the statement that the composite $f_*f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is the functor $S \otimes_R -\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$. □

Corollary 8.3.19. *The cocartesian fibration $\mathfrak{m}^*\colon \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$ from Theorem 8.3.17 is also a cartesian fibration, hence straightens to a contravariant functor*

$$\mathrm{CAlg}(\mathrm{Sp})^{\mathrm{op}} \to \mathrm{Cat}_\infty, \quad A \mapsto \mathrm{Mod}_A, \quad f \mapsto f_*.$$

Proof. It is a general fact [ref] that a cocartesian fibration $p\colon E \to C$ is also a cartesian fibration if and only if for every morphism $f\colon A \to B$ in C the induced functor $E_A \to E_B$ admits a right adjoint. In the case of $\mathfrak{m}^*\colon \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$, this is precisely the content of Proposition 8.3.18(1). □

8.4 Endomorphism algebras

This section is incomplete. It is not part of the exam material.

Let M be an abelian group, and let $\mathrm{End}(M)$ denote the set of group homomorphisms from M to itself. We may equip $\mathrm{End}(M)$ with the structure of an associative ring, where addition is given pointwise, and multiplication is given by composition of homomorphisms. Moreover, this ring admits the following universal property: for any other associative ring R, the data of a ring homomorphism $R \to \mathrm{End}(M)$ is equivalent to the data of an R-module structure on M.

In this section, we will generalize this idea by replacing the category of abelian groups by an arbitrary monoidal ∞-category C. Given an object $M \in C$, we define an *endomorphism object for M* to be an object $E \in C$ equipped with a map $a\colon E \otimes M \to M$ which satisfies the following universal property: for any other object $X \in C$, composition with a induces an equivalence $\mathrm{Hom}_C(X, E) \xrightarrow{\sim} \mathrm{Hom}_C(X \otimes M, M)$. We will show that if an endomorphism object E exists, then E admits a canonical structure of an associative algebra in C. Moreover, it is universal among associative algebras that act on M: given any associative algebra $A \in \mathrm{Alg}(C)$, there is an equivalence

$$\mathrm{Hom}_{\mathrm{Alg}(C)}(A, E) \xrightarrow{\sim} \mathrm{LMod}_A(C) \times_C \{M\}.$$

In fact, it will be convenient for applications to work in a higher generality by allowing the object M to live in an ∞-category D which is (weakly) left-tensored over C.

Our approach in this section closely follows the approach taken by Lurie in [Lur17, Section 4.7.1]:

- Given an object $M \in D$, we will construct an ∞-category $C[M]$. Informally speaking, the objects of $C[M]$ will be pairs (A, η), where A is an object of C and $\eta\colon A \otimes M \to M$ is a morphism in D.
- We will then equip $C[M]$ with the structure of a monoidal ∞-category. The tensor product on $C[M]$ is given on objects by $(A, \eta) \otimes (A', \eta') = (A \otimes A', \eta'')$, where η'' is the composite $A \otimes A' \otimes M \xrightarrow{\mathrm{id} \otimes \eta'} A \otimes M \xrightarrow{\eta} M$.
- Next, we will identify the ∞-category $\mathrm{Alg}(C[M])$ of associative algebras in $C[M]$ with the fiber $\mathrm{LMod}(D) \times_D \{M\}$.
- Finally, we will see that an endomorphism object $\mathrm{End}(M)$ (together with the map $\mathrm{End}(M) \otimes M \to M$) is precisely a terminal object of $C[M]$. Since the terminal object in a monoidal ∞-category always admits a canonical associative algebra structure, the previous point then equips $\mathrm{End}(M)$ with the structure of an associative, and M becomes a left module over $\mathrm{End}(M)$.

Notation 8.4.1. In the ∞-category $\mathcal{LM}od^{\otimes}$, we have the following two 'canonical' morphisms $(\{\mathfrak{a}\}, \{\mathfrak{m}\}) \to \{\mathfrak{m}\}$:

- The 'projection map' $\mathrm{pr}\colon (\{\mathfrak{a}\}, \{\mathfrak{m}\}) \to \{\mathfrak{m}\}$, given by the span

$$\{\mathfrak{a}\} \sqcup \{\mathfrak{m}\} \hookleftarrow \{\mathfrak{m}\} \xrightarrow{=} \{\mathfrak{m}\}.$$

- The 'action map' $\mathrm{act}\colon (\{\mathfrak{a}\}, \{\mathfrak{m}\}) \to \{\mathfrak{m}\}$, given by the span

$$\{\mathfrak{a}\} \sqcup \{\mathfrak{m}\} \xleftarrow{=} \{\mathfrak{a}\} \sqcup \{\mathfrak{m}\} \to \{\mathfrak{m}\}.$$

Construction 8.4.2 (cf. [Lur17, Definition 4.7.1.1])**.** Let C be a monoidal ∞-category and let D be an ∞-category which is weakly left-tensored over C; we let $p_D\colon D^{\otimes} \to \mathcal{LM}od^{\otimes}$ denote the associated morphism of ∞-operads. An *enriched morphism* of D is a diagram

$$N \xleftarrow{\beta} X \xrightarrow{\alpha} M$$

in $D^{\otimes}$ satisfying the following conditions:

(1) The morphism β is p_D-cocartesian and the image $p_D(\beta)$ in $\mathcal{LM}od^{\otimes}$ is the projection map $\mathrm{pr}\colon (\{\mathfrak{a}\}, \{\mathfrak{m}\}) \to \{\mathfrak{m}\}$;

(2) The image $p_D(\alpha)$ in $\mathcal{LM}od^{\otimes}$ is the action map $\mathrm{act}\colon (\{\mathfrak{a}\}, \{\mathfrak{m}\}) \to \{\mathfrak{m}\}$.

We denote by
$$\mathrm{Ar}^{\mathrm{enr}}(D) \subseteq \mathrm{Fun}_{/\mathcal{LM}od}(\ulcorner, D^{\otimes})$$
the full subcategory spanned by the enriched morphisms of D. There are two forgetful functors
$$(s, t)\colon \mathrm{Ar}^{\mathrm{enr}}(D) \to D$$
given by $s(\alpha, \beta) := N$ and $t(\alpha, \beta) := M$.

Remark 8.4.3. Letus spell out the data contained in an enriched morphism. The fiber of $D^{\otimes}$ over $(\{\mathfrak{a}\}, \{\mathfrak{m}\})$ is equivalent to $C \times D$, hence we may write $X = (A, N')$ for some $A \in C$ and $N' \in D$. Condition (1) is saying that β induces an equivalence $N' \xrightarrow{\sim} N$ in D. By condition (2), the morphism $\alpha\colon X \to M$ corresponds to a morphism $A \otimes N \to M$ in D.

Definition 8.4.4. Let D be weakly left-tensored over a monoidal ∞-category C, and let $M \in D$ be an object. We define the *endomorphism category of M* as the fiber of (s, t) over (M, M):
$$C[M] := \mathrm{Ar}^{\mathrm{enr}}(D) \times_{D \times D} \{(M, M)\}.$$
An *endomorphism object of M* is defined to be a terminal object of $C[M]$, if it exists.

We will now equip $C[M]$ with the structure of a monoidal ∞-category.

Construction 8.4.5. TO DO. I expect one can directly write down an ∞-category $C[M]^{\otimes}$ equipped with a functor to $\mathcal{A}ssoc^{\otimes}$. Here's a sketch of how I'm hoping this will work: *[but please take this with a grain of salt..!]*

- For a finite set I, consider the poset $P(I)$ of subsets of I, and consider its dual twisted arrow category $\mathrm{Tw}^r(P(I))$. The objects are pairs (I_0, I_1) of subsets of I satisfying $I_0 \subseteq I_1$. There is a morphism $(I_0, I_1) \to (I_0', I_1')$ if and only if $I_0 \subseteq I_0'$ and $I_1' \subseteq I_1$.
- Note that for $I = \langle 1 \rangle$ the one-point set, this category looks like a span $(\emptyset, \emptyset) \leftarrow (\emptyset, \langle 1 \rangle) \to (\langle 1 \rangle, \langle 1 \rangle)$.
- There should be a functor $\mathrm{Tw}(P(I)) \to \mathcal{LM}od^{\otimes}$ sending (I_0, I_1) to $(I_1 \setminus I_0, I)$. When $I = \langle 1 \rangle$ this should correspond to the span
$$(\emptyset, \{\mathfrak{m}\}) \xleftarrow{\mathrm{pr}} (\{\mathfrak{a}\}, \{\mathfrak{m}\}) \xrightarrow{\mathrm{act}} (\emptyset, \{\mathfrak{m}\}).$$
More generally, maps of the form $(I_0, I_1) \to (I_0, I_1 \sqcup J)$ should correspond to 'projection maps' $(J \sqcup (I_1 \sqcup I_0), I) \to (I_1 \sqcup I_0, I)$ in $\mathcal{LM}od^{\otimes}$, while maps of the form $(I_0 \sqcup J, I_1 \sqcup J) \to (I_0, I_1 \sqcup J)$ should correspond to 'action maps' $(J \sqcup (I_1 \sqcup I_0), I) \to (I_1 \sqcup I_0, I)$.
- We would then define the ∞-category $C[M]^{\otimes}_I$ as the ∞-category of functors $\mathrm{Tw}^r(P(I)) \to D^{\otimes}$ over $\mathcal{LM}od^{\otimes}$ sending the backwards maps $(I_0, I_1) \to (I_0, I_1 \sqcup J)$ to p_D-cocartesian morphisms.

- This should then somehow assemble into an ∞-category $C[M]^{\otimes}$ over $\mathcal{A}ssoc^{\otimes}$ whose fiber over I is $C[M]^{\otimes}_{I}$.

Conjecture 8.4.6 (cf. [Lur17, Proposition 4.7.1.30, Theorem 4.7.1.34])**.** *The functor $C[M]^{\otimes} \to \mathcal{A}ssoc^{\otimes}$ is a cocartesian fibration, equipping $C[M]$ with the structure of a monoidal ∞-category. Moreover, there exists an equivalence of ∞-categories*

$$\mathrm{Alg}(C[M]) \simeq \mathrm{LMod}(D) \times_{D} \{M\}.$$

Corollary 8.4.7 (cf. [Lur17, Corollary 4.7.1.40])**.** *Let D be weakly left-tensored over C, and let $M \in D$ be an object such that the endomorphism object $\mathrm{End}(M) \in C$ exists. Then $\mathrm{End}(M)$ admits a preferred structure of an associative algebra, and M admits a preferred structure of a left module over $\mathrm{End}(M)$. Moreover, there is an equivalence of ∞-categories making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Alg}(C)_{/\mathrm{End}(M)} \arrow[rr, dashed, "\simeq"] \arrow[dr, "\mathrm{fgt}"'] & & \mathrm{LMod}(D) \times_{D} \{M\} \arrow[dl, "\mathrm{fgt}"] \\
& \mathrm{Alg}(C). &
\end{tikzcd}$$

Proposition 8.4.8. *Let D be a stable ∞-category.*

(1) *The ∞-category D is weakly left-tensored over* Sp*;*

(2) *For every object $M \in D$, the mapping spectrum $\mathrm{map}_{D}(M, M)$ admits the structure of an associative ring spectrum.*

(3) *For every object $M \in D$, the mapping spectrum functor $\mathrm{map}_{D}(M, -)\colon D \to \mathrm{Sp}$ admits a lift*

$$\mathrm{map}_{D}(M, -)\colon D \to \mathrm{LMod}_{\mathrm{map}_{D}(M,M)}.$$

9 Properties of rings and modules

With the general theory of algebras and modules over a symmetric monoidal ∞-category in hand, we now turn our attention to the specific case of the stable ∞-category of spectra. Our goal in this chapter is to develop homotopical analogues of the fundamental concepts from classical module theory.

We begin in Section 9.1 by introducing the notion of a t-structure, which provides a powerful framework for organizing a stable ∞-category by "homological degree." This allows us to speak of connective and coconnective modules, generalizing chain complexes whose homology vanishes in negative or positive degrees, respectively, or spectra whose homotopy groups vanish in negative or positive degrees. The intersection of these two classes forms the "heart" of the t-structure, which we will see is a classical abelian category.

In Section 9.2 and Section 9.3, we study the spectral analogues of projective and flat modules. Similar to their classical counterparts, these may be characterized by exactness properties for the mapping spectrum functor $\mathrm{map}_R(P, -)$ or the tensor product functor $- \otimes_R N$.

Next, in Section 9.4, we investigate perfect modules, which are the correct notion of "small" objects in the world of modules over a ring spectrum. We will characterize them abstractly as the compact objects in the ∞-category of modules, as well as the dualizable modules. We also discuss the notion of Tor-amplitude.

Finally, in Section 9.5, we develop the theory of localization for (modules over) ring spectra. This construction generalizes the classical process of inverting elements in a ring via its ring of fractions $R[S^{-1}]$.

9.1 Connective objects and t-structures

Recall that a spectrum X is called *connective* if the homotopy groups $\pi_k(X)$ vanish for $k < 0$; we write $\mathrm{Sp}^{\geq 0} \subseteq \mathrm{Sp}$ for the full subcategory of connective spectra. Dually, we say that X is *coconnective* if $\pi_k(X)$ vanishes for $k > 0$, resulting in a subcategory $\mathrm{Sp}^{\leq 0} \subseteq \mathrm{Sp}$. Spectra that are both connective and coconnective are called *Eilenberg-MacLane spectra*: they are spectra concentrated in degree 0, and are precisely those spectra in the image of the fully faithful inclusion $\mathbf{B}^\infty \colon \mathrm{Ab} \hookrightarrow \mathrm{Sp}$.

The subcategories $\mathrm{Sp}^{\geq 0}$ and $\mathrm{Sp}^{\leq 0}$ subdivide Sp into two parts that are in some sense 'orthogonal' to each other. More specifically, these subcategories satisfy the following three properties:

(1) (Closure under shifts) The subcategory $\mathrm{Sp}^{\geq 0}$ is closed under positive shifts (i.e., suspensions) while $\mathrm{Sp}^{\leq 0}$ is closed under negative shifts (i.e., loops).

(2) (Orthogonality) If X is a connective spectrum and Y is a coconnective spectrum, then there are no non-zero maps from X to $Y[-1]$. Indeed, by the recognition principle for connective spectra, we may write $X \simeq \mathbf{B}^\infty(A)$ for some commutative group $A \in \mathrm{CGrp}(\mathrm{An})$. By adjunction, morphisms $X \to Y[-1]$ of spectra correspond to morphisms $A \to \Omega^\infty(Y[-1])$ of commutative groups. But $\Omega^\infty(Y[-1])$ is contractible since all its homotopy groups are trivial.

(3) (Decomposition) Given a spectrum X, we may always form its *connected cover* $\tau_{\geq 0}X := \mathbf{B}^\infty\Omega^\infty X$. This is a connective spectrum that comes equipped with a map $\tau_{\geq 0}X \to X$ given by the counit of the adjunction $\mathbf{B}^\infty \dashv \Omega^\infty$. If we further define $\tau_{\leq -1}X$ as the cofiber of this map, then it follows from the long exact sequence that $\tau_{\leq -1}$ is concentrated in strictly negative degrees. In particular, every spectrum can be decomposed into a part living in $\mathrm{Sp}^{\geq 0}$ and a part living in $\mathrm{Sp}^{\leq -1}$.

It turns out that structures like these are omnipresent in stable homotopy theory and homological algebra. Since these theories have similar formal properties, it is useful to introduce a general framework in which these results are phrased. The overarching notion is that of a *t-structure* on a stable ∞-category.[1] In this section, we will introduce this notion, and establish the t-structure on the module category LMod_R for a connective ring spectrum R.

Definition 9.1.1 ([BBD82])**.** A *t-structure* on a stable ∞-category C consists of a pair $(C_{\geq 0}, C_{\leq 0})$ of full subcategories of C satisfying the following conditions:

(1) (Closure under shifts) We have $C_{\geq 0}[1] \subseteq C_{\geq 0}$ and $C_{\leq 0} \subseteq C_{\leq 0}[1]$.

(2) (Orthogonality) If $X \in C_{\geq 0}$ and $Y \in C_{\leq 0}$, then $\mathrm{Hom}_C(X, Y[-1]) = 0$.

(3) (Decomposition) Every $X \in C$ sits in an exact sequence of the form

$$\tau_{\geq 0}X \to X \to \tau_{\leq -1}X,$$

with $\tau_{\geq 0}X \in C_{\geq 0}$ and $\tau_{\leq -1}X \in C_{\leq 0}[-1]$.

Objects in $C_{\geq 0}$ are called *connective* (with respect to the t-structure), while objects in $C_{\leq 0}$ are called *coconnective*. For $n \in \mathbb{Z}$ we set $C_{\geq n} := C_{\geq 0}[n]$ and $C_{\leq n} := C_{\leq 0}[n]$.

If D is another stable ∞-category equipped with a t-stucture, then an exact functor $F \colon C \to D$ is called *left t-exact* if it sends $C_{\leq 0}$ to $D_{\leq 0}$, and hence by exactness sends $C_{\leq n}$ to $D_{\leq n}$ for all $n \in \mathbb{Z}$. Similarly, F is called *right t-exact* if it sends $C_{\geq 0}$ to $D_{\geq 0}$, or equivalently sends $C_{\geq n}$ to $D_{\geq n}$ for all n. We say F is *t-exact* if it is both left and right t-exact.

Example 9.1.2. As explained in the introduction, the subcategories $\mathrm{Sp}^{\geq 0}$ and $\mathrm{Sp}^{\leq 0}$ equip Sp with a t-structure, called the *Postnikov t-structure*.

Example 9.1.3. Let $\mathcal{A}$ be a small abelian category. Then one can show that the derived ∞-category $D(\mathcal{A})$ admits a t-structure, where the connective objects are the chain complexes with trivial negative homology groups, and the coconnective objects are the claim complexes with trivial positive homology groups.

[1] This notion can already be phrased at the level of underlying triangulated categories, and is much older than the theory of ∞-categories itself.

Example 9.1.4. If C admits a t-struture $(C_{\geq 0}, C_{\leq 0})$, then C^{op} can be given the *opposite t-structure*, in which $(C^{\mathrm{op}})_{\geq 0} := (C_{\leq 0})^{\mathrm{op}}$ and $(C^{\mathrm{op}})_{\leq 0} := (C_{\geq 0})^{\mathrm{op}}$.

Example 9.1.5. If C and D both admit t-structures, then so does their product $C \times D$ by setting $(C \times D)^{\geq 0} := C^{\geq 0} \times D^{\geq 0}$ and $(C \times D)^{\leq 0} := C^{\leq 0} \times D^{\leq 0}$.

Lemma 9.1.6. *Let $(C_{\geq 0}, C_{\leq 0})$ be a t-structure on C, and let $n \in \mathbb{Z}$.*

(1) *The inclusion $C_{\geq n} \hookrightarrow C$ admits a right adjoint $\tau_{\geq n} \colon C \to C_{\geq n}$.*

(2) *The inclusion $C_{\leq n} \hookrightarrow C$ admits a left adjoint $\tau_{\leq n} \colon c \to C_{\leq n}$.*

Proof. We prove (1); the proof of (2) is dual. By shifting, we may assume that $n = 0$; in general we may then set $\tau_{\geq n}(X) := \tau_{\geq 0}(X[-n])[n]$. Using the pointwise formula for adjunctions, it suffices to show that for every object $X \in C$ there exists some $\tau_{\geq 0}X$ equipped with a morphism $\tau_{\geq 0}X \to X$ inducing equivalences $\mathrm{Hom}_C(Y, \tau_{\geq 0}X) \xrightarrow{\sim} \mathrm{Hom}_C(Y, X)$ for every $Y \in C_{\geq 0}$. For this, we consider the exact sequence $\tau_{\geq 0}X \to X \to \tau_{\leq -1}X$ provided by (3). This gives rise to a long exact sequence

$$\cdots \to \pi_{n+1} \mathrm{Hom}_C(Y, \tau_{\leq -1}X) \to \pi_n \mathrm{Hom}_C(Y, \tau_{\geq 0}X) \to \pi_n \mathrm{Hom}_C(Y, X) \to \pi_n \mathrm{Hom}_C(Y, \tau_{\leq -1}X) \to \dots$$

Since $\mathrm{Hom}_C(Y, \tau_{\leq -1}X) = 0$ by assumption, this implies that the map $\mathrm{Hom}_C(Y, \tau_{\geq 0}X) \xrightarrow{\sim} \mathrm{Hom}_C(Y, X)$ induces isomorphisms on all homotopy groups, hence is an isomorphism. □

Corollary 9.1.7. *The subcategory $C_{\leq n}$ of C is closed under all limits that exist in C. The subcategory $C_{\geq n}$ of C is closed under all colimits that exist in C.* □

Corollary 9.1.8. *Assume C admits I-indexed colimits. Then so does the subcategory $C_{\leq n}$: they are computed as $\tau_{\leq n}(\mathrm{colim}_{i \in I} X_i)$. Dually, if C admits I-indexed limits, then so does $C_{\geq n}$ via the formula $\tau_{\geq n}(\lim_{i \in I} X_i)$.* □

Warning 9.1.9. The notation $\tau_{\leq n}$ is sometimes also used for the notion of n-truncation in an ∞-category. This notation is *not* compatible with the left adjoint $\tau_{\leq n}$ from the lemma. In fact, the only n-truncated object in a stable ∞-category is the zero object.

Exercise 9.1.10. Let C and D have t-structures and let $F \colon C \rightleftarrows D \colon G$ be an adjunction. Show that F is right t-exact if and only if G is left t-exact.

Lemma 9.1.11. *Let $F \colon C \to D$ be a t-exact functor between two stable ∞-categories equipped with t-structures. Then for all $X \in C$ and $n \in \mathbb{Z}$, there are natural equivalences*

$$\tau_{\leq n}F(X) \xrightarrow{\sim} F(\tau_{\leq n}X) \qquad \text{and} \qquad F(\tau_{\geq n}X) \xrightarrow{\sim} \tau_{\geq n}F(X).$$

Proof. We proof the first equivalence; the second is dual. Since F is t-exact, applying F to the exact sequence $\tau_{\geq n+1}X \to X \to \tau_{\leq n}X$ gives another exact sequence

$$F(\tau_{\geq n+1}X) \to F(X) \to F(\tau_{\leq n}X)$$

where the first object lies in $D_{\geq n+1}$ and the last one in $D_{\leq n}$. It follows that the second map induces an isomorphism $\tau_{\leq n}F(X) \xrightarrow{\sim} F(\tau_{\leq n}X)$. □

9.1.1 t-structures on module categories

For the purpose of this chapter, our primary interest is in establishing the Postnikov t-structure on the stable ∞-category LMod_R of left modules over a connective ring spectrum R. This t-structure will be inherited from the Postnikov t-structure on Sp from Example 9.1.2. It will rely on the following compatibility between the t-structure and the monoidal structure:

Definition 9.1.12. Let C be a stably symmetric monoidal ∞-category equipped with a t-structure. We say that *the symmetric monoidal structure C is right t-exact* if the following two conditions are satisfied:

(1) The monoidal unit $\mathbb{1}$ of C is connective;

(2) For two connective objects $X, Y \in C_{\geq 0}$, the tensor product $X \otimes Y$ is again connective.

One may alternatively say that the t-structure is *compatible with the symmetric monoidal structure*. In this case, $C_{\geq 0}$ inherits a symmetric monoidal structure making the inclusion $C_{\geq 0} \hookrightarrow C$ into a symmetric monoidal functor (see Lemma 4.4.2). As a consequence, the truncation functor $\tau_{\geq 0} \colon C \to C_{\geq 0}$ is canonically lax symmetric monoidal (see Proposition 4.3.3).

Warning 9.1.13. While the property implies that the tensor product functor $- \otimes - \colon C \times C \to C$ restricts to $- \otimes - \colon C_{\geq 0} \times C_{\geq 0} \to C_{\geq 0}$, this is not actually an instance of a 'right t-exact functor' from Definition 9.1.1 as it is not exact.

Definition 9.1.14. Let C be a stable ∞-category equipped with a t-structure and a right t-exact symmetric monoidal structure. Let $R \in \mathrm{Alg}(C_{\geq 0})$ be a connective associative algebra in C. We say that a left R-module M is *connective* if its underlying object in C is connective, and similarly for coconnective modules. This defines full subcategories

$$\mathrm{LMod}_R(C)^{\geq 0}, \quad \mathrm{LMod}_R(C)^{\leq 0} \quad \subseteq \quad \mathrm{LMod}_R.$$

Lemma 9.1.15. *Assume C has a t-structure and a right t-exact symmetric monoidal structure, and let $R \in \mathrm{Alg}(C_{\geq 0})$ be a connective associative algebra in C. Then $\mathrm{LMod}_R(C)$ admits a t-structure given by the connective and coconnective left R-modules.*

Proof. We start by establishing that the inclusion $\mathrm{LMod}_R(C_{\geq 0}) \hookrightarrow \mathrm{LMod}_R(C)$ admits a right adjoint

$$\tau_{\geq 0} \colon \mathrm{LMod}_R(C) \to \mathrm{LMod}_R(C_{\geq 0})$$

given on underlying objects by the truncation functor $\tau_{\geq 0} \colon C \to C_{\geq 0}$. By Proposition 4.3.3, the adjunction between the inclusion and $\tau_{\geq 0}$ induces an adjunction of ∞-operads $\mathcal{M}_{C_{\geq 0}} \rightleftarrows \mathcal{M}_C$, which then induce adjunctions

$$\begin{tikzcd}
\mathrm{LMod}(C_{\geq 0}) \arrow[r, shift left] \arrow[d] \arrow[r, phantom, "\perp"] & \mathrm{LMod}(C) \arrow[l, shift left, "\tau_{\geq 0}"] \arrow[d] \\
\mathrm{Alg}(C_{\geq 0}) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(C) \arrow[l, shift left, "\tau_{\geq 0}"]
\end{tikzcd}$$

Passing to fibers over $R \in \mathrm{Alg}(C_{\geq 0})$ then produces the desired adjunction on module categories.

We will now prove that $\mathrm{LMod}_R(C)$ has a t-structure by checking the three conditions from Definition 9.1.1. (1) is clear, as limits and colimits in $\mathrm{LMod}_R(C)$ are computed in C. For (2), let X and Y be left R-modules such that X is connective and Y is coconnective. Using the adjunction established before, we then have

$$\mathrm{Hom}_{\mathrm{LMod}_R(C)}(X, Y[-1]) \simeq \mathrm{Hom}_{\mathrm{LMod}_R(C)}(X, \tau_{\geq 0}Y[-1]) \simeq 0,$$

where the second equality comes from the fact that $\tau_{\geq 0}Y[-1]$ is zero, as $Y[-1] \in C_{\leq -1}$. For (3), consider $X \in \mathrm{LMod}_R(C)$. Applying the right adjoint $\tau_{\geq 0}\colon \mathrm{LMod}_R(C) \to \mathrm{LMod}_R(C_{\geq 0})$ provides a connective R-module $\tau_{\geq 0}X$ equipped with a map to X. Taking $\tau_{\leq -1}X$ to be the cofiber of this map provides the desired decomposition of X. □

We now wish to apply this to the Postnikov t-structure on Sp.

Lemma 9.1.16. *The symmetric monoidal structure on* Sp *is right t-exact.*

Proof. The sphere spectrum is connective. Given two connective spectra X and Y, we need to show that $X \otimes Y$ is again connective. If we write $X_n := \Omega^{\infty - n}X$ and $Y_m := \Omega^{\infty - m}Y$ for all $n, m \geq 0$, then X_n is n-connective and Y_m is m-connective. By the classical Blakers-Massey theorem it follows that $X_n \wedge Y_m$ is $(n+m)$-connective. We then have

$$X \otimes Y \simeq \mathrm{colim}_n\, \mathrm{colim}_m (\Sigma^\infty(X_n \wedge Y_m))[-(n+m)].$$

Since each of the spectra in this colimit are connective and $\mathrm{Sp}^{\geq 0} \subseteq \mathrm{Sp}$ is closed under colimits, it follows that $X \otimes Y \in \mathrm{Sp}^{\geq 0}$, as desired. □

Corollary 9.1.17. *Let R be a connective associative ring spectrum. Then* LMod_R *admits a t-structure given by the connective and coconnective R-modules.*

Proof. In light of the previous lemma, this is an instance of Lemma 9.1.15. □

The subcategory of connective R-modules admits a simple description: it is generated by R under colimits. In fact, we have the following more refined statement:

Proposition 9.1.18 ([Lur17, Proposition 7.1.1.13(1)])**.** *Let R be a connective associative ring spectrum. Then the ∞-category* $\mathrm{LMod}_R^{\geq 0}$ *is the smallest subcategory of* LMod_R *which contains R and which is closed under colimits.*

Proof. It is clear that $\mathrm{LMod}_R^{\geq 0}$ contains R and is closed under colimits. Conversely, given a connective left R-module M, we will show that M can be written as a colimit of a diagram of left R-modules

$$M(0) \to M(1) \to M(2) \to \dots$$

satisfying the following properties:

(i) The R-module $M(0)$ is a coproduct of copies of R.

(ii) For $i \geq 0$, there is an exact sequence

$$F[i] \to M(i) \to M(i+1),$$

where F is a coproduct of copies of R.

We will construct the modules $M(i)$ inductively, together with their maps $f_i\colon M(i) \to M$. In fact, we will inductively ensure that these maps satisfy the following third property:

(iii) Let $i \geq 0$, and let $K(i)$ be a fiber of the map $M(i) \to M$. Then $\pi_j K(i) \simeq 0$ for $j < i$.

We begin by choosing $M(0)$ to be any coproduct of copies of R equipped with a map $M(0) \to M$ which induces a surjection $\pi_0 M(0) \twoheadrightarrow \pi_0 M$; for example, we can take $M(0)$ to be a coproduct of copies of R indexed by $\pi_0 M$. Observe that $K(0)$ is connective, by the long exact sequence. Let us now suppose that the map $f_i\colon M(i) \to M$ has been constructed, with $K(i) = \mathrm{fib}(f_i)$ such that $\pi_j K(i) \simeq 0$ for $j < i$. We now choose F to be a coproduct of copies of R and a map $g\colon F[i] \to K(i)$ which induces a surjection $\pi_i F \twoheadrightarrow \pi_i K(i)$. Let h denote the composite map $F[i] \to K(i) \to M(i)$, and let $M(i+1) = \mathrm{cofib}(h)$. The canonical nullhomotopy of $K(i) \to M(i) \xrightarrow{f_i} M$ defining $K(i)$ induces a factorization

$$M(i) \to M(i+1) \xrightarrow{f_{i+1}} M$$

of f. We observe that there is a canonical equivalence $\mathrm{fib}(f_{i+1}) \simeq \mathrm{cofib}(g)$, so that $\pi_j \,\mathrm{fib}(f_{i+1}) \simeq 0$ for $j \leq i$.

Let $M(\infty)$ be the colimit of the sequence $\{M(i)\}$. To show that the map $M(\infty) \to M$ is an isomorphism, we may equivalently show that its fiber K is the zero module. Note that this fiber can be identified with a colimit of the sequence $\{K(i)\}_{i \geq 0}$. Since the formation of homotopy groups preserves filtered colimits, we conclude that $\pi_j K \simeq \mathrm{colim}_i \, \pi_j K(i) \simeq 0$. We conclude that $M(\infty) \simeq M$, hence M is contained in the smallest subcategory containing R which is closed under colimits. □

Remark 9.1.19. The ∞-category LMod_R also admits a t-structure when R is not connective. In this case, the subcategory $\mathrm{LMod}_R^{\geq 0}$ may still be described as the smallest subcategory of LMod_R that contains R and is closed under colimits. The subcategory $\mathrm{LMod}_R^{\leq 0}$ is still given by those R-modules whose underlying spectrum is coconnective. We refer to [Lur17, Proposition 1.4.4.11] for a very general result for producing t-structures on presentable ∞-categories.

9.1.2 The heart of a t-structure

Given a t-structure on C, a special role is played by the objects that are both connective and coconnective.

Definition 9.1.20. Let $(C_{\geq 0}, C_{\leq 0})$ be a t-structure on C. We define the *heart* of the *t*-structure to be the intersection

$$C^{\heartsuit} := C_{\geq 0} \cap C_{\leq 0}.$$

The nice feature of t-structures is that the heart is always an abelian 1-category, and that there is a close relation between exact sequences in $C^{\heartsuit}$ (in the sense of homological algebra) and exact sequences in C (in the sense of stable homotopy theory).

Proposition 9.1.21. *Let C be a stable ∞-category equipped with a t-structure.*

(1) *The heart $C^{\heartsuit}$ is an abelian 1-category.*

(2) *A morphism $g\colon Y \to Z$ in $C^{\heartsuit}$ is an epimorphism in $C^{\heartsuit}$ if and only if its fiber in C lies in $C^{\heartsuit}$.*

(3) *A morphism $f\colon X \to Y$ in $C^\heartsuit$ is a monomorphism in $C^\heartsuit$ if and only if its cofiber in C lies in $C^\heartsuit$.*

(4) *Consider a commutative square in $C^\heartsuit$ of the form*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ W \arrow[r, "k"] & Z. \end{tikzcd}$$

(a) If g and h are epimorphisms in $C^\heartsuit$, then the square is a pullback square in $C^\heartsuit$ if and only if it is a pullback square in C.

(b) If f and k are monomorphisms in $C^\heartsuit$, then the square is a pushout square in $C^\heartsuit$ if and only if it is a pushout square in C.

Proof. For (1), first observe that $C^\heartsuit$ is an ordinary category, i.e., that the Hom animae $\mathrm{Hom}_{C^\heartsuit}(X, Y)$ are sets. Indeed, for $k \geq 1$, we have

$$\pi_k \mathrm{Hom}_{C^\heartsuit}(X, Y) \cong \pi_0 \mathrm{Hom}_{C^\heartsuit}(X, Y[-k]) = 0$$

since $X \in C_{\geq 0}$ and $Y \in C_{\leq -k} \subseteq C_{\leq -1}$. Further, since both $C_{\leq 0}$ and $C_{\geq 0}$ are closed under direct sums in C, it is clear $C^\heartsuit$ is additive. We also observe that $C^\heartsuit$ admits kernels and cokernels, which are computed as

$$\ker(Y \to Z) = \tau_{\geq 0}(\mathrm{fib}(Y \to Z)) \qquad \text{and} \qquad \mathrm{coker}(X \to Y) = \tau_{\leq 0}(\mathrm{cofib}(X \to Y)).$$

Before finishing the proof of (1), let us first address the 'only if'-directions in (2) and (3). Since replacing C by C^{op} translates (2) into (3) and vice versa, it suffices to prove the 'only if'-direction in (2). To this end, let $g\colon Y \twoheadrightarrow Z$ be an epimorphism in $C^\heartsuit$. We need to show that $\mathrm{fib}(g) \in C_{\geq 0}$. (Since $C_{\leq 0}$ is closed under limits, the condition $\mathrm{fib}(g) \in C_{\leq 0}$ is automatic.) The assumption on g implies that the commutative square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "g"'] & Z \arrow[d, "="] \\ Z \arrow[r, "="] & Z \end{tikzcd}$$

is a pushout square in $C^\heartsuit$. Since pushouts in $C^\heartsuit$ are computed by first forming the pushout in $C_{\geq 0}$ and then applying $\tau_{\leq 0}\colon C_{\geq 0} \to C^\heartsuit$, it follows that the map $Z \to Z \sqcup_Y Z$ obtained by forming the pushout in C induces an isomorphism $Z = \tau_{\geq 0} \xrightarrow{\sim} \tau_{\geq 0}(Z \sqcup_Y Z)$. Passing to cofibers then gives $\tau_{\geq 0}(\mathrm{cofib}(g)) = 0$, or equivalently $\mathrm{cofib}(g) \in C_{\geq 1}$. But then the relation $\mathrm{fib}(g) \simeq \mathrm{cofib}(g)[-1]$ implies $\mathrm{fib}(g) \in C_{\geq 0}$, which is what we needed to show..

We now prove (4). Again, (a) translates to (b) under replacing C by C^{op}, so it suffices to prove (a). By fully faithfulness of the inclusion $C^\heartsuit \hookrightarrow C$, it is clear that if the square is a pullback in C then it is also a pullback in $C^\heartsuit$. Conversely, if the square is a pullback in $C^\heartsuit$, then it in particular induces an isomorphism $\ker(h) \xrightarrow{\sim} \ker(g)$ on kernels in $C^\heartsuit$. But by the direction of (2) that we just proved, these kernels agree with the fibers in C. We deduce that the induced map $\mathrm{fib}(h) \to \mathrm{fib}(g)$ is an isomorphism in C, which implies that the square is also a pullback square in C. This finishes the proof of (4).

We now return to (1), finishing the proof that $C^\heartsuit$ is abelian. Consider a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

in $C^\heartsuit$, and assume that f is a monomorphism in $C^\heartsuit$ and that g is an epimorphism in $C^\heartsuit$. We need to show that this square is a pullback square in $C^\heartsuit$ if and only if it is a pushout square $C^\heartsuit$. But by (4) this is immediate from stability of C.

Finally, we prove the 'if'-directions in (2) and (3). As before, it suffices to do this for (2). Given a morphism $g\colon Y \to Z$ in $C^\heartsuit$ whose fiber lies in $C^\heartsuit$, we must show g is an epimorphism. Since $C^\heartsuit$ is abelian, it suffices to show that its cokernel $\operatorname{coker}(g) = \tau_{\leq 0}(\operatorname{cofib}(g))$ is zero, i.e., that $\operatorname{cofib}(g) \in C_{\geq 1}$. But this follows from the relation $\operatorname{cofib}(g) \simeq \operatorname{fib}(g)[1]$ and the assumption $\operatorname{fib}(g) \in C_{\geq 0}$. □

Corollary 9.1.22. *Let C be a stable ∞-category equipped with a t-structure. For every short exact sequence*

$$0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$$

in $C^\heartsuit$, the sequence $X \to Y \to Z$ is an exact sequence in C.

Proof. This is an immediate consequence of part (4) of Proposition 9.1.21. □

Corollary 9.1.23. *Let $F\colon C \to D$ be an exact functor between stable ∞-categories with t-structures. Assume F restricts to a functor $F\colon C^\heartsuit \to D^\heartsuit$. Then this restriction is an exact functor of abelian categories.*

Proof. Let $0 \to X \hookrightarrow Y \twoheadrightarrow Z \to 0$ be a short exact sequence in $C^\heartsuit$. By the previous corollary, the sequence $X \to Y \to Z$ is exact in C, hence induces an exact sequence $F(X) \to F(Y) \to F(Z)$ in D. Using the various parts of Proposition 9.1.21, we see:

(i) By (2), the morphism $F(Y) \to F(Z)$ is an epimorphism in $D^\heartsuit$, since its fiber $F(X)$ lies in $D^\heartsuit$;

(ii) By (3), the morphism $F(X) \to F(Y)$ is a monomorphism in $D^\heartsuit$, since its cofiber $F(Z)$ lies in $D^\heartsuit$;

(iii) By (4), the exact sequence $F(X) \to F(Y) \to F(Z)$ in D is also exact in $D^\heartsuit$.

We conclude that the sequence $0 \to F(X) \hookrightarrow F(Y) \twoheadrightarrow F(Z) \to 0$ is exact in $D^\heartsuit$, as desired. □

To identify the heart of a t-structure, the following observation is sometimes useful:

Lemma 9.1.24. *Let C be a stable ∞-category equipped with a t-structure, and let $X \in C_{\geq 0}$ be a connective object. Then X lies in the heart of C if and only if it is a 0-truncated object of $C_{\geq 0}$, in the sense that $\pi_k \operatorname{Hom}_{C_{\geq 0}}(Y, X) = 0$ for all $Y \in C_{\geq 0}$ and $k > 0$.*

Proof. First assume that $X \in C^\heartsuit$. Then for $Y \in C_{\geq 0}$ and $k > 0$ we have $\pi_k \operatorname{Hom}_C(Y, X) \simeq \pi_0 \operatorname{Hom}_C(Y, X[-k]) = 0$ as $X[-k] \in C_{\leq -k} \subseteq C_{\leq -1}$. Conversely, if X is 0-truncated in $C_{\geq 0}$, then we have $\pi_k \operatorname{Hom}_C(Y, X[-1]) \cong \pi_{k+1} \operatorname{Hom}_C(Y, X) \cong 0$ for all $Y \in C_{\geq 0}$ and $k \geq 0$. It follows that $X[-1] \in C_{\leq -1}$, and thus $X \in C^\heartsuit$. □

Corollary 9.1.25. *For a stable ∞-category equipped with a t-structure, there is an equivalence*

$$\mathcal{C}^{\heartsuit} \quad \simeq \quad \tau_{\leq 0}(\mathcal{C}_{\geq 0})$$

between the heart of $\mathcal{C}$ and the full subcategory of 0-truncated objects in $\mathcal{C}_{\geq 0}$.

Let us compute the heart of the module category LMod_R for a connective associative ring spectrum R. We start with the sphere spectrum $R = \mathbb{S}$.

Lemma 9.1.26. *The functor* $\mathbf{B}^{\infty} \colon \mathrm{Ab} \hookrightarrow \mathrm{Sp}$ *induces an equivalence*

$$\mathbf{B}^{\infty} \colon \mathrm{Ab} \xrightarrow{\sim} \mathrm{Sp}^{\heartsuit}.$$

Proof. By the recognition principle for connective spectra, the functor $\mathbf{B}^{\infty}$ induces an equivalence $\mathbf{B}^{\infty} \colon \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\sim} \mathrm{Sp}^{\geq 0}$. By Corollary 9.1.25, we may identify $\mathrm{Sp}^{\heartsuit}$ with the subcategory of $\mathrm{CGrp}(\mathrm{An})$ spanned by the 0-truncated objects. But since $\mathrm{Set} \simeq \tau_{\leq 0}(\mathrm{An})$, we get $\mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set}) \simeq \tau_{\leq}(\mathrm{CGrp}(\mathrm{An}))$. □

We may bootstrap this lemma to a computation of $\mathrm{LMod}_R^{\heartsuit}$. To this end, we observe that the adjunction $\pi_0 \dashv \mathbf{B}^{\infty}$ is a symmetric monoidal adjunction:

Lemma 9.1.27. *The functor*

$$\pi_0 \colon \mathrm{Sp}^{\geq 0} \rightleftarrows \mathrm{Ab}$$

is canonically symmetric monoidal.

Proof. We will first equip π_0 with a lax symmetric monoidal structure. The functor $\Omega^{\infty} \colon \mathrm{Sp}^{\geq 0} \to \mathrm{An}$ is lax symmetric monoidal by the definition of the symmetric monoidal structure on Sp. The functor $\pi_0 \colon \mathrm{An} \to \mathrm{Set}$ preserves finite products, hence is canonically symmetric monoidal with respect to the cartesian monoidal structures. In particular, their composite $\pi_0 \colon \mathrm{Sp}^{\geq 0} \to \mathrm{Set}$ is lax symmetric monoidal. By the universal property of the symmetric monoidal structure on $\mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set})$ from Theorem 6.5.6, we see that the lift $\pi_0 \colon \mathrm{Sp}^{\geq 0} \to \mathrm{Ab}$ is lax symmetric monoidal as well.

Next we argue that π_0 is in fact symmetric monoidal: for all connective spectra $X, Y \in \mathrm{Sp}^{\geq 0}$, the comparison map

$$\pi_0(X) \otimes \pi_0(Y) \to \pi_0(X \otimes Y)$$

is an isomorphism of abelian groups. But this is clear when $X = \mathbb{S}$, since $\pi_0(\mathbb{S}) = \mathbb{Z}$ and so both sides are $\pi_0(Y)$. Furthermore, the collection of spectra X for which this holds is closed under colimits, since both sides preserve colimits in X. It then follows from Proposition 9.1.18 (applied to $R = \mathbb{S}$) that the map is an isomorphism for all $X \in \mathrm{Sp}^{\geq 0}$. □

Observation 9.1.28. Let R be a connective associative ring spectrum. As a consequence of the lemma, we see that extracting the π_0 of a connective R-module determines a functor

$$\pi_0 \colon \mathrm{LMod}_R^{\geq 0} = \mathrm{LMod}_R(\mathrm{Sp}^{\geq 0}) \to \mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab}).$$

Furthermore, this functor is compatible with relative tensor products: given a connective right R-module M and a connective left R-module N, we have an isomorphism of abelian groups

$$\pi_0(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \xrightarrow{\sim} \pi_0(M \otimes_R N), \qquad [x] \otimes [y] \mapsto [x \otimes y].$$

Indeed, this is an immediate consequence of the description of the relative tensor product in terms of the two-sided bar construction, together with the fact that $\pi_0\colon \mathrm{Sp}^{\geq 0} \to \mathrm{Ab}$ is symmetric monoidal and preserves geometric realizations.

Lemma 9.1.29. *Let R be an associative ring spectrum. Then the functor π_0 from Observation 9.1.28 restricts to an equivalence*

$$\pi_0\colon \mathrm{LMod}_R^{\heartsuit} \xrightarrow{\sim} \mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab})$$

between the heart of LMod_R and the abelian category of discrete modules over $\pi_0(R)$.

Proof. Let us first show that the functor is essentially surjective. Every discrete $\pi_0(R)$-module M may be written as a coequalizer of free $\pi_0(R)$-modules, and since π_0 preserves colimits it suffices to show that every free $\pi_0(R)$-module $\bigoplus_I \pi_0(R)$ is in the essential image. This follows from $\pi_0(\bigoplus_I \tau_{\leq 0}R) \simeq \bigoplus_I \pi_0(\tau_{\leq 0}R) \simeq \bigoplus_I \pi_0(R)$.

We will now show that the functor is fully faithful. Let $N \in \mathrm{LMod}_R^{\heartsuit}$, and consider the subcategory $C \subseteq \mathrm{LMod}_R^{\geq 0}$ spanned by those left R-modules M such that the map

$$\pi_0 \operatorname{Hom}_{\mathrm{LMod}_R}(M, N) \to \operatorname{Hom}_{\mathrm{LMod}_{\pi_0(R)}(\mathrm{Ab})}(\pi_0(M), \pi_0(N))$$

is an isomorphism. Note that C contains R and is closed under colimits. It thus follows from Proposition 9.1.18 that $C = \mathrm{LMod}_R^{\geq 0}$, as desired. □

Notation 9.1.30. Let R be a discrete associative ring. We will from now on denote the category $\mathrm{LMod}_R(\mathrm{Ab})$ of discrete left R-modules as $\mathrm{LMod}_R^{\heartsuit}$. This notation is justified by Lemma 9.1.29: this category agrees with the heart of the t-structure on $\mathrm{LMod}_R = \mathrm{LMod}_R(\mathrm{Sp})$.

Tensor products in Ab will from now on always be denoted by $\otimes^{\heartsuit}$; we reserve $\otimes$ for the tensor product in Sp.

9.1.3 Homotopy group objects

We will now define the *homotopy group objects* of an object $X \in C$: the objects of the heart obtained by truncating X both from above and below. First observe that this operation is independent of the order of truncation:

Lemma 9.1.31. *Let C be a stable ∞-category equipped with a t-structure. For $n, m \in \mathbb{Z}$, the canonical natural transformation*

$$\tau_{\leq m}\tau_{\geq n} \to \tau_{\geq n}\tau_{\leq m}$$

is a natural isomorphism of functors $C \to C_{\leq m} \cap C_{\geq n}$.

Proof. The claim is clear if $n \geq m+1$, so assume that $n \leq m$. By the Yoneda lemma, it will suffice to show that for every $X \in C_{\leq m} \cap C_{\geq n}$ and $Y \in C$, the induced map

$$\operatorname{map}_C(X, \tau_{\leq m}\tau_{\geq n}Y) \to \operatorname{map}_C(X, \tau_{\geq n}\tau_{\leq m}Y) \simeq \operatorname{map}_C(X, \tau_{\leq m}Y)$$

is an equivalence of spectra. To this end, consider the two exact sequences

$$\tau_{\geq m+1}Y \to Y \to \tau_{\leq m}Y \qquad \text{and} \qquad \tau_{\geq m+1}\tau_{\geq n}Y \to \tau_{\geq n}Y \to \tau_{\leq m}\tau_{\geq n}Y.$$

Applying $\mathrm{map}_C(X,-)$, this gives the following diagram of spectra, with exact horizontal rows:

$$\begin{tikzcd}
\mathrm{map}_C(X,\tau_{\geq m+1}\tau_{\geq n}Y) \arrow[r] \arrow[d] & \mathrm{map}_C(X,\tau_{\geq n}Y) \arrow[r] \arrow[d] & \mathrm{map}_C(X,\tau_{\leq m}\tau_{\geq n}Y) \arrow[d] \\
\mathrm{map}_C(X,\tau_{\geq m+1}Y) \arrow[r] & \mathrm{map}_C(X,Y) \arrow[r] & \mathrm{map}_C(X,\tau_{\leq m}Y).
\end{tikzcd}$$

The left vertical map is an isomorphism since $\tau_{\geq m+1}\tau_{\geq n}Y \simeq \tau_{\geq \max m+1,n}Y = \tau_{\geq m+1}Y$, and the middle vertical map is an isomorphism by adjunction. It thus follows that also the right vertical map is an isomorphism, as desired. $\square$

Definition 9.1.32. Let C be a stable ∞-category equipped with a t-structure and let X be an object of C. For every $n \in \mathbb{Z}$, we define the *homotopy group objects* $\pi_n(X) \in C^{\heartsuit}$ by

$$\pi_n(X) := \tau_{\geq 0}\tau_{\leq 0}(X[-n]) \quad \in \quad C^{\heartsuit}.$$

Remark 9.1.33. In the context of homological algebra, for example when C is the derived ∞-category of an abelian category, the objects $\pi_n(X)$ are often denoted by $H_n(C)$ instead, emphasizing the analogy with the homology groups of a chain complex rather than the analogy with homotopy groups of a spectrum.

Lemma 9.1.34. *Let $F\colon C \to D$ be a t-exact functor between two stable ∞-categories equipped with t-structures. Then for all $X \in C$ and $n \in \mathbb{Z}$, there is a natural isomorphism $F(\pi_n X) \xrightarrow{\sim} \pi_n F(X)$.*

Proof. By Lemma 9.1.11 we have natural isomorphisms

$$F(\pi_n X) = F(\tau_{\geq 0}\tau_{\leq 0}(X[-n])) \xrightarrow{\sim} \tau_{\geq 0}\tau_{\leq 0}(F(X)[-n]) = \pi_n(F(X)). \qquad \square$$

Just like for spectra, exact sequences in C induce long exact sequences on homotopy groups:

Proposition 9.1.35 (Long exact sequence of homotopy groups)**.** *Let C be a stable ∞-category equipped with a t-structure. For every exact sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ in C, we obtain a long exact sequence of homotopy groups in $C^{\heartsuit}$:*

$$\cdots \to \pi_{n+1}(Z) \xrightarrow{\partial} \pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \xrightarrow{\pi_n(g)} \pi_n(Z) \xrightarrow{\partial} \pi_{n-1}(X) \to \dots$$

Proof. Since every such exact sequence in C induces exact sequences $Z[-1] \to X \to Y$ and $Y \to Z \to X[1]$ and we have $\pi_n(Z[-1]) = \pi_{n+1}(Z)$ and $\pi_n(X[1]) = \pi_{n-1}(X)$, it suffices by induction to show that the sequence

$$\pi_0(X) \xrightarrow{\pi_0(f)} \pi_0(Y) \xrightarrow{\pi_0(g)} \pi_0(Z)$$

is exact in $C^{\heartsuit}$. We will prove this in progressively more general cases:

Case 1: Assume that $X, Y, Z \in C_{\geq 0}$. We claim that in this case, the sequence

$$\pi_0(X) \to \pi_0(Y) \to \pi_0(Z) \to 0$$

is exact. To see this, it suffices to show that for every $W \in C^{\heartsuit}$, the induced sequence of abelian groups

$$0 \to \mathrm{Hom}_{C^{\heartsuit}}(\pi_0(Z), W) \to \mathrm{Hom}_{C^{\heartsuit}}(\pi_0(Y), W) \to \mathrm{Hom}_{C^{\heartsuit}}(\pi_0(X), W)$$

is exact. Since $Z \in C_{\geq 0}$, we have $\pi_0(Z) = \tau_{\leq 0}Z$, so by adjunction we obtain isomorphisms

$$\mathrm{Hom}_{C^\heartsuit}(\pi_0(Z), W) \cong \pi_0 \mathrm{Hom}_C(\tau_{\geq 0}Z, W) \cong \pi_0 \mathrm{Hom}_C(Z, W) \cong \pi_0 \mathrm{map}_C(Z, W),$$

and similarly for X and Y. Furthermore, we have $\pi_1 \mathrm{map}_C(X, W) \cong \pi_0 \mathrm{map}_C(X[1], W) \cong 0$ as $\tau_{\leq 0}(X[1]) \cong 0$. The claim now follows from the long exact sequence on homotopy groups associated to the exact sequence of mapping spectra

$$\mathrm{map}_C(Z, W) \to \mathrm{map}_C(Y, W) \to \mathrm{map}_C(X, W).$$

Case 2: Assume that $X \in C_{\geq 0}$, but Y and Z are arbitrary. Since $\tau_{\leq -1} \colon C \to C_{\leq -1}$ preserves cofiber sequences, we get a cofiber sequence

$$0 \cong \tau_{\leq -1}X \to \tau_{\leq -1}Y \to \tau_{\leq -1}Z,$$

showing that $\tau_{\leq -1}Y \cong \tau_{\leq -1}Z$. Consider now the following diagram:

$$\begin{tikzcd}
\tau_{\geq 0}X \arrow[r] \arrow[d, "\cong"'] & \tau_{\geq 0}Y \arrow[r] \arrow[d] & \tau_{\geq 0}Z \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & \tau_{\leq -1}Y \arrow[r, "\cong"] & \tau_{\leq -1}Z.
\end{tikzcd}$$

Since all three columns are exact and the bottom two rows are exact, it follows that the top row is also exact. Since $\pi_0(X) \cong \pi_0(\tau_{\geq 0}X)$, and similarly for Y and Z, the exactness of $\pi_0(X) \to \pi_0(Y) \to \pi_0(Z) \to 0$ has thus been reduced to Case 1.

Case 3: Assume that $Z \in C_{\leq 0}$, but X and Y are arbitrary. Using an argument dual to that of Cases 1 and 2, we obtain an exact sequence

$$0 \to \pi_0(X) \to \pi_0(Y) \to \pi_0(Z).$$

Case 4: Finally, we consider the case for general X, Y, Z. Let Q denote the cofiber of the composite $\tau_{\geq 0}X \to X \to Y$. We then have the following commutative diagram:

$$\begin{tikzcd}
\tau_{\geq 0}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & Q \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
\tau_{\leq -1}X \arrow[r] & 0 \arrow[r, "\cong"] & (\tau_{\leq -1}X)[1].
\end{tikzcd}$$

The three rows are exact, as are the left two columns, and hence so is the third column. Applying Cases 2 and 3, we get exact sequences

$$\pi_0(X) \to \pi_0(Y) \twoheadrightarrow \pi_0(Q) \to 0 \qquad \text{and} \qquad 0 \to \pi_0(Q) \hookrightarrow \pi_0(Z) \to \pi_{-1}(X).$$

Gluing them together then gives the desired exact sequence $\pi_0(X) \to \pi_0(Y) \to \pi_0(Z)$. $\square$

9.1.4 Complete and bounded t-structures

One of the primary advantages of having a t-structure on a given stable ∞-category C is that it often allows us to reduce questions about arbitrary objects of X to questions about objects of the heart $C^\heartsuit$, which may be easier to address. In order for this strategy to work, we need to ensure that the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ retain enough information about the objects of C. For this purpose, we now introduce completeness and boundedness conditions on C.

Definition 9.1.36. A t-structure on C is *left complete* if every infinitely connective object $X \in C_{\geq\infty} = \bigcap_{n\in\mathbb{Z}} C_{\geq n}$ is zero. Dually we say C is *right complete* if every infinitely coconnective object $X \in \cap_{n\in Z} C_{\leq n}$ is zero.

Lemma 9.1.37. *Let R be a connective associative ring spectrum. Then the Postnikov t-structure on* LMod_R *is both left and right complete.*

Proof. Let M be a left R-module. If $M \in C_{\geq n}$ for all $n \in \mathbb{Z}$, then in particular $\pi_n(M) = 0$ for all n, which implies $M = 0$. □

Lemma 9.1.38. *Let C admit a left complete t-structure that admits countable products, and assume that $C_{\geq 0}$ is closed under countable products. Then for every object $X \in C$, there is an equivalence*

$$X \quad \simeq \quad \lim(\cdots \to \tau_{\leq n+1}X \to \tau_{\leq n}X \to \cdots \to \tau_{\leq 1}X \to \tau_{\leq 0}X).$$

Dually, if the t-structure on C is left complete, admits countable coproducts and $C_{\leq 0}$ is closed under countable coproducts, then we have

$$X \quad \simeq \quad \operatorname{colim}(\tau_{\geq 0}X \to \tau_{\geq -1}X \cdots \to \tau_{\geq -n}X \to \tau_{\geq -n-1}X \to \dots).$$

Proof. We prove the first case, as the second case is dual. First note that every tower of objects

$$\cdots \to Y_{n+1} \to Y_n \to \dots$$

admits a limit, given by the fiber of the map $\mathrm{id} - \mathrm{shift}\colon \prod_{n\in\mathbb{Z}} Y_n \to \prod_{n\in\mathbb{Z}} Y_n$, where shift is induced by the maps $Y_n \to Y_{n-1}$. In particular, if we have $Y_n \in C_{\geq 0}$, then $\lim_n Y_n \in C_{\geq -1}$.

Now, to show the equivalence in question, let K denote the fiber of the map $X \to \lim_n \tau_{\leq n}X$. We wish to show that $K = 0$. As C is left complete, it suffices to show that $K \in C_{\geq m}$ for all $m \in \mathbb{Z}$. Since limits commute with fibers, we have $K \simeq \lim_n \operatorname{fib}(X \to \tau_{\leq n}X) \simeq \lim_n \tau_{\geq n+1}X$. By cofinality, we may compute this limit as a limit over only those n satisfying $n \geq m$. But then $\tau_{\geq n+1}X \in C_{\geq m+1}$, hence their limit K lies in $C_{\geq m+1}$. This finishes the proof. □

Definition 9.1.39. A t-structure on C is *right bounded* if every object $X \in C$ is contained in $C_{\geq n}$ for some $n \in \mathbb{Z}$. It is *left bounded* if every $X \in C$ is contained in $C_{\leq n}$ for some $n \in \mathbb{Z}$. It is *bounded* if it is both left and right bounded.

Definition 9.1.40. Let C be a stable ∞-category with a t-structure. We define the subcategory of *bounded objects* by

$$C^\flat := \bigcup_{n\in\mathbb{N}} (C_{\geq -n} \cap C_{\leq n}) \subseteq C.$$

Observe that $C^{\flat}$ inherits a t-structure from C which by construction is bounded. One may similarly define subcategories

$$C^{-} := \bigcup_{n\in\mathbb{N}} C_{\geq -n} \subseteq C \qquad \text{and} \qquad C^{+} := \bigcup_{n\in\mathbb{N}} C_{\leq n} \subseteq C$$

of *bounded below* and *bounded above* objects, respectively.

We will state without proof a useful universal property for the bounded derived ∞-category of an abelian category:

Theorem 9.1.41 (Lurie [Lur17, Theorem HA.1.3.3.2])**.** *Let $\mathcal{A}$ be an abelian category with enough projectives, let C be a stable ∞-category with a left complete t-structure, and let*

$$\mathrm{Fun}'(D(\mathcal{A})^{-}, C) \subseteq \mathrm{Fun}(D(\mathcal{A})^{-}, C)$$

denote the full subcategory of functors $D(\mathcal{A})^{-} \to C$ that are right t-exact and send projective objects of $\mathcal{A}$ to $C^{\heartsuit}$. Then restriction along the inclusion $i\colon \mathcal{A} \hookrightarrow D(\mathcal{A})^{-}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}'(D(\mathcal{A})^{-}, C) \xrightarrow{\simeq} \mathrm{Fun}^{\mathrm{rex}}(\mathcal{A}, C^{\heartsuit}).$$

9.2 Projective objects and Ext-groups

In classical algebra, projective modules are the "well-behaved" building blocks essential for homological algebra. Their definition makes sense in an arbitrary abelian category $\mathcal{A}$: an object P is called *projective* if the Hom functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathrm{Ab}$ is an exact functor of abelian categories. This definition has an immediate analogue in an arbitrary t-structure:

Definition 9.2.1. Let C be a stable ∞-category equipped with a t-structure. An object $P \in C$ is called *t-projective* if the mapping spectrum functor $\mathrm{map}_C(P, -)\colon C \to \mathrm{Sp}$ is t-exact.

Observation 9.2.2. Every t-projective object P is connective. To see this, consider the truncation $\tau_{\leq -1}P$. By exactness, the mapping spectrum $\mathrm{map}_C(P, \tau_{\leq -1}P)$ lies in $\mathrm{Sp}^{\leq -1}$, and it follows that

$$0 = \pi_0 \,\mathrm{map}_C(P, \tau_{\leq -1}P) \cong \pi_0 \,\mathrm{Hom}_C(\tau_{\leq -1}P, \tau_{\leq -1}P),$$

where the last identification holds by adjunction. In particular, the identity on $\tau_{\leq -1}P$ is homotopic to the zero map, forcing $\tau_{\leq -1}P = 0$, hence $P \in C_{\geq 0}$.

Observation 9.2.3. Conversely, if $P \in C_{\geq 0}$ is a connective object, then $\mathrm{map}_C(P, -)\colon C \to \mathrm{Sp}$ is always left t-exact: if $X \in C_{\leq 0}$, then for $n < 0$ we have

$$\pi_n(\mathrm{map}_C(P, X)) \cong \pi_0 \,\mathrm{Hom}_C(P, X[n]) = 0,$$

since $P \in C_{\geq 0}$ while $X \in C_{\leq -1}$. So P is t-projective if and only if $\mathrm{map}_C(P, -)$ is also *right* t-exact.

9.2.1 Ext-groups

We will next give a formulation of t-projectivity in terms of the Ext-groups in $\mathcal{C}$.

Definition 9.2.4 (Ext-groups)**.** Let $\mathcal{C}$ be a stable ∞-category. For objects $X, Y \in \mathcal{C}$ and $n \geq 0$, we define the *n-th* Ext-*group* to be

$$\mathrm{Ext}^n_{\mathcal{C}}(X,Y) := \pi_{-n}\,\mathrm{map}_{\mathcal{C}}(X,Y) \simeq \pi_0\,\mathrm{Hom}_{\mathcal{C}}(X, Y[n]).$$

Remark 9.2.5. Assume $\mathcal{C}$ comes equipped with a t-structure. For objects $M, N \in \mathcal{C}^\heartsuit$, the Ext-groups $\mathrm{Ext}^n_{\mathcal{C}}(M,N)$ can be described in terms of *iterated extensions*, explaining the terminology:

- For $n < 0$, we have $\mathrm{Ext}^n_{\mathcal{C}}(M,N) = 0$, since $X \in \mathcal{C}_{\geq 0}$ while $Y[n] \in \mathcal{C}_{\leq n} \subseteq \mathcal{C}_{\leq -1}$;
- For $n = 0$, the set $\mathrm{Ext}^0_{\mathcal{C}}(M,N) = \mathrm{Hom}_{\mathcal{C}^\heartsuit}(M,N)$ is the set of morphisms of R-modules from M to N;
- For $n = 1$, the set $\mathrm{Ext}^1_{\mathcal{C}}(M,N) = \pi_0\,\mathrm{Hom}_{\mathcal{C}}(M, N[1])$ is the set of isomorphism classes of *extensions of M by N*, i.e. short exact sequences in $\mathcal{C}^\heartsuit$ of the form

  $$0 \to N \hookrightarrow E \twoheadrightarrow M \to 0.$$

  Indeed, given a morphism $\varphi\colon M \to N[1]$, we may form the fiber $E := \mathrm{fib}(\varphi)$. The long exact sequence on homotopy groups (Proposition 9.1.35) for the fiber sequence $N \to E \to M$ shows that E is contained in $\mathcal{C}^\heartsuit$ as well, and it follows from Proposition 9.1.21 that this exact sequence in $\mathcal{C}$ corresponds to a short exact sequence in $\mathcal{C}^\heartsuit$. Reversing this reasoning shows that the homotopy class of φ is determined by E as well.
- If $\mathcal{C}^\heartsuit$ has enough projectives[2], then one can describe $\mathrm{Ext}^n_{\mathcal{C}}(M,N)$ more generally in terms of *degree n Yoneda extensions*, defined as exact sequences in $\mathcal{C}^\heartsuit$ of the form

  $$0 \to N \to E_n \to E_{n-1} \to \cdots \to E_2 \to E_1 \to M \to 0.$$

  This will follow from Proposition 9.2.10 below.

Observation 9.2.6 (Ext long exact sequence)**.** Observe that every exact sequence $Y' \to Y \to Y''$ in $\mathcal{C}$ induces an exact sequence of mapping spectra $\mathrm{map}_{\mathcal{C}}(X,Y') \to \mathrm{map}_{\mathcal{C}}(X,Y) \to \mathrm{map}_{\mathcal{C}}(X,Y'')$, thus giving rise to a long exact sequence of homotopy groups of the form

$$\begin{tikzcd}
 & \dots \arrow[r] & \mathrm{Ext}^{n-2}_{\mathcal{C}}(X,Y'') \arrow[dll] \\
\mathrm{Ext}^{n-1}_{\mathcal{C}}(X,Y') \arrow[r] & \mathrm{Ext}^{n-1}_{\mathcal{C}}(X,Y) \arrow[r] & \mathrm{Ext}^{n-1}_{\mathcal{C}}(X,Y'') \arrow[dll] \\
\mathrm{Ext}^{n}_{\mathcal{C}}(X,Y') \arrow[r] & \mathrm{Ext}^{n}_{\mathcal{C}}(X,Y) \arrow[r] & \mathrm{Ext}^{n}_{\mathcal{C}}(X,Y'') \arrow[dll] \\
\mathrm{Ext}^{n+1}_{\mathcal{C}}(X,Y') \arrow[r] & \dots &
\end{tikzcd}$$

[2] An abelian category is said to *have enough projectives* if for every object M there exists an epimorphism $P \twoheadrightarrow M$ from a projective object P.

Similarly, every exact sequence $X' \to X \to X''$ in C induces a long exact sequence of the form

$$\begin{tikzcd} & \dots \arrow[r] & \mathrm{Ext}^{n-1}_C(X', Y) \arrow[dll, out=0, in=180] \\ \mathrm{Ext}^n_C(X'', Y) \arrow[r] & \mathrm{Ext}^n_C(X, Y) \arrow[r] & \mathrm{Ext}^n_C(X', Y) \arrow[dll, out=0, in=180] \\ \mathrm{Ext}^{n+1}_C(X'', Y) \arrow[r] & \dots & \end{tikzcd}$$

As a special case, given an object $N \in C^\heartsuit$ and a short exact sequence $M' \hookrightarrow P \twoheadrightarrow M$ in $C^\heartsuit$, the negative Ext-groups vanish, giving a one-sided long exact sequence of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{C^\heartsuit}(M, N) \arrow[r] & \mathrm{Hom}_{C^\heartsuit}(P, N) \arrow[r] & \mathrm{Hom}_{C^\heartsuit}(M', N) \arrow[dll, out=0, in=180] \\ & \mathrm{Ext}^1_C(M, N) \arrow[r] & \mathrm{Ext}^1_C(P, N) \arrow[r] & \mathrm{Ext}^1_C(M', N) \arrow[dll, out=0, in=180] \\ & \mathrm{Ext}^2_C(M, N) \arrow[r] & \mathrm{Ext}^2_C(P, N) \arrow[r] & \mathrm{Ext}^2_C(M', N) \arrow[dll, out=0, in=180] \\ & \dots & & \end{tikzcd}$$

Proposition 9.2.7 ([Lur17, Proposition 7.2.2.6]). *Let C be a stable ∞-category equipped with a t-structure, and let $P \in C_{\geq 0}$ be a connective object. Then the following conditions are equivalent:*

(1) *The object P is t-projective;*

(2) *For every $Q \in C_{\geq 0}$ and every $i > 0$, the abelian group $\mathrm{Ext}^i_C(P, Q)$ is zero;*

(3) *For every $Q \in C_{\geq 0}$, the abelian group $\mathrm{Ext}^1_C(P, Q)$ is zero;*

(4) *Given an exact sequence*
$$N' \to N \to N''$$
in C with $N', N, N'' \in C_{\geq 0}$, the map $\mathrm{Ext}^0_C(P, N) \to \mathrm{Ext}^0_C(P, N'')$ is surjective.

If C is left complete, this is further equivalent to:

(5) *For every $Q \in C^\heartsuit$ and every $i > 0$, the abelian group $\mathrm{Ext}^i_C(P, Q)$ is zero;*

Proof. Recall from Observation 9.2.3 that P is t-projective if and only if the mapping spectrum functor $\mathrm{map}_C(P, -)\colon C \to \mathrm{Sp}$ is right t-exact. By definition, this means that for any object $Q \in C_{\geq 0}$, the mapping spectrum $\mathrm{map}_C(P, Q)$ has vanishing negative homotopy groups $\pi_{-i} \mathrm{map}_C(P, Q) = \mathrm{Ext}^i_C(P, Q)$ for $i > 0$. This shows the equivalence (1) $\Leftrightarrow$ (2).

It is clear that (2) implies (3), and the implication (3) $\Rightarrow$ (2) follows by replacing Q by $Q[i]$ for all $i \geq 0$.

We next show that (3) $\Leftrightarrow$ (4). It is clear that (3) implies (4): the exact sequence $N' \to N \to N''$ induces a long exact sequence in $C^\heartsuit$ of the form

$$\dots \to \mathrm{Ext}^0_C(P, N) \to \mathrm{Ext}^0_C(P, N'') \to \mathrm{Ext}^1_C(P, N) \to \dots,$$

so the vanishing of $\mathrm{Ext}^1_C(P,N)$ implies that the first map must be surjective. Assume now that (4) holds. Given a class $\eta \in \mathrm{Ext}^1(P,Q) = \pi_0 \mathrm{Hom}_C(P,Q[1])$, we obtain an exact sequence

$$P' \to P \xrightarrow{\eta} Q[1]$$

and thus a long exact sequence

$$\cdots \to \mathrm{Ext}^0_C(P,P') \to \mathrm{Ext}^0_C(P,P) \xrightarrow{\eta\circ -} \mathrm{Ext}^1_C(P,Q) \to \ldots$$

Since the first map is assumed surjective, the second map is the zero map. Since it sends id_P to η, we conclude that $\eta = 0$. This shows that (3) $\Leftrightarrow$ (4).

Finally, we show that (2) $\Leftrightarrow$ (5) whenever C is left complete. It is clear that (2) implies (5), even without left completeness. Conversely, assume that (5) holds. For any object $Q \in C_{\geq 0}$ and $n \geq 0$, we may consider the exact sequence $(\pi_{n+1}Q)[n+1] \to \tau_{\leq n+1}Q \to \tau_{\leq n}Q$ in C, which by Observation 9.2.6 induces a long exact sequence of Ext-groups

$$\cdots \to \mathrm{Ext}^{i+n+1}_C(P,\pi_{n+1}Q) \to \mathrm{Ext}^i_C(P,\tau_{\leq n+1}Q) \to \mathrm{Ext}^i_C(P,\tau_{\leq n}Q) \to \mathrm{Ext}^{n+i+2}_C(P,\pi_{n+1}Q) \to \ldots$$

Assumption (5) then shows that the two outer terms vanish whenever $n \geq -i$, and in particular the middle map is an isomorphism whenever $n \geq 0$. Since C is left complete, we have $Q \simeq \lim_{n\in N} \tau_{\leq n}Q$, and thus

$$\mathrm{Ext}^i_C(P,Q) \simeq \lim_n \mathrm{Ext}^i_C(P,\tau_{\leq n}Q) \cong \mathrm{Ext}^i_C(P,\tau_{\leq 0}Q) = 0,$$

where the last equivalence holds by (5) as $\tau_{\leq 0}Q \in C^{\heartsuit}$. This shows that (2) $\Leftrightarrow$ (5), finishing the proof. □

Remark 9.2.8. Let R be a discrete associative ring. Applying Proposition 9.2.7 to $C = \mathrm{LMod}_R$, we see that a discrete left R-module $P \in \mathrm{LMod}^{\heartsuit}_C$ is t-projective in the sense of Definition 9.2.11 if and only if it is projective in the sense of classical homological algebra.

As a consequence of the previous proposition, we may compute Ext-groups in terms of projective resolutions.

Definition 9.2.9. Let C be a stable ∞-category equipped with a left-complete t-structure, and let $M \in C^{\heartsuit}$ be an object in the heart of C. A *projective resolution* of M is a long exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \twoheadrightarrow M \to 0$$

in $C^{\heartsuit}$ such that each object P_n is t-projective.

Proposition 9.2.10 (Ext-groups via projective resolutions)**.** *Let C be a stable ∞-category equipped with a left-complete t-structure, and let $P_\bullet$ be a projective resolution of $M \in C^{\heartsuit}$. Then for every other object $N \in C^{\heartsuit}$ and every $n \in \mathbb{Z}$ there is an isomorphism of abelian groups*

$$\mathrm{Ext}^n_C(M,N) \quad\cong\quad C^n(\mathrm{Hom}_{C^{\heartsuit}}(P_\bullet,N))$$

between the n-th Ext group and the n-th cohomology group of the cochain complex $\mathrm{Hom}_{C^{\heartsuit}}(P_\bullet,N)$.

Proof. The claim is clear for $n < 0$ as both sides are zero: $\mathrm{Ext}^n_C(M, N) = \pi_0 \operatorname{Hom}_C(M, N[n]) = 0$ since $M \in C_{\geq 0}$ and $N[n] \in C_{\leq n} \subseteq C_{\leq -1}$.

Next, consider the auxiliary objects $M_{-1} := M$, $M_0 := \ker(P_0 \twoheadrightarrow M)$ and $M_n := \ker(P_n \to P_{n-1})$ for $n \geq 1$. Each map $P_{n+1} \to P_n$ factors through M_n, and we obtain a decomposition of the projective resolution $P_\bullet$ into a collection of short exact sequences in $C^\heartsuit$:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
 & & M_2 \arrow[ur] \arrow[dr, tail] & & & & M_0 \arrow[ur] \arrow[dr, tail] & & & \\
\cdots \arrow[r, dashed] & P_3 \arrow[ur, two heads] \arrow[rr, dashed] & & P_2 \arrow[dr, two heads] \arrow[rr, dashed] & & P_1 \arrow[ur, two heads] \arrow[rr, dashed] & & P_0 \arrow[dr, two heads] & & \\
\cdots \arrow[ur, tail] & & & & M_1 \arrow[ur, tail] \arrow[dr] & & & & M \arrow[dr] & \\
 & & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

For each of these short exact sequences, we obtain a long exact sequence on Ext-groups from Observation 9.2.6. Since the higher Ext-groups of the projective objects P_n vanish by Proposition 9.2.7, this long exact sequence takes the form

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{C^\heartsuit}(M_{n-1}, N) \arrow[r] & \operatorname{Hom}_{C^\heartsuit}(P_n, N) \arrow[r] & \operatorname{Hom}_{C^\heartsuit}(M_n, N) \arrow[dll, out=-10, in=170] \\
 & \mathrm{Ext}^1_C(M_{n-1}, N) \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^1_C(M_n, N) \arrow[dll, out=-10, in=170] \\
 & \mathrm{Ext}^2_C(M_{n-1}, N) \arrow[r] & 0 \arrow[r] & \mathrm{Ext}^2_C(M_n, N) \arrow[dll, out=-10, in=170] \\
 & \cdots & &
\end{tikzcd}$$

Using the initial part of this exact sequence, we see that $\operatorname{Hom}_{C^\heartsuit}(M_{n-1}, N)$ is isomorphic to the kernel of the map $\operatorname{Hom}_{C^\heartsuit}(P_n, N) \to \operatorname{Hom}_{C^\heartsuit}(M_n, N)$, and by injectivity of $\operatorname{Hom}_{C^\heartsuit}(M_n, N) \rightarrowtail \operatorname{Hom}_{C^\heartsuit}(P_{n+1}, N)$ we get an isomorphism

$$\operatorname{Hom}_{C^\heartsuit}(M_{n-1}, N) \quad \cong \quad \ker(\operatorname{Hom}_{C^\heartsuit}(P_n, N) \to \operatorname{Hom}_{C^\heartsuit}(P_{n+1}, N)).$$

Note that for $n = 0$, the left-hand side is $\mathrm{Ext}^0_C(M, N)$ while the right-hand side is $C^0(\operatorname{Hom}_{C^\heartsuit}(P_\bullet, N))$, showing the statement of the proposition for $n = 0$.

Using the rest of the long exact sequence, we conclude that $\mathrm{Ext}^{i+1}_C(M_{n-1}, N) \cong \mathrm{Ext}^i_C(M_n, N)$ for all $i \geq 1$, hence by induction $\mathrm{Ext}^n_C(M, N) \cong \mathrm{Ext}^1_C(M_{n-2}, N)$ for all $n \geq 1$. Furthermore, we see that $\mathrm{Ext}^1_C(M_{n-2}, N)$ is isomorphic to the cokernel of the map $\operatorname{Hom}_{C^\heartsuit}(P_{n-1}, N) \to \operatorname{Hom}_{C^\heartsuit}(M_{n-1}, N)$. Together with the identification of $\operatorname{Hom}_{C^\heartsuit}(M_{n-1}, N)$ as a kernel from before, this provides isomorphisms

$$\mathrm{Ext}^n_C(M, N) \quad \cong \quad \operatorname{coker}(\operatorname{Hom}_{C^\heartsuit}(P_{n-1}, N) \to \ker(\operatorname{Hom}_{C^\heartsuit}(P_n, N) \to \operatorname{Hom}_{C^\heartsuit}(P_{n+1}, N)))$$

for all $n \geq 1$. Since the right-hand side is isomorphic to the n-th cohomology group of the cochain complex $\operatorname{Hom}_{C^\heartsuit}(P_\bullet, N)$, this finishes the proof. $\square$

9.2.2 Projective modules

If C is the module category LMod_R for some ring spectrum R, we may characterize the projective modules as the retracts of free modules, just as in classical algebra.

Definition 9.2.11 (Projective module)**.** Let R be a connective associative ring spectrum. A left R-module P is *projective* if it is is a t-projective object of LMod_R.

Definition 9.2.12 (Free module)**.** Let R be an associative ring spectrum. A left R-module M is *free* if M is a (possibly infinite) coproduct of (unshifted) copies of R, viewed as a left module over itself. A free left R-module is *finitely generated* if it is equivalent to a finite coproduct of copies of R.

Warning 9.2.13. This notion of freeness is distinct from the notion of free R-module from Section 8.1.2: the free R-modules in the sense of Definition 9.2.12 are those R-modules of the form $R \otimes X$ where $X = \bigoplus_{i \in I} \mathbb{S}$ is a (possibly infinite) coproduct of copies of the sphere spectrum.

Proposition 9.2.14 ([Lur17, Proposition 7.2.2.7])**.** *Let R be a connective associative ring spectrum, and let P be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module P is projective.*

(2) *There exists a free R-module M such that P is a retract of M.*

Proof. Suppose first that P is projective. We can choose a map of left R-modules $p\colon N \to P$, where N is free, such that the induced map $\pi_0(p)\colon \pi_0(N) \to \pi_0(P)$ is surjective. (For instance, we can take N to be the free module generated by the set $\pi_0(P)$.) Letting N' be the fiber of p, it follows from the long exact sequence on homotopy groups that N' is connective as well. By part (4) of Proposition 9.2.7 the map $p\colon N \to P$ then admits a section $s\colon P \to N$, exhibiting P as a retract of a free R-module.

For the converse, we observe that the collection of projective left R-modules is stable under retracts. It will therefore suffice to show that every free left R-module is projective. We check criterion (2) from Proposition 9.2.7. Since $\mathrm{Ext}^i_R(\bigoplus_I R, Q) \cong \prod_I \mathrm{Ext}^i_R(R, Q)$, we may assume $P = R$ is free on a single generator. But then $\mathrm{Ext}^i_R(R, Q) = \pi_0 \operatorname{Hom}_R(R, Q[i]) \simeq \pi_0 \operatorname{Hom}_{\mathrm{Sp}}(\mathbb{S}, Q[i]) = \pi_{-i} Q$. Since Q is connective, this is zero for $i > 0$, as desired. □

Remark 9.2.15. If R is a discrete associative ring, it follows from the proposition that the projective R-modules in the sense of Definition 9.2.11 are precisely the projective R-modules in the classical sense.

9.2.3 Projective objects and geometric realizations

In left complete t-structures, there is an alternative characterization of t-projective objects in terms of geometric realizations:

Definition 9.2.16. Let D be an ∞-category with geometric realizations. An object $X \in D$ is called *projective* if the functor $\operatorname{Hom}_D(X, -)\colon D \to \mathrm{An}$ preserves geometric realizations.

Lemma 9.2.17. *Assume that the t-structure on C is left complete. Then $C_{\geq 0}$ admits geometric realizations. If C' is another stable ∞-category with a left complete t-structure, then any right-exact functor $F\colon C_{\geq 0} \to C'_{\geq 0}$ preserves geometric realizations.*

Proof sketch. The assumption that C is left complete implies that the truncation functors $\tau_{\leq n}$ induce an equivalence $C_{\geq 0} \to \lim_n (C_{\geq 0})_{\leq n}$. Since each ∞-category $(C_{\geq 0})_{\leq n}$ is a n-category admitting finite colimits, it admits geometric realizations. (In an n-category, a geometric realization may be computed as a colimit over $(\mathbf{\Delta}_{\leq n+1})^{\mathrm{op}}$, which by [Lur17, Lemma 1.2.4.17] may be computed as a finite colimit. In fact, the indexing diagram $(\mathbf{\Delta}_{\leq n+1})^{\mathrm{op}}$ itself is already finite: see [Cal+23, Example 6.5.3].) Each of the truncation functors $\tau_{\leq n}$ preserves finite colimits, hence geometric realizations. We conclude that $C_{\geq 0}$ admits geometric realizations as well.

The second part is proved in a similar fashion. □

Lemma 9.2.18. *Let C be a stable ∞-category equipped with a left-complete t-structure, and let $P \in C_{\geq 0}$ be a connective object of C. Then P is t-projective if and only if it is a projective object in $C_{\geq 0}$.*

Proof. First assume that P is projective in $C_{\geq 0}$. We will verify condition (2) from Proposition 9.2.7. Given an object $Q \in C_{\geq 0}$, observe that the Čech nerve of the morphism $0 \to Q[1]$ is given by the simplicial diagram

$$\cdots \; Q \oplus Q \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; Q \;\substack{\longrightarrow\\[-1ex]\longrightarrow}\; 0.$$

This is a diagram in $C_{\geq 0}$, so by projectivity of P we see that

$$\mathrm{Ext}^1_C(P,Q) = \pi_0 \operatorname{Hom}_C(P, Q[1]) \simeq \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \pi_0 \operatorname{Hom}_C(P, Q^n).$$

It follows that $\mathrm{Ext}^1_C(P,Q)$ is a quotient of $\pi_0 \operatorname{Hom}_C(P, Q^0) = 0$, hence is zero itself. This shows that P is t-projective.

For the converse, assume now that P is t-projective, giving a t-exact functor $\operatorname{map}_C(P,-)\colon C \to \mathrm{Sp}$. We get from Lemma 9.2.17 that the restricted functor $\operatorname{map}_C(P,-)\colon C_{\geq 0} \to \mathrm{Sp}^{\geq 0}$ preserves geometric realizations. Since $\Omega^\infty\colon \mathrm{Sp}^{\geq 0} \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An}) \to \mathrm{An}$ preserves geometric realizations as well, it follows that so does $\operatorname{Hom}_C(P,-)\colon C_{\geq 0} \to \mathrm{An}$, showing that P is projective in $C_{\geq 0}$. □

Warning 9.2.19. It is not reasonable to ask for P be projective in the ∞-category C itself, since the only projective objects in C are the zero objects. To see this, assume that P is a projective object of C. For any other object $N \in C$, we may write its shift $N[1]$ as the geometric realization of the following diagram:

$$\cdots \; N \oplus N \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; N \;\substack{\longrightarrow\\[-1ex]\longrightarrow}\; 0.$$

We then obtain equivalences

$$\operatorname{Hom}_C(M[-1], N) \simeq \operatorname{Hom}_C(M, N[1]) \simeq |[n] \mapsto \operatorname{Hom}_C(M, N^i)|.$$

Since the right-hand side is connected, we see that $\pi_0 \operatorname{Hom}_C(M[-1], N) = 0$. Replacing N by $N[-n]$ for all $n \in \mathbb{N}$ then gives that $\pi_n \operatorname{Hom}_C(M[-1], N) = 0$ for all n, hence $\operatorname{Hom}_C(M[-1], N) = 0$, showing that M is the zero object.

Corollary 9.2.20. *For a connective associative ring spectrum R, a left module R is projective if and only if it is connective and it is a projective object of $\mathrm{LMod}_R^{\geq 0}$ in the sense of Definition 9.2.16.*

9.3 Flat modules and Tor groups

The notion of projective R-modules has a counterpart, known as *flat modules*, where we work with Tor-groups rather than Ext-groups.

Definition 9.3.1 (Tor-groups)**.** Let R be an associative ring spectrum. For a right R-module M and a left R-module N, we define their *Tor groups* as

$$\mathrm{Tor}_i^R(M,N) \quad := \quad \pi_i(M \otimes_R N).$$

Just like for Ext, we obtain long exact sequences for Tor:

Observation 9.3.2 (Tor long exact sequence)**.** Given an exact sequence $N' \to N \to N''$ of left R-modules, we obtain an exact sequence $M \otimes_R N' \to M \otimes_R N \to M \otimes_R N''$ of spectra, and thus a long exact sequence of homotopy groups of the form

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{n+1}(M,N'') \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}^R_n(M,N') \arrow[r] & \mathrm{Tor}^R_n(M,N) \arrow[r] & \mathrm{Tor}^R_n(M,N'') \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}^R_{n-1}(M,N') \arrow[r] & \cdots &
\end{tikzcd}$$

Similarly, an exact sequence $M' \to M \to M''$ of right R-modules induces a long exact sequence of the form

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{n+1}(M'',N) \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}^R_n(M',N) \arrow[r] & \mathrm{Tor}^R_n(M,N) \arrow[r] & \mathrm{Tor}^R_n(M'',N) \arrow[dll, out=-10, in=170] \\
\mathrm{Tor}^R_{n-1}(M',N) \arrow[r] & \cdots &
\end{tikzcd}$$

Exercise 9.3.3. Let $R = \mathbb{Z}$ be the ring of integers, and let M be an abelian group. Using the short exact sequence

$$0 \to Z \xrightarrow{n} Z \to Z/n\mathbb{Z},$$

show that $\mathrm{Tor}^1(M, \mathbb{Z}/n\mathbb{Z})$ is the subgroup of M given by the n-torsion elements, i.e. those $x \in M$ satisfying $nx = 0$.

Observation 9.3.4. Recall that for a connective right R-module M and a connective R-module N, the relative tensor product $M \otimes_R N$ is again connective: this is a consequence of Lemma 9.1.16. Furthermore, in Observation 9.1.28 we observed that there is an isomorphism $\pi_0(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \cong \pi_0(M \otimes_R N)$. In particular, if R is a discrete ring and M and N are discrete R-modules, we have

$$M \otimes^{\heartsuit}_R N \quad \cong \quad \pi_0(M \otimes_R N) \quad = \quad \mathrm{Tor}^R_0(M,N).$$

We start by recalling the definition of flatness in the discrete setting.

Definition 9.3.5. Let R be a discrete associative ring and let N be a discrete left R-module. Then N is called *flat* if the functor $- \otimes_R^{\heartsuit} N \colon \mathrm{RMod}_R^{\heartsuit} \to \mathrm{Ab}$ is exact: given a short exact sequence

$$0 \to M' \hookrightarrow M \twoheadrightarrow M'' \to 0$$

of discrete right R-modules, the induced sequence of abelian groups

$$0 \to M' \otimes_R^{\heartsuit} N \hookrightarrow M \otimes_R^{\heartsuit} N \twoheadrightarrow M'' \otimes_R^{\heartsuit} N \to 0$$

is again exact.

For the purpose of generalizing this definition to non-discrete rings and modules, we provide a characterization of flatness in terms of not-necessarily-discrete R-modules:

Proposition 9.3.6. *Let R be a discrete associative ring and let $N \in \mathrm{LMod}_R^{\heartsuit}$ be a discrete left R-module. Then the following conditions are equivalent:*

(1) *The left R-module N is flat;*

(2) *For every injection $M' \hookrightarrow M$ of discrete right R-modules, the morphism $M' \otimes_R^{\heartsuit} N \to M \otimes_R^{\heartsuit} N$ is again injective;*

(3) *For every discrete right R-module M, the abelian group $\mathrm{Tor}_1^R(M, N)$ is zero;*

(4) *For every coconnective right R-module M and every $i > 0$, the abelian group $\mathrm{Tor}_i^R(M, N)$ is zero;*

(5) *The tensor product functor $- \otimes_R N \colon \mathrm{RMod}_R \to \mathrm{Sp}$ is left t-exact (hence t-exact).*

Proof. The implication (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), consider a short exact sequence $0 \to M' \hookrightarrow M \twoheadrightarrow M'' \to 0$ of discrete right R-modules. By Corollary 9.1.22, the sequence $M' \to M \to M''$ is exact in RMod_R, hence by Observation 9.3.2 and Observation 9.3.4 we obtain a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_1^R(M, N) \to \mathrm{Tor}_1^R(M'', N) \to M' \otimes_R^{\heartsuit} N \hookrightarrow M \otimes_R^{\heartsuit} N \twoheadrightarrow M'' \otimes_R^{\heartsuit} N \to 0 \to 0 \to \dots .$$

It follows that (2) implies (1). From this long exact sequence, it also immediate that (3) implies (2). For (2) $\Rightarrow$ (3), let $F = \bigoplus_I R$ be a free R-module with a surjection $F \twoheadrightarrow M$ (for example, we may take I to be the underlying set of M). Taking $M' := \ker(F \twoheadrightarrow M)$, the long exact sequence implies that the map $\mathrm{Tor}_1^R(F, N) \to \mathrm{Tor}_1^R(M, N)$ is surjective. But since $F \otimes_R N \cong \bigoplus_I N$ is a discrete R-module, we have $\mathrm{Tor}_1^R(F, N) = 0$, implying that also $\mathrm{Tor}_1^R(M, N) = 0$.

The implication (4) $\Rightarrow$ (3) is clear. For (3) $\Rightarrow$ (4), the same argument from before shows by induction that if M is discrete we have $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$. If M is coconnective but not necessarily discrete, we consider the exact sequence $\tau_{\geq n+1} M \to \tau_{\geq n} M \to \pi_n(M)[n]$, which results in a long exact sequence of Tor-groups

$$\cdots \to \mathrm{Tor}_{i-n+1}^R(\pi_n(M), N) \to \mathrm{Tor}_i^R(\tau_{\geq n+1} M, N) \to \mathrm{Tor}_i^R(\tau_{\geq n} M, N) \to \mathrm{Tor}_{i-n}^R(\pi_n(M), N) \to \dots .$$

Since $\pi_n(M)$ is discrete, the outer two terms vanish whenever $i > n$. In particular, the middle map is an isomorphism whenever $n < 0$. But since $M \simeq \mathrm{colim}_n \tau_{\geq n} M$, it follows that

$$\mathrm{Tor}_i^R(M, N) \simeq \mathrm{colim}_n \mathrm{Tor}_i^R(\tau_{\geq n} M, N) \cong \mathrm{Tor}_i^R(\tau_{\geq 0} M, N).$$

But the latter is zero since $\tau_{\geq 0} M$ is discrete.

Finally, the equivalence (4) $\Leftrightarrow$ (5) is clear: both conditions are saying that if M is a coconnective R-module, then $M \otimes_R N$ is a coconnective spectrum. (The functor $- \otimes_R N$ is always right t-exact by Observation 9.3.4.) $\square$

We will now generalize the definition of flatness to the non-discrete setting.

Definition 9.3.7 (Flat module)**.** Let R be an associative ring spectrum. A left R-module N is said to be *flat* over R if the following conditions are satisfied:

(1) The homotopy group $\pi_0(N)$ is a flat left $\pi_0(R)$-module in the classical sense (Definition 9.3.5).

(2) For each integer n, the map

$$\pi_n(R) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \to \pi_n(N), \qquad [r] \otimes [n] \mapsto [rn]$$

is an isomorphism of abelian groups.

Observation 9.3.8. The following facts are easily verified:

(a) The subcategory of LMod_R spanned by the flat R-modules is closed under coproducts, retracts and filtered colimits.

(b) Every free module is flat. Combining this with (1) and using Proposition 9.2.14, it follows that every projective module is flat.

(c) If R is a connective ring spectrum, then any flat left R-module is also connective.

(d) If R is a discrete ring, then a left R-module N is flat in the sense of Definition 9.3.7 if and only if N is discrete and $\pi_0 N$ is a flat $\pi_0 R$-module in the classical sense.

In the connective setting, the flatness of N is still equivalent to t-exactness of the functor $(-) \otimes_R N$:

Proposition 9.3.9 ([Lur17, Proposition 7.2.2.13, Theorem 7.2.2.15])**.** *Let R be a connective associative ring spectrum and let N be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module N is flat.*

(2) *For every right R-module M and $n \in \mathbb{Z}$, the following map is an isomorphism:*

$$\pi_n(M) \otimes^{\heartsuit}_{\pi_0(R)} \pi_0(N) \xrightarrow{\sim} \pi_n(M \otimes_R N), \qquad [x] \otimes [y] \mapsto [x \otimes y].$$

(3) *The functor* $(-) \otimes_R N \colon \mathrm{RMod}_R \to \mathrm{Sp}$ *is t-exact.*

(4) *If M is a discrete right R-module, then $M \otimes_R N$ is discrete. In particular,* $\mathrm{Tor}^i_R(M, N) = 0$ *for $i > 0$.*

Proof. The implication (2) $\Rightarrow$ (3) is clear: if $\pi_k M = 0$, then also $\pi_k(M \otimes_R N) \cong \pi_k(M) \otimes_{\pi_0(R)} \pi_0(N) = 0$. Conversely, assuming (3), then by Lemma 9.1.34 the t-exact functor $(-) \otimes_R N$ commutes with taking n-th homotopy groups. Furthermore, it follows from Observation 9.3.4 that on the heart $\mathrm{RMod}_R^\heartsuit \simeq \mathrm{RMod}_{\pi_0(R)}^\heartsuit$ it is given by the functor $\pi_0(-) \otimes^\heartsuit_{\pi_0(R)} \pi_0(N)$. We conclude that we get a natural isomorphism

$$\pi_0(M) \otimes^\heartsuit_{\pi_0(R)} \pi_0(N) = \pi_n(M) \otimes_R N \xrightarrow{\sim} \pi_n(M \otimes_R N).$$

This shows (2) $\Leftrightarrow$ (3). The implication (3) $\Rightarrow$ (4) is clear. Conversely, for (4) $\Rightarrow$ (3), note that $(-) \otimes_R N$ is always right t-exact (Observation 9.3.4). For left t-exactness, consider $M \in \mathrm{LMod}_R^{\leq 0}$. Since $\tau_{\geq 0} M = \pi_0 M$ is discrete, $\tau_{\geq 0} M \otimes_R N$ is discrete as well by (4). Similarly, (4) implies that the right-hand term in the exact sequence

$$\tau_{\geq n} M \otimes_R N \to \tau_{\geq n-1} M \otimes_R N \to (\pi_{n-1} M[n-1]) \otimes_R N$$

is concentrated in degree $n-1$, so by induction we see that $\tau_{\geq n} M \otimes_R N$ is contained in $\mathrm{Sp}^{\leq 0}$ for all $n \leq 0$. Writing M as $M \simeq \mathrm{colim}_{n \leq 0} \tau_{\geq n} M$ gives an equivalence $M \otimes_R N \simeq \mathrm{colim}_{n \leq 0}(\tau_{\geq n} M) \otimes_R N$, implying that also $M \otimes_R N \in \mathrm{Sp}^{\leq 0}$. This shows that (3) $\Leftrightarrow$ (4).

We now prove (3) $\Rightarrow$ (1). Assuming (3), the functor $- \otimes_R N \colon \mathrm{RMod}_R \to \mathrm{Sp}$ restricts by Corollary 9.1.23 to an exact functor $\mathrm{RMod}_R^\heartsuit \to \mathrm{Ab}$, which by Observation 9.3.4 is given by the functor $\pi_0(-) \otimes_{\pi_0(R)} \pi_0(N)$. But under the equivalence $\mathrm{RMod}_R^\heartsuit \simeq \mathrm{RMod}_{\pi_0 R}^\heartsuit$ from Lemma 9.1.29, this exact functor corresponds to the functor $- \otimes_{\pi_0(R)} \pi_0(N)$. We conclude that $\pi_0(N)$ is a flat $\pi_0(R)$-module. The fact that the map $\pi_n(M) \otimes_{\pi_0(R)} \pi_0(N) \to \pi_n(M \otimes_R N)$ is an isomorphism for $M = R$ is a special case of (2), which we showed follows from (3). This shows (3) $\Rightarrow$ (1).

Finally, we show that (1) implies (2). Assuming (1), we show that the map in (2) is an isomorphism for progressively more general M. The isomorphism holds by assumption when $M = R$, and so it follows more generally when $M = F[i]$ is the shift of a free right R-module F. We now observe that given an exact sequence $M' \to M \to M''$ of right R-modules, if condition (2) is satisfied for two of the three modules, then it is also satisfied for the third. Indeed, since $\pi_0(N)$ is a flat $\pi_0(R)$-module, the functor $- \otimes_{\pi_0(R)} \pi_0(N)$ is exact, hence the long exact sequence on homotopy groups for $M' \to M \to M''$ stays exact after tensoring. Comparing this with the long exact sequence for $M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N$ and using the five lemma then gives the claim. This gives the claim for an arbitrary connective R-module M: in the proof of Proposition 9.1.18 we saw that M can be written as a colimit of a sequence $M(0) \to M(1) \to M(2) \to \dots$ such that $M(0)$ is a free R-module, and such that the fiber of each $M(i) \to M(i+1)$ is a shift of a free R-module. By induction, condition (2) then holds for each $M(i)$, hence for $M = \mathrm{colim}_i M(i)$. Finally, an arbitrary module M may be written as a colimit of the modules $\tau_{\geq m} M$, each of which is a shift of a connective module. □

Remark 9.3.10. Let $R \in \mathrm{CAlg}(\mathrm{Sp}^{\geq 0})$ be a connective commutative ring spectrum. Then the tensor product of two flat R-modules is again flat. In particular, the subcategory $\mathrm{Mod}_R^\flat \subseteq \mathrm{Mod}_R^{\geq 0}$ spanned by the flat R-modules inherits a symmetric monoidal structure (Lemma 4.4.2).

Just like the vanishing of higher Ext-groups for t-projective objects allowed for the computation of Ext-groups via projective resolutions (see Proposition 9.2.10), the vanishing of higher Tor-groups for flat R-modules lets us compute Tor-groups via flat resolutions.

Definition 9.3.11. Let R be a connective associative ring spectrum and let N be a discrete left R-module. A *flat resolution* of N is a long exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \twoheadrightarrow N \to 0$$

in $\mathrm{LMod}_R^{\heartsuit}$ such that each object F_n is a flat R-module.

Observation 9.3.12. Note that by Observation 9.3.8(b) every projective resolution is in particular a flat resolution.

Proposition 9.3.13 (Tor-groups via flat resolutions)**.** *Let R be a connective associative ring spectrum, and let $F_\bullet$ be a flat resolution of $N \in \mathrm{LMod}_R^{\heartsuit}$. Then for every object $M \in \mathrm{RMod}_R^{\heartsuit}$ and every $n \in \mathbb{Z}$ there is an isomorphism of abelian groups*

$$\mathrm{Tor}_n^R(M,N) \quad \cong \quad C_n(F_\bullet \otimes_R^{\heartsuit} N)$$

between the n-th Tor-group and the n-th homology group of the chain complex $F_\bullet \otimes_R^{\heartsuit} N$.

Proof. The proof is entirely dual to Proposition 9.2.10; we will therefore be rather brief and leave the details to the reader. The flat resolution may be split up into short exact sequences of the form

$$0 \to N_n \rightarrowtail F_n \twoheadrightarrow N_{n-1} \to 0,$$

providing a long-exact sequence on Tor-groups by Observation 9.3.2, in which the higher Tor-groups of F_n vanish by Proposition 9.3.9. One then concludes that

$$M \otimes_R^{\heartsuit} N_{n-1} \quad \cong \quad \mathrm{coker}(M \otimes_R^{\heartsuit} F_{n+1} \to M \otimes_R^{\heartsuit} F_n)$$

for all $n \geq 0$. Finally, one uses an induction to show that

$$\mathrm{Tor}_n^R(M,N) \quad \cong \quad \ker(M \otimes N_{n-1} \to M \otimes F_{n-1}).$$

Combining all these statements then proves the claim. $\square$

For connective ring spectra, flat modules admit a characterization analogous to Lazard's theorem in ordinary algebra, showing that they are precisely the filtered colimits of finitely generated free modules.

Theorem 9.3.14 (Lazard's theorem for spectra, [Lur17, Theorem 7.2.2.15])**.** *Let R be a connective associative ring spectrum and let N be a connective left R-module. The following conditions are equivalent:*

(1) *The left R-module N can be obtained as a filtered colimit of finitely generated free left R-modules.*

(2) *The left R-module N can be obtained as a filtered colimit of finitely generated projective left R-modules.*

(3) *The left R-module N is flat.*

Proof sketch. The implication (1) $\Rightarrow$ (2) is clear: every free module is projective. The implication (2) $\Rightarrow$ (3) follows from the fact that projective modules are flat and the class of flat modules is closed under filtered colimits. For (3) $\Rightarrow$ (1), let $C_{/N} \subseteq (\mathrm{LMod}_R)_{/N}$ denote the full subcategory spanned by those maps $M \to N$ such that M is a finitely generated free left R-module. There is a canonical comparison map $\operatorname{colim}_{M \in C_{/N}} M \to N$. It thus remains to show the following two claims:

- This comparison map is an isomorphism;
- The ∞-category $C_{/N}$ is filtered.

The first claim is a formal consequence of the fact that $\mathrm{LMod}_R^{\geq 0}$ is generated under colimits by the finitely generated free left R-modules. The second claim is more tricky. Lurie ultimately reduces it to an instance of the classical Lazard's theorem for modules over a discrete ring. □

9.4 Perfect modules

In classical algebra, perfect complexes are those that are quasi-isomorphic to a bounded complex of finitely generated projective modules. They represent a notion of "finiteness" or "smallness" in derived categories, analogous to compact objects in topology. In this section, we will introduce the analogous notion for modules over a ring spectrum, and show that these perfect modules may alternatively be characterized as the compact objects in the module category, and in the commutative case as the dualizable objects in the module category. We will also discuss the notion of 'finite Tor-amplitude'.

Definition 9.4.1. Let R be an associative ring spectrum. We denote by

$$\mathrm{Perf}(R) \subseteq \mathrm{LMod}_R$$

the smallest thick subcategory that contains R (i.e. it is a stable subcategory closed under retracts). A left R-module is called *perfect* if it is contained in $\mathrm{Perf}(R)$.

Lemma 9.4.2. *Let R be a connective associative ring spectrum and let M be a perfect R-module. Then M is bounded below: there exists an integer n such that $\pi_i(M) = 0$ for all $i < n$.*

Proof. This is clear from the fact that R is bounded below and that the bounded below R-modules are closed under shifts, retracts and cofibers, hence form a thick subcategory of LMod_R. □

9.4.1 Compact objects

We start by relating perfect modules to compact objects in LMod_R.

Definition 9.4.3 (Compact object)**.** Let C be an ∞-category with filtered colimits. An object $X \in C$ is called *compact* if the functor $\mathrm{Hom}_C(X, -)\colon C \to \mathrm{An}$ preserves filtered colimits. We write $C^\omega \subseteq C$ for the full subcategory of compact objects.

Corollary 9.4.4. *Let C be a stable ∞-category with filtered colimits. Then C^ω is a thick subcategory of C (i.e. it is a stable subcategory closed under retracts).*

Proof. Closure under retracts is clear. For closure under finite colimits, we note that $\mathrm{Hom}_C(\mathrm{colim}_i X_i, -) \simeq \lim_i \mathrm{Hom}_C(X_i, -)$, so the claim follows from the fact that finite limits commute with filtered colimits in An. It remains to show that C^ω is closed under negative shifts, which is clear since $\mathrm{Hom}_C(X[-n], -) \simeq \mathrm{Hom}_C(X, (-)[n])$, and $(-)[n]\colon C \xrightarrow{\sim} C$ preserves all colimits. □

Definition 9.4.5. An ∞-category C is called *compactly generated* if C has small colimits, and there is a *set of compact generators* $\mathcal{G}$, meaning that each object $X \in \mathcal{G}$ is compact and that the objects of $\mathcal{G}$ generate C under filtered colimits.[3]

Proposition 9.4.6 (cf. Hovey, Palmieri, and Strickland [HPS97, Corollary 2.3.12])**.** *Let C be a stable ∞-category with small colimits, and let $\mathcal{G}$ be a set of compact objects of C satisfying the condition that the functors* $\mathrm{Hom}_C(Y[n], -)\colon C \to \mathrm{An}$ *for $Y \in \mathcal{G}$ and $n \in \mathbb{Z}$ are jointly conservative. Then the following two statements hold:*

(1) *Every object of C can be written as a filtered colimit of compact objects. In particular, C is compactly generated.*

(2) *An object $X \in C$ is compact if and only if it lies in the smallest thick subcategory of C containing the objects of $\mathcal{G}$.*

Proof. For the purpose of the proof, let us say that an object $X \in C$ is *$\mathcal{G}$-finite* if it lies in the smallest thick subcategory of C containing the objects of $\mathcal{G}$. It is clear that all $\mathcal{G}$-finite objects are compact: by Corollary 9.4.4 the subcategory $C^\omega \subseteq C$ of compact objects is a thick subcategory containing all the compact generators $\mathcal{G}$.

For part (1), let $X \in C$ be an arbitrary object. Consider the full subcategory $I_X \subseteq C_{/X}$ spanned by the morphisms $Y \to X$ in C such that Y is $\mathcal{G}$-finite. Note that I_X is closed under finite colimits, hence in particular is a filtered ∞-category. The maps $Y \to X$ are by definition natural in I_X, giving rise to a map $\varphi\colon\ \mathrm{colim}_{Y\in I_X} Y \to X$. We will show that φ is an isomorphism in C, which in particular shows (1). By the assumption on $\mathcal{G}$, it suffices to show that for every $\mathcal{G}$-finite object Y' the induced map

$$\varphi \circ -\colon\ \mathrm{Hom}_C(Y', \mathrm{colim}_{Y\in I_X} Y) \to \mathrm{Hom}_C(Y', X)$$

is an equivalence. We will show that it induces isomorphisms on homotopy groups π_k for all $k \geq 0$. Replacing Y' by $Y'[k]$, it suffices to do this for $k = 0$. Since Y' is compact, this amounts to showing that the map

$$\mathrm{colim}_{Y\in I_X} \pi_0 \mathrm{Hom}_C(Y', Y) \to \pi_0 \mathrm{Hom}_C(Y', X)$$

is an isomorphism of abelian groups. Surjectivity is clear: given a map $f\colon Y' \to X$, we may regard the pair (Y', f) as an object of I_X, and f is the image of the class $[\mathrm{id}_{Y'}] \in \pi_0 \mathrm{Hom}_C(Y', Y')$. For injectivity, consider an element of the kernel of this map. Since the left-hand side is a filtered colimit, this element comes from some stage $(Y, f) \in I_X$ of the colimit, given by a morphism $g\colon Y' \to Y$ in C. The condition that it is sent to zero on the right-hand side means that $f \circ g\colon Y' \to X$ is nullhomotopic, hence factors through the cofiber $Y'' := \mathrm{cofib}(g\colon Y' \to Y)$ via some map $h\colon Y'' \to X$. Since Y'' is again $\mathcal{G}$-finite, the pair (Y'', h) lies in I_X. The class $[g] \in \pi_0 \mathrm{Hom}_C(Y', Y)$ then gets mapped to zero in $\pi_0 \mathrm{Hom}_C(Y', Y'')$, and it in particular becomes zero in $\mathrm{colim}_{Y\in I_X} \pi_0 \mathrm{Hom}_C(Y', Y)$, as desired. This finishes the proof of (1).

[3] More precisely: the smallest full subcategory of C which contains $\mathcal{G}$ and is closed under filtered colimits is C itself.

For part (2), we already argued that all $\mathcal{G}$-finite objects are compact. To show that every compact object X is $\mathcal{G}$-finite, we use the colimit-description of X we just established. Since I_X is filtered and X is compact, the identity map $X \to X \simeq \operatorname{colim}_{Y \in I_X} Y$ must factor through some morphism $Y \to X$ in the diagram, exhibiting X as a retract of Y. As Y is $\mathcal{G}$-finite and $\mathcal{G}$-finite objects are closed under retracts, it follows that X is also $\mathcal{G}$-finite, as desired. □

Lemma 9.4.7 ([Lur17, Proposition 7.2.4.4])**.** *Let R be an associative ring spectrum.*

(1) *The object R is compact in the module category* LMod_R.

(2) *The ∞-category* LMod_R *is compactly generated R.*

(3) *A left R-module M is perfect if and only if it is a compact of* LMod_R.

Proof. For (1), note that the mapping spectrum $\mathrm{map}_R(R, M)$ for a left R-module M is equivalent to the underlying spectrum of M, hence $\mathrm{Hom}_{\mathrm{LMod}_R}(R, -)$ is given by the composite of two functors

$$\mathrm{LMod}_R \xrightarrow{\mathrm{fgt}} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{An}.$$

Since both functors preserve filtered colimits, R is compact as a left R-module. Parts (2) and (3) are an instance of Proposition 9.4.6 applied to the ∞-category LMod_R: we must show that the functors

$$\mathrm{Hom}_{\mathrm{LMod}_R}(R[n], -)\colon \mathrm{LMod}_R \to \mathrm{An}$$

are jointly conservative for $n \in \mathbb{Z}$, which is clear since we have $\pi_n(M) \cong \pi_0 \mathrm{Hom}(R[n], M)$. □

9.4.2 Dualizable modules

Recall from last semester that an object X in a symmetric monoidal ∞-category C is dualizable if and only if there is another object $X^\vee$, called its *dual*, and maps $\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1}$ and $\mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee$ such that the two composites

$$X \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} X \otimes X^\vee \otimes X \; \mathrm{id} \otimes \mathrm{ev} X \qquad \text{and} \qquad X^\vee \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} X^\vee$$

are homotopic to the respective identities. We will now relate perfect R-modules to dualizable objects in LMod_R.

Lemma 9.4.8. *Let C be a stably symmetric monoidal ∞-category. Then the collection of dualizable objects forms a thick subcategory of C, i.e. it is a stable subcategory closed under retracts.*

Proof. Recall that an object X is dualizable if and only if the natural map

$$Z \otimes \underline{\mathrm{Hom}}(X, \mathbb{1}) \to \underline{\mathrm{Hom}}(X, Z)$$

is an equivalence for every $Z \in C$. Since both sides are exact functors in X, it follows that the collection of objects X for which it is an equivalence forms a stable subcategory of C. It is also closed under retracts since isomorphisms in C are closed under retracts. □

Lemma 9.4.9. *Let C be a symmetric monoidal ∞-category with geometric realizations, and assume that the tensor product in C preserves geometric realizations in both variables. If the monoidal unit $\mathbb{1} \in C$ is compact, then every dualizable objects is compact.*

Proof. Let X be a dualizable object in C, and let $X^\vee$ denote its dual. Then the functor $X \otimes -: C \to C$ admits a right adjoint of the form $X^\vee \otimes -: C \to C$. In particular, we have $\mathrm{Hom}_C(X, -) \simeq \mathrm{Hom}_C(X \otimes \mathbb{1}, -) \simeq \mathrm{Hom}_C(\mathbb{1}, X^\vee \otimes -)$, and the right-hand side preserves geometric realizations by the assumptions on C. □

In practice, it frequently happens that the converse direction is also true: all compact objects are dualizable.

Lemma 9.4.10 (cf. Hovey, Palmieri, and Strickland [HPS97, Corollary 2.3.12])**.** *Let C be a compactly generated symmetric monoidal ∞-category such that the tensor product preserves colimits in both variables. Let $\mathcal{G}$ be a set of compact generators of C, and assume that each object $Y \in \mathcal{G}$ is dualizable. Then every compact object in C is dualizable.*

Proof. We showed in Proposition 9.4.6 that an object in C is compact if and only if it is $\mathcal{G}$*-finite*, i.e. contained in the thick subcategory of C generated by $\mathcal{G}$. The claim thus immediately follows from the observation that the Part (2) is an immediate consequence of part (1). Indeed, by part (1) we need to show that every $\mathcal{G}$-finite object is dualizable. By Lemma 9.4.8, the subcategory of dualizable objects is a thick subcategory of C. It contains the objects of $\mathcal{G}$ by assumption, and thus it contains all $\mathcal{G}$-finite objects, finishing the proof. □

The situation in which the compact objects agree with the dualizable objects occurs so frequently that it is useful to give it a name:

Definition 9.4.11. A compactly generated stable symmetric monoidal ∞-category C is called *rigid* if the class of compact objects agrees with the class of strongly dualizable objects.

Corollary 9.4.12. *Let C be a presentably symmetric monoidal ∞-category C and assume that C is stable and that it is compactly generated with compact generators $\mathcal{G}$. Then C is rigid if and only if the unit $\mathbb{1}$ is compact and every object $G \in \mathcal{G}$ is dualizable.*

Proof. This is an immediate consequence of Lemma 9.4.9 and Lemma 9.4.10. □

We now apply this to the module category Mod_R for a commutative ring spectrum R.

Lemma 9.4.13 ([Lur17])**.** *Let R be a connective commutative ring spectrum. A left R-module M in* Mod_R *is perfect if and only if it is a dualizable object in the symmetric monoidal ∞-category* $(\mathrm{Mod}_R, \otimes_R)$.

Proof. We showed last semester that the subcategory of Mod_R spanned by the dualizable objects is a stable subcategory closed under retracts. Since the monoidal unit R is dualizable, it follows that every perfect module is dualizable. Conversely, let M be a dualizable module. Then the functor $(-) \otimes_R M: \mathrm{Mod}_R \to \mathrm{Mod}_R$ has a right adjoint given by $- \otimes_R M^\vee$, where $M^\vee$ is the dual of M, giving a natural equivalence

$$\mathrm{Hom}_{\mathrm{Mod}_R}(M, -) \simeq \mathrm{Hom}_{\mathrm{Mod}_R}(R, - \otimes_R M^\vee).$$

Since the right-hand side preserves filtered colimits, M is compact. ◻

9.4.3 Tor-amplitude

We will introduce the notion of Tor-amplitude.

Definition 9.4.14 (Tor-amplitude)**.** Let R be an associative ring spectrum and let M be a left R-module. We say that M has *Tor-amplitude contained in* $[a, b]$ for integers $a \leq b$ if for every discrete right R-module N, the homotopy groups of the relative tensor product $N \otimes_R M$ are concentrated in degrees $[a, b]$, that is, $\mathrm{Tor}_i^R(N, M) = \pi_i(N \otimes_R M) = 0$ whenever $i < a$ or $i > b$. If such a and b exist, M is said to have *finite Tor-amplitude*.

Proposition 9.4.15 ([AG14, Proposition 2.13])**.** *Let R be a connective associative ring spectrum and let M and N be left R-module spectra.*

(1) *If M has Tor-amplitude contained in $[a, b]$, then the shift $M[k]$ has Tor-amplitude contained in $[a + k, b + k]$ for all $k \in \mathbb{Z}$.*

(2) *If M is perfect, then M has finite Tor-amplitude.*

(3) *If R happens to be a commutative ring spectrum and $S \in \mathrm{CAlg}_R^{\geq 0}$ is a connective commutative R-algebra and M has Tor-amplitude contained in $[a, b]$, then the left S-module $S \otimes_R M$ has Tor-amplitude contained in $[a, b]$.*

(4) *If M and N have Tor-amplitude contained in $[a, b]$, then the fiber and cofiber of a map $M \to N$ have Tor-amplitude contained in $[a - 1, b]$ and $[a, b + 1]$.*

(5) *If M is perfect with Tor-amplitude contained in $[0, b]$, then M is connective and $\pi_0 M \cong \pi_0(\pi_0 R \otimes_R M)$.*

(6) *If M is perfect with Tor-amplitude contained in $[a, a]$, then M is equivalent to $P[a]$ for a finitely generated projective left R-module P.*

(7) *If M is perfect with Tor-amplitude contained in $[a, b]$, then there exists an exact sequence $P[a] \to M \to Q$ with P finitely generated projective and Q perfect with Tor-amplitude contained in $[a + 1, b]$.*

Proof. Part (1) is obvious. For (2), since R clearly has finite Tor-amplitude, it remains to show that the collection of modules with finite Tor-amplitude is a thick subcategory of Mod_R. It is clearly closed under shifts and retracts, and closure under cofibers follows from the long exact sequence of Tor groups (Observation 9.3.2). Part (3) is immediate from the isomorphism $N \otimes_S (S \otimes_R M) \cong N \otimes_R M$. Part (4) is immediate from the long exact sequence of Tor groups. TO DO: finish proof. ◻

9.5 Localizations of ring spectra

In classical algebra, given a commutative ring R and a multiplicative subset $S \subseteq R$, one can form the localization $R[S^{-1}]$, a new ring where all elements of S become invertible. This construction is universal: any ring homomorphism $R \to R'$ that sends elements of S to units in R' factors uniquely through the localization map $R \to R[S^{-1}]$. In this section, we will develop an analogous construction for ring spectra.

Observation 9.5.1. If R is a ring spectrum, then its homotopy groups $\pi_*(R)$ form a graded ring, where the multiplication of a class $[x\colon \mathbb{S}^n \to R] \in \pi_n(R)$ and a class $[y\colon \mathbb{S}^m \to R] \in \pi_m(R)$ is the composite

$$[\mathbb{S}^{n+m} \simeq \mathbb{S}^n \otimes \mathbb{S}^m \xrightarrow{x\otimes y} R \otimes R \xrightarrow{m} R] \in \pi_{n+m}(R).$$

Similarly, if M is a left R-module, then $\pi_*(M)$ becomes a graded left $\pi_*(R)$-module. While one can check this by hand, it also follows formally from the fact that the functor $\pi_*\colon \mathrm{Sp} \to \mathrm{Ab}^{\mathbb{Z}}$ from spectra to graded abelian groups is lax monoidal. To see this, we first note that the functor $\pi_0\colon \mathrm{Sp} \to \mathrm{Ab}$ is lax symmetric monoidal. Indeed, $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$ is lax symmetric monoidal, and $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ preserves finite products hence is also canonically symmetric monoidal. Since $\mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set})$, it follows from Theorem 6.5.6 that the lift $\pi_0\colon \mathrm{Sp} \to \mathrm{Ab}$ is lax symmetric monoidal as well.

To deduce that $\pi_*\colon \mathrm{Sp} \to \mathrm{Ab}^{\mathbb{Z}}$ is lax monoidal, we equip the category $\mathrm{Ab}^{\mathbb{Z}} = \mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$ of graded abelian groups with the Day convolution monoidal structure with respect to addition on the set $\mathbb{Z}$. Under (un)currying, the functor π_* corresponds to the functor $\mathbb{Z} \to \mathrm{Fun}(\mathrm{Sp}, \mathrm{Ab})$ given by the composite

$$\mathbb{Z} \xrightarrow{n \mapsto \Omega^n} \mathrm{Fun}(\mathrm{Sp}, \mathrm{Sp}) \xrightarrow{\pi_*} \mathrm{Fun}(\mathrm{Sp}, \mathrm{Ab}),$$

so by the universal property of Day convolution it remains to check both functors are monoidal. But the first functor is monoidal (as $\mathbb{Z}$ is the free monoid on one generator) while the second one is lax monoidal as previously argued.

Definition 9.5.2. Let R be a ring spectrum, and let $S \subseteq \pi_*(R)$ be a subset of homogeneous elements (i.e., every $x \in S$ lies in some $\pi_n(R)$). We say that S is *multiplicative* if it contains the unit $1 \in \pi_0(R)$, and if for every $x, y \in S$ their product xy is also in S.

Definition 9.5.3 (Localization of a ring spectrum)**.** Let R be a ring spectrum and let $S \subseteq \pi_* R$ be a multiplicative subset. The *localization of R at S*, denoted $R[S^{-1}]$, is an R-algebra equipped with a map $\eta\colon R \to R[S^{-1}]$ that satisfies the following property: for every ring spectrum A, precomposition with η induces a monomorphism of animae

$$- \circ \eta\colon \mathrm{Hom}_{\mathrm{Alg}}(R[S^{-1}], A) \hookrightarrow \mathrm{Hom}_{\mathrm{Alg}}(R, A)$$

whose image consists of those commutative R-algebras A for which the image of every element of S under the induced map $\pi_* R \to \pi_* A$ is invertible.

Note that $R[S^{-1}]$ is unique, if it exists. In an analogous way, we may define localization at the level of modules. Let us first introduce some terminology:

Definition 9.5.4. Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset.

(1) A left R-module M is called *S-nilpotent* if for every element $x \in \pi_n(M)$, there exists an element $s \in S$ such that $s \cdot x = 0$ in $\pi_*(M)$.

(2) A left R-module N is called *S-local* if for every element $s \in S$, left multiplication by s induces an isomorphism of graded abelian groups $s \cdot -\colon \pi_*(N) \xrightarrow{\sim} \pi_*(N)$.

We denote by $\mathrm{LMod}_R^{S\text{-nil}}$ and $\mathrm{LMod}_R^{\mathrm{Loc}(S)}$ the full subcategories of LMod_R spanned by the S-nilpotent and S-local modules, respectively.

Observation 9.5.5. The subcategories $\mathrm{LMod}_R^{S\text{-nil}}$ and $\mathrm{LMod}_R^{\mathrm{Loc}(S)}$ are stable subcategories closed under colimits.

Definition 9.5.6. Let R be a ring spectrum and let $S \subseteq \pi_* R$ be a multiplicative subset. Given a left R-module M, its *S-localization* is an S-local left R-module $S^{-1}M$ equipped with a morphism of left R-modules $\eta \colon M \to S^{-1}M$ satisfying the property that for every other S-local left R-module N, precomposition with η induces an equivalence of animae

$$- \circ \eta \colon \mathrm{Hom}_{\mathrm{LMod}_R}(S^{-1}M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LMod}_R}(M, N).$$

Again, $S^{-1}M$ is unique whenever it exists. In the non-commutative setting, this abstract operation of formally inverting elements can be poorly behaved. To ensure a good theory, we need an additional condition on the set S, known as the Ore condition.

Definition 9.5.7 ([Lur17, Definition 7.2.3.1])**.** Let R be a classical associative ring and let $S \subseteq R$ be a multiplicative subset. We say that S satisfies the *left Ore condition* if the following hold:

(a) For every pair of elements $x \in R$ and $s \in S$, there exist elements $y \in R$ and $t \in S$ such that $tx = ys$.

(b) If $xs = 0$ for some $x \in R$ and $s \in S$, then there exists some $t \in S$ such that $tx = 0$.

This seemingly purely algebraic condition can be conveniently rephrased in terms of S-nilpotent modules. For an element $s \in S$ of degree d, we denote by R/Rs the cofiber of the map of left R-modules $\cdot s \colon R[d] \to R$ given by right multiplication by s, i.e., the cofiber of the following composite

$$\cdot s \colon R[d] \simeq R \otimes \mathbb{S}^d \xrightarrow{\mathrm{id} \otimes s} R \otimes R \xrightarrow{m} R.$$

Lemma 9.5.8 ([Lur17, Lemma 7.2.3.11])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset. The following conditions are equivalent:*

(1) *The set S satisfies the left Ore condition.*

(2) *For every element $s \in S$, the left R-module R/Rs is S-nilpotent.*

Proof. Assume first that (1) is satisfied. Let $x \in \pi_n(R/Rs)$; we wish to show that it is annihilated by some element of S. Using the exact sequence

$$\pi_n(R/Rs) \xrightarrow{\varphi} \pi_{n-d-1}(R) \xrightarrow{\cdot s} \pi_{n-1}(R),$$

we deduce that $\varphi(x) \in \pi_{n-d-1}(R)$ is annihilated by right multiplication by s. Using condition (b) of the left Ore condition, we deduce that there exists an element $t \in S$ of degree d' such that $0 = t\varphi(x) = \varphi(tx) \in \pi_{n+d'-d-1}(R)$. Using the exactness of the sequence

$$\pi_{n+d'}(R) \xrightarrow{\psi} \pi_{n+d'}(R/Rs) \xrightarrow{\varphi} \pi_{n+d'-d-1}(R),$$

we conclude that $tx = \psi(y)$ for some $y \in \pi_{n+d'}(R)$. Using condition (a) of the left Ore condition, we can find an element $u \in S$ of degree d'' and an element $z \in \pi_{n+d'+d''-d}(R)$ such that $uy = zs$. It follows that the image of uy in $\pi_{n+d'+d''}(R/Rs)$ vanishes, so that $(ut)x = u(tx) = u\psi(y) = \psi(uy) = 0$. This completes the proof that (1) implies (2).

Now suppose that (2) is satisfied. We will show that S satisfies conditions (a) and (b) of the left Ore condition. For (a), suppose that $x \in \pi_*(R)$ is a homogeneous element of degree n and $s \in S$ has degree d. We wish to show that there exist $y \in \pi_*(R)$ and $t \in S$ such that $tx = ys$. By S-nilpotence, the image of x in $\pi_n(R/Rs)$ is annihilated by multiplication by some homogeneous element $t \in S$ of degree d'. It follows that tx belongs to the kernel of the map $\pi_{n+d'}(R) \to \pi_{n+d'}(R/Rs)$, and therefore to the image of the map $\cdot s\colon \pi_{n+d'-d}(R) \to \pi_{n+d'}(R)$ given by right multiplication by s.

We now verify (b). It suffices to show that if $x \in \pi_n(R)$ is annihilated by right multiplication by some element $s \in S$ of degree d, then $tx = 0$ for some $t \in S$. The exactness of the sequence

$$\pi_{n+d+1}(R/Rs) \xrightarrow{\varphi} \pi_n(R) \xrightarrow{\cdot s} \pi_{n+d}(R)$$

shows that $x = \varphi(y)$ for some $y \in \pi_{n+d+1}(R/Rs)$. Since R/Rs is S-nilpotent, there exists an element $t \in S$ such that $ty = 0$, from which it follows immediately that $tx = t\varphi(y) = \varphi(ty) = 0$. □

Proposition 9.5.9 ([Lur17, Proposition 7.2.3.14]). *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. Let N be a left R-module. Then the following conditions are equivalent:*

(1) *The module N is S-local.*

(2) *For every element $s \in S$ and every integer n, the Hom anima $\mathrm{Hom}_{\mathrm{LMod}_R}(R/Rs[n], N)$ is contractible.*

(3) *For every S-nilpotent module $M \in \mathrm{LMod}_R$, the Hom anima $\mathrm{Hom}_{\mathrm{LMod}_R}(M, N)$ is contractible.*

Proof. Let $s \in S$ be an element of degree d. The exact sequence

$$R[d] \xrightarrow{\cdot s} R \to R/Rs$$

induces an exact sequence of spectra upon applying the mapping spectrum functor $\mathrm{map}_{\mathrm{LMod}_R(-,N)}$, which results in a long exact sequence in homotopy groups=:

$$\cdots \to \pi_{-n}N \xrightarrow{s\cdot} \pi_{d-n}N \to \mathrm{Ext}^{n+1}_R(R/Rs, N) \to \pi_{-n-1}N \to \ldots$$

Here, $\mathrm{Ext}^k_R(M, N) := \pi_k \mathrm{Hom}_{\mathrm{LMod}_R}(M, N)$. Condition (1) states that the map $s\cdot$ is an isomorphism for all n, which is equivalent to the vanishing of the Ext groups for all n. This, in turn, is equivalent to the Hom anima $\mathrm{Hom}_{\mathrm{LMod}_R}(R/Rs, N)$ being contractible. Since this holds for any shift, the equivalence (1) $\Leftrightarrow$ (2) follows.

To prove the implication (3) $\Rightarrow$ (2), we observe that since S satisfies the left Ore condition, the module R/Rs is S-nilpotent for every $s \in S$, as established in Lemma 9.5.8. Thus, condition (3) applied to $M = R/Rs[n]$ immediately yields condition (2).

We now prove the implication (2) $\Rightarrow$ (3). Let M be an S-nilpotent left R-module. A key result, which follows from the left Ore condition, is that any such M can be constructed as the colimit of a sequence of left R-modules

$$0 = M_0 \to M_1 \to M_2 \to \cdots$$

where for each $i \geq 0$, the fiber of the map $M_i \to M_{i+1}$ is a coproduct of modules of the form $(R/Rs_\alpha)[n_\alpha]$ for some elements $s_\alpha \in S$. We refer to [Lur17, Lemma 7.2.3.13] for a proof. Since Hom animae turn colimits in the first variable into limits, we have

$$\mathrm{Hom}_{\mathrm{LMod}_R}(M, N) \xrightarrow{\sim} \lim_i \mathrm{Hom}_{\mathrm{LMod}_R}(M_i, N).$$

It will therefore suffice to show that each Hom anima $\mathrm{Hom}_{\mathrm{LMod}_R}(M_i, N)$ is contractible. We proceed by induction on i. The case $i = 0$ is clear, since $M_0 = 0$. For the inductive step, assume that $\mathrm{Hom}_{\mathrm{LMod}_R}(M_i, N)$ is contractible. Let K_i denote the fiber of the map $M_i \to M_{i+1}$. By construction, we have an exact sequence of modules $K_i \to M_i \to M_{i+1}$. Applying the functor $\mathrm{Hom}_{\mathrm{LMod}_R}(-, N)$ yields a fiber sequence of animae

$$\mathrm{Hom}_{\mathrm{LMod}_R}(M_{i+1}, N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(M_i, N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(K_i, N).$$

By the inductive hypothesis, the middle term is contractible. It therefore suffices to show that the right-hand term, $\mathrm{Hom}_{\mathrm{LMod}_R}(K_i, N)$, is also contractible. By construction, K_i is a coproduct of modules of the form $(R/Rs_\alpha)[n_\alpha]$. Since Hom animae send coproducts in the first variable to products, we have

$$\mathrm{Hom}_{\mathrm{LMod}_R}(K_i, N) \xrightarrow{\sim} \prod_\alpha \mathrm{Hom}_{\mathrm{LMod}_R}((R/Rs_\alpha)[n_\alpha], N).$$

By condition (2), each factor in this product is contractible. It follows that the entire product is contractible, which completes the inductive step and the proof. □

We will now show the existence of the S-localizized R-modules $S^{-1}M$ when S satisfies the left Ore condition, by showing that any left R-module M decomposes into an S-nilpotent part and an S-local part:

Proposition 9.5.10 ([Lur17, Proposition 7.2.3.17, Remark 7.2.3.18])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. Then for every left R-module M, there exists an exact sequence in* LMod_R

$$M' \to M \to M'',$$

where M' is S-nilpotent and M'' is S-local.

Sketch of proof. The collection of S-nilpotent modules $\mathrm{LMod}_R^{S\text{-nil}}$ forms a presentable ∞-category, generated under colimits by the set of objects $\{(R/Rs)[n] \mid s \in S, n \in \mathbb{Z}\}$. Since the inclusion functor $\mathrm{LMod}_R^{S\text{-nil}} \hookrightarrow \mathrm{LMod}_R$ preserves colimits, it follows from the adjoint functor theorem that it admits a right adjoint G. For any module M, the counit of this adjunction gives a map $G(M) \to M$. We define $M' := G(M)$ and we let M'' be the cofiber of this map. It remains to show that M'' is S-local, for which we use criterion (3) from Proposition 9.5.9. Let N be an S-nilpotent left R-module. Applying $\mathrm{map}_{\mathrm{LMod}_R}(N, -)$ provides an exact sequence of spectra

$$\mathrm{map}_{\mathrm{LMod}_R}(N, M') \to \mathrm{map}_{\mathrm{LMod}_R}(N, M) \to \mathrm{map}_{\mathrm{LMod}_R}(N, M'').$$

By adjunction, the first map is an equivalence of spectra, hence it follows that $\mathrm{map}_{\mathrm{LMod}_R}(N, M'') = 0$, as desired. □

Remark 9.5.11. The previous two results show that the ∞-category LMod_R admits a t-structure $(\mathrm{LMod}_R^{S\text{-nil}}, \mathrm{LMod}_R^{\mathrm{Loc}(S)})$ consisting of the S-nilpotent modules and the S-local modules.

Corollary 9.5.12. *Let $S \subseteq \pi_*(R)$ satisfy the left Ore condition. For every left R-module M, there exists an S-localization $S^{-1}M$. In particular, the inclusion $\mathrm{LMod}_R^{\mathrm{Loc}(S)} \hookrightarrow \mathrm{LMod}_R$ admits a left adjoint*

$$S^{-1}(-)\colon \mathrm{LMod}_R \to \mathrm{LMod}_R^{\mathrm{Loc}(S)}.$$

Proof. By Proposition 9.5.10 there is an exact sequence $M' \to M \to M''$ of left R-modules such that M' is S-nilpotent and M'' is S-local. We claim that M'' is an S-localization. Indeed, for any other S-local left R-module N, mapping into N provides a fiber sequence of animae

$$\mathrm{Hom}_{\mathrm{LMod}_R}(M'', N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(M, N) \to \mathrm{Hom}_{\mathrm{LMod}_R}(M', N).$$

But since M' is S-nilpotent, the anima $\mathrm{Hom}_{\mathrm{LMod}_R}(M', N)$ is contractible by part (3) of Proposition 9.5.9. It follows that the first map is an equivalence, as desired. □

The existence of S-localizations of R-modules can be used to deduce the existence of localizations of ring spectra. Regarding R as a module over itself, applying the S-localization functor gives an S-local R-module $S^{-1}R \in \mathrm{LMod}_R$. This object can be endowed with the structure of an associative ring spectrum, which we denote by $R[S^{-1}]$.

Proposition 9.5.13 ([Lur17, Remark 7.2.3.26, Proposition 7.2.3.27])**.** *Let R be an associative ring spectrum and let $S \subseteq \pi_*(R)$ be a multiplicative subset satisfying the left Ore condition. The S-localization $S^{-1}R$ admits the structure of an associative ring spectrum, which we denote by $R[S^{-1}]$. Furthermore, the canonical map $R \to S^{-1}R$ can be enhanced to a morphism of associative ring spectra*

$$\varphi\colon R \to R[S^{-1}].$$

This morphism exhibits $R[S^{-1}]$ as an S-localization of R, in the sense of Definition 9.5.3.

Proof sketch. The idea is to define $R[S^{-1}]$ as the endomorphism spectrum of $S^{-1}R$ in the stable ∞-category $\mathrm{LMod}_R^{\mathrm{Loc}(S)}$, in the sense of Proposition 8.4.8. Part (3) of that result also defines a functor

$$\mathrm{map}(S^{-1}R, -)\colon \mathrm{LMod}_R^{\mathrm{Loc}(S)} \to \mathrm{LMod}_{R[S^{-1}]},$$

which by definition sends $S^{-1}R$ to $R[S^{-1}]$. One can show that this functor is an equivalence. Composing it with the S-localization functor thus provides a colimit-preserving functor

$$\mathrm{LMod}_R \to \mathrm{LMod}_{R[S^{-1}]}.$$

A classification result for such functors then gives that this must be of the form f^* for some algebra map $f\colon R \to R[S^{-1}]$. □

Finally, this abstract homotopical construction recovers the classical ring of fractions when applied to a discrete associative ring. The homotopy groups of the S-localization $S^{-1}M$ can be computed using a graded version of the classical construction of fractions.

Proposition 9.5.14 ([Lur17, Proposition 7.2.3.20, Proof of 7.2.3.5])**.** *Let R be an associative ring and let $S \subseteq R$ be a subset satisfying the left Ore condition. Let M be a (discrete) left R-module. The S-localization $S^{-1}M$ is a discrete R-module spectrum, and its zeroth homotopy group $\pi_0(S^{-1}M)$ is naturally isomorphic to the classical left module of fractions, constructed as the set of equivalence classes of symbols $s^{-1}x$ for $s \in S$ and $x \in M$.*

10 Deformation theory

Our goal in this chapter is to develop an ∞-categorical analogue of the module of Kähler differentials Ω_A, which for a classical commutative ring A classifies derivations into any A-module. Recall that a derivation in an A-module M is an A-module homomorphism $d\colon A \to M$ satisfying the *Leibniz rule*: $d(ab) = ad(b) + bd(a)$. Derivations are used to studying "infinitesimal" deformations of A.

In the setting of higher algebra, it is usually not a good idea to define structures by writing down explicit equations: the equalities ought to be replaced by homotopies, which should themselves satisfy higher coherences. To get around this issue, it is convenient to work with a reformulation of the definition of derivation in terms of *split square-zero extensions*. Observe that the abelian group $A \oplus M$ admits a canonical commutative ring structure whose multiplication is given by $(a, m) \cdot (a', m') := (aa', am' + a'm)$; this ring is called the split square-zero extension of A by M. Every derivation d then defines a ring map $\eta := (\mathrm{id}_A, d)\colon A \to A \oplus M$ with respect to this ring structure. Conversely, for any ring map $A \to A \oplus M$ whose first component is id_A, the second component is necessarily a derivation $d\colon A \to M$. This allows us to formulate the data of a derivation purely in terms of morphisms between commutative rings.

We will use this description of derivations in the setting of ring spectra: given a commutative ring spectrum A and an A-module M, we will construct its split square-zero extension $A \ltimes M$, and define the anima of A-derivations in M as $\mathrm{Der}(A, M) := \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M)$. The higher analogue of the module of Kähler differentials is called the *cotangent complex* L_A of A, which classifies derivations in the sense that for any A-module M there is a natural equivalence

$$\mathrm{Hom}_{\mathrm{LMod}_A}(L_A, M) \simeq \mathrm{Der}(A, M).$$

For a morphism of commutative ring spectra $f\colon A \to B$, there is also a relative cotangent complex $L_{B/A}$, which is a B-module. It fits into a fundamental cofiber sequence

$$B \otimes_A L_A \to L_B \to L_{B/A}.$$

A key application, which we will build towards, is that f is an étale morphism if and only if $L_{B/A} \simeq 0$.

To do: expand with explanation of deformation theory

10.1 Split square-zero extensions and the tangent bundle

We start by constructing an analogue of the split square-zero extension $A \oplus M$ in the world of ring spectra. To this end, let us have a closer look at this construction in the classical setting for a commutative ring A and an A-module M. The ring $A \oplus M$ defined in the introduction comes equipped with various ring homomorphisms:

- There is a 'projection map' $\mathrm{pr}\colon A \oplus M \to A, \quad (a,m) \mapsto a$;
- There is a 'zero map' $0\colon A \to A \oplus M, \quad a \mapsto (a,0)$;
- There is an 'addition map' $+\colon (A \oplus M) \times_A (A \oplus M) \cong A \oplus M \oplus M \to A \oplus M, \quad (a, m_1, m_2) \mapsto (a, m_1 + m_2)$;
- There is a 'minus map' $-\colon A \oplus M \to A \oplus M, \quad (a,m) \mapsto (a, -m)$.

The maps 0, + and − are compatible with the projection maps to A, in the sense that they form morphisms in $\mathrm{CRing}_{/A}$. Moreover, they behave like an abelian group structure on the ring $A \oplus M$: for $x, y, z \in A \oplus M$, the relations $0 + x = x = x + 0$, $(x+y)+z = x+(y+z)$ and $x + (-x) = 0 = (-x) + x$ are satisfied. In particular, the assignment $M \mapsto A \oplus M$ refines to a functor

$$A \oplus -\colon \mathrm{Mod}_A \to \mathrm{Ab}(\mathrm{CRing}_{/A}),$$

where the right-hand side denotes the category of 'abelian group objects' in the category of commutative rings over A. Conversely, given an object $R \in \mathrm{Ab}(\mathrm{CRing}_{/A})$, we in particular have a ring map $p\colon R \to A$, and a zero map $0\colon A \to R$ over A. The latter turns R into an A-module, and the map $p\colon R \to A$ into a morphism of A-modules. We may then define $M := \ker(p\colon R \to A)$, which is then also an A-module. It turns out that these two assignments are inverse equivalences:

Exercise 10.1.1. Show that the functor $A \oplus -\colon \mathrm{Mod}_A \to \mathrm{Ab}(\mathrm{CRing}_{/A})$ is an equivalence of categories, with inverse given by sending $R \to A$ to $\ker(R \to A)$.

This equivalence gives us a recipe for a construction of split square-zero extensions in the context of ring spectra. For a commutative ring spectrum A, we may consider the slice ∞-category $\mathrm{CAlg}_{/A}$ of commutative ring spectra over A. Rather than taking abelian group objects in this slice, we may consider its *spectrum objects*, i.e. its stabilization. Just like in the classical setting, the resulting ∞-category turns out to be equivalent to the ∞-category of A-modules.

Construction 10.1.2. Consider an object of $\mathrm{Sp}(\mathrm{CAlg}_{/A})$. Then R in particular has an underlying pointed object R in $\mathrm{CAlg}_{/A}$, i.e. there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[dr, equal] & R \arrow[d, "p"] \\ & A. \end{tikzcd}$$

The map s turns R into a commutative A-algebra, hence via restriction of scalars into an A-module. Similarly, the map p becomes a morphism of A-modules, and in particular the fiber of p inherits the structure of an A-module. These constructions can be made functorial, resulting in a functor

$$\mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{Mod}_A.$$

Theorem 10.1.3 ([Lur17, Corollary 7.3.4.14])**.** *Let A be a commutative ring spectrum. Then the functor* $\mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{Mod}_A$ *is an equivalence of ∞-categories.*

Taking this result as a black box, we immediately obtain a definition of split square-zero extensions of A-modules:

Definition 10.1.4. Let A be a commutative ring spectrum. We define the *square-zero extension functor* as the composite

$$A \ltimes -\colon \mathrm{Mod}_A \xrightarrow{\sim} \mathrm{Sp}(\mathrm{CAlg}_{/A}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/A}.$$

Given an A-module, we refer to the resulting commutative algebra $A \ltimes M$ as the *square-zero extension of A by M*. The ring map $A \ltimes M \to A$ is called the *augmentation* of $A \ltimes M$.

Remark 10.1.5. Let us provide an informal description of the commutative ring spectrum $A \ltimes M$:

- It comes with morphisms $\mathrm{pr}\colon A \ltimes M \to A$ and $\eta_0\colon A \to A \ltimes M$ in CAlg fitting in a commutative triangle

$$\begin{tikzcd} A \arrow[r, "\eta_0"] \arrow[dr, equal] & A \ltimes M \arrow[d, "\mathrm{pr}"] \\ & A. \end{tikzcd}$$

  The fiber of pr is the given A-module M.

- We in particular obtain a split exact sequence $M \to A \ltimes M \to A$ of A-modules. It follows that the underlying A-module of $A \ltimes M$ is equivalent to the direct sum $A \oplus M$.

- Under this equivalence, the unit map $\mathbb{S} \to A \ltimes M$ is the composite $S \xrightarrow{1} A \xrightarrow{\eta_0} A \to A \oplus M$.

- Moreover, under this equivalence the multiplication map $(A \ltimes M) \times (A \ltimes M) \to A \ltimes M$ corresponds to a map

$$(A \otimes M) \otimes (A \otimes M) \simeq (A \otimes A) \oplus (A \otimes M) \oplus (M \otimes A) \oplus (M \otimes M) \to A \oplus M$$

  which may be described as follows:

  (i) On the summand $A \otimes A$, it is given by the multiplication map $A \otimes A \to A$.

  (ii) On the summands $A \otimes M$ and $M \otimes A$, it is given by the A-module structure on M.

  (iii) On the summand $M \otimes M$, it is nullhomotopic.

  Claim (i) follows from the fact that the map $(\mathrm{id}_A, 0)\colon A \to A \oplus M$ is a morphism in CAlg. Claim (ii) follows from the fact that M agrees with the fiber of $A \oplus M \to A$ *as an A-module*. For claim (iii), we use the naturality of this map in M, giving a map $\psi_N\colon N \otimes N \to N$ for every A-module N. Let M' and M'' be copies of the A-module M, which we will distinguish notationally for clarity, and let $\nabla\colon M' \oplus M'' \to M$ denote the "fold" map which is the identity on each factor. Invoking the functoriality of ψ, we deduce that the map $\psi_M\colon M \otimes M \to M$ factors as a composition

$$M \otimes M = M' \otimes M'' \to (M' \oplus M'') \otimes (M' \oplus M'') \xrightarrow{\psi_{M' \oplus M''}} M' \oplus M'' \xrightarrow{\nabla} M.$$

  Consequently, to prove that ψ_M is nullhomotopic, it will suffice to show that $\varphi = \psi_{M' \oplus M''}|_{(M' \otimes M'')}$ is nullhomotopic. Let $\pi_{M'}\colon M' \oplus M'' \to M'$ and $\pi_{M''}\colon M \oplus M'' \to M''$ denote the projections onto the first and second factor, respectively. To prove that $\psi_{M' \oplus M''}$ is nullhomotopic, it suffices to show that $\pi_{M'} \circ \varphi$ and $\pi_{M''} \circ \varphi$ are nullhomotopic. We now invoke functoriality once more to deduce that $\pi_{M'} \circ \varphi$ is homotopic to the composition

$$M' \otimes M'' \xrightarrow{(id,0)} M' \otimes M' \xrightarrow{\psi_{M'}} M'.$$

This composition is nullhomotopic, since the first map factors through $M' \otimes 0 \simeq 0$. The same argument shows that $\pi_{M''} \circ \varphi$ is nullhomotopic, as desired.

The equivalence from Theorem 10.1.3 can be made functorial in the commutative ring spectrum A. Observe that both sides admit canonical refinements to contravariant functors $\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Cat}_\infty$:

- The assignment $A \mapsto \mathrm{Mod}_A$ is contravariantly functorial by Corollary 8.3.19.
- The assignment $A \mapsto \mathrm{Sp}(\mathrm{CAlg}_{/A})$ inherits its functoriality from the slice functor $\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{lex}}$, $A \mapsto \mathrm{CAlg}_{/A}$ by composition with the stabilization functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty$.

Theorem 10.1.6 ([Lur17, Theorem 7.3.4.18])**.** *The equivalence* $\mathrm{Sp}(\mathrm{CAlg}_{/A}) \simeq \mathrm{Mod}_A$ *from Theorem 10.1.3 refined to a natural equivalence*

$$\mathrm{Sp}(\mathrm{CAlg}_{/-}) \simeq \mathrm{Mod}_{(-)}$$

of functors $\mathrm{CAlg}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

For the definition of the cotangent complex L_A for a commutative ring spectrum, it is convenient to work in a certain ∞-category whose objects are pairs (A, M) consisting of a commutative ring spectrum A and an A-module M. This will be a special case of a construction known as the *tangent ∞-category*.

Construction 10.1.7 (Tangent ∞-category)**.** Let C be an ∞-category with finite limits. We will construct an ∞-category T_C together with a commutative diagram

$$\begin{tikzcd} T_C \arrow[rr, "\Omega^\infty"] \arrow[dr, "\pi_C"'] & & \mathrm{Fun}([1], C) \arrow[dl, "\mathrm{ev}_1"] \\ & C. & \end{tikzcd}$$

To this end, recall from Example 2.1.7 that the target functor $\mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$ is a cartesian fibration, hence straightens to a functor $C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which is given on objects by sending $X \in C$ to the slice $C_{/X}$ and on morphisms by forming pullbacks in C. Observe that each slice $C_{/X}$ admits a terminal object (X, id_X), and that it inherits pullbacks from C. In particular, the assignment $X \mapsto C_{/X}$ refines to a functor

$$C_{/-}\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty^{\mathrm{lex}}$$

into the ∞-category of left exact ∞-categories and left exact functors. Composing with the stabilization functor $\mathrm{Sp}\colon \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty$ thus provides a functor

$$\mathrm{Sp}(C_{/-})\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty.$$

We denote by $\pi_C\colon T_C \to C$ the cartesian unstraightening of $\mathrm{Sp}(C_{/-})$. The counit of the adjunction provides a natural transformation $\Omega^\infty\colon \mathrm{Sp}(C_{/-}) \to C_{/-}$ of functors $C^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which under (un)straightening corresponds to a cartesian functor $\Omega^\infty\colon T_C \to \mathrm{Fun}([1], C)$ over C. We refer to T_C as the *tangent ∞-category of C* and to the functor $T_C \to C$ as the *tangent bundle of C*.

Remark 10.1.8. The terminology 'tangent bundle' is inspired by Goodwillie calculus. Goodwillie [Goo90; Goo92; Goo03] developed a formalism for taking the 'derivative' of a functor between suitable ∞-categories, in analogy to the derivative of a smooth map of manifolds. Under this analogy, the stabilization of an ∞-category may be thought of as a 'linear approximation' of the ∞-category. In particular, for an ∞-category C with finite limits, we may loosely think of the stabilization $\mathrm{Sp}(C_{/X})$ as the 'tangent space of C at the object X'.

Example 10.1.9. Let $C := \mathrm{CAlg}$ be the ∞-category of commutative ring spectra. There is an equivalence of ∞-categories

$$T_{\mathrm{CAlg}} \quad \simeq \quad \mathrm{Mod}(\mathrm{Sp})$$

between the tangent ∞-category of CAlg and the ∞-category of modules in Sp. Indeed, by Theorem 10.1.6, the functor $\mathrm{Sp}(\mathrm{CAlg}_{/-})$ is naturally equivalent to the funtor $\mathrm{Mod}_{(-)}$. As a result, the ∞-category T_{CAlg} is equivalent to the unstraightening of the functor $A \mapsto \mathrm{Mod}_A$. But the latter is by definition the forgetful functor $\mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp})$.

In particular, we have the following informal description of T_{CAlg}:

- The objects of T_{CAlg} are pairs (A, M), where $A \in \mathrm{CAlg}$ and $M \in \mathrm{Mod}_A$;
- A morphism $(A, M) \to (B, N)$ consists of a morphism $f \colon A \to B$ in CAlg together with a morphism $M \to f_*N$ of A-modules.

Under this description, the functor $\Omega^\infty \colon T_{\mathrm{CAlg}} \to \mathrm{Fun}([1], \mathrm{CAlg})$ sends the pair (A, M) to the augmentation $A \ltimes M \to A$.

Corollary 10.1.10. *Let $f \colon B \to A$ be a morphism of commutative ring spectra, and let M be an A-module. Then there is a canonical pullback square in* CAlg *of the form*

$$\begin{tikzcd} B \ltimes f_*M \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \ltimes M \arrow[d] \\ B \arrow[r, "f"] & A. \end{tikzcd}$$

Proof. The forgetful functor $\mathrm{Mod}(\mathrm{Sp}) \to \mathrm{CAlg}$ is a cartesian fibration by Corollary 8.3.19, and the B-module f_*M is obtained by picking a cartesian lift $(B, f_*M) \to (A, M)$ of the morphism $f \colon B \to A$. The functor $\Omega^\infty \colon \mathrm{Mod}(\mathrm{Sp}) \to \mathrm{Fun}([1], \mathrm{CAlg})$ from Construction 10.1.7 is by construction a cartesian functor over CAlg, so the image of the morphism $(B, f_*M) \to (A, M)$ in $\mathrm{Fun}([1], \mathrm{CAlg})$ is cartesian with respect to the evaluation functor $\mathrm{ev}_1 \colon \mathrm{Fun}([1], \mathrm{CAlg}) \to \mathrm{CAlg}$. But this is precisely expressing that the corresponding commutative square in CAlg is a pullback square, see Example 2.1.7. □

10.2 Derivations and the cotangent complex

Equipped with a spectral analogue of split square-zero extensions, we will now return to the definition of the cotangent complex L_A.

Definition 10.2.1 (Derivation)**.** Let A be a commutative ring spectrum, and let M be an A-module. An *M-valued derivation of A* is a morphism $\eta\colon A \to A \ltimes M$ in $\mathrm{CAlg}_{/A}$. We denote by

$$\mathrm{Der}(A, M) \quad := \quad \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M)$$

the anima of M-valued derivations of A. More generally, if A is a commutative B-algebra for some $B \in \mathrm{CAlg}$, we may regard $A \ltimes M$ as a commutative B-algebra as well via the composite

$$B \to A \to A \ltimes M.$$

We then define an *B-linear M-valued derivation of A* to be a morphism $\eta\colon A \to A \ltimes M$ in $(\mathrm{CAlg}_B)_{/A}$. We denote by

$$\mathrm{Der}_B(A, M) \quad := \quad \mathrm{Hom}_{(\mathrm{CAlg}_B)_{/A}}(A, A \ltimes M)$$

the anima of B-linear M-valued derivations of A. We may visualize such map as follows:

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \ltimes M \arrow[d, "\mathrm{pr}"] \\ A \arrow[r, equal] \arrow[ur, dashed, "\eta"] & A. \end{tikzcd}$$

Example 10.2.2 (Zero derivation)**.** By construction, the commutative ring spectrum $A \ltimes M$ comes equipped with a ring homomorphism $\eta_0\colon A \to A \ltimes M$. This is a section of $\mathrm{pr}\colon A \ltimes M \to A$, hence corresponds to an M-valued derivation of A, called the *zero derivation*. Under the equivalence $A \ltimes M \simeq A \oplus M$ of A-modules from Remark 10.1.5, this map corresponds to $(\mathrm{id}_A, 0)\colon A \to A \oplus M$. Similarly, since $A \ltimes M$ lies in the image of the functor $\mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{CAlg}_{/A}$, it comes equipped with a canonical structure of a commutative group object. Since $\mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, -)\colon \mathrm{CAlg}_{/A} \to \mathrm{An}$ preserves finite products, it follows that the anima $\mathrm{Der}(A, M)$ inherits a canonical structure of a commutative group object in An, whose zero element is η_0. In particular, given two derivations η_1 and η_2 we may form their sum $\eta_1 + \eta_2$ and difference $\eta_1 - \eta_2$.

Remark 10.2.3. Observe that the assignment $M \mapsto \mathrm{Der}_B(A, M)$ refines to a functor $\mathrm{Mod}_A \to \mathrm{An}$: it is given by the composite

$$\mathrm{Mod}_A \xrightarrow{A \ltimes -} (\mathrm{CAlg}_B)_{/A} \xrightarrow{\mathrm{Hom}_{(\mathrm{CAlg}_B)_{/A}}(A, -)} \mathrm{An}.$$

Definition 10.2.4 ((Relative) Cotangent complex)**.** Let $B \to A$ be a morphism of commutative ring spectra. An B-linear derivation $\eta_{A/B}\colon A \to A \ltimes L_{A/B}$ valued in some A-module $L_{A/B}$ is called *universal* if for every other A-module M the induced map

$$\mathrm{Hom}_A(L_A, M) \to \mathrm{Der}_B(A, M), \quad f \mapsto ((A \ltimes f) \circ \eta_{A/B})$$

is an equivalence of animae. This condition determines the A-module $L_{A/B}$ up to contractible choice, if it exists, in which case we refer to it as the *relative cotangent complex of A over B*. When $B = \mathbb{S}$ is the sphere spectrum, we write $L_A := L_{A/\mathbb{S}}$, and refer to it as the *(absolute) cotangent complex of A*.

We will now show that the relative cotangent complex $L_{A/B}$ exists for every morphism $B \to A$. We start with the absolute case L_A. Note that in this case, the A-module L_A should classify morphisms $A \to A \ltimes M$ in $\mathrm{CAlg}_{/A}$, so the existence is guaranteed if we knew that the functor $A \ltimes -\colon \mathrm{LMod}_A \to \mathrm{CAlg}_{/A}$ admits a left adjoint. Due to the equivalence $\mathrm{LMod}_A \simeq \mathrm{Sp}(\mathrm{CAlg}_{/A})$ from Theorem 10.1.3, this is equivalent to the functor $\Omega^\infty\colon \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{CAlg}_{/A}$ admitting a left adjoint $\Sigma^\infty_+\colon \mathrm{CAlg}_{/A} \to \mathrm{Sp}(\mathrm{CAlg}_{/A})$. This in fact holds in much larger generality:

Proposition 10.2.5. *Let C be an ∞-category with finite limits and small colimits, and assume that $\Omega\colon C_* \to C_*$ preserves sequential colimits. Then the functor $\Omega^\infty\colon \mathrm{Sp}(C) \to C$ admits a left adjoint $\Sigma^\infty_+\colon C \to \mathrm{Sp}(C)$.*

Proof. Recall from Section 6.2 the equivalence $\mathrm{Sp}(C) \simeq \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C)$ between the stabilization of C and the ∞-category of reduced excisive functors from finite pointed animae to C. Under this equivalence, the functor Ω^∞ corresponds to the evaluation functor

$$\mathrm{ev}_{S^0}\colon\ \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \to C.$$

Recall from Proposition 6.4.3 that the inclusion $\mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, C) \hookrightarrow \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C)$ admits a left adjoint P_1. It thus remains to show that also $\mathrm{ev}_{S^0}\colon \mathrm{Fun}(\mathrm{An}_*^{\mathrm{fin}}, C) \to C$ admits a left adjoint. But this is given by left Kan extension along $S^0\colon * \to \mathrm{An}_*^{\mathrm{fin}}$, which exists since C admits small colimits. □

Corollary 10.2.6. *The functor $\Omega^\infty\colon \mathrm{Sp}(\mathrm{CAlg}_{/A}) \to \mathrm{CAlg}_{/A}$ admits a left adjoint.*

Proof sketch. We will verify the conditions of Proposition 10.2.5. The fact that $\mathrm{CAlg}_{/A}$ admits finite limits follows from the existence of finite limits in CAlg, and similarly for colimits. These in turn follow from the following general facts for a symmetric monoidal ∞-category C:

(i) Assume that C admits I-indexed limits for some ∞-category I. Then also $\mathrm{CAlg}(C)$ admits I-indexed limits, and these are preserved by the forgetful functor $\mathrm{CAlg}(C) \to C$. A reference is [Lur17, Corollary 3.2.2.5] applied to the commutative ∞-operad;

(ii) Assume that C admits I-indexed colimits for a sifted ∞-category I, and that the tensor product on C commutes with I-indexed colimits in both variables. Then also $\mathrm{CAlg}(C)$ admits I-indexed colimits, and these are preserved by the forgetful functor $\mathrm{CAlg}(C) \to C$. A reference is [Lur17, Corollary 3.2.3.2];

(iii) The ∞-category $\mathrm{CAlg}(C)$ always has finite coproducts: the initial object is the monoidal unit $\mathbb{1}$ (Corollary 8.1.15) while the coproduct of A and B is given by the tensor product $A \otimes B$. A reference is [Lur17, Corollary 3.2.4.8].

(iv) In particular, if C admits small operadic colimits, $\mathrm{CAlg}(C)$ again admits small colimits.

Applying this to $C = \mathrm{Sp}$ shows that CAlg admits small limits and colimits. The functor $\Omega\colon (\mathrm{CAlg}_{/A})_* \to (\mathrm{CAlg}_{/A})_*$ preserves sequential colimits since both Ω and the colimit are computed on underlying spectra, where the claim holds. □

Proposition 10.2.7. *The relative cotangent complex $L_{A/B}$ exists for every morphism $B \to A$ of commutative ring spectra.*

Proof. Step 1: We first show that the absolute cotangent complex L_A exists (i.e. the claim when $B = \mathbb{S}$ is the sphere spectrum). In this case, the composite

$$A \ltimes -\colon \mathrm{Mod}_A \simeq \mathrm{Sp}(\mathrm{CAlg}_{/A}) \xrightarrow{\Omega^\infty} \mathrm{CAlg}_{/A}$$

admits a left adjoint by the previous corollary, and we define L_A to be the image of id_A under this left adjoint. The unit of the adjunction takes the form of a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "\eta_A"] \arrow[dr, equal] & & A \ltimes M \arrow[dl] \\ & A, & \end{tikzcd}$$

and the adjunction equivalence says that for every A-module M the induced composite

$$\mathrm{Hom}_A(L_A, M) \to \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A \ltimes L_A, M) \xrightarrow{-\circ\eta_A} \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M) = \mathrm{Der}_A(A, M)$$

is an equivalence. This is precisely the defining universal property of L_A.

Step 2: For a morphism $f\colon B \to A$ in CAlg, we claim that the A-module f^*L_B corepresents the functor $M \mapsto \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(B, A \ltimes M)$. Indeed, for an A-module M, there are natural equivalences

$$\mathrm{Hom}_A(f^*L_B, M) \overset{(1)}{\simeq} \mathrm{Hom}_B(L_B, f_*M) \overset{(2)}{\simeq} \mathrm{Hom}_{\mathrm{CAlg}_{/B}}(B, B \ltimes f_*M) \overset{(3)}{\simeq} \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(B, A \ltimes M),$$

where (1) holds by adjunction, (2) by the defining property of L_B, and (3) because of the pullback square

$$\begin{tikzcd} B \ltimes f_*M \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \ltimes M \arrow[d] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

from Corollary 10.1.10. Note that precomposition with f defines a natural map

$$\mathrm{Hom}_A(L_A, M) \simeq \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M) \to \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(B, A \ltimes M) \simeq \mathrm{Hom}_A(f^*L_B, M),$$

so that by Yoneda we obtain a morphism $f^*L_B \to L_A$ of A-modules.

Step 3: For an arbitrary morphism $f\colon B \to A$ in CAlg, we define $L_{A/B}$ via the following cofiber sequence of A-modules:

$$f^*L_B \to L_A \to L_{A/B}.$$

We have to show that for every A-module M there is a natural equivalence $\mathrm{Hom}_A(L_{A/B}, M) \simeq \mathrm{Der}_B(A, M)$. To this end, consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_A(L_{A/B}, M) \arrow[r] \arrow[d, dashed, "\simeq"'] & \mathrm{Hom}_A(L_A, M) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_A(f^*L_B, M) \arrow[d, "\simeq"] \\ \mathrm{Hom}_{(\mathrm{CAlg}_B)_{/A}}(A, A \ltimes M) \arrow[r] & \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M) \arrow[r, "-\circ f"] & \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(B, A \ltimes M). \end{tikzcd}$$

The top sequence is a fiber sequence, due to the definition of $L_{A/B}$ as a cofiber. The middle vertical map is an equivalence by definition of L_A and the right vertical map is the equivalence from Step 2. The right square commutes by definition of the map $f^*L_B \to L_A$. Now, since we may identify the ∞-category $(\mathrm{CAlg}_B)_{/A}$ with the slice category $(\mathrm{CAlg}_{/A})_{B/}$, the formula for Hom animae in slice categories tells us that also the bottom sequence is a fiber sequence. This results in a natural equivalence

$$\mathrm{Hom}_A(L_{A/B}, M) \simeq \mathrm{Hom}_{(\mathrm{CAlg}_B)_{/A}}(A, A \ltimes M) = \mathrm{Der}_B(A, M).$$

By picking $\eta_{A/B} \colon A \to A \ltimes L_{A/B}$ to be the B-linear derivation corresponding to $\mathrm{id}_{L_{A/B}}$ under this equivalence, we deduce that $L_{A/B}$ is a relative cotangent bundle for $f \colon B \to A$. $\square$

We record the following consequence from the proof:

Corollary 10.2.8. *For a morphism $f \colon B \to A$ of commutative ring spectra, there is an exact sequence of A-modules*

$$f^* L_B \to L_A \to L_{A/B}.$$ $\square$

Let us record the basic interaction of relative cotangent complexes with base change and composition:

Proposition 10.2.9. *(1) (Base change) For every pushout square of commutative ring spectra*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B' \arrow[d] \\ A \arrow[r, "g"'] \arrow[ur, phantom, "\ulcorner" near end] & A' \end{tikzcd}$$

we have an isomorphism of A'-modules $L_{A'/B'} \simeq g^* L_{A/B}$.

(2) (Composites) For every pair of composable morphisms of commutative ring spectra $C \xrightarrow{g} B \xrightarrow{f} A$, we have an exact sequence of A-modules

$$f^* L_{B/C} \to L_{A/C} \to L_{A/B}.$$

Proof. (1) For every A'-module M we have natural equivalences

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_{A'}}(L_{A'/B'}, M) &\overset{(1)}{\simeq} \mathrm{Hom}_{(\mathrm{CAlg}_{B'})_{/A'}}(A', A' \ltimes M) \overset{(2)}{\simeq} \mathrm{Hom}_{(\mathrm{CAlg}_{B})_{/A'}}(A, A' \ltimes M) \\ &\overset{(3)}{\simeq} \mathrm{Hom}_{(\mathrm{CAlg}_{B})_{/A}}(A, A \ltimes g_* M) \overset{(4)}{\simeq} \mathrm{Hom}_{\mathrm{Mod}_{A}}(L_{A/B}, g_* M) \\ &\overset{(5)}{\simeq} \mathrm{Hom}_{\mathrm{Mod}_{A'}}(g^* L_{A/B}, M), \end{aligned}$$

providing the desired equivalence via the Yoneda lemma. Here (1) and (4) hold by universal properties, (5) by the adjunction $g^* \dashv g_*$, (2) by the fact that A' is a pushout of A and B' over B, and (3) by the fact that $A \ltimes g_* M$ is a pullback of $A' \ltimes M$ along $A \to A'$ (Corollary 10.1.10).

(2) The desired cofiber sequence is obtained by passing to vertical cofibers in the following diagram of cofiber sequences:

$$\begin{tikzcd} g^*(f^* L_C) \arrow[r, "\simeq"] \arrow[d] & (gf)^* L_C \arrow[r] \arrow[d] & 0 \arrow[d] \\ g^* L_B \arrow[r] & L_A \arrow[r] & L_{A/B}. \end{tikzcd}$$

Here we use Corollary 10.2.8 three times: once to conclude that the bottom sequence is exact, $\square$

10.3 Square-zero extensions and deformations

Let A be a classical commutative ring. A *square-zero extension* of A is a commutative ring A' together with a surjective ring homomorphism $\varphi\colon A' \to A$ satisfying the property that the product of any two elements in the kernel $M := \ker(\varphi)$ is zero. We may think of the ring A' as an 'infinitesimal thickening' of the ring A: its elements can be thought of as elements of A plus an 'infinitesimal' part lying in M. In this situation, there is a close connection between (flat) A-algebras and (flat) A'-algebras. Given an A'-algebra B', we may form the A-algebra $B := A \otimes_{A'} B'$ via extentension of scalars. In good situations, we can go in the other direction as well: every A-algebra B arises in this form for some A'-algebra B'. The classification of all such choices of B' is the topic of *deformation theory*.

The classification of square-zero extensions of A is closely related to derivations and the module of Kähler differentials Ω_A. To see this, first observe that for every A-module M, the split square-zero extension $A \ltimes M$ defined in the introduction of this chapter is indeed a square-zero extension. In the other direction, observe that for any square-zero extension $\pi\colon A' \to A$, the kernel M of π can be equipped with the structure of an A-module by setting $a \cdot m := a'm$ for any $a' \in A'$ satisfying $\varphi(a') = a$; this is independent of the choice of a' due to the fact that $(a'_1 - a'_2)m = 0$ whenever $\varphi(a'_1) = \varphi(a'_2)$. We say that the extension π is *split* if it admits a section ('splitting') $\sigma\colon A \to A'$. In this case, the map $A' \to A \ltimes M$, $a' \mapsto (\pi(a), a' - \sigma(\pi(a')))$ is an isomorphism of commutative rings over A. In this way, splittings of π correspond precisely to identifications of the extension A' with the 'trivial' extension $A \ltimes M$.

Not every square-zero extension is split; for example, the morphism $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\,\mathbb{Z}$ does not admit a splitting. Furthermore, if A' is split, the section σ may not be unique. However, as discussed before, the set of sections of the split square-zero extension $A \ltimes M \to A$ is precisely the set $\mathrm{Der}(A, M)$ of M-valued derivations of A. In particular, we deduce that the *automorphism group* of the trivial square zero extension of A by M can be identified with the group of R-module homomorphisms $\mathrm{Ext}^0_R(\Omega_A, M)$.

In a similar way, the group $\mathrm{Ext}^1_A(\Omega_A, M)$ is related to the set of *isomorphism classes* of square-zero extensions of A. Given a class $\eta \in \mathrm{Ext}^1_A(\Omega_A, M) = \pi_0 \operatorname{Hom}_A(\Omega_A, M[1])$, we may form the corresponding extension of Ω_A by M, i.e. the short exact sequence

$$0 \to M \hookrightarrow M^\eta \overset{f}{\twoheadrightarrow} \Omega_A \to 0$$

of A-modules, where $M^\eta := \mathrm{fib}(\eta)$. We may now consider the pullback

$$\begin{tikzcd} A^\eta \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "d"] \\ M^\eta \arrow[r, two heads] & \Omega_A \end{tikzcd}$$

in the category of abelian groups, where d is the universal derivation on A. The elements of A^η are pairs (a, m') with $a \in A$ and $m' \in M^\eta$ satisfy the equation $f(m') = da$. We may equip the abelian group A^η with a commutative ring structure given by

$$(a_1, m'_1) \cdot (a_2, m'_2) := (a_1 a_2, a_1 m'_2 + a_2 m'_1).$$

It is then clear that the morphism $A^\eta \twoheadrightarrow A$ is a square-zero extension of A by M. However, not every square-zero extension of A by M can be obtained from this construction. In order to obtain *all* square-zero extensions of A, we need to replace the module of Kähler differentials Ω_A by the cotangent complex L_A.

In this section, we will generalize this discussion to the setting of higher algebra, following [Lur17, Section 7.4].

10.3.1 Square-zero extensions

Let us start by introducing the 'higher analogue' of a square-zero extension. The primary example will be the maps $\tau_{\leq n}A \to \tau_{\leq n-1}A$ appearing in the Postnikov tower of a connective commutative ring spectrum A.

Definition 10.3.1. Let A be a commutative ring spectrum and let M be a left A-module. Given an $M[1]$-valued derivation $\eta\colon A \to A \ltimes M[1]$ of A, we define the commutative ring spectrum A^η as the following pullback in CAlg:

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta"] \\ A \arrow[r, "\eta_0"] & A \ltimes M[1]. \end{tikzcd}$$

Here the bottom horizontal map is the zero derivation from Example 10.2.2. We say a morphism $A' \to A$ in CAlg is a *square-zero extension by M* if it is obtained as the left vertical map in such a diagram.

Note that the fiber of the map $A^\eta \to A$ is M, so that A^η is indeed an extension of A by M.

Remark 10.3.2. If A is a discrete ring and M is a discrete module, then the ring A^η is also discrete and agrees with the previous definition of A^η.

Let us make precise the way in which the fiber of the map $A^\eta \to A$ 'squares to zero'.

Construction 10.3.3. Let $f\colon A' \to A$ be a morphism in CAlg, and let $M := \ker(f)$ denote its fiber. By regarding f as a morphism in $\mathrm{Mod}_{A'}$, M inherits a canonical A'-module structure. If we regard A' and A as commutative algebras in $\mathrm{Mod}_{A'}$, we get a commutative diagram

$$\begin{tikzcd} M \otimes_{A'} M \arrow[r] \arrow[d, dashed] & A' \otimes_{A'} A' \arrow[r, "f \otimes_{A'} f"] \arrow[d, "\simeq"] & A \otimes_{A'} A \arrow[d] \\ M \arrow[r] & A' \arrow[r, "f"] & A, \end{tikzcd}$$

where the right vertical map is the multiplication on A. Since the top composite factors through $0 \otimes_{A'} 0$, we obtain a 'multiplication map' $M \otimes_{A'} M \to M$.

Remark 10.3.4. Since the composite $A' \otimes_{A'} A' \to A \otimes_{A'} A \to A$ agrees with $A' \otimes_{A'} A \to A' \otimes_{A'} A \xrightarrow{\sim} A$, we see that the multiplication map on M may alternatively be described as the composite

$$M \otimes_{A'} M \to A' \otimes_{A'} M \xrightarrow{\sim} M.$$

Lemma 10.3.5. *Let $A' \to A$ be a square-zero extension, and let $M := \mathrm{fib}(A' \to A)$. Then the multiplication map $M \otimes_{A'} M \to M$ is nullhomotopic.*

Proof. We may assume that $A' = A^\eta$ for some derivation $\eta\colon A \to A \ltimes M[1]$, i.e. that A' sits in a pullback square of commutative ring spectra as follows:

$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\eta"] \\ A \arrow[r, "\eta_0"] & A \ltimes M[1]. \end{tikzcd}$$

The vertical fibers in this diagram are both M, Regarding this as a diagram of commutative A'-algebras, the multiplication maps on M coming from the left vertical map multiplication map coming from the right vertical map, in the sense that it factors as follows:

$$M \otimes_{A'} M \to M \otimes_A M \to M.$$

It thus suffices to show that the second map is zero. By Remark 10.3.4, this map may be factored as $M \otimes_A M \to A \otimes_A M \xrightarrow{\sim} M$, hence it suffices to show that $M \to A$ is the zero morphism. But this holds true since it is the fiber of the map $\eta\colon A \to A \ltimes M[1]$, which admits a retraction $\mathrm{pr}\colon A \ltimes M[1] \to A$. □

In general, a square-zero extension $A' \to A$ of A by M may not come from a *unique* M-valued derivation on A. However, this is satisfied under some mild assumptions.

Definition 10.3.6. Let $f\colon A' \to A$ be a morphism of connective commutative ring spectra, and let $n \geq 0$. We say that f is an *n-small extension* if $\mathrm{fib}(f)$ has homotopy groups concentrated in degrees $[n, 2n]$ and the multiplication map $\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f) \to \mathrm{fib}(f)$ is zero.

We say that a derivation $\eta\colon A \to A \oplus M[1]$ is *n-small* if the associated square-zero extension $A^\eta \to A$ is n-small.

Remark 10.3.7. Let $f\colon A' \to A$ be a morphism in $\mathrm{CAlg}^{\geq 0}$ and assume that $\mathrm{fib}(f)$ is concentrated in degrees $[n, 2n]$. Then we in particular have $\mathrm{fib}(f) \in \mathrm{Mod}_{A'}^{\geq n}$. By connectivity of A', it then follows that $\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f) \in \mathrm{Mod}_{A'}^{\geq 2n}$. Since we also assume $\mathrm{fib}(f) \in \mathrm{Mod}_{A'}^{\leq 2n}$, we get an isomorphism

$$\begin{aligned} \mathrm{Hom}_{A'}(\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f), \mathrm{fib}(f)) &\simeq \mathrm{Hom}_{A'}(\tau_{\leq 2n}(\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f)), \tau_{\geq 2n}(\mathrm{fib}(f))) \\ &\simeq \mathrm{Hom}_{A'}(\pi_n \mathrm{fib}(f) \otimes^\heartsuit \pi_n \mathrm{fib}(f), \pi_{2n} \mathrm{fib}(f)). \end{aligned}$$

We conclude that the multiplication map $\mathrm{fib}(f) \otimes_{A'} \mathrm{fib}(f) \to \mathrm{fib}(f)$ is zero if and only if a certain map

$$\pi_n \mathrm{fib}(f) \otimes^\heartsuit \pi_n \mathrm{fib}(f) \to \pi_{2n} \mathrm{fib}(f)$$

is zero in $\mathrm{Mod}^\heartsuit_{\pi_0(A')}$.

In particular, this condition is trivially satisfied if $\pi_{2n} \mathrm{fib}(f) = 0$.

Construction 10.3.8. The assignment $(A, M) \mapsto \mathrm{Der}(A, M)$ defines a functor $\mathrm{Mod} \to \mathrm{An}$ given as the composite

$$\mathrm{Mod} \xrightarrow{(A,M)\mapsto(A\ltimes M\to A)} \mathrm{Fun}([1], \mathrm{CAlg}) \xrightarrow{(B\to A)\mapsto \mathrm{Hom}_{/A}(A,B)} \mathrm{An}.$$

We let $\mathrm{Der} \to \mathrm{Mod}$ denote the associated cocartesian unstraightening. We identify the objects of Der with triples (A, M, η) consisting of a commutative ring spectrum A, an A-module M and an M-valued derivation η of A. A morphism $(A, M, \eta) \to (B, N, \xi)$ consists of a morphism of commutative ring spectra $f\colon A \to B$, a morphism of A-modules $\varphi\colon M \to f_*N$, and a commutative triangle in $\mathrm{CAlg}_{/A}$ of the form

$$\begin{tikzcd} & A \ltimes M \arrow[d, "A\ltimes\varphi"] \\ A \arrow[ur, "\eta"] \arrow[r, "f_*(\xi)"'] & A \ltimes f_*N. \end{tikzcd}$$

By functoriality of pullbacks, the assignment $(A, M, \eta) \mapsto (A^\eta \to A)$ refines to a functor

$$\Phi\colon \mathrm{Der} \to \mathrm{Fun}([1], \mathrm{CAlg}).$$

Theorem 10.3.9 ([Lur17, Theorem 7.4.1.26])**.** *The functor Φ restricts to an equivalence*

$$\Phi_{n\text{-sm}}\colon \mathrm{Der}_{n\text{-sm}} \xrightarrow{\sim} \mathrm{Fun}_{n\text{-sm}}([1], \mathrm{CAlg})$$

between the full subcategory $\mathrm{Der}_{n\text{-sm}} \subseteq \mathrm{Der}$ spanned by the n-small derivations and the subcategory $\mathrm{Fun}_{n\text{-sm}}([1], \mathrm{CAlg}) \subseteq \mathrm{Fun}([1], \mathrm{CAlg})$ by the n-small extensions.

Proof. *[Proof omitted, though it may be added in at some later point.]* □

Corollary 10.3.10. *Every n-small extension is a square-zero extension.*

Proof. By the theorem every n-small extension is in the essential image of Φ_n, hence in particular in the essential image of Φ. □

As mentioned, one of the primary sources of examples of square-zero extensions are Postnikov towers:

Lemma 10.3.11. *Let $A \in \mathrm{CAlg}^{\geq 0}$ be a connective commutative ring spectrum. For each $n > 0$, the map $\tau_{\leq n}A \to \tau_{\leq n-1}A$ is a square-zero extension.*

Proof. By Corollary 10.3.10, it will suffice to show that the morphism $\tau_{\leq n}A \to \tau_{\leq n-1}A$ is n-small. The fiber is $\pi_n(A)[n]$, which is concentrated in degree n, which is contained in $[n, 2n]$. In light of Remark 10.3.7, checking that the map is n-small then reduces to checking that a certain map on homotopy groups vanishes, which is trivial in this case since $\pi_{2n}(\pi_n(A)[n]) = 0$. □

10.3.2 Deformations of commutative ring spectra

We will now introduce an analogue of deformation theory in the setting of higher algebra.

Definition 10.3.12 (Deformation)**.** Let $A' \to A$ be a square-zero extension of a commutative ring spectrum A by an A-module M. Given a commutative B-algebra $B \in \mathrm{CAlg}_A$, then a *deformation of B to A'* is a pair (B', α) where $B' \in \mathrm{CAlg}_{A'}$ and where α is an equivalence $B \simeq B' \otimes_{A'} A$ of commutative ring spectra. In other words, B' fits into a pushout square in CAlg as follows:

$$\begin{tikzcd} A' \arrow[r, dashed] \arrow[d] & B' \arrow[d, dashed] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & B. \end{tikzcd}$$

In classical deformation theory, the A-algebra B is usually assumed flat over A, and similarly B' is assumed flat over A'. This condition will be automatically taken care of, in light of the following lemma:

Lemma 10.3.13. *Let $f\colon A' \to A$ be a square-zero extension, and consider a deformation B' of B to A', as in the previous definition. Then B is flat as an A-module if and only if B' is flat as an A'-module.*

Proof. First assume that B' is flat as an A'-module, so that the functor $B' \otimes_{A'} -\colon \mathrm{Mod}_{A'} \to \mathrm{Sp}$ is t-exact. By associativity of relative tensor products, tensoring with $B = B' \otimes_{A'} A$ is given by $B \otimes_A M \simeq B' \otimes_{A'} M$, i.e. the functor $B \otimes_A -$ factors as

$$\mathrm{Mod}_A \xrightarrow{f_*} \mathrm{Mod}_{A'} \xrightarrow{B' \otimes_{A'} -} \mathrm{Sp}.$$

Since the t-structures on Mod_A and $\mathrm{Mod}_{A'}$ are inherited from Sp, it is clear that f_* is t-exact, hence so is $B \otimes_A -$, showing that B is flat as an A-module.

Conversely, assume now that B is flat over A. We will show that B' is flat over A'. By Proposition 9.3.9, it will suffice to show that for any discrete A'-module $N \in \mathrm{Mod}_{A'}^{\heartsuit} \simeq \mathrm{Mod}_{\pi_0(A')}^{\heartsuit}$, the relative tensor product $B' \otimes_{A'} N$ is again discrete. To this end, let $I \subseteq \pi_0 A'$ be the kernel of the surjective map $\pi_0 A' \twoheadrightarrow \pi_0 A$, giving rise to a short exact sequence of modules over $\pi_0 A'$:

$$0 \to IN \hookrightarrow N \twoheadrightarrow N/IN \to 0.$$

Tensoring with B' in $\mathrm{Mod}_{A'}$ gives an exact sequence, and it will suffice to show that $B' \otimes_{A'} IN$ and $B' \otimes_{A'} N/IN$ are discrete. Since $A' \to A$ is a square-zero extension, we have $I^2 = 0$ in $\pi_0 A'$, and it follows that I acts trivially both on IN and N/IN. Replacing N by IN or N/IN, we can thus reduce to the case where $IN = 0$. This implies that N has the structure of a discrete module over $\pi_0(A')/I \cong \pi_0(A)$. We then conclude that $B' \otimes_{A'} N$ is isomorphic to $B \otimes_A N$, which is discrete by the assumption that B is flat over A. □

In the situation of Definition 10.3.12, the assumption that A' is a square-zero extension of A by M implies that $A' \simeq A^\eta$ for some A-linear morphism $\eta\colon L_A \to M[1]$. Given a morphism $A \to B$ of commutative ring spectra, this induces a morphism of B-modules $\eta_B\colon f^* L_A \to f^* M[1]$. If we assume that A, B and M are all connective, then the problem of finding deformations of B to A' may be expressed in terms of this map η_B as follows:

(*) The data of a deformation of B to A' is equivalent to the data of a factorization of η_B through the morphism $f^* L_A \to L_B$:

$$\begin{tikzcd} f^* L_A \arrow[r, "\eta_B"] \arrow[d] & f^* M[1] \\ L_B. \arrow[ur, dashed, "\eta'"'] & \end{tikzcd}$$

The deformation B' corresponding to such a factorization η' is the square-zero extension $B^{\eta'}$ of B.

Since the fiber of the vertical map is $L_{B/A}[-1]$ (see Corollary 10.2.8), the data of such a factorization η' is equivalent to the data of a nullhomotopy for the composite

$$L_{B/A}[-1] \to B \otimes_A L_A \xrightarrow{\eta_B} B \otimes_A M[1].$$

Let us now give a precise formulation of this claim.

Construction 10.3.14. Recall from Construction 10.3.8 the ∞-category Der of derivations, whose objects may be thought of as triples (A, M, η) where A is a commutative ring spectrum, M is an A-module and $\eta\colon A \to A \ltimes M[1]$ is an $M[1]$-valued derivation of A. We say a derivation is *connective* if both A and M are connective. We define the (non-full) subcategory $\mathrm{Der}^+ \subseteq \mathrm{Der}$ as follows:

- The objects of Der^+ are the connective derivations;
- The morphisms of Der^+ are those morphisms $(f, \varphi, \alpha)\colon (A, M, \eta) \to (B, N, \xi)$ such that the morphism $f^*M \to N$ induced by φ is an isomorphism of B-modules.

Similarly, we denote by

$$\mathrm{Fun}^+([1], \mathrm{CAlg}) \subseteq \mathrm{Fun}([1], \mathrm{CAlg})$$

the (non-full) subcategory spanned by those morphisms $A' \to A$ such that A' and A are both connective and the fiber $\mathrm{fib}(A' \to A)$ is connective, and by those morphisms in $\mathrm{Fun}([1], \mathrm{CAlg})$ that correspond to pushout squares in CAlg.

Given an connective derivation (A, M, η), a morphism in Der^+ out of this object is essentially given by the data of a morphism $f\colon A \to B$ in CAlg together with a derivation $\eta'\colon B \to B \ltimes f^*M[1]$ of B such that the following diagram commutes:

$$\begin{tikzcd} & A \ltimes M[1] \arrow[d, "A\ltimes\mathrm{unit}"] \\ A \arrow[ur, "\eta"] \arrow[r, "f_*(\eta)"'] & A \ltimes f_*f^*M[1]. \end{tikzcd}$$

Note that the composite along the top corresponds to the A-linear morphism $L_A \xrightarrow{\eta} M[1] \xrightarrow{\mathrm{unit}} f_*f^*M[1]$, which under the adjunction $f^* \dashv f_*$ is precisely the map $\eta_B\colon f^*L_A \to f^*M[1]$. Similarly, if we identify η' with a B-linear morphism $\eta'\colon L_B \to f^*M[1]$, then the derivation $f_*(\eta)$ of A corresponds to the composite morphism $L_A \to f_*L_B \xrightarrow{f_*\eta'} f_*f^*M[1]$, which in turn corresponds to the composite $f^*L_A \to L_B \xrightarrow{\eta'} f^*M[1]$.

If A and M are connective, then so is A^η in light of the fiber sequence $M \to A^\eta \to A$. In particular, the functor $\Phi\colon \mathrm{Der} \to \mathrm{Fun}([1], \mathrm{CAlg})$ from Construction 10.3.8 restricts to a functor $\Phi^+\colon \mathrm{Der}^+ \to \mathrm{Fun}^+([1], \mathrm{CAlg})$.

Theorem 10.3.15 ([Lur17, Proposition 7.4.2.5, Theorem 7.4.2.7])**.** *The functor* $\Phi^+\colon \mathrm{Der}^+ \to \mathrm{Fun}^+([1], \mathrm{CAlg})$ *is a left fibration.*

Proposition 10.3.16. *Let (A, M, η) be a connective derivation. Then the functor Φ^+ induces an equivalence of ∞-categories*

$$\mathrm{Der}^+_{(A,M,\eta)/} \xrightarrow{\sim} \mathrm{CAlg}^{\geq 0}_{A^\eta}, \qquad (B, N[1], \eta') \mapsto B^{\eta'}.$$

Proof. Since Φ^+ is a left fibration, it follows from Proposition 2.3.6 that it induces equivalences

$$\mathrm{Der}^+_{(A,M,\eta)/} \xrightarrow{\sim} (\mathrm{Fun}^+([1], \mathrm{CAlg}))_{(A^\eta \to A)/}, \qquad (B, f^*M, \eta') \mapsto (B^{\eta'} \to B)$$

on slice categories. It will thus suffice to show that the forgetful functor

$$(\mathrm{Fun}^+([1], \mathrm{CAlg}))_{(A^\eta \to A)/} \to \mathrm{CAlg}^{\geq 0}_{A^\eta/}$$

is an equivalence. Indeed, an inverse to this functor is given by sending a morphism $A^\eta \to B'$ to the pushout square

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & B' \arrow[d] \\ A \arrow[r] & B. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

□

10.3.3 Connectivity results

[To do.]

In the case of classical commutative rings, the cotangent complex encompasses the module of Kähler differentials:

Lemma 10.3.17. *Let A be a classical commutative ring. Then there is an isomorphism $\pi_0 L_A \cong \Omega_A$ of A-modules.*

Proof. By the Yoneda lemma, it suffices to show that both objects of $\mathrm{Mod}_A^\heartsuit$ corepresent the same functor. To this end, note that the fully faithful inclusion $\mathrm{CRing} \hookrightarrow \mathrm{CAlg}$ of commutative rings into commutative ring spectra induces an equivalence

$$\mathrm{Der}^\heartsuit(A, M) \cong \mathrm{Hom}_{\mathrm{CRing}_{/A}}(A, A \ltimes M) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}_{/A}}(A, A \ltimes M) = \mathrm{Der}(A, M)$$

between the set of classical M-valued derivations of A and the anima of M-valued derivations of A in the sense of Definition 10.2.1. In particular, we get for every $M \in \mathrm{Mod}_A^\heartsuit$ a natural equivalence

$$\mathrm{Hom}_A^\heartsuit(L_A, M) \cong \mathrm{Der}^\heartsuit(A, M) \xrightarrow{\sim} \mathrm{Der}(A, M) \simeq \mathrm{Hom}_A(L_A, M) \simeq \mathrm{Hom}_A^\heartsuit(\pi_0 L_A, M).$$

Here the last equivalence holds since L_A is connective [ref] and M is discrete. □

10.4 Classification of étale algebras

In this section, we introduce a homotopical analogue of the notion of an étale morphism between commutative rings, and show that étale algebras over a connective commutative ring spectrum A can be classified in terms of the classical étale algebras over $\pi_0(A)$.

Definition 10.4.1 (Étale morphism)**.** A morphism $f\colon A \to B$ of commutative rings is called *étale* if the following conditions are satisfied:

(1) The commutative ring B is finitely presented as an A-algebra;

(2) The map f exhibits B as a flat A-module;

(3) The multiplication map $B \otimes_A B \to B$ is the projection onto a summand, i.e. there exists another morphism of commutative rings $q\colon B \otimes_A B \to R$ such that p and q induce an isomorphism $B \otimes_A B \xrightarrow{\sim} B \times R$.

From a geometric perspective, the étale maps correspond to the (not necessarily surjective) covering maps in topology.

Definition 10.4.2. Let A be a commutative ring spectrum. A commutative A-algebra B is called:

- *Finitely generated and free* if there exists a finitely generated free A-module M and an equivalence $B \simeq \mathrm{Free}(M)$, where $\mathrm{Free}\colon \mathrm{Mod}_A \to \mathrm{CAlg}_A$ denotes a left adjoint to the forgetful functor.
- *Finitely presented* if B is contained in the full subcategory of CAlg_A that contains all finitely generated and free commutative A-algebras and is closed under finite colimits.

Definition 10.4.3. A morphism $f\colon A \to B$ of commutative ring spectra is called *étale* if the following conditions are satisfied:

(1) The commutative A-algebra B is finitely presented;

(2) The underlying A-module of B is flat;

(3) The morphism $\pi_0 f\colon \pi_0 A \to \pi_0 B$ is étale in the sense of Definition 10.4.1.

We denote by $\mathrm{CAlg}_A^{\text{ét}} \subseteq \mathrm{CAlg}_A$ the full subcategory spanned by the étale commutative A-algebras.

Let us state without prove some basic properties of étale maps:

Proposition 10.4.4 ([Lur17, Remark 7.5.1.7])**.** *Consider a commutative triangle*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B \arrow[rr, "h"] & & C \end{tikzcd}$$

of commutative ring spectra. If f is étale, then g is étale if and only if h is étale.

Proof. Omitted. □

Proposition 10.4.5 ([Lur17, Corollary 7.5.4.5])**.** *Let $f\colon A \to B$ be an étale morphism of connective commutative ring spectra. Then the relative cotangent complex $L_{B/A}$ vanishes. In particular, the morphism $f^*L_A \to L_B$ is an equivalence of B-modules.*

From this result, we deduce the following classification of étale algebras over square-zero extensions:

Proposition 10.4.6. *Let $f\colon A' \to A$ be a square-zero extension of connective commutative ring spectra. Then the relative tensor product functor*

$$- \otimes_{A'} A\colon \mathrm{CAlg}_{A'} \to \mathrm{CAlg}_A$$

induces an equivalence

$$- \otimes_{A'} A\colon \mathrm{CAlg}^{\text{ét}}_{A'} \xrightarrow{\sim} \mathrm{CAlg}^{\text{ét}}_A$$

between étale commutative A'-algebras and étale commutative A-algebras.

Proof. Since A' is a square-zero extension of A, it is of the form $A' \simeq A^{\eta}$ for some $M[1]$-valued deformation $\eta\colon A \to A \ltimes M[1]$ of A. Recall from Proposition 10.3.16 that the functor $\Phi\colon \mathrm{Der} \to \mathrm{Fun}([1], \mathrm{CAlg}), (B, N, \xi) \mapsto B^{\xi}$ induces an equivalence

$$\mathrm{Der}^{+}_{(A,M,\eta)/} \xrightarrow{\sim} \mathrm{CAlg}^{\geq 0}_{A^{\eta}}, \qquad (B, N, \eta') \mapsto B^{\eta'}.$$

Since every étale morphism is flat, it follows that any flat commutative A^{η}-algebra is connective, and hence there is a fully faithful inclusion

$$\mathrm{CAlg}^{\text{ét}}_{A^{\eta}} \hookrightarrow \mathrm{CAlg}^{\geq q0}_{A^{\eta}}.$$

We may similarly consider the full subcategory

$$\mathrm{Der}^{\text{ét}}_{(A,M,\eta)/} \hookrightarrow \mathrm{Der}^{+}_{(A,M,\eta)/}$$

spanned by those (B, N, η') such that $A \to B$ is an étale map. For every such map, we have a pushout square

$$\begin{array}{ccc} A^{\eta} & \longrightarrow & B^{\eta'} \\ \downarrow & & \downarrow \\ A & \longrightarrow & B, \end{array}$$

and similar to Lemma 10.3.13 one can show that $A \to B$ is an étale morphism if and only if $A^{\eta} \to B^{\eta'}$ is. In particular, the above equivalence restricts to an equivalence

$$\mathrm{Der}^{\text{ét}}_{(A,M,\eta)/} \xrightarrow{\sim} \mathrm{CAlg}^{\text{ét}}_{A^{\eta}}.$$

It remains to show that the forgetful functor

$$\mathrm{Der}^{\text{ét}}_{(A,M,\eta)/} \xrightarrow{\sim} \mathrm{CAlg}^{\text{ét}}_{A}$$

is an equivalence. Since this functor is obtained from the left fibration $\mathrm{Der}^{\text{ét}} \to \mathrm{CAlg}^{\text{ét}}$ by passing to slices, it is again a left fibration, see Lemma 2.3.5. By Corollary 2.3.2, it suffices to show that the fiber over every commutative étale A-algebra $f\colon A \to B$ is contractible. Unwinding definitions, we see that this fiber equivalent to the fiber over η_B of the map

$$\mathrm{Der}(B, f^*M) \to \mathrm{Der}(A, f_*f^*M), \qquad \eta' \mapsto f_*\eta'.$$

Equivalently, this is the fiber of the map

$$\mathrm{Hom}_B(L_B, f^*M) \to \mathrm{Hom}_B(f^*L_A, f^*M)$$

given by precomposition with $f^*L_A \to L_B$. But since f is étale, this map is an equivalence by Proposition 10.4.5. This finishes the proof. $\square$

In particular, we obtain the following important theorem:

Theorem 10.4.7 ([Lur17, Theorem 7.5.0.6])**.** *Let A be a connective commutative ring spectrum. Then the functor*

$$\pi_0\colon \mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}_{\pi_0 A}^{\text{ét}}$$

is an equivalence.

Proof. We claim that an inverse is given by the base change functor

$$(-)\otimes_{\pi_0 A} A\colon \mathrm{CAlg}_{\pi_0 A}^{\text{ét}} \to \mathrm{CAlg}_A^{\text{ét}}.$$

Recall from Lemma 10.3.11 that each map $\tau_{\leq n}A \to \tau_{\leq n-1}A$ is a square-zero extension, hence by the previous proposition induces an equivalence on étale algebras. It follows that the functor

$$(-)\otimes_{\pi_0 A} \tau_{\leq n}A\colon \mathrm{CAlg}_{\pi_0 A}^{\text{ét}} \to \mathrm{CAlg}_{\tau_{\leq n}A}^{\text{ét}}$$

is an equivalence for all $n \geq n$, with inverse given by forming π_0. In particular, the functor

$$\pi_0\colon\ \lim_n \mathrm{CAlg}_{\tau_{\leq n}A}^{\text{ét}} \to \mathrm{CAlg}_{\pi_0 A}^{\text{ét}}$$

is an equivalence. But also the functor $(\tau_{\leq n})\colon \mathrm{CAlg}_A^{\text{ét}} \to \lim_n \mathrm{CAlg}_{\tau_{\leq n}A}^{\text{ét}}$ is an equivalence since $B \simeq \lim_n \tau_{\leq n}B$ for all B. $\square$

A Exercises

Here is a collection of exercises related to the content of the course. These exercises do not need to be handed in.

Classical operad theory

Exercise A.1 (Equivariance conditions)**.** In the definition of an operad, we have not been precise about the precise way in which the permutation Σ_n-action on $\mathcal{O}(n)$ is supposed to interact with the composition structure. In this exercise, you will formulate the two missing conditions one should put on an operad.

Let $\mathcal{O}$ be a non-colored operad, and consider operations $P \in \mathcal{O}(k)$ and $Q_i \in \mathcal{O}(n_i)$ for $i = 1, \ldots, k$.

(1) Consider a permutation $\sigma \in \Sigma_k$. Using the motivating example of the endomorphism operad $\mathcal{O} = \mathcal{E}nd_{\mathcal{C}}(A)$, write down a rule for expressing the operation

$$(P\sigma)(Q_1, \ldots, Q_k)$$

as an operation of the form $P(R_1, \ldots, R_k)\tau$ for certain operations R_i and a certain permutation $\tau \in \Sigma_n$, where $n := \sum_{i=1}^k n_i$.

(2) Consider permutations $\sigma_i \in \Sigma_{n_i}$ for all $i = 1, \ldots, k$. Again using $\mathcal{O} = \mathcal{E}nd_{\mathcal{C}}(A)$ as motivating example, write down a rule for expressing the operation

$$P(Q_1\sigma_1, \ldots, Q_k\sigma_k)$$

as an operation of the form $P(Q_1, \ldots, Q_k)\tau$ for a certain permutation $\tau \in \Sigma_n$, where $n := \sum_{i=1}^k n_i$.

Exercise A.2 (The associative operad)**.** Write out the composition maps and the Σ_n-actions in the case of the associative operad $\mathcal{A}ssoc$.

Exercise A.3 (Lie operad)**.** Look op the definition of the Lie operad $\mathcal{L}ie$ over a given field K. Give a precise description of the sets $\mathcal{L}ie(n)$ for all n and of the composition structure of the operad.

Bonus question: Look up the definition of an *enriched* operad, and explain in what sense the Lie operad over the field K is enriched in K-vector spaces. Also explain why its algebras in the symmetric monoidal category Vect_K are precisely the Lie algebras.

Exercise A.4. Let $\mathcal{C}$ be a symmetric monoidal category. Show that:

- $\mathcal{T}riv$-algebras are simply objects of $\mathcal{C}$;

- $\mathcal{E}_0$-algebras are 'pointed objects' of C, i.e., objects A equipped with a 'unit map' $\eta\colon \mathbb{1} \to A$.

- $\mathcal{C}omm$-algebras are objects A equipped with maps $\eta\colon \mathbb{1} \to A$ and $m\colon A \otimes A \to A$ which are unital, associative and commutative.

- $\mathcal{A}ssoc$-algebras are similar but without commutativity.

Exercise A.5 (Initial and terminal operads)**.** Show that the commutative operad $\mathcal{C}omm$ is the *terminal* operad and that the trivial operad $\mathcal{T}riv$ is the *initial* operad.

Exercise A.6 (Little cubes operad)**.** Spend some time understanding the 1-dimensional little cubes operad $\mathcal{E}_1$. Explain how we may think of $\mathcal{E}_1$ as a *topological* operad, i.e., each $\mathcal{E}_1(n)$ is a topological space and all the structure maps are continuous. Show that there is an operad morphism $\mathcal{E}_1 \to \mathcal{A}ssoc$ which induces homotopy equivalences $\mathcal{E}_1(n) \to \mathcal{A}ssoc(n)$ for all $n \geq 0$.

Remark: In more classical setups, for example when working with model categories, the homotopically correct operad to work with is often $\mathcal{E}_1$. This is the reason that in older literature you will often find the phrase '$\mathcal{E}_1$-algebra' in places where we would say 'associative algebra'.

Exercise A.7 (Left module operad)**.** Spell out the composition maps in the left module operad $\mathcal{LM}od$.

Exercise A.8. Let $\mathcal{O}$ be a colored operad, and let C be a symmetric monoidal category. Recall that we defined an *$\mathcal{O}$-algebra* as a morphism of colored operads $A\colon \mathcal{O} \to \mathcal{M}_C$. Spell out in explicit terms all the data contained in such an $\mathcal{O}$-algebra. In the special case of $\mathcal{O} = \mathcal{LM}od$, verify that this structure precisely encodes a pair (A, M) of an associative algebra A in C together with a left module M over it.

Remark: Saying 'left' versus 'right' is just a convention; the same operad also encodes right actions.

Exercise A.9. Compute the envelope $\mathrm{Env}(\mathcal{A}ssoc)$ of the associative operad, and show that it may be described as follows:

- The objects are finite sets I;

- The morphisms $I \to J$ are tuples $(f, \preceq_j)$ consisting of a morphism $f\colon I \to J$ of finite sets and a collection of linear orders $\preceq_j$ on each of the preimages $f^{-1}(j)$ of f;

- The symmetric monoidal structure is disjoint union.

Exercise A.10. Show that the assignment $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ is functorial: every morphism of operads $\mathcal{O} \to \mathcal{P}$ induces a functor $\mathrm{Env}(\mathcal{O}) \to \mathrm{Env}(\mathcal{P})$.

Exercise A.11. Show that the assignment $\mathcal{O} \mapsto \mathrm{Env}(\mathcal{O})$ is left adjoint to the assignment $C \mapsto \mathcal{M}_C$: for an operad $\mathcal{O}$ and a symmetric monoidal category C, there is a bijection between operad morphisms $\mathcal{O} \to \mathcal{M}_C$ and symmetric monoidal functors $\mathrm{Env}(\mathcal{O}) \to C$.

Cocartesian fibrations and straightening/unstraightening

Exercise A.12. Let C be an ∞-category and consider $p := \mathrm{ev}_1\colon \mathrm{Fun}([1], C) \to C$.

(1) Show that a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & Y' \end{tikzcd}$$

is p-cocartesian (when regarded as a morphism in $\mathrm{Fun}([1], C)$ from $\binom{X}{\downarrow \atop Y}$ to $\binom{X'}{\downarrow \atop Y'}$) if and only if the map $X \to X'$ is an isomorphism.

(2) Conclude that p is always a cocartesian fibration.

(3) Show that a commutative square like the one above is p-cartesian if and only if it is a pullback square in C. (Since pullback squares are also called 'cartesian squares', this explains the name 'cartesian morphisms').

(4) Conclude that p is a cartesian fibration if and only if C admits pullbacks.

Dualize this statement by showing that ev_0 is always a cartesian fibration and that it is additionally a *co*cartesian fibration if and only if C admits pushouts.

Exercise A.13. Consider the category VectBund whose objects are pairs (X, E) consiting of a topological space X and a real vector bundle $p\colon E \to X$, and whose morphisms are commutative squares

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

of topological such that F induces $\mathbb{R}$-linear maps $E_x \to E'_{f(x)}$ on fibers. Show that the forgetful functor VectBund $\to$ Top, $(X, E) \mapsto X$ is a cartesian fibration.

Exercise A.14. Let $p\colon D \to C$ and $q\colon E \to D$ be two cocartesian fibrations.

(1) Show that $pq\colon E \to C$ is again a cocartesian fibration.

(2) Show that the functor p is a cocartesian functor over C.

Exercise A.15. Consider a pullback square of ∞-categories of the form

$$\begin{tikzcd} E' \arrow[r, "F"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ C' \arrow[r] & C, \end{tikzcd}$$

where p is a cocartesian fibration. Show that p' is a cocartesian fibration as well, and a morphism $f\colon x \to y$ in E' is a p'-cocartesian morphism if and only if $G(f)$ is a p-cocartesian morphism.

Exercise A.16. For a functor $p\colon E \to C$, consider the full subcategory

$$\{p\text{-cocart}\} \subseteq \mathrm{Fun}([1], C)$$

spanned by the p-cocartesian morphisms.

(1) Show that the functor

$$\{p\text{-cocart}\} \to E \times_{p,C,\mathrm{ev}_0} \mathrm{Fun}([1], C), \quad (\varphi\colon e \to e') \mapsto (e, p\varphi)$$

is fully faithful.

(2) Deduce that p is a cocartesian fibration if and only if this functor is an equivalence.

(3) Use this equivalence to show that for every morphism $f\colon x \to y$, the assignment $e \mapsto f_! e$ from Notation 2.1.3 can be enhanced to a cocartesian transport functor

$$f_!\colon E_x \to E_y.$$

Exercise A.17. A morphism $f\colon Y \to X$ of animae *is called* a *covering* if all its fibers are sets, i.e., if all their higher homotopy groups vanish. Letting $\mathrm{Cov}(X) \subseteq \mathrm{An}_{/X}$ denote the full subcategory spanned by the coverings, deduce that the straightening equivalence induces an equivalence

$$\mathrm{Cov}(X) \xrightarrow{\sim} \mathrm{Fun}(\Pi_1(X), \mathrm{Set})$$

between the category of covering spaces over X and the category of functors $\Pi_1(X) \to \mathrm{Set}$. Explain how this provides a high-tech version of the classical classification of covering spaces.

Exercise A.18. Let I and C ∞-categories, and let $F = \mathrm{const}_C\colon I \to \mathrm{Cat}_\infty$ be the constant functor with value C.

- Show that the colimit of F is the product $C \times \|I\|$, where $\|I\| := C[\mathrm{all}^{-1}]$ is the geometric realization of I (obtained by inverting arbitrary morphisms).
- Show that the limit of F is $\mathrm{Fun}(\|I\|, C)$.
- Conclude that for an anima X we have $\mathrm{colim}_X * \simeq X$.

Exercise A.19. Use the formula for colimits in Cat_∞ to compute the pushout in Cat_∞ of the following diagram:

$$\begin{tikzcd} * \arrow[r, "0"] \arrow[d, "1"'] & {[1]} \\ {[1]}. & \end{tikzcd}$$

(You may take for granted that for a 1-category I the unstraightening of a functor $I \to \mathrm{Cat}_\infty$ that happens to land in the subcategory $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ of 1-categories is again a 1-category, computed by the Grothendieck construction.)

Categories as complete Segal animae

Exercise A.20. Show that composition in a Segal anima is unital and associative.

Exercise A.21. Consider the functor $\mathbf{\Delta} \to \mathbf{\Delta}$ which sends $[n]$ to the poset $[n]^{\mathrm{op}} \star [n] \cong [2n+1]$ that has $2(n+1)$ elements $\overline{n}, \overline{n-1}, \dots, \overline{1}, \overline{0}, 0, 1, \dots, n$ with partial order given by

$$\overline{n} \le \overline{n-1} \le \cdots \le \overline{1} \le \overline{0} \le 0 \le 1 \le \cdots \le n.$$

- Show that this assignment extends to a well-defined functor $D\colon \mathbf{\Delta} \to \mathbf{\Delta}$;

- Show that for every ∞-category C, the simplicial anima $\mathrm{NTw}(C) \in \mathrm{sAn}$ defined as the composite
$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{D^{\mathrm{op}}} \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\infty}(-,C)} \mathrm{An}$$
is a complete Segal anima, hence is equivalent to the nerve $N(\mathrm{Tw}(C))$ of some ∞-category $\mathrm{Tw}(C)$.

- Produce natural transformations $s\colon [n]^{\mathrm{op}} \hookrightarrow [n]^{\mathrm{op}} \star [n]$ and $t\colon [n] \hookrightarrow [n]^{\mathrm{op}} \star [n]$ given by $s(i) = \bar{i}$ and $t(i) = i$. Deduce that there are functors
$$s\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \qquad \text{and} \qquad t\colon \mathrm{Tw}(C) \to C.$$

- Show that the fiber of $(s,t)\colon \mathrm{Tw}(C) \to C^{\mathrm{op}} \times C$ over (x,y) is equivalent to $\mathrm{Hom}_C(x,y)$.

In the following two exercises, you may freely use that filtered colimits commute with finite limits in An.

Exercise A.22. Show that the subcategory $\mathrm{CSeg}(\mathrm{An}) \subseteq \mathrm{sAn}$ is closed under limits and filtered colimits. Deduce that the nerve functor $N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{sAn}$ preserves filtered colimits and that finite limits commute with filtered colimits in Cat_∞.

Exercise A.23. Show that, for every finite category I, the functor $\mathrm{Hom}_{\mathrm{Cat}_\infty}(I,-)\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits.

Applying this exercise to $I = *$ in particular shows that the core functor $(-)^{\simeq}\colon \mathrm{Cat}_\infty \to \mathrm{An}$ preserves filtered colimits.

Extensive categories

Exercise A.24. Show that the category Fin_G of finite G-sets is extensive for every finite group G.

Exercise A.25. Show that the categories Top and Sch of topological spaces and schemes, respectively, are extensive.

Exercise A.26. Show that An is extensive.

Exercise A.27. Let C be an ∞-category with finite coproducts. Show that C is extensive if and only if it satisfies the following two conditions:

- *Coproducts in C are disjoint*: for objects X and Y, the commutative square
$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd}$$
is a pullback square.

- *Coproduct decompositions are stable under pullback*: given a morphism $f\colon Z \to X \sqcup Y$, the map $Z_X \sqcup Z_Y \to Z$ is an isomrphism, where Z_X and Z_Y are defined by the following two pullback squares:

$$\begin{tikzcd} Z_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] \\ X \arrow[r, hook] & X \sqcup Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "f"] \\ Y \arrow[r, hook] & X \sqcup Y. \end{tikzcd}$$

Exercise A.28. Show that C is extensive if and only if it has finite coproducts and for all $n \geq 0$ and objects $X_1, \ldots, X_n$ the coproduct functor

$$\prod_{i=1}^{n} C_{/X_i} \to C_{/\bigsqcup_i X_i}$$

is an equivalence.

Span categories

Exercise A.29. Use the results on commutative monoids in semiadditive ∞-categories from last semester to prove that Span(Fin) is the free semiadditive ∞-category on a single generator: for every semiadditive ∞-category C, evaluation at the one-point set $\langle 1 \rangle \in \mathrm{Fin}$ induces an equivalence

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), C) \xrightarrow{\sim} C$$

of ∞-categories.

∞-operads

Exercise A.30. Show that the following functors are examples of ∞-operads:

- $\mathcal{C}omm = (\mathrm{Span}(\mathrm{Fin}), \mathrm{id})$;
- $\mathcal{T}riv = (\mathrm{Fin}^{\mathrm{op}}, \text{inclusion})$;
- $\mathcal{T}riv_{\emptyset} = (*, \text{inclusion})$;
- $\mathcal{E}_0 = (\mathrm{Span}_{\mathrm{all,inj}}(\mathrm{Fin}), \text{inclusion})$.

In each case, compute the multimorphism sets and compute the corresponding composition of multimorphisms, and show that these ∞-operads do indeed agree with their classical counterparts discussed in Section 1.1.

Exercise A.31. Show that $\mathcal{C}omm$ is a terminal object in Op_{∞}. Show that $\mathcal{T}riv_{\emptyset}$ is an initial object in Op_{∞}.

Exercise A.32. Given a symmetric monoidal ∞-category C and objects $x_1, \ldots, x_n, y \in C^{\simeq}$, show that the anima $\mathcal{M}_C((x_1, \ldots, x_n); y)$ is naturally equivalent to $\mathrm{Hom}_C(x_1 \otimes \cdots \otimes x_n, y)$.

Exercise A.33. Let $\mathcal{O}$ be an ∞-operad. Show that $\mathcal{O}$ is (the multimorphism operad of) a symmetric monoidal ∞-category if and only if for every $n \geq 0$ and every tuple $(x_1, \dots, x_n)$ of colors, there exists a cocartesian lift $(x_1, \dots, x_n) \to x_1 \otimes \cdots \otimes x_n$ over the span $\langle n \rangle \xleftarrow{=} \langle n \rangle \to \langle 1 \rangle$.

Similarly, given symmetric monoidal ∞-categories C and D, show that an operad map $f\colon \mathcal{M}_C \to \mathcal{M}_D$ lies in the image of the functor $\mathcal{M}\colon \mathrm{Cat}_\infty^\otimes \to \mathrm{Op}_\infty$ if and only if the map $f^\otimes\colon C^\otimes \to D^\otimes$ preserves the cocartesian lifts $(x_1, \dots, x_n) \to x_1 \otimes \cdots \otimes x_n$ of the span $\langle n \rangle \xleftarrow{=} \langle n \rangle \to \langle 1 \rangle$ for every $n \geq 0$.

Exercise A.34. Let I be an ∞-category. In this exercise we will construct the ∞-operad $\mathcal{T}riv_I$.

(1) Let $p\colon \mathrm{Fin}(I) \to \mathrm{Fin}$ denote the cartesian unstraightening of the functor $I^{(-)}\colon \mathrm{Fin}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ right Kan extended from the point. Show that $\mathrm{Fin}(I)$ admits finite products and is extensive.

(2) Show that the composite

$$\mathrm{Fin}(I)^{\mathrm{op}} \xrightarrow{p^{\mathrm{op}}} \mathrm{Fin}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$$

is an ∞-operad. This ∞-operad is denoted $\mathcal{T}riv_I$ and called the *trivial ∞-operad on I*.

(3) Let $\mathcal{O}$ be an ∞-operad. Show that restriction along the inclusion $I^{\mathrm{op}} \hookrightarrow \mathrm{Fin}(I)^{\mathrm{op}}$ induces an equivalence

$$\mathrm{Fun}_{\mathrm{Op}_\infty}(\mathcal{T}riv_I, \mathcal{O}) \xrightarrow{\sim} \mathrm{Fun}(I^{\mathrm{op}}, \mathcal{O}^\otimes_{\langle 1 \rangle}).$$

(4) Conclude that for a symmetric monoidal ∞-category C, $\mathcal{T}riv_I$-algebras in C are functors $I^{\mathrm{op}} \to C$:

$$\mathrm{Alg}_{\mathcal{T}riv_I}(C) \xrightarrow{\sim} \mathrm{Fun}(I^{\mathrm{op}}, C).$$

Exercise A.35. Let $\mathcal{O}$ be an ∞-operad. Show that there is an equivalence

$$\mathcal{T}riv \times \mathcal{O} \simeq \mathcal{T}riv_{\mathcal{O}_{\langle 1 \rangle}}$$

between the product of $\mathcal{O}$ with the trivial operad and the trivial ∞-operad on $\mathcal{O}_{\langle 1 \rangle}$.

Exercise A.36. During the lecture, we saw the following two statements:

(1) The right adjoint R of a symmetric monoidal functor $L\colon C \to D$ admits a canonical lax symmetric monoidal structure.

(2) The left adjoint L of a lax symmetric monoidal functor $R\colon D \to C$ admits a canonical symmetric monoidal structure whenever certain comparison maps

$$L\Big(\bigotimes_{i=1}^{n} X_i\Big) \to \bigotimes_{i=1}^{n} L(X_i)$$

are isomorphisms in D.

Formulate and prove versions of these statements for $\mathcal{O}$-monoidal ∞-categories for an arbitrary ∞-operad $\mathcal{O}$.

In the lecture, we proved a criterion for a morphism in a span category to be a cocartesian morphism. The main application of this result is Barwick's *unfurling construction*, which we will now formulate as a series of exercises.

Definition A.37. Let (C, C_L, C_R) be an adequate triple. A functor $p\colon E \to C$ is called a *Beck-Chevalley fibration* with respect to (C_L, C_R) if the following three conditions are satisfied:

(1) For every morphism $l\colon X \to Y$ in C_L and every object $\tilde{Y}$ of E lifting Y, there exists a p-cartesian lift $\tilde{l}\colon \tilde{X} \to \tilde{Y}$ of l;

(2) For every morphism $r\colon X \to Y$ in C_R and every object $\tilde{X}$ of E lifting X, there exists a p-cocartesian lift $\tilde{r}\colon \tilde{X} \to \tilde{Y}$ of r;

(3) Consider a commutative square

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\tilde{r}'"] \arrow[d, "\tilde{l}'"'] & \tilde{X} \arrow[d, "\tilde{l}"] \\ \tilde{Y}' \arrow[r, "\tilde{r}"] & \tilde{Y} \end{tikzcd}$$

in E whose image in C is a pullback square satisfying $p(\tilde{l}) \in C_L$ and $p(\tilde{r}') \in C_R$, and for which $\tilde{r}$ is p-cocartesian while $\tilde{l}'$ is p-cartesian. Then the morphism $\tilde{r}'$ is p-cocartesian if and only if $\tilde{l}$ is p-cartesian.

We let $\mathrm{Fib}^{\mathrm{BC}}_{L,R}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$ denote the (non-full) subcategory spanned by the Beck-Chevalley fibrations $p\colon E \to C$ with respect to (C_L, C_R) and by those functors $E \to E'$ over C which preserve p-cartesian morphisms over C_L and p-cocartesian morphisms over C_R.

Remark A.38. The three conditions may be interpreted as follows:

(1) For $l\colon X \to Y$ in C_L, we get a functor $l^*\colon E_Y \to E_X$ on fibers.

(2) For $r\colon X \to Y$ in C_R, we get a functor $r_!\colon E_x \to E_Y$ on fibers.

(3) For a pullback square

$$\begin{tikzcd} X' \arrow[r, "r'"] \arrow[d, "l'"'] & X \arrow[d, "l"] \\ Y' \arrow[r, "r"] & Y \end{tikzcd}$$

in C with $l \in C_L$ and $r \in C_R$, the 'canonical' transformation $r'_! l'^* \to l^* r_!$ is a natural isomorphism, making the following square commute:

$$\begin{tikzcd} E_{X'} \arrow[r, "r'_!"] & E_X \\ E_{Y'} \arrow[r, "r_!"] \arrow[u, "l'^*"] & E_Y . \arrow[u, "l^*"'] \end{tikzcd}$$

Exercise A.39. Prove the equivalence between the two versions of condition (3).

Exercise A.40. Let (C, C_L, C_R) be an adequate triple and let $p\colon E \to C$ be a Beck-Chevalley fibration with respect to (C_L, C_R). Show that the functor

$$\mathrm{Span}(p)\colon\ \mathrm{Span}(E, p^{-1}(C_L) \cap E^{p\text{-cart}}, p^{-1}(C_R)) \to \mathrm{Span}(C, C_L, C_R)$$

is a cocartesian fibration.

Exercise A.41. Let $F\colon C \to \mathrm{Cat}_\infty$ be a functor with cocartesian unstraightening $p\colon E := \mathrm{Un}^{\mathrm{cc}}(F) \to C$, and let $C_L \subseteq C$ be a wide subcategory closed under base change (i.e. (C, C_L, C) is an adequate triple). Show that p is a Beck-Chevalley fibration with respect to (C, C_L) if and only if the functor $l_! := F(l)$ admits a right adjoint l^* for every morphism l in C_L and if for every pullback square

$$\begin{tikzcd} X' \arrow[r, "r'"] \arrow[d, "l'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "l"] \\ Y' \arrow[r, "r"] & Y \end{tikzcd}$$

in C with $l \in C_L$ the canonical ‘Beck-Chevalley transformation

$$r'_! l'^* \xrightarrow{\eta} l^* l_! r'_! l'^* \simeq l^* r_! l'_! l'^* \xrightarrow{\varepsilon} l^* r_!$$

is a natural isomorphism of functors $F(Y') \to F(X)$.

In particular, the combination of the previous two exercises shows that the functor F we started with admits a canonical extension

$$\mathrm{Unf}(F)\colon\ \mathrm{Span}_{L,\mathrm{all}}(C) \to \mathrm{Cat}_\infty$$

which on backwards maps is given by the right adjoints. This extension is called the *unfurling construction of* F, originally due to Barwick [Bar17]. The situation for left adjoints is dual:

Exercise A.42. Let $F\colon C^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a functor with cartesian unstraightening $p\colon E := \mathrm{Un}^{\mathrm{ct}}(F) \to C$, and let $C_R \subseteq C$ be a wide subcategory closed under base change (i.e. (C, C, C_R) is an adequate triple).

(1) Provide conditions on F such that p is a Beck-Chevalley fibration with respect to (C_R, C).

(2) Conclude that F extends to a functor

$$\mathrm{Span}_{\mathrm{all},R}(C) \to \mathrm{Cat}_\infty$$

which on forward maps implements the left adjoints to $r^* := F(R)$.

B Recollections on higher category theory

This course is a follow-up of the previous course *Introduction to stable homotopy theory*, and we will build upon the theory developed in that course. In particular we will freely use the 'model-independent' language of ∞-categories, functors, and natural isomorphisms as introduced in that course. Lecture notes may be found here. For the convenience of the reader, we will quickly recap the main properties and definitions.

Throughout this course, we will think of ∞-categories as a *primitive* notion, just like sets are taken to be primitive in ordinary mathematics. What this means is that we take for granted that we may speak of ∞-categories, functors and natural isomorphisms, without concerning ourselves with any specific choice of implementation of these notions. In particular, all of our theory will be developed in this 'model-independent' language.

There are various ways of constructing new categories from old ones, described by the following rules:

(B.1) A *terminal ∞-category* $*$: for every ∞-category C there is a functor $p_C : C \to *$, and for any two functors $F, G : C \to *$ there is a natural isomorphism $F \cong G$.

(B.2) An *initial ∞-category* $\emptyset$: for every C there is a functor $\emptyset \to C$, and for any two functors $F, G : \emptyset \to C$ there is a natural isomorphism $F \cong G$.

(B.2') The ∞-category $\emptyset$ is strictly initial: every functor $C \to \emptyset$ is an equivalence.

(B.3) For every two ∞-categories C and D a *product* $C \times D$ with functors $\mathrm{pr}_C : C \times D \to C$ and $\mathrm{pr}_D : C \times D \to D$: functors $E \to C \times D$ are specified by giving the two components $E \to C$ and $E \to D$, and natural isomorphisms between functors $E \to C \times D$ are specified by giving natural isomorphisms between the components.

(B.4) Similarly there is a *coproduct* $C \sqcup D$ with functors $i_C : C \sqcup D$ and $i_D : D \to C \sqcup D$: functors $C \sqcup D \to E$ are specified by giving the two components $C \to E$ and $D \to E$, and natural isomorphisms between functors $C \sqcup D \to E$ are specified by giving natural isomorphisms between the components.

(B.4') Coproducts are universal: a functor $E \to C \sqcup D$ into a coproduct can essentially uniquely be written in the form $E_C \sqcup E_D \to C \sqcup D$ for two functors $E_C \to C$ and $E_D \to D$.

(B.5) For functors $F: C \to E$ and $G: D \to E$ there exists a *pullback* $C \times_E D$ with functors $\mathrm{pr}_C: C \times_E D \to C$ and $\mathrm{pr}_D: C \times_E D \to D$ and a natural isomorphism $F \circ \mathrm{pr}_C \cong G \circ \mathrm{pr}_D$. Functors $T \to C \times_E D$ are specified by giving two components $t: T \to C$ and $s: T \to D$ together with a natural isomorphism $F \circ t \cong G \circ s$. One may similarly specify natural isomorphisms between two functors $T \to C \times_E D$.

(B.6) For ∞-categories C and D there is a *functor category* $\mathrm{Fun}(C, D)$. Every functor $F: E \times C \to D$ gives rise to a *curried functor* $F_c: E \to \mathrm{Fun}(C, D)$ and conversely every functor $G: E \to \mathrm{Fun}(C, D)$ gives rise to an *uncurried functor* $G^u: E \times C \to D$, satisfying $F \cong (F_c)^u$ and $G \cong (G^u)_c$. We may similarly curry/uncurry natural isomorphisms between functors. Currying is demanded to be functorial in E.

For every $n \geq 0$, there is a poset $[n]\{0 \leq 1 \leq \cdots \leq n\}$. We may regard this poset as a category (hence an ∞-category) in which there is a (unique) arrow from i to j if and only if $i \leq j$:

$$[n] \qquad = \qquad (0 \to 1 \to \ldots \to n).$$

The category $[0]$ is equivalent to the terminal category. We are especially interested in the categories $[0]$, $[1]$ and $[2]$ as they allow us to define objects, morphisms and commutative triangles:

Definition B.1. Let C be an ∞-category.

(1) An *object* in C is a functor $x: * \to C$;

(2) A *morphism* in C is a functor $f: [1] \to C$. Composing with $0: * \to [1]$ and $1: * \to [1]$ define its *source* $f(0)$ and its *target* $f(1)$. We will generally write $f: x \to y$ to indicate that the source of f is (isomorphic to) x and the target is (isomorphic to) y.

(3) For objects x and y of C, we define the *Hom anima* of C as the following pullback:

$$\begin{tikzcd} \mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1], C) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ * \arrow[r, "(x,y)"] & C \times C. \end{tikzcd}$$

Note that the objects of $\mathrm{Hom}_C(x, y)$ are precisely the morphisms $f: x \to y$ in C. Given functors $F, G: C \to D$, we write

$$\mathrm{Nat}(F, G) := \mathrm{Hom}_{\mathrm{Fun}(C,D)}(F, G)$$

for the Hom anima in the functor category. Its objects $\alpha: F \to G$ are called *natural transformations from F to G*.

(4) A *commutative triangle* in C is a functor $\sigma: [2] \to C$. We will generally draw commutative triangles as

$$\sigma \quad = \quad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z, \end{tikzcd}$$

where $f = \sigma \circ d_2$, $g = \sigma \circ d_0$ and $h = \sigma \circ d_1$.

To guarantee that these notions behave as expected, we impose the following three properties:

(D) *Commutative square axiom*: the data of a commutative square $[1] \times [1] \to C$ is the same as that of two commutative triangles $[2] \to C$ that agree along their diagonal:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "l"] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

(E) *Segal axiom*: for every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$, there is an essentially unique commutative triangle $\sigma\colon [2] \to C$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & & z. \end{tikzcd}$$

More precisely, the restriction functor

$$(d_0^*, d_2^*)\colon \operatorname{Fun}([2], C) \to \operatorname{Fun}([1], C) \times_C \operatorname{Fun}([1], C), \qquad \sigma \mapsto (g, f)$$

is an equivalence of ∞-categories. Inverting this map and passing to fibers results in a *composition map*

$$- \circ -\colon \operatorname{Hom}_C(y, z) \times \operatorname{Hom}_C(x, y) \to \operatorname{Hom}_C(x, z)$$

for all $x, y, z \in C$. One can show that composition is unital and associative: there are natural isomorphisms

$$\mathrm{id}_y \circ f \cong f \cong f \circ \mathrm{id}_x \qquad \text{and} \qquad h \circ (g \circ f) \cong (h \circ g) \circ f,$$

where $f\colon x \to y$, $g\colon y \to z$ and $h\colon z \to w$.

(F) *Rezk axiom*: We say a morphism $f\colon x \to y$ in C is *invertible*, or an *isomorphism*, if it has both a left-inverse and a right-inverse, i.e. there are morphisms $g\colon y \to x$ and $h\colon y \to x$ satisfying $gf \cong \mathrm{id}_x$ and $fh \cong \mathrm{id}_y$. This defines a subcategory $\operatorname{Iso}(C) \subseteq \operatorname{Fun}([1], C)$ consisting of the invertible morphisms. We then demand that every isomorphism in C is isomorphic to an identity morphism; or more precisely that the map

$$C \to \operatorname{Iso}(C), \quad x \mapsto \mathrm{id}_x$$

is an equivalence of ∞-categories.

Definition B.2. The *simplex category* $\mathbf{\Delta}$ is the category of non-empty finite linearly ordered sets and order-preserving maps between them. A *simplicial object* in an ∞-category C is a functor $\mathbf{\Delta}^{\mathrm{op}} \to C$. We write

$$\mathrm{s}C := \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$$

for the ∞-category of simplicial objects in C.

Definition B.3. An ∞-category C is called an *anima*[1] or *(∞-)groupoid* if every morphism in C is invertible.

Proposition B.4. *For objects x and y of an ∞-category C, the Hom anima* $\mathrm{Hom}_C(x, y)$ *is an anima.*

(G) *Groupoid core axiom*: For every ∞-category C, there is an anima $C^{\simeq}$ called the *(groupoid) core of C*, which comes equipped with a functor $\gamma_C \colon C^{\simeq} \to C$. Every functor $F \colon X \to C$ from an anima X factors through γ_C, and for functors $G, H \colon X \to C^{\simeq}$ every natural isomorphism $\gamma_C \circ G \cong \gamma_C \circ H$ may be lifted to a natural isomorphism $G \cong H$.

Furthermore, the groupoid core of $[1]$ is equivalent to $* \sqcup *$.

Definition B.5. For ∞-categories C and D, we write $\mathrm{Map}(C, D)$ for the anima $\mathrm{Fun}(C, D)^{\simeq}$.

Definition B.6. A functor $F \colon C \to D$ is called *fully faithful* if for every pair of objects x and y of C, the induced map of animae

$$F \colon \mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(Fx, Fy)$$

is an equivalence.

Definition B.7. A functor $F \colon C \to D$ is called *essentially surjective* if for every object y of D there exists an object x of C together with an isomorphism $Fx \cong y$.

Theorem B.8 (Joyal, [Lan21, Theorem 2.3.20], [Cis+24, Theorem 6.3.1])**.** *A functor $F \colon C \to D$ is an equivalence if and only if it is fully faithful and essentially surjective.*

Theorem B.9 (Joyal, [Lan21, Theorem 2.2.1], [Cis+24, Theorem 6.2.10])**.** *Let $F, G \colon C \to D$ be two functors and let $\alpha \colon F \Rightarrow G$ be a natural transformation. Then α is a natural isomorphism if and only if for every object x of C the morphism $\alpha(x) \colon F(x) \to G(x)$ is an isomorphism in D.*

(H) For every ∞-category C there is an *opposite category* C^{op}. Similarly every functor $F \colon C \to D$ induces an opposite functor $F^{\mathrm{op}} \colon C^{\mathrm{op}} \to D^{\mathrm{op}}$, and this process respects composition of functors. We have equivalences $(C^{\mathrm{op}})^{\mathrm{op}} \simeq C$ and $(F^{\mathrm{op}})^{\mathrm{op}} \simeq F$.

If C and D are ordinary categories then C^{op} and F^{op} agree with the usual constructions of opposite categories.

If X is an anima then we have an equivalence $X^{\mathrm{op}} \simeq X$.

Since functors $[1] \to C^{\mathrm{op}}$ correspond to functors $[1] \simeq [1]^{\mathrm{op}} \to (C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, we see that morphisms in C^{op} are just morphisms in C but with source and target flipped: for objects x and y there is an equivalence

$$\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) \simeq \mathrm{Hom}_C(y, x).$$

Also, since $[2] \simeq [2]^{\mathrm{op}}$, this correspondence between morphisms in C and morphisms in C^{op} preserves composition.

Definition B.10 (Limit/colimit)**.** Let I and C be ∞-categories and let $F \colon I \to C$ be a functor.

[1]The plural of 'anima' is either 'anima' or 'animae', depending on taste.

(1) A *cone on F* is an object X in C together with a natural transformation $\varepsilon\colon \mathrm{const}_X \to F$ of functors $I \to C$.

(2) A cone (X, ε) is called a *limit cone* if for every other object Y of C the composite

$$\mathrm{Hom}_C(Y, X) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_Y, \mathrm{const}_X) \xrightarrow{\varepsilon\circ -} \mathrm{Nat}(\mathrm{const}_Y, F)$$

is an equivalence.

In this case, the object X is called the *limit* of F. It is unique up to contractible choice, and will be denoted either by $\lim_I F$ or $\lim_{i\in I} F(i)$.

Dually, one defines a *cocone* to be a natural transformation $\eta\colon F \to \mathrm{const}_X$. It is a *colimit cone* if for every Y the composite

$$\mathrm{Hom}_C(X, Y) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_X, \mathrm{const}_Y) \xrightarrow{-\circ\eta} \mathrm{Nat}(F, \mathrm{const}_Y)$$

is equivalence.

Definition B.11. We say an ∞-category C *admits all I-indexed (co)limits* if for every functor $F\colon I \to C$ there exists a (co)limit (co)cone for F. If $G\colon C \to D$ is a functor between ∞-categories that admit I-indexed (co)limits, then we say that G *preserves I-indexed (co)limits* if for every $F\colon I \to D$, the functor G sends (co)limit (co)cones of F in C to (co)limit (co)cones of GF in D.

Limits and colimits of the unique functor $\emptyset \to C$ can be characterized concretely due to the fact that $\mathrm{Fun}(\emptyset, C) \simeq *$:

Definition B.12 (Initial/terminal objects)**.** An object X of an ∞-category C is called *terminal* if for every other object Y the Hom anima $\mathrm{Hom}_C(Y, X)$ is contractible. We say that Y is *initial* if for every other Y the Hom anima $\mathrm{Hom}_C(X, Y)$ is contractible.

Definition B.13. We define the ∞-category $\lrcorner$ via the following pushout square of ∞-categories:

$$\begin{tikzcd} {[0]} \arrow[r, "1"] \arrow[d, "1"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \lrcorner . \end{tikzcd}$$

In particular, a functor $F\colon\ \lrcorner\ \to C$ by definition corresponds to a pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in C. One can show that a cone $\mathrm{const}_W \to F$ on F corresponds to a commutative square in C of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z. \end{tikzcd}$$

We say that such a square is a *pullback square* if the corresponding cone $\mathrm{const}_W \to F$ is a limit cone. In this case we also write $W = X \times_Z Y$.

The story for pushouts is entirely dual, giving a notion of a *pushout square*: a commutative square

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] & W \end{tikzcd}$$

whose associated cocone $F \to \mathrm{const}_W$ is a colimit cocone. In this case we write $W = X \sqcup_Z Y$.

Definition B.14. A functor $F\colon C \to D$ is called a *monomorphism*, or an *embedding*, if the commutative square

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "F"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

is a pullback square, i.e., the functor $\Delta_F := (\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \times_D C$ is an equivalence of ∞-categories.

Definition B.15. Let C be an ∞-category. A *collection of morphisms in C* is a monomorphism $M \hookrightarrow \mathrm{Map}([1], C)$. We say it is *closed under composition* if the following two conditions are satisfied:

- For a morphism $f\colon x \to y$ in C, if $f \in M$ then also $\mathrm{id}_x \in M$ and $\mathrm{id}_y \in M$;
- For morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, if $g, f \in M$ then also $g \circ f \in M$.

(I) Consider an ∞-category C equipped with a collection of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1], C)$ of C which is closed under composition. Then there exists an ∞-category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ which is universal among functors $D \to C$ whose induced map $\mathrm{Map}([1], D) \to \mathrm{Map}([1], C)$ on morphisms factors through M.

We refer to $\langle M\rangle_C$ as the *(non-full) subcategory spanned by the morphisms in M*.

We may also form *full* subcategories in which we only select a collection of objects and then take *all* morphisms between those objects:

Construction B.16. Consider an embedding $\Gamma \hookrightarrow C^{\simeq}$. We define M_Γ as the following pullback:

$$\begin{tikzcd} M_\Gamma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1], C) \arrow[d, "(s,t)"] \\ \Gamma \times \Gamma \arrow[r, hook] & C^{\simeq} \times C^{\simeq}. \end{tikzcd}$$

We refer to $\langle M_\Gamma\rangle_C$ as the *full subcategory spanned by the objects in* Γ.

Another important ∞-categorical construction is that of *localizations*, where we formally invert a collection of morphisms in an ∞-category:

(I) For every ∞-category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W: it sends the morphisms in W to isomorpisms and is universal with this property.

Example B.17. The functor $p_{[1]}\colon [1] \to *$ exhibits $*$ as a localization of $[1]$ at all its morphisms.

In the case where W consists of *all* morphisms in C, we denote the localization by $\|C\|$ and call it the *geometric realization of* C.

(J) For every ∞-category C, the geometric realization $\|C\|$ is an anima.

(K) There is an ∞-category Cat_∞ whose objects are called *small ∞-categories*. Every object $c \in \mathrm{Cat}_\infty$ defines an ∞-category C. For objects $c, d \in \mathrm{Cat}_\infty$ corresponding to ∞-categories C and D, there is an equivalence of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(c, d) \xrightarrow{\sim} \mathrm{Map}(C, D),$$

and in particular morphisms in Cat_∞ correspond to functors of ∞-categories. Composition in Cat_∞ corresponds to composition of functors.

The ∞-category Cat_∞ admits all I-limits and I-colimits for every small category I.

The ∞-category Cat_∞ is closed under all constructions of ∞-categories introduced so far: initial/terminal ∞-categories, (co)products, pullbacks, functor categories, groupoid cores, subcategories, and localizations. The ∞-categories $[n]$ are small for all $n \geq 0$.

Definition B.18. We define the *∞-category of animae* as the full subcategory

$$\mathrm{An} \subseteq \mathrm{Cat}_\infty$$

spanned by the animae.

The composite functor

$$\Pi_\infty\colon \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \subseteq \mathrm{sAn} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An}) \xrightarrow{|-|:=\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}} \mathrm{An}$$

exhibits the ∞-category An as a localization of the 1-category of topological spaces at the weak homotopy equivalences. We may therefore think of An as the ∞-category that captures the weak homotopy types of topological spaces (and indeed in the literature it is often called the '∞-category of spaces'). It is often much more convenient to formulate constructions directly in An rather than trying to directly model them at the level of Top.

When X is a topological space, we refer to the anima $\Pi_\infty(X)$ as its *fundamental ∞-groupoid* or *underlying anima*.

For an ∞-category C, the Hom animae $\mathrm{Hom}_C(X, Y)$ assemble into a *Hom functor*

$$\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}.$$

In particular every object $X \in C$ defines a functor $Y_X := \mathrm{Hom}_C(-, X)\colon C^{\mathrm{op}} \to \mathrm{An}$.

Theorem B.19 (Yoneda lemma)**.** *For a functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$ and an object $X \in C$ there is an equivalence*

$$\mathrm{Nat}(Y_X, F) \xrightarrow{\sim} F(X), \qquad (\varphi\colon Y_X \to F) \mapsto \varphi(\mathrm{id}_X).$$

In particular the Yoneda embedding

$$Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}), \qquad X \mapsto \mathrm{Hom}_C(-, X)$$

is fully faithful.

Theorem B.20. *The functor* $\mathrm{Hom}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$ *preserves limits in both variables separately.*

The final aspect of category theory that I wish to introduce is the notion of an *adjunction*:

Definition B.21. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors. An *adjunction* between $\mathcal{C}$ and $\mathcal{D}$ consists of a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(-),-) \cong \mathrm{Hom}_{\mathcal{C}}(-,G(-))$$

of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism has been given.

As in ordinary category theory, adjunctions may equivalently be characterized in terms of a unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ satisfying the triangle identities $(\varepsilon F)(F\eta) \cong \mathrm{id}_F$ and $(G\varepsilon)(\eta G) \cong \mathrm{id}_G$. An important feature of adjunctions is that they can be characterized *pointwise*: given a functor $F\colon \mathcal{C} \to \mathcal{D}$, if one can provide for every object Y of $\mathcal{D}$ a suitable object $G(Y)$ of $\mathcal{C}$ equipped with a map $\varepsilon_Y\colon FG(Y) \to Y$ satisfying a suitable adjointness condition, then these objects automatically assemble into a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$, and the maps ε_Y assemble into a counit ε for the adjunction.

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