Lie groups and representation theory

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Contents

Introduction

Lie groups are an important area of mathematics that studies continuous symmetries and transformations. They originated from the work of Sophus Lie in the late 1800s, who wanted to unify the different notions of continuous transformation groups appearing in various mathematical and physical contexts at the time. Lie was inspired by the geometric symmetries present in solutions to differential equations, which are very important in physics. As the theory matured over time, this led to the definition of a Lie group:

Definition. A *Lie group* is a group G equipped with the structure of a smooth manifold such that the multiplication map $m\colon G\times G \to G$ and the inverse map $i\colon G \to G$ are both smooth.

Lie groups aren't just abstract math - they are ubiquitous in physics. The symmetry groups related to equations of motion determine conservation laws and fundamental particle interactions in theories like quantum mechanics and general relativity. Lie groups also find applications in engineering, computer graphics and more, anywhere understanding symmetries is useful for solving problems efficiently.

Lie group theory is intimately connected to many other areas of math like differential geometry, representation theory, algebraic topology, and mathematical physics. During the course, we will also touch on some interesting interactions with analysis and algebra. The theory of Lie groups and their representations is a rich subject, too rich to properly address in a single course, and hence we will restrict ourselves to the following topics:

(1) We will start by introducing Lie groups and discussing various examples. We will review some basic concepts from differential geometry, and use that to define the *exponential map* of a Lie group. We will discuss *invariant integration*, which provides a method of 'averaging' over a compact Lie group.

(2) We will introduce the notion of a *representation* of a Lie group: a vector space on which the Lie group acts linearly. The basic aspects of representation theory will be covered, including characters and orthogonality relations.

(3) We will discuss the *Peter-Weyl theorem*, which will involve some digressions into analysis. As a consequence, we will show that every compact Lie group is a subgroup of a matrix group. We will also see that every irreducible representation of a compact Lie group is finite-dimensional.

(4) We will discuss the uniqueness (up to conjugacy) of *maximal tori* in a compact connected Lie group. We will see that various questions about Lie groups can be reduced to questions about their maximal tori.

(5) We will discuss the connection between Lie groups and Lie algebras, and describe how this results in a complete classification of the compact connected Lie groups, making use of the notion of a *root system*.

Lecture notes:

The lecture notes of this course you are reading right now are based on lecture notes from a previous iteration of this course I taught in the winter term of 2024. There may be updates as the semester progresses.

Recommended literature:

Main reference: Theodor Bröcker, Tammo tom Dieck: *Representations of Compact Lie Groups* [Bt95];

Other books:

- Daniel Bump: *Lie groups* [Bum13];
- J. Frank Adams: *Lectures on Lie groups* [Ada69];
- Luiz A. B. San Martin: *Lie Groups* [San21];
- Wulf Rossmann: *Lie groups: an introduction through linear groups* [Ros06];
- Brian C. Hall: *Lie groups, Lie algebras, and representations* [Hal15];
- Anthony W. Knapp: *Lie Groups - Beyond an Introduction* [Kna02].

1 Lie groups

In this chapter, we recall some basic aspects from the theory of smooth manifolds and introduce the notion of a Lie group. We discuss various classical examples of Lie groups, construct the exponential map of a Lie group, and study subgroups and quotients of Lie groups. Our treatment closely follows Chapter 1 of the book *Representations of compact Lie groups* by Bröcker and tom Dieck [Bt95], though some definitions are instead taken from the book *Lectures on Lie groups* by Adams [Ada69].

1.1 The concept of a Lie group and the classical examples

Recall from the introduction the definition of a Lie group:

Definition 1.1.1. A *Lie group* is a smooth manifold G equipped with a group structure such that the multiplication map $m\colon G \times G \to G$ and the inverse map $i\colon G \to G$ are smooth maps.

Given another Lie group H, a *morphism of Lie groups* $f\colon H \to G$ is a smooth group homomorphism.

Remark 1.1.2. It in fact suffices to assume that only the multiplication map $m\colon G \times G \to G$ is smooth: the smoothness of the inverse map $i\colon G \to G$ is then automatic as a consequence of the inverse function theorem (Exercise 1.1).

We will assume that the reader has at least some familiarity with smooth manifolds. That being said, we will now provide a quick overview of the relevant definitions. We shall use the coordinate-free perspective on these definitions, following [Ada69].

Definition 1.1.3 (Smooth map)**.** Let V and W be finite-dimensional real vector spaces. Let U be an open subset[1] of V, $f\colon U \to W$ a map, and x a point of U. We say that f is *differentiable at x* if there is a linear map $f'(x)\colon V \to W$ such that

$$f(x+h) = f(x) + (f'(x))(h) + o(h),$$

meaning that we have

$$\lim_{h \to 0} \frac{||f(x+h) - f(x) - (f'(x))(h)||}{||h||} = 0$$

for some (hence any) metric on V. We say that f *is differentiable on U* if it is differentiable at each point of U. In this case, we have a function

$$f'\colon U \to \mathrm{Hom}(V, W)$$

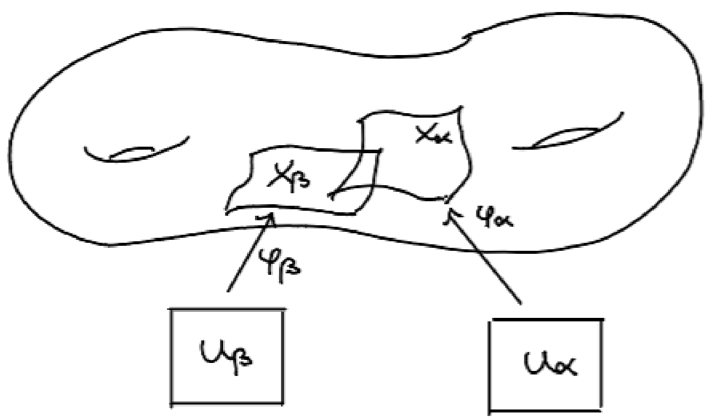
and we may ask this to be differentiable. We say that f is *smooth on U* if each function $f, f', f'', \dots$ is differentiable on U.

[1] There is a unique topology on V such that any linear isomorphism $V \cong \mathbb{R}^n$ is a homeomorphism.

Definition 1.1.4 (Smooth atlas)**.** Let X be a topological space and let V be a finite-dimensional vector space. A *chart* is a homeomorphism $\varphi_\alpha\colon U_\alpha \to X_\alpha$, where $U_\alpha \subseteq V$ and $X_\alpha \subseteq X$ are open subspaces. An *atlas* is a collection of charts $\{\varphi_\alpha\}$ such that $X = \bigcup_\alpha X_\alpha$. The atlas is called *smooth* if for all α, β the function $\varphi_\beta^{-1}\varphi_\alpha$, defined on $\varphi_\alpha^{-1}(X_\alpha \cap X_\beta)$, is smooth.

If X and Y are topological spaces equipped with smooth atlases $\{\varphi_\alpha\}$ and $\{\psi_\beta\}$, we say that a map $f\colon X \to Y$ is *smooth* if for all α, β the map $\psi_\beta^{-1} f \varphi_\alpha$, defined on $\varphi_\alpha^{-1}(X_\alpha \cap f^{-1}Y_\beta)$, is smooth. It is called a *diffeomorphism* if it admits a smooth inverse $f^{-1}\colon Y \to X$.

Notice that the composite of two maps is smooth, and the identity map of a space with atlas is smooth. Also notice that if we equip U_α with the smooth atlas consisting of only the identity map $\mathrm{id}\colon U_\alpha \to U_\alpha$, then every chart $\varphi_\alpha\colon U_\alpha \to X_\alpha$ is smooth.



Definition 1.1.5 (Smooth manifold)**.** A smooth atlas $\{\varphi_\alpha\}$ is called *maximal* if every chart $\varphi\colon U \to X$ which is smooth with respect to the atlas is already contained in the atlas. A *smooth manifold* is a Hausdorff second-countable[2] topological space X equipped with a maximal smooth atlas $\{\varphi_\alpha\}$.

Notice that every smooth atlas on X determines a unique maximal atlas, consisting of all charts on X which are smooth with respect to the given smooth atlas. Two smooth atlases determine the same smooth structure on X if and only if they have the same associated maximal atlas.

Proposition 1.1.6. *Let X and Y be smooth manifolds. Then the cartesian product $X \times Y$ can be given the structure of a smooth manifold in a unique way which satisfies the following two properties:*

(i) The projection maps $\mathrm{pr}_1\colon X \times Y \to X$ *and* $\mathrm{pr}_2\colon X \times Y \to Y$ *are smooth maps;*

(ii) A map $f\colon Z \to X \times Y$ is smooth if and only if both composites $\mathrm{pr}_1 f$ *and* $\mathrm{pr}_2 f$ *are smooth.*

Proof. Given charts φ_α in X and ψ_β in Y, we may form the chart $\varphi_\alpha \times \psi_\beta$ in $X \times Y$. Doing this for every pair (α, β) determines an atlas of $X \times Y$. We leave it to the reader to check that this is a smooth atlas. □

Remark 1.1.7. We now have everything that is required to understand Definition 1.1.1: since G is a smooth manifold, the cartesian product $G \times G$ is also a smooth manifold in a canonical way, and we may ask the structure maps $m\colon G \times G \to G$ and $i\colon G \to G$ to be smooth.

[2] This means that X admits a countable basis for its topology.

Let us now discuss the classical examples of Lie groups.

Example 1.1.8 (Euclidean space)**.** Every finite-dimensional vector space V is a Lie group if we equip it with its additive group structure. Up to isomorphism, this gives a Lie group $\mathbb{R}^n$ for every $n \geq 0$.

Example 1.1.9. The *circle group* S^1 is defined as

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\},$$

with the group structure given by complex multiplication. It has an atlas given by the two subsets $S^1 \setminus \{1\}$ and $S^1 \setminus \{-1\}$.

The function $\mathbb{R} \to S^1$ sending $t \in \mathbb{R}$ to $e^{2\pi i t}$ is smooth and is a group homomorphism, hence is a morphism of Lie groups. Since its kernel is precisely the integers $\mathbb{Z} \subseteq \mathbb{R}$, we obtain an isomorphism of Lie groups $\mathbb{R}/\mathbb{Z} \cong S^1$.

Example 1.1.10 (Tori)**.** For $n \geq 0$, we define the *torus* T^n as the quotient group $\mathbb{R}^n/\mathbb{Z}^n$ of the euclidean space $\mathbb{R}^n$ by the subgroup $\mathbb{Z}^n$. The topology on T^n is given as the quotient topology of $\mathbb{R}^n$. Note that there is a homeomorphism

$$T^n \cong (\mathbb{R}/\mathbb{Z})^n, \quad [(x_1, \ldots, x_n)] \mapsto ([x_1], \ldots, [x_n]),$$

and since $\mathbb{R}/\mathbb{Z} \cong S^1$ is a smooth manifold, it follows from Proposition 1.1.6 that also T^n is a smooth manifold. It is again a Lie group by the following example:

Lemma 1.1.11. *If G and H are Lie groups, then the product group $G \times H$ is again a Lie group when equipped with the smooth structure from Proposition 1.1.6.*

Proof. We must check that the maps $G \times H \times G \times H \to G \times H$, $(g, h, g', h) \mapsto (gg', hh')$ and $G \times H \to G \times H$, $(g, h) \mapsto (g^{-1}, h^{-1})$ are smooth. By definition of the smooth structure on $G \times H$, it suffices to check that both of the components of these expressions are smooth. But this is immediate from the fact that G and H are Lie groups. $\square$

Remark 1.1.12. Combining the previous examples, we obtain for all integers $n, m \geq 0$ a Lie group $\mathbb{R}^n \times T^m$. Notice that all these Lie groups are *abelian*. It turns out that in fact all connected abelian Lie groups are of this form, see Theorem 1.3.8.

Example 1.1.13 (Discrete groups)**.** Let G be a countable group equipped with the discrete topology, for example a *finite* group. Then G admits a smooth atlas given by the collection of points $\{g\}$ for $g \in G$, which turns it into a Lie group. A lot of things we will say about representations of Lie groups are still interesting for finite groups.

For the purpose of this course, the most important examples of finite groups are the following:

- The *symmetric group* S_n, i.e. the group of all permutations of the set $\{1, \ldots, n\}$;
- The *alternating group* A_n, i.e. the group of all *even* permutations of the set $\{1, \ldots, n\}$;
- The *cyclic group* $C_n = \mathbb{Z}/n\mathbb{Z}$ of order n, or equivalently the subgroup of S_n given by the *cyclic* permutations.

Example 1.1.14 (General linear group)**.** Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. The set $\mathrm{End}(V)$ of linear endomorphisms of V is again a vector space, and hence is a smooth manifold. The subset

$$\mathrm{Aut}(V) = \{f\colon V \to V \mid f \text{ is an isomorphism}\} = \{f\colon V \to V \mid \det(f) \neq 0\} \subseteq \mathrm{End}(V)$$

is an open subset, since the determinant is a continuous function. It follows that also $\mathrm{Aut}(V)$ is a smooth manifold. Composition of linear maps provides $\mathrm{Aut}(V)$ with a group structure.

When $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$, we write

$$\mathrm{GL}_n(\mathbb{R}) := \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^n) \qquad \text{and} \qquad \mathrm{GL}_n(\mathbb{C}) := \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n).$$

These groups are called the *general linear groups*. Since linear maps $\mathbb{R}^n \to \mathbb{R}^k$ may be described by $(k \times n)$-matrices, we may identify $\mathrm{GL}_n(\mathbb{R})$ with the group of invertible $(n \times n)$-matrices, with the group structure given by matrix multiplication. Since matrix multiplication is given in terms of algebraic operations, this is a smooth operation. Similarly one argues that inverting a matrix is smooth. It follows that $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ are Lie groups.

It follows that $\mathrm{Aut}(V)$ is a Lie group for arbitrary V, since every finite-dimensional real (or complex) vector space is isomorphic to $\mathbb{R}^n$ (or $\mathbb{C}^n$) for some $n \geq 0$.

Example 1.1.15. The map $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^* = \mathbb{R}\setminus\{0\}$ is a morphism of Lie groups. We let $\mathrm{GL}_n^+(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$ denote the subgroup of matrices with positive determinant. Note that this subgroup is both open and closed.

Exercise 1.1.16. The Lie group $\mathrm{GL}_n^+(\mathbb{R})$ is connected. *Hint:* use elementary row and column operations.

We will now introduce several groups which are defined as closed subgroups or quotients of the general linear groups. While it is possible to prove on a case by case basis that each of these groups is a Lie group, it will also follow from two general results proved in Section 1.3 and Section 1.4:

- Any closed subgroup of a Lie group inherits a Lie group structure (Theorem 1.3.17);
- Any quotient of a Lie group by a closed normal subgroup inherits a Lie group structure (Corollary 1.4.12).

Example 1.1.17 (Special linear group)**.** We may consider the closed subgroups

$$\mathrm{SL}_n(\mathbb{R}) := \{A \in \mathrm{GL}_n(\mathbb{R}) \mid \det(A) = 1\} \qquad \text{and} \qquad \mathrm{SL}_n(\mathbb{C}) := \{A \in \mathrm{GL}_n(\mathbb{C}) \mid \det(A) = 1\},$$

called the *special linear groups* over $\mathbb{R}$ and $\mathbb{C}$. We similarly have the *projective groups*

$$\mathrm{PGL}_n(\mathbb{R}) := \mathrm{GL}_n(\mathbb{R})/\mathbb{R}^* \qquad \text{and} \qquad \mathrm{PGL}_n(\mathbb{C}) := \mathrm{GL}_n(\mathbb{C})/\mathbb{C}^*,$$

where $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ are embedded as the subgroups of scalar multiples of the identity matrix.

The groups GL_n, SL_n and PGL_n are not compact. Throughout most of the course, we will in fact focus on *compact* Lie groups. So let's now recall some of the famous compact subgroups of $\mathrm{GL}_n(\mathbb{R})$.

Definition 1.1.18 (Orthogonal/unitary group)**.** We define the *n-th orthogonal group* $\mathrm{O}(n)$ as

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A^T \cdot A = I\},$$

where A^T denotes the transpose of the matrix A and I is the identity matrix. In other words, the n column vectors in A are *orthonormal* to each other.

Similarly, we define the *n-th unitary group* $\mathrm{U}(n)$ as

$$\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) \mid A^* \cdot A = I\},$$

where $A^* = \overline{A}^T$ is the *conjugate transpose* of A.

The groups $\mathrm{O}(n)$ and $U(n)$ have a very geometric meaning as *length-preserving linear transformations*. Recall that $\mathbb{R}^n$ has a canonical inner product given by the *standard euclidean scalar product*:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

Similarly, on $\mathbb{C}^n$ we have the *standard Hermitian product*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Then $\mathrm{O}(n)$ consists of those automorphisms $A\colon \mathbb{R}^n \to \mathbb{R}^n$ that preserve this inner product, i.e. we have

$$\langle Ax, Ay\rangle = \langle x, y\rangle$$

for all $x, y \in \mathbb{R}^n$. Similarly for $\mathrm{U}(n)$.

Definition 1.1.19 (Special orthogonal/unitary group)**.** Inside $\mathrm{O}(n)$ and $\mathrm{U}(n)$ we have the transformations with determinant one:

$$\mathrm{SO}(n) := \{A \in \mathrm{O}(n) \mid \det(A) = 1\};$$
$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) \mid \det(A) = 1\}.$$

The group $\mathrm{SO}(n)$ turns out to be path-connected: this follows from the path-connectedness of $\mathrm{GL}_n^+(\mathbb{R})$ and the fact that there is a continuous way to turn every matrix into an orthogonal matrix (called Gram-Schmidt orthogonalization).

Lemma 1.1.20. *The Lie groups* $\mathrm{O}(n)$*,* $\mathrm{U}(n)$*,* $\mathrm{SO}(n)$ *and* $\mathrm{SU}(n)$ *are compact.*

Proof. Since all the column vectors have unit length, they form (closed and) bounded subspaces of the finite-dimensional vector space $\mathrm{End}(\mathbb{R}^n)$ or $\mathrm{End}(\mathbb{C}^n)$, hence they are compact. □

Example 1.1.21. Let's do some special cases. For $n = 1$, we get

$$\mathrm{SO}(1) = \{1\}, \qquad \mathrm{O}(1) = \{\pm 1\}.$$
$$\mathrm{SU}(1) = \{1\}, \qquad \mathrm{U}(1) = \{x \in \mathbb{C} \mid |x| = 1\} = S^1.$$

For $n = 2$, we see that

$$\mathrm{SO}(2) = \{\text{rotations in } \mathbb{R}^2\} \cong S^1.$$

The group $\mathrm{O}(2)$ also contains the reflections, and sits in a short exact sequence

$$1 \to \mathrm{SO}(2) \hookrightarrow \mathrm{O}(2) \xrightarrow{\det} \mathbb{Z}/2 \to 1.$$

Example 1.1.22 (Quaternions)**.** Apart from the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$, there is also the *quaternion algebra* $\mathbb{H}$:

$$\mathbb{H} := \{a + bj \mid a, b \in \mathbb{C}\} \cong \mathbb{C}^2$$

with multiplication given by

$$(a_1 + b_1 j)(a_2 + b_2 j) = (a_1 a_2 - b_1 \overline{b_2}) + (a_1 b_2 + \overline{a_2} b_1) j.$$

In other words: the multiplication is $\mathbb{R}$-linear in both variables, satisfies $j^2 = -1$ and satisfies $jz = \overline{z}j$ for all $z \in \mathbb{C}$. **Warning:** $\mathbb{H}$ is not commutative!

Equivalently, we may identify $\mathbb{H}$ with the ring of complex (2×2)-matrices of the form

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix},$$

with matrix addition and multiplication. Under this correspondence, we have

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a real vector space, $\mathbb{H}$ has a basis given by the four vectors 1, i, j and $k := ij$, and multiplication is determined by

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

The quaternions come with a *conjugation map*

$$\overline{} \colon \mathbb{H} \to \mathbb{H}, \quad h = a + bj \mapsto \overline{h} = \overline{a} - bj, \qquad a, b \in \mathbb{C}.$$

This leads to an inner product on $\mathbb{H}^n$, called the *standard symplectic scalar product*:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

In particular, this defines a *norm* on $\mathbb{H}^n$:

$$|x|^2 = \langle x, x \rangle = \sum_{i=1}^{n} x_i \overline{x_i}.$$

Under the identification of $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ as complex vector spaces, the two standard norms on $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ correspond.

For $n \geq 0$, we define the $\mathbb{H}$-*linear group* as

$$\mathrm{GL}_n(\mathbb{H}) = \mathrm{Aut}_{\mathbb{H}}(\mathbb{H}^n),$$

the $\mathbb{H}$-linear automorphisms $A \colon \mathbb{H}^n \to \mathbb{H}^n$, meaning that $A(\sum_{i=1}^n h_i x_i) = \sum_{i=1}^n h_i A(x_i)$ for $h_i \in \mathbb{H}$ and $x_i \in \mathbb{H}^n$. As is the case over $\mathbb{R}$ and over $\mathbb{C}$, such a linear automorphism is fully determined on the n basis vectors $e_1, \dots, e_n$, so that A is fully determined by the $(n \times n)$-matrix (A_{ij}) determined by

$A(e_i) = \sum_{j=1}^{n} A_{ij} e_j$. Consequently, we may identify $\mathrm{GL}_n(\mathbb{H})$ with the group of invertible $(n \times n)$-matrices with coefficients in $\mathbb{H}$, and hence it is a Lie group.

We define the *n-th symplectic group* as

$$\mathrm{Sp}(n) := \{A \in \mathrm{GL}_n(\mathbb{H}) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \text{ for all } x, y \in \mathbb{H}^n\},$$

i.e. those $\mathbb{H}$-linear transformations that preserve the symplectic scalar product.

Exercise 1.1.23. Show that an $\mathbb{H}$-linear automorphism $A \colon \mathbb{H}^n \to \mathbb{H}^n$ preserves the symplectic scalar product if and only if it preserves the norm:

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}_n(\mathbb{H}) \mid |A(x)| = |x| \text{ for all } x \in \mathbb{H}^n\}.$$

In particular, we have the *quaternion group*

$$\mathrm{Sp}(1) = \{h \in \mathbb{H} \mid |h| = 1\}.$$

Remark 1.1.24. Every $\mathbb{H}$-linear map $\mathbb{H}^n \to \mathbb{H}^n$ is in particular $\mathbb{C}$-linear. If we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ as complex vector spaces, we thus see that we get an inclusion

$$\mathrm{GL}_n(\mathbb{H}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}).$$

The image consists precisely of those invertible endomorphisms of $\mathbb{C}^n \oplus \mathbb{C}^n$ of the form

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \qquad\qquad A, B \in \mathrm{End}_C(\mathbb{C}^n).$$

Since the norms on $\mathbb{H}^n$ and $\mathbb{C}^{2n}$ correspond, this in particular means that a quaternionic matrix $A \in \mathrm{GL}_n(\mathbb{H})$ is symplectic if and only if the corresponding complex matrix $A \in \mathrm{GL}_{2n}(\mathbb{C})$ is unitary.

Exercise 1.1.25. Show that this assignment induces an isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$.

Remark 1.1.26. We have defined the three subgroups

$$\begin{aligned} \mathrm{O}(n) &\subseteq \mathrm{GL}_n(\mathbb{R}), && \text{Euclidean scalar product} \\ \mathrm{U}(n) &\subseteq \mathrm{GL}_n(\mathbb{C}), && \text{Hermitian scalar product} \\ \mathrm{Sp}(n) &\subseteq \mathrm{GL}_n(\mathbb{H}), && \text{symplectic scalar product} \end{aligned}$$

in a completely analogous way. More generally, given a bilinear map

$$\langle -, - \rangle \colon V \times V \to H$$

of a finite-dimensional real vector space V into a real vector space H, we may associate the Lie group

$$G = \{A \in \mathrm{Aut}(V) \mid \langle Av, Aw \rangle = \langle v, w \rangle \text{ for all } v, w \in V\}.$$

Many important Lie groups with a geometric flavor arise in this way, for example the *Lorentz group*, which comes from the scalar product on $\mathbb{R}^4$:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4.$$

Let us finish the section by giving an overview of all the linear groups we have introduced, together with some of their inclusions:

$$\begin{tikzcd}
 & & \mathrm{GL}_{2n}^+(\mathbb{R}) & \\
\mathrm{GL}_n^+(\mathbb{R}) \arrow[r, hook] & \mathrm{GL}_n(\mathbb{R}) \arrow[r, hook] & \mathrm{GL}_n(\mathbb{C}) \arrow[r, hook] \arrow[u, hook] & \mathrm{GL}_n(\mathbb{H}) \\
\mathrm{SO}(n) \arrow[r, hook] \arrow[u, hook] & \mathrm{O}(n) \arrow[r, hook] \arrow[u, hook] & \mathrm{U}(n) \arrow[r, hook] \arrow[u, hook] \arrow[d, hook] & \mathrm{Sp}(n) \arrow[u, hook] \arrow[d, hook] \\
 & & \mathrm{SO}(2n) & \mathrm{U}(2n).
\end{tikzcd}$$

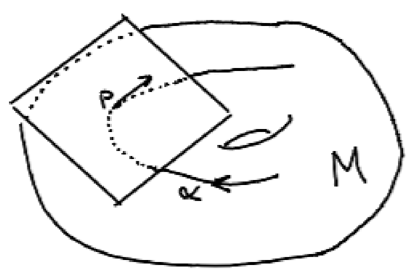
Exercise 1.1.27. Construct all the inclusion maps displayed in this diagram.

1.2 Left-invariant vector fields and one-parameter groups

Next, we will discuss tangent spaces of manifolds and see what they look like for Lie groups. The material in this section is taken from Bröcker and tom Dieck [Bt95, Section 1.2].

Recap on tangent spaces

Given a smooth manifold M that is embedded into some euclidean space $\mathbb{R}^n$, a *tangent vector* of M at some point $p \in M$ is the velocity vector $\dot{\alpha}(0)$ of some smooth curve $\alpha\colon \mathbb{R} \to M$ that satisfies $\alpha(0) = p$. (See the picture below.)



We would like to make sense of this without having to choose an embedding of M into some euclidean space. The idea is as follows: instead of directly specifying what a tangent vector X at p is, we specify what it means to *take a directional derivative* in the direction of X. The input of such a directional derivative should be a function on M that is defined locally around p, and the derivative of such a function should really only depend on arbitrary small open neighborhoods of p. This naturally leads to the following notion of a *germ*:

Definition 1.2.1 (Germs)**.** Let M and N be smooth manifolds and let $p \in M$.

- A *locally defined smooth map from* (M, p) *to* N is a pair (U, f), where $U \subseteq M$ is an open subspace containing p, and $f\colon U \to N$ is a smooth map.

- Given another locally defined smooth map (V, g), we say that f and g have *equal germs at p* if there exists an open subset $p \in W \subseteq U \cap V$ such that $f|_W = g|_W$. This defines an equivalence relation on the set of all locally defined smooth maps.
- An equivalence class of the above equivalence relation is called a *germ*, and is denoted $f\colon (M, p) \to N$. If we have fixed another point $q \in N$ and the germ satisfies $f(p) = q$ (for some, hence any, representative f) then we write $f\colon (M, p) \to (N, q)$.
- We denote by $C^\infty_p(M)$ the set of germs of real-valued functions $\varphi\colon (M, p) \to \mathbb{R}$. It admits a natural $\mathbb{R}$-algebra structure: given two locally defined smooth maps (U, φ) and (V, ψ), we may define their sum and product as

$$(U,\varphi)+(V,\psi) := (W, \varphi|_W + \psi|_W) \qquad \text{and} \qquad (U,\varphi)\cdot(V,\psi) := (W, \varphi|_W \cdot \psi|_W),$$

where $W := U \cap V$. One may similarly define the scalar multiplication by $\lambda \cdot (U, \varphi) = (U, \lambda \cdot \varphi)$. One checks that this respects the equivalence relation, and thus induces an $\mathbb{R}$-algebra structure on the set $C^\infty_p(M)$.

Remark 1.2.2. For the reader who is familiar with the language of sheaves: the set $C^\infty_p(M)$ of real-valued germs is precisely the stalk at $p \in M$ of the structure sheaf $C^\infty(-)$ on M, i.e. the sheaf given by sending an open subset U of M to the set $C^\infty(U)$ of real-valued smooth maps $U \to \mathbb{R}$.

Definition 1.2.3 (Tangent vector)**.** Given a smooth manifold M, a *tangent vector* at $p \in M$ is an $\mathbb{R}$-linear map $X\colon C^\infty_p(M) \to \mathbb{R}$ satisfying the *Leibniz rule*:

$$X(\varphi \cdot \psi) = X(\varphi) \cdot \psi(p) + \varphi(p) \cdot X(\psi).$$

(Such linear maps are also called *derivations* of the $\mathbb{R}$-algebra $C^\infty_p(M)$.) As mentioned before, we think of $X(\varphi)$ as the *directional derivative* of φ in the direction X.

The set T_pM of all tangent vectors at p is a real vector space in a natural way, and is called the *tangent space* of M at p.

Construction 1.2.4 (Functoriality)**.** The definition of the tangent space T_pM is *functorial* in (M, p): if we are given another smooth manifold N and a point $q \in N$, then any germ $f\colon (M, p) \to (N, q)$ induces a homomorphism of $\mathbb{R}$-algebras

$$f^*\colon C^\infty_q(N) \to C^\infty_p(M), \qquad \varphi \mapsto \varphi \circ f,$$

and hence we obtain the *tangent map* (or *differential*) of f at p:

$$T_pf\colon T_pM \to T_qN, \qquad X \mapsto X \circ f^*.$$

In other words, we have $T_pf(X)\varphi = X(\varphi \circ f)$. The fact that this is functorial means that we have

$$T_p\,\mathrm{id}_M = \mathrm{id}_{T_pM}\colon T_pM \to T_pM \qquad \text{and} \qquad T_pg \circ T_pf = T_p(g \circ f)\colon T_pM \to T_rL,$$

where $g\colon (N, q) \to (L, r)$ is another germ.

Remark 1.2.5. We say that a germ $f\colon (M,p) \to (N,q)$ is *invertible* if there is another germ $g\colon (N,q) \to (M,p)$ such that $g \circ f = \mathrm{id}_{(M,p)}$ and $f \circ g = \mathrm{id}_{(N,q)}$ *as germs*, i.e. f is a diffomorphism locally around p. It follows immediately from functoriality that an invertible germ induces an invertible map on tangent spaces:

$$T_p f\colon T_p M \xrightarrow{\cong} T_q N.$$

In particular, for any open subspace $U \subseteq M$ we have a canonical isomorphism $T_p U \xrightarrow{\cong} T_p M$. As a consequence, given any chart $\varphi_\alpha\colon U_\alpha \xrightarrow{\cong} M_\alpha \subseteq M$, for $U_\alpha \subseteq V_\alpha$ an open subset of a finite-dimensional vector space V_α, we obtain an isomorphism of tangent spaces

$$T_q\varphi_\alpha\colon T_q V_\alpha \cong T_q U_\alpha \xrightarrow{\cong} T_p M_\alpha \cong T_p M.$$

The tangent space $T_q V_\alpha$ is easily understood:

Proposition 1.2.6. *If V is a finite-dimensional real vector space, then $T_p V$ is canonically isomorphic to V for all $p \in V$.*

We will frequently use this to simply identify a vector space V with its tangent space.

Proof. We define a homomorphism $V \to T_p V$ by sending the vector v to the derivation $X_v\colon C_p^\infty(V) \to \mathbb{R}$ given by taking the derivative in the direction of v:

$$X_v(\varphi) = \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi(p+tv).$$

Since $C_p^\infty(V)$ in particular contains the dual V^* consisting of the *linear* maps $V \to \mathbb{R}$, the map $V \to T_p V$ is injective. It will thus remain to show it is surjective.

Without loss of generality, we may assume that $(V,p) = (\mathbb{R}^n, 0)$. Then the map $\mathbb{R}^n \to T_0\mathbb{R}^n$ sends the standard basis vector e_i to the derivation $\frac{\partial}{\partial x_i}$. In particular, any linear combination of the $\frac{\partial}{\partial x_i}$ lies in the image. We will show that the derivations $\frac{\partial}{\partial x_i}$ in fact form a *basis* of $T_0\mathbb{R}^n$.

Consider the collection of elements of $C_0^\infty(\mathbb{R}^n)$ given by the coordinate functions $\mathbb{R}^n \to \mathbb{R}$, $x \mapsto x_i$, $1 \le i \le n$. Given an *arbitrary* derivation $X \in T_0\mathbb{R}^n$, we then obtain elements $a_i := X(x_i)$. We are now done if we can show that $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$. By defining $Z := X - \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, this follows from the following general claim:

Claim: Any tangent vector $Z \in T_p(\mathbb{R}^n)$ satisfying $Z(x_i) = 0$ for all $1 \le i \le n$ is zero.

Proof: First note that Z vanishes on constant germs: we have $Z(1) = Z(1 \cdot 1) = Z(1) + Z(1)$, hence $Z(1) = 0$, and thus $Z(c) = cZ(1) = 0$ for all $c \in \mathbb{R}$. Now, given any germ $\varphi\colon (\mathbb{R}^n, 0) \to \mathbb{R}$, we might use a form of Taylor expansion to write φ around 0 as

$$\varphi(x) = \varphi(0) + \sum_{i=1}^n \varphi_i(x) \cdot x_i, \qquad \text{where} \qquad \varphi_i(x) = \int_{t=0}^1 \frac{\partial \varphi}{\partial x_i}(tx)\,dt.$$

But then using linearity of Z and applying the Leibniz rule, we get

$$Z(\varphi) = Z(\varphi(0)) + \sum_{i=1}^n Z(\varphi_i(x)) \cdot 0 + \sum_{i=1}^n \varphi_i(0) \cdot Z(x_i) = 0 + 0 + 0 = 0,$$

as desired. □

Let us briefly discuss how this definition of derivative relates to the notion we are used to from analysis by looking at the case of germs $f\colon (\mathbb{R}^m,0) \to (\mathbb{R}^n,0)$ between euclidean spaces. By definition, we may calculate its derivative $T_0 f\colon T_0\mathbb{R}^m \to T_0\mathbb{R}^n$ as follows:

$$T_0 f(\frac{\partial}{\partial x_j})(\varphi) = \frac{\partial}{\partial x_j}\bigg|_0 (\varphi \circ f) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j}(0) \cdot \frac{\partial}{\partial y_i}\bigg|_0 (\varphi),$$

and thus $T_0 f(\frac{\partial}{\partial x_j}) = \sum_{i=1}^n \frac{\partial f_i}{\partial x_j}(0)\cdot \frac{\partial}{\partial y_i}$. So with respect to the standard bases of $T_0\mathbb{R}^m$ and $T_0\mathbb{R}^n$ provided by the (proof of the) proposition, $T_0 f$ is completely described by its *Jacobian matrix*

$$Df = \left(\frac{\partial f_i}{\partial x_j}(0)\right)_{i,j}.$$

Definition 1.2.7 (Tangent bundle)**.** For a smooth manifold M, we define its *tangent bundle* as the disjoint union

$$TM = \coprod_{p\in M} T_p M,$$

together with the map $\pi\colon TM \to M$ sending $v \in T_p M$ to p.

Construction 1.2.8 (Functoriality)**.** For a smooth map $f\colon M \to N$ of smooth manifolds, we obtain a *tangent map*

$$Tf\colon TM \to TN$$

given by assembling the collection of maps $T_p M \xrightarrow{T_p f} T_{f(p)} N \hookrightarrow TN$ for all $p \in M$. Note that $T(\mathrm{id}_M) = \mathrm{id}_{TM}$ and that $T(g\circ f) = Tg \circ Tf$.

In particular, every open subspace $U \subseteq M$ induces an inclusion $TU \subseteq TM$.

Construction 1.2.9 (Smooth structure on TM)**.** The set TM admits a canonical smooth structure. In the $M = V$ is a finite-dimensional real vector space, this is clear: by Proposition 1.2.6 there is a canonical bijection $TV \cong V\times V$, and thus we may give TV the smooth manifold structure corresponding to the vector space structure on $V\times V$. For arbitrary M, we pick charts $\varphi_\alpha\colon U_\alpha \xrightarrow{\cong} M_\alpha \subseteq M$, where U_α is an open subset of a vector space V_α. By the previous construction, we obtain inclusions $TM_\alpha \subseteq TM$, and we have $TM = \bigcup_\alpha TM_\alpha$. Similarly we obtain maps maps

$$TM_\alpha \cong TU_\alpha \subseteq TV_\alpha \cong V_\alpha \times V_\alpha,$$

and in particular each TM_α inherits a smooth manifold structure from $V_\alpha\times V_\alpha$. If we now define the topology on TM by saying that $U \subseteq TM$ is open if and only if every $U\cap TM_\alpha \subseteq TM_\alpha$ is open, then one can verify that the isomorphisms $TU_\alpha \xrightarrow{\cong} TM_\alpha$ define charts for a smooth manifold structure on TM, as desired. Note that the dimension of TM is twice the dimension of M.

This construction also shows that the map $\pi\colon TM \to M$ is a *vector bundle*: if we restrict the bundle to the open cover M_α, the resulting maps $\pi|_{M_\alpha}\colon \pi^{-1}(M_\alpha) \to M_\alpha$ are isomorphic to the trivial bundles $M_\alpha \times \mathbb{R}^n \to M_\alpha$, where $n = \dim(V_\alpha)$.

Observation 1.2.10. When $M = G$ is a Lie group, its tangent bundle turns out to be trivial: given any $g \in G$, we may consider the *left translation function*

$$l_g\colon G \to G, \quad h \mapsto gh.$$

This is a diffomorphism, and hence induces an isomorphism on tangent spaces $T_e l_g \colon T_e G \xrightarrow{\cong} T_g G$, where $e \in G$ is the unit. Doing this for all $g \in G$ provides an isomorphism of vector bundles

$$\begin{tikzcd} G \times T_e G \arrow[rr, "\cong"] \arrow[dr, "\mathrm{pr}_1"'] & & TG \arrow[dl, "\pi"] \\ & G & \end{tikzcd} \qquad (g, X) \mapsto T_e l_g(X)$$

Definition 1.2.11. The vector space $\mathrm{L}G := T_e G$ is called the *Lie algebra* of G. (So far, we have not defined any 'algebra' structure on $\mathrm{L}G$, but we will do this soon.) Any homomorphism of Lie groups $f \colon G \to H$ induces a homomorphism $\mathrm{L}f = T_e f \colon \mathrm{L}G \to \mathrm{L}H$ of Lie algebras in a functorial way.

Immersions and submersions

We will now recall the notions of *immersions* and *submersions*.

Definition 1.2.12. Let $f \colon M \to N$ be a smooth map of smooth manifolds. We say that:

- f is a *submersion* if for every $x \in M$ the tangent map $T_x f \colon T_x M \to T_{f(x)} N$ is surjective;
- f is an *immersion* if for every $x \in M$ the tangent map $T_x f \colon T_x M \to T_{f(x)} N$ is injective;
- f has *constant rank* if there is a natural number $k \in \mathbb{N}$ such that for every $x \in M$ the tangent map $T_x f \colon T_x M \to T_{f(x)} N$ has rank k.

Note that both submersions and immersions have constant rank.

We recall without proof the following theorem about smooth maps of constant rank:

Theorem 1.2.13 (Constant rank theorem, Lee [Lee02, Theorem 5.13])**.** *Let $f \colon M \to N$ be a smooth map of constant rank k. For every point $p \in M$, we may pick a chart of M around p with coordinates $(x_1, \ldots, x_m)$ and a chart of N around $f(p)$ with coordinates $(y_1, \ldots, y_n)$ such that in these coordinates f takes the form*

$$f(x_1, \ldots, x_m) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

Corollary 1.2.14. *Let $f \colon M \to N$ be a smooth bijection of constant rank. Then f is a diffeomorphism.*

Proof. Write $m := \dim(M)$, $n := \dim(N)$ and $k := \mathrm{rank}(f)$. By Theorem 1.2.13, there exists for every $x \in M$ a chart U_x around x and a chart V_x around $f(x)$ such that with respect to the local coordinates determined by these charts, $f|_{U_x} \colon U_x \to V_x$ corresponds to the map $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_k, 0, \ldots, 0)$. Since f is a bijection, this in particular means that $k = m$, as for $k < m$ this map is not injective. We will show that also $m = n$; this will finish the proof, as the inverse function theorem will then give us that the inverse $f^{-1} \colon N \to M$ is a smooth map around every point, so that f is a diffeomorphism.

Suppose for contradiction that we had $m < n$. Picking charts U_x and V_x as above, let K_x be the subset of U_x corresponding to the closed unit ball of $\mathbb{R}^m$. Then $f(K_x)$ is a compact subset of N with empty interior, using $m < n$. Since M is second countable, it has a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ of open sets each of which is contained in some such K_x. But then

$$f(M) = f\left(\bigcup_{n=0}^{\infty} \overline{B_n}\right) = \bigcup_{n=0}^{\infty} f(\overline{B_n})$$

is a countable union of compact sets with empty interior. Since N is locally compact, it follows that $f(M)$ has empty interior, and in particular f is not surjective. But this contradicts the assumption that f is a bijection. □

Corollary 1.2.15. *Every bijective homomorphism of Lie groups is an isomorphism of Lie groups.*

Proof. Since the inverse of a bijective group homomorphism is automatically a group homomorphism, it remains to show that $f\colon G \to H$ is a diffeomorphism. By Corollary 1.2.14, it will suffice to show that f has constant rank. Let k be the rank of the map $T_e f\colon T_e G \to T_e H$. We claim that f has constant rank k. Indeed, for any $g \in G$ we may consider the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "l_g"'] & H \arrow[d, "l_{f(g)}"] \\ G \arrow[r, "f"] & H, \end{tikzcd}$$

and so by passing to tangent maps we obtain a commutative diagram

$$\begin{tikzcd} T_e G \arrow[r, "T_e f"] \arrow[d, "T_e l_g"'] & T_{f(e)} H \arrow[d, "T_e l_{f(g)}"] \\ T_g G \arrow[r, "T_g f"] & T_{f(g)} H. \end{tikzcd}$$

Since the vertical maps are linear isomorphisms, it follows that the rank of $T_g f$ agrees with that of $T_e f$, showing that f has constant rank k. □

Vector fields, integral curves and flows

Definition 1.2.16. Let M be a smooth manifold. A (smooth) *vector field* on M is a smooth section $X\colon M \to TM$ of the tangent bundle $\pi\colon TM \to M$, i.e. a smooth map satisfying $\pi \circ X = \mathrm{id}_M$. In other words, X determines a smooth family of tangent vectors $X_p \in T_p M$.

In the case of $M = \mathbb{R}^n$, with coordinates $(x_1, \dots, x_n)$, every vector field may be written in the form

$$x \mapsto \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i},$$

and it is smooth if and only if the functions $x \mapsto a_i(x)$ are smooth. By choosing charts on M, this determines how smooth vector fields on M look like locally around every point.

Definition 1.2.17. If $M = G$ is a Lie group, we say a vector field X on G is *left-invariant* if for every $g \in G$ the diagram

$$\begin{tikzcd} TG \arrow[r, "Tl_g"] & TG \\ G \arrow[u, "X"] \arrow[r, "l_g"'] & G \arrow[u, "X"'] \end{tikzcd}$$

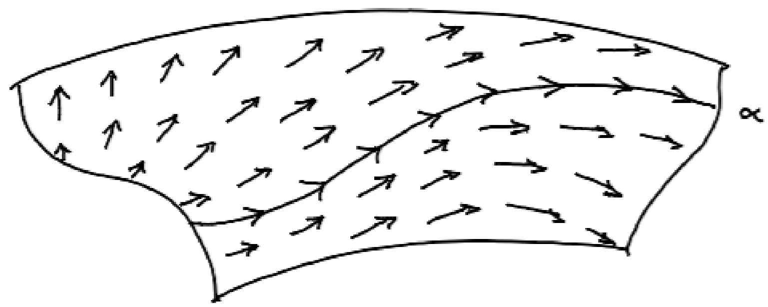
commutes. Every vector $v \in \mathrm{L}G$ determines a left-invariant vector field, given by the composition

$$G \xrightarrow{g \mapsto (g,v)} G \times \mathrm{L}G \xrightarrow{\cong} TG.$$

It is easy to see that every left-invariant vector field X on G must be of this form: the vector $X_g \in T_g G$ must be given by $Tl_g X_e$, hence is the vector field associated to $v = X_e$.

In what follows, we will identify $\mathrm{L}G$ with the space of left-invariant vector fields on G. We will denote a left-invariant vector field on G by $X \in \mathrm{L}G$.

A vector field X on a smooth manifold M determines a *flow* on M.



Let's make this precise.

Definition 1.2.18 (Integral curve)**.** Consider a smooth curve $\alpha\colon (a,b) \to M$, with $a,b \in \mathbb{R}$. For $t \in (a,b)$, let $p := \alpha(t) \in M$. We define the *velocity vector at time* t, denoted $\dot{\alpha}(t) \in T_p M$, as the derivation

$$\dot{\alpha}(t)\colon C^\infty_p(M) \to \mathbb{R}\colon \varphi \mapsto \left.\frac{\partial}{\partial t'}\right|_{t'=t} \varphi(\alpha(t')).$$

In other words, it is the image of the standard generator $\frac{\partial}{\partial t} \in T_t(a,b) \cong \mathbb{R}$ under the tangent map $T_t\alpha\colon T_t(a,b) \to T_p M$. We say that α is an *integral curve* of a vector field X on M if it satisfies

$$\dot{\alpha}(t) = X(\alpha(t)) \qquad \text{for all } a < t < b.$$

We say that α *starts at* p if $0 \in (a,b)$ and $\alpha(0) = p$.

The theory of differential equations tells us that for every point $p \in M$, there is exactly one maximal integral curve $\alpha_p\colon (a_p, b_p) \to M$ starting at p, where 'maximal' means that it cannot be extended to any bigger interval. Assembling these integral curves together, we obtain the (local) *flow* of the vector field:

Theorem 1.2.19 (Lang [Lan83, Chapter 6])**.** *There is an open set $A \subseteq \mathbb{R} \times M$ and a smooth map*

$$\Phi\colon A \to M, \qquad (t,p) \mapsto \alpha_p(t),$$

such that for every $p \in M$ the intersection $A \cap (\mathbb{R} \times \{p\})$ is an open interval (a_p, b_p) containing 0, and the restriction of Φ to this interval is the unique maximal integral curve of X starting at p.

The flow Φ automatically satisfies the following *flow equations*:

$$\Phi(0,p) = p \qquad \text{and} \qquad \Phi(s,\Phi(t,p)) = \Phi(s+t,p),$$

whenever the left-hand side is defined. The first equation is clear. For the second, note that if we fix t then the smooth curves

$$s \mapsto \Phi(s,\Phi(t,p)) \qquad \text{and} \qquad s \mapsto \Phi(s+t,p)$$

are both integral curves of X starting in $\Phi(t,p)$. If we write $\Phi_t(p) := \Phi(t,p)$, we may rewrite these equations as

$$\Phi_0 = \mathrm{id}_M \qquad \text{and} \qquad \Phi_s \circ \Phi_t = \Phi_{s+t}.$$

We may recover the vector field X from the flow Φ as follows:

$$X_p = \frac{\partial}{\partial t}\bigg|_0 \Phi(t,p) \in T_pM \qquad \text{i.e.} \qquad X_p(\varphi) = \frac{\partial}{\partial t}\bigg|_0 \varphi(\Phi(t,p)).$$

One-parameter groups

Let us now return to the case of a Lie group G, where the whole situation simplifies a lot. If X is a left-invariant vector field on G, and $\alpha_e\colon (a,b) \to G$ is an integral curve of X starting at $e \in G$, then for every $g \in G$ also the curve $l_g\alpha\colon (a,b) \to G$ is an integral curve of X which starts at g. By the uniqueness of such curve, we must therefore have

$$\alpha_g = l_g \alpha_e.$$

In particular, if α_e is defined on some interval $(-\varepsilon,\varepsilon)$, then also α_g can be defined on the interval $(-\varepsilon,\varepsilon)$, so that α can always be extended by time $\pm\varepsilon$. It follows that the flow Φ is in fact globally defined on $\mathbb{R}\times G$, and that we have

$$\Phi(t,g) = g \cdot \alpha_e^X(t),$$

where α_e^X is the integral curve for the field $X \in \mathrm{L}G$ starting at $\alpha^X(0) = e$.

Definition 1.2.20. A *one-parameter group* of a Lie group G is a homomorphism of Lie groups

$$\alpha\colon \mathbb{R} \to G$$

(really the homomorphism α, not just its image!)

Proposition 1.2.21. *The assignment $\alpha \mapsto \dot\alpha(0) \in \mathrm{L}G$ defines a bijection between the set of one-parameter groups of G and the Lie algebra of G.*

Proof. Interpreting $\mathrm{L}G$ as the space of left-invariant vector fields on G, the inverse map is given by

$$X \mapsto \alpha^X = \text{ integral curve of } X \text{ starting at } e.$$

To see that α^X is indeed a homomorphism, write Φ for the flow associated to X. We then have

$$\alpha^X(s+t) = \Phi_{t+s}(e) = \Phi_t(\Phi_s(e)) \qquad \text{and} \qquad \Phi_t(ge) = g\Phi_t(e),$$

where the first equality is the flow equation, and the second equality follows from the left-invariance of X. Taking $g = \Phi_s(e)$, we thus get

$$\alpha^X(s+t) = \Phi_t(\Phi_s(e)) = \Phi_t(\Phi_s(e)\cdot e) = \Phi_s(e)\cdot\Phi_t(e) = \alpha^X(s)\cdot\alpha^X(t),$$

showing that α^X is a homomorphism.

The velocity vector $\dot\alpha^X(0)$ is by definition equal to X, so $X \mapsto \alpha^X \mapsto \dot\alpha^X(0)$ is the identity. Conversely, given an arbitrary one-parameter group α and defining $X := \dot\alpha(0)$, we need to show that $\alpha = \alpha^X$. To see this, consider the following flow on G:

$$\Phi\colon\ \mathbb{R}\times G \to G, \qquad (t,g) \mapsto g\cdot\alpha(t).$$

This satisfies $\frac{\partial}{\partial t}\big|_0 \Phi(g,t) = Tl_g(\dot\alpha(0))$, and hence this must be the same flow as the one corresponding to the left-invariant vector field X. In particular, the integral curves of these flows starting at e must agree, showing that $\alpha = \alpha^X$. $\square$

Example 1.2.22. Let $G = V$ be a finite-dimensional real vector space, equipped with addition. Then $\mathrm{L}V \cong V$, and the one-parameter group corresponding to $v \in V = \mathrm{L}V$ is $\alpha^v(t) = tv$.

Example 1.2.23. The n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ has $\mathbb{R}^n$ as its Lie algebra, and the one-parameter group corresponding to $v \in \mathbb{R}^n$ is $\alpha^v(t) = (tv \mod \mathbb{Z}^n)$.

Example 1.2.24. Since $\mathrm{Aut}(V)$ is an open submanifold of $\mathrm{End}(V)$, the vector space of all linear endomorphisms of V, we see that

$$\mathrm{L\,Aut}(V) \xrightarrow{\cong} \mathrm{L\,End}(V) \xrightarrow{\cong} \mathrm{End}(V).$$

The one-parameter group corresponding to $A \in \mathrm{End}(V)$ is

$$\alpha^A\colon\ \mathbb{R} \to \mathrm{Aut}(V), \qquad t \mapsto \exp(tA) = \sum_{n=0}^{\infty} \frac{1}{n!}(tA)^n,$$

which is known as the *matrix exponential*. This is an illustrative example that we will return to various times in the future.

The Lie bracket on LG

So far, we have examined what the basic constructions from manifold theory mean for Lie groups. Before turning to a more detailed study of one-parameter groups, we should say a few words about the 'algebra' structure on the Lie algebra of a Lie group. Although we will not use it a lot in this course, this structure is an important fundamental concept.

Consider two left-invariant vector fields $X, Y \in \mathrm{L}G$. For every point $g \in G$, X and Y yield derivations X_g and Y_g on function germs defined around g. Given a smooth map $\varphi\colon U \to \mathbb{R}$ for $U \subseteq G$ open, applying X gives a new smooth map $X(\varphi)\colon U \to \mathbb{R}$ given by $g \mapsto X_g(\varphi)$, and then applying Y gives $Y(X(\varphi)) \in \mathbb{R}$. Similarly, every germ φ determines a real number $X(Y(\varphi)) \in \mathbb{R}$. The *Lie bracket* $[X,Y]$ of X and Y is defined as the derivation given by

$$[X,Y](\varphi) = X(Y(\varphi)) - Y(X(\varphi)).$$

One can show by explicit calculation that this satisfies the Leibniz rule (in contrast to the individual two summands) and hence it is a derivation.

The group structure on G provides an alternative description of the Lie bracket. For an element $g \in G$, consider the inner automorphism

$$c_g \colon G \to G, \qquad h \mapsto ghg^{-1}.$$

By taking the derivative at the unit $e \in G$, we obtain a linear automorphism $\mathrm{L}c_g \colon \mathrm{L}G \to \mathrm{L}G$.

Definition 1.2.25. The *adjoint representation* Ad is the group homomorphism

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{L}G), \qquad g \mapsto \mathrm{L}c_g.$$

By taking derivatives, we obtain a morphism of Lie algebras

$$\mathrm{ad} := \mathrm{L}\,\mathrm{Ad}\colon \mathrm{L}G \to \mathrm{L}\,\mathrm{Aut}(\mathrm{L}G) = \mathrm{End}(\mathrm{L}G).$$

Proposition 1.2.26. *The map* ad *computes the Lie bracket:*

$$\mathrm{ad}(X)(Y) = [X,Y].$$

Proof. We will compute the left-invariant vector field of $\mathrm{ad}(X)(Y)$ and show that it agrees with that of $[X,Y]$. For $X,Y \in \mathrm{L}G$, we have

$$c_{\alpha^X(s)}(\alpha^Y(t)) = \alpha^X(s)\cdot\alpha^Y(t)\cdot(\alpha^X(s))^{-1} = \alpha^X(s)\cdot\alpha^Y(t)\cdot\alpha^X(-s).$$

Setting $a(s,t) := c_{\alpha^X(s)}(\alpha^Y(t))$, we then have

$$\mathrm{Ad}(\alpha^X(s))Y = \frac{\partial}{\partial t}\bigg|_0 a(s,t) \in \mathrm{L}G,$$

and thus

$$\mathrm{ad}(X)(Y) = \frac{\partial}{\partial s}\bigg|_0 \frac{\partial}{\partial t}\bigg|_0 a(s,t) \in \mathrm{L}G,$$

where we identified $\mathrm{L}G$ with its tangent space. A computation shows that the derivation associated to this tangent vector acts on a germ of functions $\varphi\colon (G,e) \to (\mathbb{R},0)$ via

$$(\mathrm{ad}(X)Y)\varphi = \frac{\partial^2}{\partial s \partial t}\bigg|_0 \varphi(a(s,t)).$$

To compute this derivative, we use the chain rule, where we rewrite the map $(s,t) \mapsto \varphi(a(s,t))$ as a composite

$$\begin{array}{ccccc} (s,t) & \longmapsto & (s,t,-s) & & \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & & (s_1,t,s_2) & \longmapsto & \varphi(\alpha^X(s_1)\cdot\alpha^Y(t)\cdot\alpha^X(s_2)) \end{array}$$

The result is

$$\frac{\partial^2}{\partial s\partial t}\bigg|_0 \varphi(a(s,t)) = \frac{\partial^2}{\partial s_1 \partial t}\bigg|_0 \varphi(\alpha^X(s_1)\cdot\alpha^Y(t)) - \frac{\partial^2}{\partial s_2\partial t}\bigg|_0 \varphi(\alpha^Y(t)\cdot\alpha^X(s_2)).$$

We now claim that the first term is $XY\varphi$ while the second term is $YX\varphi$. By symmetry, it suffices to show this for $XY\varphi$. Since the flow corresponding to Y is given by $(t,g)\mapsto g\cdot\alpha^Y(t)$, we get

$$Y\varphi(g) = \frac{\partial}{\partial t}\bigg|_0 \varphi(g\cdot\alpha^Y(t)).$$

Applying the same reasoning to X, we get

$$XY\varphi(g) = \frac{\partial}{\partial s}\bigg|_0 \frac{\partial}{\partial t}\bigg|_0 \varphi(g\cdot\alpha^X(s)\cdot\alpha^Y(t)).$$

Plugging in $g=e$ now gives the claim. □

Lemma 1.2.27. *The Lie bracket* $[-,-]\colon \mathrm{L}G\times\mathrm{L}G\to\mathrm{L}G$ *satisfies the following properties:*

(i) It is bilinear, i.e. $[X,-]\colon \mathrm{L}G\to\mathrm{L}G$ *and* $[-,Y]\colon\mathrm{L}G\to\mathrm{L}G$ *are* $\mathbb{R}$*-linear;*

(ii) We have $[X,X]=0$ *for all* X*, and thus* $[X,Y]=-[Y,X]$ *for all* X,Y*;*

(iii) It satisfies the Jacobi identity*:*

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

Proof. Parts (i) and the beginning of (ii) are clear. The second part of (ii) is then automatic:

$$[X,Y]+[Y,X] = [X,X]+[X,Y]+[Y,X]+[Y,Y] = [X+Y,X+Y] = 0.$$

For part (iii) we simply write out the definition of the left-hand side, which gives us 12 terms of the form $X(Y(Z\varphi))$, where each of the six orderings of X, Y and Z appears precisely twice with once a positive sign and once a negative sign. □

Definition 1.2.28 (Lie algebra)**.** A *Lie algebra* is a real vector space $\mathfrak{g}$ equipped with a map $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying properties (i)-(iii).

In the remainder of this section, we will determine the Lie algebras associated to the Lie groups introduced in Section 1.1.

Example 1.2.29. If G is a discrete Lie group, then $\mathrm{L}G=T_eG=0$ is the trivial vector space with the trivial Lie algebra structure.

Example 1.2.30. If G is an *abelian* Lie group, then each conjugation map $c_g\colon G\to G$ is the identity, and thus the adjoint representation $\mathrm{Ad}\colon G\to\mathrm{Aut}(\mathrm{L}G)$ is trivial. Since $[X,Y]=\mathrm{ad}(X)(Y)$ by Proposition 1.2.26, it follows that $[X,Y]=0$ for all $X,Y\in\mathrm{L}G$.

We now come to the matrix groups.

Example 1.2.31. For $G=\mathrm{Aut}(V)$, we have $\mathrm{L}G=\mathrm{End}(V)$ by Example 1.2.24, where the one-parameter group to $X\in\mathrm{End}(V)$ is given by

$$\alpha^X(s) = I+sX+\frac{1}{2}s^2X^2+\dots.$$

Up to higher order terms, we thus get

$$\begin{aligned} a(s,t) &= \alpha^X(s)\alpha^Y(t)\alpha^X(-s) = (I+sX)(I+tY)(I-sX) \\ &= I + tY + st(XY - YX) \mod (s^2, t^2). \end{aligned}$$

In particular, we get

$$[X,Y] = \left.\frac{\partial}{\partial s}\right|_0 \left.\frac{\partial}{\partial t}\right|_0 a(s,t) = XY - YX \quad \in \quad \mathrm{End}(V).$$

Exercise 1.2.32. Let A be an associative $\mathbb{R}$-algebra. Then the bracket $[X,Y] := XY - YX$ equips A with the structure of a Lie algebra.

The above Lie bracket on $\mathrm{L}\,\mathrm{Aut}(V) = \mathrm{End}(V)$ also gives the Lie bracket for each of the subgroups of $\mathrm{Aut}(V)$ that we have seen.

Example 1.2.33. We define the Lie algebra $\mathrm{so}(n)$ as

$$\mathrm{so}(n) := \mathrm{L}\,\mathrm{SO}(n) \subseteq \mathrm{End}(\mathbb{R}^n).$$

We claim that $\mathrm{so}(n)$ is precisely the subspace of *skew-symmetric matrices*, and so in particular $\dim(\mathrm{SO}(n)) = \dim(\mathrm{so}(n)) = \frac{1}{2}n(n-1)$. Indeed, given $X \in \mathrm{so}(n) \subseteq \mathrm{End}(\mathbb{R}^n)$, let α^X be the associated one-parameter group in $\mathrm{SO}(n)$. Then up to higher order terms we have

$$I = (\alpha^X(s))^T \cdot \alpha^X(s) = (I+sX)^T(I+sX) = I + s(X^T + X) \mod s^2,$$

and so taking derivatives with respect to s gives $X^T + X = 0$, i.e. X is skew-symmetric. Conversely, if $X \in \mathrm{End}(\mathbb{R}^n)$ is skew-symmetric, then we have

$$(\alpha^X(s))^T\alpha^X(s) = \exp(sX^T)\exp(sX) = \exp(-sX)\exp(sX) = \exp(0) = I,$$

and thus α^X lies entirely in $\mathrm{SO}(n)$.

Example 1.2.34. One may show in a completely analogous way that the Lie algebra

$$\mathrm{u}(n) := \mathrm{L}\,\mathrm{U}(n) \subseteq \mathrm{End}(\mathbb{C}^n)$$

is the Lie algebra of skew-Hermitian matrices, i.e. those satisfying $X^* + X = 0$. In particular we get $\dim(\mathrm{U}(n)) = n^2$.

Example 1.2.35. We claim that the Lie algebra $\mathrm{sl}(n) := \mathrm{L}\,\mathrm{SL}(n)$ consists of the matrices in $\mathrm{End}(\mathbb{R}^n)$ with zero trace. Indeed, if we reason up to higher order terms as before, we see that for $X \in \mathrm{sl}(n)$ we get

$$1 = \det(\alpha^X(s)) = \det(I + sX) = 1 + s\,\mathrm{tr}(X) \mod s^2,$$

and thus $\mathrm{tr}(X) = 0$. Conversely, if $X \in \mathrm{End}(\mathbb{R}^n)$ has zero trace, then

$$\left.\frac{d}{ds}\right|_0 \det(\exp((t+s)X)) = \det(\exp(tX)) \cdot \left.\frac{d}{ds}\right|_0 \det(\exp(sX)) = \det(\exp(tX)) \cdot \mathrm{tr}(X) = 0,$$

where we use that the derivative of $\det(\exp(sX))$ is $\mathrm{tr}(X)$ as we see from the above computation. Since the function $t \mapsto \det(\exp(tX))$ has zero derivative, it must be constant. But since $\det(\exp(0)) = \det(I) = 1$, it follows that $\exp(tX) \in \mathrm{SL}(n)$ for all t, and thus $X \in \mathrm{sl}(n)$.

Example 1.2.36. Combining the previous two examples, we see that the Lie algebra $\mathrm{su}(n) := \mathrm{L\,SU}(n) \subseteq \mathrm{End}(\mathbb{C}^n)$ consists of the skew-Hermitian matrices with trace zero.

Example 1.2.37. Similar to $\mathrm{so}(n)$ and $\mathrm{u}(n)$, the Lie algebra $\mathrm{sp}(n) := \mathrm{LSp}(n)$ consists of the skew-Hermitian matrices in $\mathrm{End}(\mathbb{H}^n) \subseteq \mathrm{End}(\mathbb{C}^{2n})$. We may identify these with the complex $(2n \times 2n)$-matrices of the form

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \qquad \text{with} \qquad \overline{A}^T = -A \qquad \text{and} \qquad B^T = B.$$

We see that $\dim(\mathrm{Sp}(n)) = 2n^2 + n$.

Remark 1.2.38. The adjoint representation of the linear groups $\mathrm{Aut}(V)$ is given by

$$\mathrm{Ad}(A)\colon \mathrm{End}(V) \to \mathrm{End}(V), \qquad X \mapsto AXA^{-1}.$$

To see this, note that

$$c_A(\alpha^X(t)) = A\exp(tX)A^{-1} = \exp(tAXA^{-1}) = \alpha^{AXA^{-1}}(t),$$

so differentiating at $t = 0$ gives $\mathrm{Ad}(A)(X) = AXA^{-1}$. This shows that the action of $\mathrm{Aut}(V)$ on $\mathrm{L\,Aut}(V) = \mathrm{End}(V)$ is by conjugation.

1.3 The exponential map

In the examples of one-parameter groups given in the previous section, the matrix exponential $\exp\colon \mathrm{End}(V) \to \mathrm{Aut}(V)$ played an important role. In this section, we will see that this exponentiation operation admits a generalization to an *arbitrary* Lie group G, in which case it takes the form $\exp\colon \mathrm{L}G \to G$. Recall from the previous section that every vector $X \in \mathrm{L}G$ determines a left-invariant vector field of G, and thus determines a one-parameter group $\alpha^X\colon \mathbb{R} \to G$ with $\dot{\alpha}^X(0) = X$.

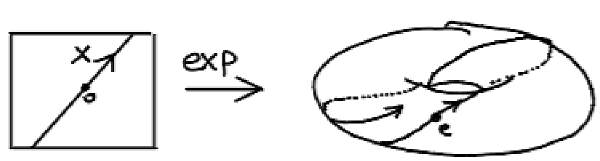
Proposition 1.3.1. *The map*

$$\exp\colon \mathrm{L}G \to G, \qquad X \mapsto \alpha^X(1),$$

*called the **exponential map**, is smooth. Furthermore, its differential at the origin*

$$\mathrm{L}\exp\colon \mathrm{L}G = \mathrm{LL}G \to \mathrm{L}G$$

is the identity. In particular, exp *is a local diffeomorphism.*



Proof. Let us assume for a moment that exp is a smooth map. Note that for fixed t, the one-parameter groups

$$s \mapsto \alpha^{tX}(s) \qquad \text{and} \qquad s \mapsto \alpha^{X}(ts)$$

correspond to the same vector $tX \in \mathrm{L}G$, and hence they are equal. Applying this to $s = 1$ gives

$$\exp(tX) = \alpha^{tX}(1) = \alpha^{X}(t).$$

It follows that $\frac{\partial}{\partial t}\big|_0 \exp(tX) = X$, and hence $T_0 \exp = \mathrm{id}_{\mathrm{L}G}$ as claimed.

To show that exp is smooth, we will show it can be obtained from a certain flow on the Lie group $G \times \mathrm{L}G$. Consider the following map:

$$\mathbb{R} \times G \times \mathrm{L}G \to G \times \mathrm{L}G, \qquad (t, g, X) \mapsto (g \cdot \alpha^{X}(t), X).$$

The derivative of this map with respect to t is $(Tl_g X, 0)$, hence this map is precisely the flow on $G \times \mathrm{L}G$ of the vector field $(g, X) \mapsto (X_g, 0)$, and in particular this map is smooth. But then also the restriction

$$\{1\} \times \{e\} \times \mathrm{L}G \to G, \qquad (1, e, X) \mapsto \alpha^{X}(1)$$

is smooth, finishing the proof. $\square$

From now on, we will always express the one-parameter group α^X as $t \mapsto \exp(tX)$. The exponential map plays a central role in the theory of Lie groups: it describes how the Lie algebra determines the structure of the corresponding Lie group.

Lemma 1.3.2 (Naturality). *A homomorphism of Lie groups $f\colon G \to H$ induces a commutative diagram*

$$\begin{tikzcd} \mathrm{L}G \arrow[r, "Lf"] \arrow[d, "\exp"'] & LH \arrow[d, "\exp"] \\ G \arrow[r, "f"] & H. \end{tikzcd}$$

Proof. Consider a vector $X \in LG$. Then the composite $f \circ \alpha^X\colon \mathbb{R} \to H$ satisfies $\frac{\partial}{\partial t}\big|_0 f(\alpha^X(t)) = T_e f(\dot{\alpha}^X(0)) = \mathrm{L}f(X)$, so it is a one-parameter group $\alpha^{\mathrm{L}f(X)}$ associated to $\mathrm{L}f(X)$. It follows that

$$f(\exp(X)) = f(\alpha^X(1)) = \alpha^{Lf(X)}(1) = \exp(Lf(X))$$

as desired. $\square$

The exponential map is a local diffeomorphism around the origin $0 \in LG$: this is a consequence of the inverse function theorem, since its differential at 0 is the identity map of LG.

Example 1.3.3. For $G = \mathrm{Aut}(V)$, we have $\mathrm{L}G = \mathrm{End}(V)$ (see Example 1.2.24) and the exponential map is given by the matrix exponential:

$$\exp\colon \mathrm{End}(V) \to \mathrm{Aut}(V), \qquad A \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

By naturality of the exponential map, it directly follows that the same formula holds for *all* the linear groups, i.e. for $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, etcetera.

Lemma 1.3.4. *Let G and H be Lie groups, and assume that G is connected. Then a homomorphism $f\colon G \to H$ is determined by its differential at the unit* $\mathrm{L}f\colon \mathrm{L}G \to \mathrm{L}H$.

Proof. By naturality of the exponential map, we know that the composite $\mathrm{L}G \xrightarrow{\exp} G \xrightarrow{f} H$ is completely determined by $Lf\colon LG \to LH$. Since the map $\exp\colon \mathrm{L}G \to G$ is a local diffeomorphism at $0 \in \mathrm{L}G$, this means that $\mathrm{L}f$ completely determined f on an open neighborhood U of $e \in G$. It then follows that f is completely determined on the subgroup of G generated by U. But this is all of G, in light of the following lemma: $\square$

Lemma 1.3.5. *Let G be a connected topological group and let $e \in U \subseteq G$ be an open neighborhood of the identity. Then G is generated by U.*

Proof. Write $U^{-1} := \{u^{-1} \mid u \in U\} \subseteq G$, which is again an open neighborhood of e. By replacing U by $U \cap U^{-1}$, we may assume that $U = U^{-1}$. Now consider

$$U^n := \{u_1 \cdot \dots \cdot u_n \mid u_i \in U\}.$$

The union $V := \bigcup_{n=1}^{\infty} U^n \subseteq G$ is then an open subset of G which is also a *subgroup*. It follows that also all the cosets gV are open subspaces of G. But then it follows that V is also *closed*, since its complement is the union of all the cosets gV that are distinct from V and hence is open. Since G is connected, it follows that $G = V$, finishing the proof. $\square$

Remark 1.3.6. We will occasionally need to consider other smooth maps $(LG, 0) \to (G, e)$ whose differential at the origin is the identity. The multiplication map $m\colon G \times G \to G$ has a differential at the point (e, e), which takes the form

$$T_{(e,e)}m\colon \mathrm{L}G \times \mathrm{L}G \to \mathrm{L}G, \qquad (X, Y) \mapsto X + Y.$$

To see this, we observe that this map is linear, and that it restricts to the identity whenever we set $X = 0$ or whenever we set $Y = 0$. So when the Lie algebra $\mathrm{L}G$ can be written in some way as a direct sum of vector spaces

$$\mathrm{L}G = V_1 \oplus V_2 \oplus \cdots \oplus V_k,$$

the map

$$\mathrm{L}G \to G, \qquad (X_1, \dots, X_k) \mapsto \exp(X_1) \cdot \ldots \cdot \exp(X_k)$$

has differential id_{LG} at the origin, and in particular it is locally invertible. (It need not agree with $\exp(X_1 + \cdots + X_k)$!)

The exponential map is not necessarily surjective, even when the Lie group is connected. We will show in a future lecture that in the case of a *compact* connected Lie group, the exponential map is always surjective. Furthermore, the exponential map of a connected Lie group is, in general, only a group homomorphism when restricted to one-dimensional subvector spaces of $\mathrm{L}G$. In fact, it is a homomorphism on all of $\mathrm{L}G$ if and only if G is abelian:

Lemma 1.3.7. *The exponential map* $\exp\colon \mathrm{L}G \to G$ *is a homomorphism of Lie groups if and only if G is abelian.*

Proof. First assume that exp is a homomorphism of Lie groups. The map exp is a bijection on a neighborhood of the origin, and as we have seen in Lemma 1.3.5 such a neighborhood generates G. Since $\mathrm{L}G$ is commutative as an additive group, it follows that multiplication of elements in the open neighborhood is commutative, but then this implies that G is already abelian.

Conversely, if G is abelian, then the multiplication map $m\colon G\times G\to G$ is a group homomorphism, which induces the map $\mathrm{L}G\times\mathrm{L}G\to\mathrm{L}G$ given by $(X,Y)\mapsto X+Y$ as we saw above. The statement then follows from the naturality of the exponential map, Lemma 1.3.2. ◻

Theorem 1.3.8. *Any connected abelian Lie group G is the product of a torus and a vector space: $G\cong T^k\times\mathbb{R}^m$.*

Proof. As we saw in Lemma 1.3.7, the exponential map $\exp\colon \mathrm{L}G\to G$ is a group homomorphism, and since its image contains a set of generators by Lemma 1.3.5 it follows that it is surjective. If we write $K\subseteq\mathrm{L}G$ for the kernel of exp, then we see that K must be a *discrete* subgroup, since the exponential map is a local bijection at the origin. It then follows, as we will show in Lemma 1.3.9 below, that K is generated by linearly independent vectors $g_1,\dots,g_k\in\mathrm{L}G$. We may complete this to a basis of $\mathrm{L}G$ by adding vectors $g_{k+1},\dots,g_n$, which determines an isomorphism $\mathrm{L}G\cong\mathbb{R}^n$ such that $K\subseteq\mathrm{L}G$ corresponds to the subset

$$\mathbb{Z}^k\times\{0\}\subseteq\mathbb{R}^k\times\mathbb{R}^{n-k}=\mathbb{R}^n\,.$$

It then follows that the quotient $\mathrm{L}G/K$ is isomorphic to $\mathbb{R}^n/(\mathbb{Z}^k\times\{0\})\cong T^k\times\mathbb{R}^{n-k}$. The resulting homomorphism

$$T^k\times\mathbb{R}^{n-k}\cong\mathrm{L}G/K\to G$$

is a bijective local diffeomorphism, and hence it is an isomorphism of Lie groups. ◻

We still need to prove the following lemma:

Lemma 1.3.9. *A discrete subgroup B of a finite-dimensional vector space V is generated by linearly independent vectors $g_1,\dots,g_k$.*

Proof. We proceed by induction on $n=\dim(V)$. For $n=1$ we may assume $V=\mathbb{R}$, and then either $B=0$ or B is generated by its smallest positive element. Now let $n>1$ and $B\neq 0$. Choose a euclidean metric on V and an element $g_1\in B$ of smallest positive norm. Then there is an orthogonal splitting

$$V=\mathbb{R}\cdot g_1\oplus W,$$

where W is the orthogonal complement of $\mathbb{R}\cdot g_1$. Consider the projection $p\colon B\subseteq\mathbb{R}\cdot g_1\oplus W\to W$. We claim that the group $p(B)\subseteq W$ does not contain a nonzero element of norm smaller than $|g_1|/2$. Indeed, if there exists $g\in B$ such that $0<|p(g)|<|g_1|/2$, then there exists $m\in\mathbb{Z}$ such that the projection of $g+mg_1$ onto $\mathbb{R}\cdot g_1$ has norm at most $|g_1|/2$. But then we have an element $g+mg_1\in B$ satisfying $0<|g+mg_1|<|g_1|/\sqrt{2}$, contradicting the choice of g_1.

We thus conclude that $p(B)$ is a discrete subgroup of W, and by the induction hypothesis it is generated in W by linearly independent vectors $h_2,\dots,h_k$ with $k\le n$. The kernel of p is generated by g_1, so we have a short exact sequence

$$0\to\langle g_1\rangle\to B\xrightarrow{p}\langle h_2,\dots,h_k\rangle\to 0.$$

If we now pick elements $g_2, \dots, g_k \in B$ satisfying $p(g_i) = h_i$, then we obtain a section of the map $B \to \langle h_2, \dots, h_k \rangle$. It follows that B is generated by the elements $g_1, \dots, g_k$, and they are linearly independent as $p(g_2), \dots, p(g_k)$ are linearly independent in W and g_1 lies in the orthogonal complement of W. $\square$

As a corollary of the theorem, we get:

Corollary 1.3.10. *A compact abelian Lie group is isomorphic to the product of a torus and a finite abelian group.*

Proof. During this proof, we will use *additive* notation for all Lie groups that appear. By Theorem 1.3.8, the connected component of the unit of a compact abelian Lie group G is a torus T (where T might be $T^0 = \{e\}$!). So we have a short exact sequence

$$0 \to T \xrightarrow{i} G \xrightarrow{p} B \to 0.$$

We want to show that $G \cong T \times B$. For this, it will suffice to show that the map $p\colon G \to B$ admits a section $s\colon B \to G$ (i.e. $ps = \mathrm{id}_B$), since then we obtain homomorphisms of Lie groups

$$T \times B \to G, (t,b) \mapsto i(t) + s(b) \qquad \text{and} \qquad G \to T \times B, g \mapsto (g - s(p(g)), p(g))$$

which are inverse to each other. To see that the section s exists, note that the group B is discrete, since it is given as the quotient of G by the open subgroup T. Since B is also compact, it must be a finite group, and hence we may write it as a product of cyclic groups:

$$B \cong \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_k.$$

Let $b_1, \dots, b_k \in B$ denote the corresponding generators of B. Since the map p is surjective, we may find elements $c_1, \dots, c_k \in G$ such that $p(c_i) = b_i$ for all i. Note that $n_i \cdot c_i \in T = \ker(p)$, since $p(n_i \cdot c_i) = n_i \cdot p(c_i) = n_i \cdot b_i = 0$. Since $T = \mathbb{R}^m/\mathbb{Z}^m$ is a torus, we may find an element $d_i \in \mathbb{R}^m$ which maps to $n_i c_i$ under the map $\mathbb{R}^m \to T$, and we may consider $[d_i/n_i] \in T \subseteq G$, which by definition satisfies $n_i \cdot [d_i/n_i] = [d_i] = n_i \cdot c_i$. If we now take $g_i := c_i - [d_i/n_i]$, we see that $p(g_i) = p(c_i) - p([d_i/n_i]) = p(c_i) = b_i$, while $n_i g_i = 0$. In particular, we obtain a homomorphism $s\colon B \to G$ sending b_i to g_i which is a section of $p\colon B \to G$, finishing the proof. $\square$

Subgroups and submanifolds

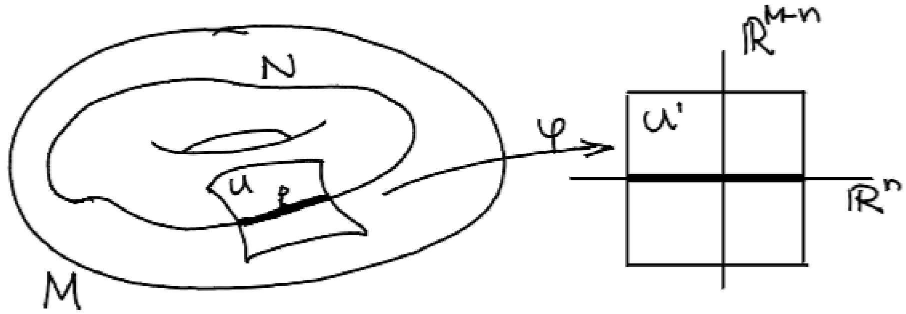
We will now use the exponential map to study the question of when an abstract subgroup of a Lie group is again a Lie group. This will give us a way to finally prove that the examples of matrix groups provided in Section 1.1 are indeed Lie groups.

We start by recalling the definition of a *submanifold*:

Definition 1.3.11. Let M be a smooth manifold of dimension m. For $0 \le n \le m$, we say that a subset $N \subseteq M$ is an *n-dimensional submanifold* if for every point $p \in N$ there exists a chart of M around p of the form

$$\varphi\colon (U,p) \xrightarrow{\cong} (U',0) \subseteq \mathbb{R}^m \cong \mathbb{R}^n \oplus \mathbb{R}^{m-n},$$

with $p \in U \subseteq M$ open, such that $\varphi(N \cap U) = \mathbb{R}^n \cap U'$, where $\mathbb{R}^n = \mathbb{R}^n \oplus 0 \subseteq \mathbb{R}^m$.



In other words: we may choose the charts of M in such a way that the restriction of N to those charts always corresponds to the first n coordinates in the euclidean space $\mathbb{R}^m$.

If we pick charts φ_α like this around every point in N, then these charts restrict to maps $N \cap U \to \mathbb{R}^n \cap U'$ which provide an atlas for N and hence give N the structure of a smooth manifold.

Definition 1.3.12. A map $f : M \to N$ of smooth manifolds is called an *embedding* if the following three conditions are satisfied:

- The map f is injective;
- The map f is an *immersion*: for every $x \in M$, the induced map $T_x f : T_x M \to T_{f(x)} N$ is injective;
- The bijection $f : M \xrightarrow{\cong} f(M)$ is a homeomorphism (where $f(M)$ has the subspace topology).

It turns out (but we will not show) that f is an embedding if and only if the image $f(M) \subseteq N$ is a submanifold and the map $f : M \to f(M)$ is a diffeomorphism.

Next up we consider the possible notions of *subgroups* of a Lie group G:

Definition 1.3.13. A *Lie subgroup* of a Lie group G is a Lie group H together with an injective homomorphism of Lie groups

$$f : H \hookrightarrow G.$$

An *abstract subgroup* of a Lie group G is a subset $H \subseteq G$ which is a subgroup of G in the group-theoretic sense, forgetting the smooth structure on G.

Lemma 1.3.14. *Let G be a Lie group and let $H \subseteq G$ be an abstract subgroup. If H is a submanifold of G, then it is also a Lie subgroup.*

Proof. The multiplication map of H is simply the restriction of the multiplication map of G and hence is still smooth. One may argue similarly for the inversion map, or one may use exercise 1.1 which says that if the multiplication map is smooth then the inversion map is automatically also smooth. □

Note that every injective one-parameter group is a subgroup. Also, the inclusions of the classical linear groups provide many examples of subgroups. We emphasize that we do not assume a subgroup to be an embedding of manifolds. Nevertheless, we do always have:

Lemma 1.3.15. *Let $f\colon H \hookrightarrow G$ be a Lie subgroup of a Lie group G. Then f is an immersion.*

Proof. We have to show that the tangent map $T_h f\colon T_h H \to T_{f(h)} G$ is injective for every $h \in H$. By using the commutative diagram

$$\begin{tikzcd} H \arrow[r, hook, "f"] \arrow[d, "l_h"', "\cong"] & G \arrow[d, "l_{f(h)}", "\cong"'] \\ H \arrow[r, hook, "f"] & G \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \mathrm{L}H = T_e H \arrow[r, hook, "\mathrm{L}f"] \arrow[d, "T_e l_h"', "\cong"] & T_e G = \mathrm{L}G \arrow[d, "T_e l_{f(h)}", "\cong"'] \\ T_h H \arrow[r, hook, "T_h f"] & T_{f(h)} G, \end{tikzcd}$$

it suffices to show that the map $\mathrm{L}f\colon \mathrm{L}H \to \mathrm{L}G$ is injective. But since the exponential maps for H and G are local diffeomorphisms, this follows directly from the injectivity of $f\colon H \hookrightarrow G$ using the following commutative diagram:

$$\begin{tikzcd} \mathrm{L}H \arrow[r, "\mathrm{L}f"] \arrow[d, "\exp"'] & \mathrm{L}G \arrow[d, "\exp"] \\ H \arrow[r, hook, "f"] & G. \end{tikzcd}$$ □

Warning 1.3.16. While every Lie subgroup $f\colon H \hookrightarrow G$ is both an injection and an immersion, it need not be an embedding! In fact, $f(H)$ may be a dense proper subset of G. An example for which this happens is the group homomorphism

$$\mathbb{Z} \to S^1, \qquad n \mapsto e^{in}.$$

This is a subgroup with dense image which is not an embedding.

The following result characterizes those subgroups that are embeddings:

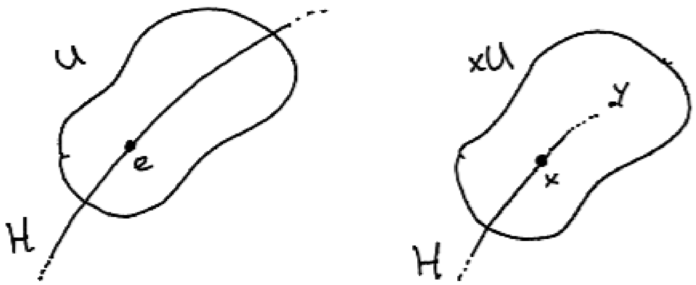
Theorem 1.3.17. *An abstract subgroup H of a Lie group G is a submanifold of G if and only if H is closed in G.*

Proof. We first show the 'only if', so assume that H is a submanifold. Given a point $y \in \overline{H}$, we have to show that $y \in H$. Since H is a submanifold, there is an open neighborhood U of the unit $e \in G$ such that $H \cap U$ is closed in U. Then the set

$$yU^{-1} := \{yu^{-1} \mid u \in U\} \subseteq G$$

is an open neighborhood of y, and since $y \in \overline{H}$ it must intersect H non-trivially, i.e. we get an $x \in H$ with $x \in yU^{-1}$, and thus $y \in xU$. But then $y \in \overline{H} \cap xU$ and thus $x^{-1}y \in \overline{H} \cap U = H \cap U$. Hence $y \in H$, showing that H is closed.

Conversely, assume that the abstract subgroup H is a closed subset of G. By the usual translation argument, it suffices to find a neighborhood V of the unit of G such that $H \cap V$ is a submanifold of V. We will do by first finding a subspace of $\mathrm{L}G$ which is a likely candidate for $\mathrm{L}H$, and then using the exponential map to locally map $\mathrm{L}G$ into V and $\mathrm{L}H$ into $H \cap V$, showing that $H \cap V$ is a submanifold.



Since $\exp\colon \mathrm{L}G \to G$ is a local diffeomorphism, we may find open neighborhoods $e \in U \subseteq G$ and $0 \in U' \subseteq \mathrm{L}G$ such that $\exp|_{U'}\colon U' \to U$ is a diffeomorphism. We write $\log\colon U \to U'$ for the inverse of $\exp|_{U'}$. Define

$$H' := \log(H \cap U) \subseteq \mathrm{L}G.$$

The proof now proceeds via the following three claims:

Claim 1: Equip the vector space $\mathrm{L}G$ with a euclidean metric, and let $(h_n)_{n\in\mathbb{N}}$ be a sequence in H' which converges to zero in $\mathrm{L}G$. If the sequence $(h_n/|h_n|)_{n\in\mathbb{N}}$ converges to some vector $X \in \mathrm{L}G$, then $\exp(tX) \in H$ for all $t \in \mathbb{R}$.

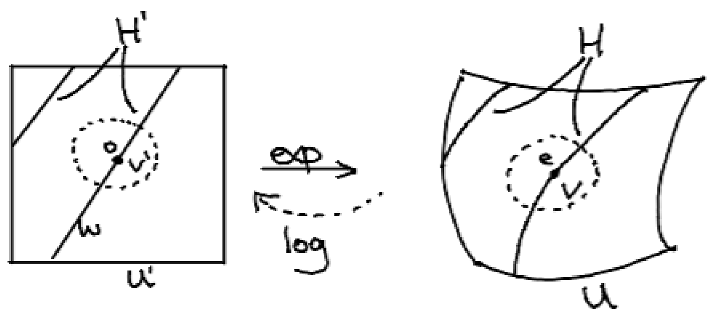
Claim 2: Consider the set

$$W := \{sX \mid X = \lim_n(h_n/|h_n|), h_n \in H', h_n \to 0, s \in \mathbb{R}\} \subseteq \mathrm{L}G.$$

Then W is a linear subspace of $\mathrm{L}G$.

Claim 3: The image $\exp(W) \subseteq H$ is an open neighborhood of the unit $e \in H$.

We may picture the situation as follows:



Assuming for a moment that we have proved these three claims, let us see why this finishes the proof. Since W is a linear subspace of $\mathrm{L}G$, it is clearly a submanifold. Since the image $\exp(W)$ in H is an open neighborhood of $e \in H$, we must have

$$\exp(W) \cap V = H \cap V$$

for some open neighborhood $e \in V \subseteq U$. Let $V' := \exp^{-1}(U')$. Then the map $\exp|_{V'}: V' \to V$ is a diffeomorphism satisfying $\exp(W \cap V') = H \cap V$. It follows that $H \cap V \subseteq V$ is a submanifold, which is what we needed to prove.

Let us now prove the three claims:

Proof of claim 1: The sequence $(t/|h_n|) \cdot h_n$ converges to tX in $\mathrm{L}G$. Since $|h_n| \to 0$, there are integers $m_n \in \mathbb{Z}$ such that $(m_n \cdot |h_n|) \to t$, and thus we have

$$\exp(m_n \cdot h_n) = \exp(m_n \cdot |h_n| \cdot (h_n/|h_n|)) \to \exp(tX).$$

Since H is closed, it will thus suffice to show that $\exp(m_n \cdot h_n) \in H$ for all n. But this is clear, since

$$\exp(m_n \cdot h_n) = \exp(h_n)^{m_n}.$$

Proof of claim 2: Let $X, Y \in W$, and define $h(t) := \log(\exp(tX) \cdot \exp(tY)) \in H'$. We saw in Remark 1.3.6 that the sequence $h(t)/t$ converges to $X + Y$ as $t \to 0$. It thus follows that the sequence

$$h(t)/|h(t)| = h(t)/t \cdot t/|h(t)|$$

converges to $(X+Y)/|X+Y|$, and thus $X + Y \in W$ as desired.

Proof of claim 3: Let $D \subseteq \mathrm{L}G$ be the orthogonal complement of W in $\mathrm{L}G$. Then the map

$$W \oplus D \to G, \qquad (X, Y) \mapsto \exp(X) \cdot \exp(Y)$$

is a local diffeomorphism around the origin, see Remark 1.3.6. If claim 3 were false, we could find $(X_n, Y_n) \in W \oplus D$ with

$$\exp(X_n) \cdot \exp(Y_n) \in H, \qquad Y_n \neq 0, \qquad \text{and} \qquad (X_n, Y_n) \to 0 \text{ as } n \to \infty.$$

Since D is a (closed) subspace, the sequence $Y_n/|Y_n|$ admits a subsequence converging to some vector $Y \in D$. Note that $|Y| = 1$ and in particular $Y \neq 0$. But since $\exp(X_n) \in H$ and H is a subgroup, we also have $\exp(Y_n) \in H$, and it follows that $Y \in W$. So $Y \in W \cap D = \{0\}$, giving a contradiction. □

Proposition 1.3.18. *Let $f: G \to H$ be a group homomorphism between Lie groups which is continuous as a map between manifolds. Then f is smooth, and hence is a Lie group homomorphism.*

Proof. Consider the graph

$$\Gamma_f := \{(g, f(g)) \mid g \in G\} \subseteq G \times H$$

of f. Note that Γ_f is a closed subgroup of the Lie group $G \times H$: it is the preimage of the diagonal in H under the continuous map $f \times \mathrm{id}_H: G \times H \to H \times H$, and the diagonal of H is closed since H is Hausdorff. It thus follows from the theorem that Γ_f is a Lie subgroup. The projection $p := \mathrm{pr}_1|_{\Gamma_f}: \Gamma_f \to G$ is a bijective homomorphism of Lie groups, and hence is an isomorphism by Corollary 1.2.15. Since $f = \mathrm{pr}_2 \circ p^{-1}$, we conclude that also f is a smooth map. □

Corollary 1.3.19. *A topological group G admits at most one Lie group structure.*

Proof. Given two Lie group structures on G, we may apply the proposition to the identity map $\mathrm{id}_G: G \to G$ to see that the identity map is smooth, and also its inverse is smooth. We conclude that the two smooth structures on G must agree. □

1.4 Homogeneous spaces and quotient groups

Now that we have a solid grip on subgroups of Lie groups, we move to quotient groups. We start by introducing/recalling a lot of basic terminology:

Definition 1.4.1 (G-space)**.** Let G be a Lie group. A (left) *G-space* is a topological space X equipped with a continuous *left action* of G on X, i.e. a map $G \times X \to X$ denoted $(g,x) \mapsto gx$ (or $(g,x) \mapsto g \cdot x$) satisfying unitality and associativity:

$$ex = x \qquad \text{and} \qquad (gh)x = g(hx) \qquad\qquad \text{for all } x \in X,\ g,h \in G.$$

A continuous map $f\colon X \to Y$ of G-spaces is called *equivariant*, or a *G-map*, if it preserves the G-action:

$$f(gx) = gf(x) \qquad\qquad \text{for all } x \in X,\ g \in G.$$

A subset $A \subseteq X$ is called *invariant* if $ga \in A$ for all $a \in A$ and $g \in G$. An invariant point is called a *fixed point*. The subgroup

$$G_x := \{g \in G \mid gx = x\}$$

is called the *isotropy group* of the point $x \in X$. (Note that G_x is a closed subgroup if X is Hausdorff.) The invariant subspace

$$Gx := \{gx \mid g \in G\} \subseteq X$$

is called the *orbit* of x. The set of all orbits is denoted by X/G, which comes equipped with a canonical projection

$$X \to X/G, \quad x \mapsto Gx.$$

We equip X/G with the quotient topology induced by this map, and call it the *orbit space*. If X/G is a single point (i.e. if X consists of only a single orbit), we say that the action is *transitive*.

Remark 1.4.2. There is a dual notion of a *right action* $X \times G \to X$ which instead satisfies $xe = x$ and $x(gh) = (xg)h$. Every right action determines a left action via the formula $gx = xg^{-1}$ and vice versa. When the direction is not mentioned, we are talking about *left* actions.

Definition 1.4.3. A *smooth G-manifold* is a smooth manifold M equipped with a smooth G-action $G \times M \to M$.

Example 1.4.4 (G-representations)**.** Let V be a finite-dimensional vector space and let $G = \mathrm{Aut}(V)$. Then V admits a canonical left action

$$\mathrm{Aut}(V) \times V \to V, \qquad (\varphi, v) \mapsto \varphi(v),$$

turning V into a smooth G-manifold. More generally, given a continuous (hence smooth, by Proposition 1.3.18) homomorphism $\varphi\colon G \to \mathrm{Aut}(V)$, we obtain a left G-action on V:

$$G \times V \to V, \qquad (g, v) \mapsto \varphi(g)(v).$$

We will encounter such actions a lot in the next chapter: they are called *G-representations*.

Example 1.4.5 (Homogeneous spaces)**.** Let $H \subseteq G$ be a closed subgroup, so that H is itself a Lie group by Theorem 1.3.17. Then the group multiplication defines a *right* action $G \times H \to G, (g,h) \mapsto gh$, turning G into a smooth H-manifold. We refer to the orbit space G/H of right H-cosets of G as the *homogeneous space* associated to H. The group G acts on G/H on the left via

$$G \times G/H \to G/H, \qquad (g, xH) \mapsto gxH.$$

It will turn out that the quotient G/H admits a canonical smooth manifold structure and that the left G-action on it turns it into a smooth G-manifold. In fact, the quotient map $G \to G/H$ is a *principal bundle*, a fundamental concept to the theory of Lie group actions that we will now introduce:

Definition 1.4.6 (Principal bundle)**.** Let H be a Lie group. The *trivial H-principal bundle* on a topological space X is the projection map $X \times H \to X$, where we equip $X \times H$ with the canonical right H-action given by $(x,h)h' := (x, hh')$.

An *H-principal bundle* is a continuous map $\pi\colon P \to B$ such that P is equipped with a right H-action such that:

(1) The right H-action on P preserves the fibers of π: we have $\pi(ph) = \pi(p)$ for all $p \in P$ and $h \in H$;

(2) The map π is a *locally trivial bundle*: around every point $b \in B$ there is an open neighborhood $b \in U \subseteq B$ such that the restriction $\pi^{-1}(U) \to U$ of π is a trivial H-principal bundle, i.e. there exists an H-equivariant homeomorphism $\varphi\colon U \times H \xrightarrow{\cong} \pi^{-1}(U)$ over U:

$$\begin{tikzcd} U \times H \arrow[r, "\varphi", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & \pi^{-1}(U) \arrow[r, hook] \arrow[d, "\pi|_U"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\ & U \arrow[r, hook] & B. \end{tikzcd}$$

We refer to P as the *total space*, to H as the *structure group*, and to B as the *base space*.

Given another H-principal bundle $\pi'\colon P' \to B'$, a *morphism of principal bundles* is a commutative square

$$\begin{tikzcd} P \arrow[r, "F"] \arrow[d, "\pi"'] & P' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B', \end{tikzcd}$$

where the map F is H-equivariant.

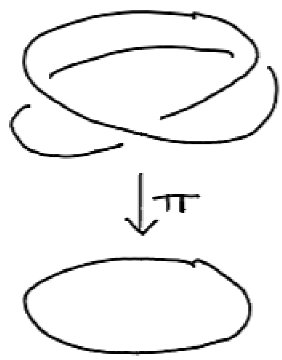
Remark 1.4.7. Note that by (1) there is a canonical map $P/H \to B$, and by (2) this is a homeomorphism. So in a sense, all information about the principal bundle is contained in the total space P. Similarly, the morphism $f\colon B \to B'$ is uniquely determined by F under the condition $f(\pi(p)) = \pi'(F(p))$.

While an H-principal bundle *locally* just looks like a product $U \times H$, it might not *globally* be a product since there might be non-trivial 'gluings'. Specifically: if $\varphi'\colon U' \times H \xrightarrow{\cong} \pi^{-1}(U')$ is another local trivialization, we obtain a map

$$\begin{tikzcd}
(U\cap U')\times H \arrow[r, "\varphi'", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & \pi^{-1}(U\cap U') \arrow[r, "\varphi^{-1}", "\cong"'] \arrow[d, "\pi"] & (U\cap U')\times H \arrow[dl, "\mathrm{pr}_1"] \\
& U\cap U', &
\end{tikzcd}$$

and the top composite must be of the form $(u,h)\mapsto(u,\gamma(u)\cdot h)$, where $\gamma\colon U\cap U'\to H$ is defined by $\varphi^{-1}\varphi'(u,e)=(u,\gamma(u))$. In other words: the function γ determines how much the trivialization over U' is 'twisted' with respect to the trivialization over U.

Example 1.4.8. One of the easiest examples of non-trivial principal bundle is the two-fold covering map $S^1\to S^1, z\mapsto z^2$. This is a $\mathbb{Z}/2$-principal bundle.



Definition 1.4.9. An H-principal bundle $\pi\colon P\to B$ is called *smooth* if P is a smooth H-manifold, B is a smooth manifold, and the local trivializations $U\times H\xrightarrow{\cong}\pi^{-1}(U)$ can be chosen to be diffeomorphisms.

Observation 1.4.10. The smooth structure on B is uniquely determined by the smooth structure on P: for another smooth manifold M, a map $f\colon B\to M$ is smooth if and only if the composite $f\pi\colon P\to M$ is smooth. Indeed, the property of being smooth is a *local* property, meaning that it only depends on arbitrary small open neighborhoods around all points of B. By picking local trivializations, we may thus reduce to the case of a trivial bundle $U\times H\to U$. But there it is clear, since this map has a smooth section $U\to U\times H,\ u\mapsto(u,e)$.

It also follows from this reasoning that the map $\pi\colon P\to B$ is a smooth submersion (i.e. the map $T_p\pi\colon T_pP\to T_{\pi(p)}B$ is surjective for all $p\in P$.)

Theorem 1.4.11. *Let G be a Lie group and let $H\subseteq G$ be a closed subgroup. Then the homogeneous space G/H is a smooth manifold and the quotient map $\pi\colon G\to G/H$ is a smooth H-principal bundle (where we equip G with the canonical H-action given by right multiplication).*

Proof. Step 1: We start by proving that the quotient topology on G/H is Hausdorff, which is a bit more subtle than one might naively expect. Let $xH\neq yH$ be two distinct points in G/H. Since G is locally compact, we may find a compact neighborhood K of x disjoint from the (closed) coset yH. Then KH is a neighborhood of xH in G/H which is disjoint from yH, and so it remains to show that KH is closed in G. But note that $x\in KH$ if and only if $k^{-1}x\in H$ for some $k\in K$, so if we define $f\colon K\times G\to G$ by $f(k,g)=k^{-1}g$ we see that $KH=\mathrm{pr}_2(f^{-1}H)$. Since H is closed, also $f^{-1}H$ is closed, and since K is compact the projection $\mathrm{pr}_2\colon K\times G\to G$ is (proper, hence) closed. This finishes step 1.

Next, we need to describe the local product structure of G as an H-space. Choose an inner product on $\mathrm{L}G$, and let V be the orthogonal complement of $\mathrm{L}H$ in $\mathrm{L}G$, so that $\mathrm{L}G=V\oplus\mathrm{L}H$. For $\varepsilon>0$, define

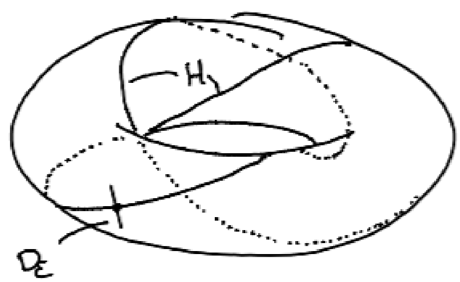
$$V_\varepsilon := \{X\in V \mid |X|<\varepsilon\} \qquad \text{and} \qquad D_\varepsilon := \exp(V_\varepsilon)\subseteq G.$$

We call D_ε a *slice* to H at e. We show:

Step 2: For sufficiently small $\varepsilon>0$, the map

$$\mu\colon D_\varepsilon\times H\to G, \qquad (g,h)\mapsto gh$$

is an open embedding.



Proof of step 2: The differential of μ at the point (e,e) is the identity on both summands $T_eD_\varepsilon=V$ and $T_eH=\mathrm{L}H$ of $T_{(e,e)}(D_\varepsilon\times H)$. Thus if we choose ε sufficiently small, μ gives a diffeomorphism $D_\varepsilon\times U\to D_\varepsilon\cdot U$ for some open neighborhood $e\in U\subseteq H$. This implies that μ is a local diffeomorphism everywhere: for $h\in H$ we have $\mu|_{D_\varepsilon\times Uh}=h\circ(\mu|_{D_\varepsilon\times Uh})\circ h^{-1}$.

It thus remains to show that we can make μ injective by choosing ε small enough. Since $U\subseteq H$ is open, it is of the form $U=U'\cap H$ for some open $e\in U'\subseteq G$. Pick ε small enough such that $d_1^{-1}d_2\in U'$ for all $d_1,d_2\in D_\varepsilon$. Now, assume that we have $d_1,d_2\in D_\varepsilon$ and $h_1,h_2\in H$ with $d_1h_1=d_2h_2$. Setting $h:=h_1h_2^{-1}\in H$ we see that $d_1h=d_2$, and thus $h=d_1^{-1}d_2\in U'\cap H=U$. Since μ is injective on $D_\varepsilon\times U$, this means that $d_1=d_2$ and $h=e$, i.e. $h_1=h_2$. It follows that μ is injective on all of $D_\varepsilon\times H$ as claimed.

Step 3: We now use this to provide the trivializations for the quotient map $\pi\colon G\to G/H$. For $g\in G$, define

$$U_g := gD_\varepsilon\cdot H.$$

Then U_g is invariant under the right H-action, and the subsets $U_g/H\subseteq G/H$ form an open cover of G/H. Charts for G/H are given by the homeomorphisms $D_\varepsilon\xrightarrow{\cong}U_g/H$ given as follows:

$$D_\varepsilon = D_\varepsilon\times\{e\}\hookrightarrow D_\varepsilon\times H\xrightarrow{\mu}D_\varepsilon\cdot H\xrightarrow{g}gD_\varepsilon\cdot H=U_g\xrightarrow{\pi}U_g/H.$$

This shows that G/H is a smooth manifold, and that the map $\pi\colon G\to G/H$ is smooth. Finally, we see that π is a G-principal bundle since

$$\pi^{-1}(U_g/H)=U_g\cong D_\varepsilon\times H\cong U_g/H\times H. \qquad \square$$

Corollary 1.4.12. *Let $N \subseteq G$ be a closed normal subgroup of a Lie group. Then the quotient G/N, with its canonical smooth and group structures, is itself a Lie group. Given a homomorphism $f\colon G \to H$ with $N \subseteq \ker(f)$, there is a unique factorization of morphisms of Lie groups:*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Proof. To show that the group multiplication $m\colon G/N \times G/N \to G/N$ is smooth, it suffices by Observation 1.4.10 to show this after composing with the projection $\pi\times\pi\colon G\times G \to G/N \times G/N$. But then it follows from the fact that π is a group homomorphism:

$$\begin{tikzcd} G\times G \arrow[r, "m"] \arrow[d, "\pi\times\pi"'] & G \arrow[d, "\pi"] \\ G/N\times G/N \arrow[r, "m"] & G/N. \end{tikzcd}$$

If $N \subseteq \ker(f)$, we obtain a group homomorphism $G/N \to H$ of groups whose composite with $G \to G/N$ is f, and again it follows from Observation 1.4.10 that this map must also be smooth. □

Remark 1.4.13. We will often write

$$H \to P \to B$$

to indicate that the map $P \to B$ is an H-principal bundle. For example, we have

$$H \to G \to G/H.$$

We will now give an alternative definition of homogeneous spaces:

Definition 1.4.14. A *homogeneous space* of a Lie group G is a smooth G-manifold M such that the action of G on M is transitive.

Proposition 1.4.15. *Let M be a homogeneous space of the Lie group G, let $p \in M$, and let $G_p \subseteq G$ be the isotropy group of p. Then the map*

$$f_p\colon G/G_p \to M, \qquad gG_p \mapsto gp$$

is well-defined and is a diffeomorphism.

Proof. The map f_p is well-defined and injective by the very definition of G_p. Since M is transitive, f_p is also surjective. The map f_p is smooth since the composite $G \to G/G_p \to M, g \mapsto gp$ is smooth. So f_p is a smooth bijection, and so by Corollary 1.2.14 it will suffice to show that f_p has constant rank. But since G acts transitively on G and f_p is G-equivariant, we see that the rank of $T_g f_p\colon T_g G/G_p \to T_{f_p(g)}M$ agrees with $T_e f_p\colon T_e G/G_p \to T_p M$ for every $g \in G$, so that f_p does indeed have constant rank. □

On the one hand, we may use this proposition to recognize certain well-known manifolds as homogeneous spaces. On the other hand, we might use this proposition to *construct* interesting smooth manifold structures: if we are given a set M on which a Lie group G acts transversely (as a group without smooth structure) and the isotropy group G_p of some point $p \in M$ is closed, then we may use the bijection $G/G_p \cong M$ to provide M with the structure of a smooth manifold with this bijection becoming a diffeomorphism. We will now give various classical examples.

Example 1.4.16. The group $\mathrm{SO}(n)$ acts linearly on $\mathbb{R}^n$. Since it consists of the length-preserving matrices, its action restricts to the unit sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\},$$

and in fact $\mathrm{SO}(n)$ acts *transitively* on S^{n-1} since every point on the sphere may be reached from $e_n = (0, 0, \dots, 1)$ via a rotation. Note that the isotropy group of e_n is the subgroup $\mathrm{SO}(n-1) \subseteq \mathrm{SO}(n)$ of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

with $A \in \mathrm{SO}(n-1)$, and thus the proposition provides a diffeomorphism

$$\mathrm{SO}(n)/\mathrm{SO}(n-1) \to S^{n-1}, \qquad [A] \mapsto Ae_n.$$

In particular we obtain an $\mathrm{SO}(n-1)$-principal bundle

$$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}.$$

Incidentally, this yields an inductive proof that $\mathrm{SO}(n)$ is connected for all n: if a bundle has connected fiber and connected base space, then also its total space is connected.

Example 1.4.17. The action of $\mathrm{O}(n)$ on S^{n-1} similarly leads to a principal bundle

$$\mathrm{O}(n-1) \to \mathrm{O}(n) \to \mathrm{O}(n)/\mathrm{O}(n-1) \cong S^{n-1}.$$

Example 1.4.18. The unitary group $\mathrm{U}(n)$ acts linearly on $\mathbb{C}^n$ and as in the previous example it restricts to a transitive action on the unit sphere

$$S^{2n-1} = \{x + iy \in \mathbb{C}^n \mid |x|^2 + |y|^2 = 1\}.$$

The isotropy group of the vector $e_n = (0, \dots, 0, 1) \in \mathbb{C}^n$ is $\mathrm{U}(n-1)$, hence we obtain a $\mathrm{U}(n-1)$-principal bundle

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to \mathrm{U}(n)/\mathrm{U}(n-1) \cong S^{2n-1}.$$

As above, we find as a consequence that $\mathrm{U}(n)$ is connected for all n.

Remark 1.4.19. We have seen that bundles can tell us something about connectedness of topological spaces, but in fact they are useful more generally to give interesting topological information that relate the total space, the base space and the fiber. For example, Exercise 4.2 on the next sheet has a bonus exercise in which you may calculate the fundamental group of $\mathrm{SO}(n)$ for all n.

Example 1.4.20. Using the methods from above, we also obtain principal bundles

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$$

and

$$\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to S^{4n-1}.$$

Example 1.4.21. Since the group $\mathrm{SU}(n)$ acts linearly on $\mathbb{C}^n$, it has an induced operation on the projective space $\mathbb{C}P^{n-1}$ of lines through the origin. Since this space is a quotient of the unit sphere S^{2n-1}, this action is transitive. In this case, the isotropy group of the point $p = \mathbb{C}\cdot e_n \in \mathbb{C}P^{n-1}$ is the subgroup of $\mathrm{SU}(n)$ consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix},$$

which is precisely the image of the embedding

$$\mathrm{U}(n-1) \to \mathrm{SU}(n), \qquad A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & \alpha \end{smallmatrix}\right), \qquad \alpha := \det(A)^{-1}.$$

We thus obtain a $\mathrm{U}(n-1)$-principal bundle

$$\mathrm{U}(n-1) \to \mathrm{SU}(n) \to \mathbb{C}P^{n-1}.$$

The same construction over the real numbers yields an embedding $\mathrm{O}(n-1) \to \mathrm{SO}(n)$ and a principal bundle over the real projective space $\mathbb{R}P^{n-1}$ of lines through the origin in $\mathbb{R}^n$:

$$\mathrm{O}(n-1) \to \mathrm{SO}(n) \to \mathbb{R}P^{n-1}.$$

Example 1.4.22 (The Hopf fibration)**.** An important special case of the previous example is the case $n = 2$. In this case we have $\mathrm{U}(1) \cong S^1$ and $\mathrm{SU}(2) \cong S^3$, so that we obtain a principal bundle

$$S^1 \to S^3 \to S^2$$

known as the *Hopf fibration*. Using the induced long exact sequence of higher homotopy groups, Heinz Hopf used this fibration to find the first nontrivial higher homotopy group of a sphere, which was a groundbreaking result in algebraic topology. To this day, higher homotopy groups of spheres are an active area of investigation.

Example 1.4.23. Recall the *Stiefel manifold* $V_k(\mathbb{R}^n)$:

$$V_k(\mathbb{R}^n) := \{(v_1, \dots, v_k) \in (\mathbb{R}^n)^k \mid |v_i| = 1, v_i \perp v_j \text{ for } i \neq j\}.$$

Elements of $V_k(\mathbb{R}^n)$ are called *orthonormal* k*-frames* in $\mathbb{R}^n$. The orthogonal group $\mathrm{O}(n)$ acts transitively on $V_k(\mathbb{R}^n)$ by

$$A \cdot (v_1, \dots, v_k) = (Av_1, \dots, Av_k).$$

If we consider the k-frame $p = (e_{n-k+1}, \dots, e_n)$ consisting of the last k unit vectors in $\mathbb{R}^n$, then we see that the isotropy group of p is the subgroup $\mathrm{O}(n-k) \subseteq \mathrm{O}(n)$, so we get a bijection

$$\mathrm{O}(n)/\mathrm{O}(n-k) \xrightarrow{\cong} V_k(\mathbb{R}^n),$$

and in particular this gives $V_k(\mathbb{R}^n)$ the structure of a smooth manifold. We also obtain a principal bundle

$$\mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n).$$

Example 1.4.24. Recall the *Grassmann manifold* $\mathrm{Gr}_k(\mathbb{R}^n)$:

$$\mathrm{Gr}_k(\mathbb{R}^n) := \{V \subseteq \mathbb{R}^n \mid V \text{ is a } k\text{-dimensional linear subspace }\}.$$

Elements of $\mathrm{Gr}_k(\mathbb{R}^n)$ are sometimes called *k-planes* in $\mathbb{R}^n$. The group $\mathrm{O}(n)$ acts transitively on $\mathrm{Gr}_k(\mathbb{R}^n)$ and the isotropy group of the k-plane $\mathbb{R}^k \subseteq \mathbb{R}^n$ is the subgroup $\mathrm{O}(k) \times \mathrm{O}(n-k) \subseteq \mathrm{O}(n)$, via the embedding

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

In particular, we may equip $\mathrm{Gr}_k(\mathbb{R}^n)$ with a smooth structure such that

$$\mathrm{Gr}_k(\mathbb{R}^n) \cong \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k)).$$

We obtain a principal bundle

$$\mathrm{O}(k) \times \mathrm{O}(n-k) \to \mathrm{O}(n) \to \mathrm{Gr}_k(\mathbb{R}^n).$$

Analogously there are Grassmann manifolds

$$\mathrm{Gr}_k(\mathbb{C}^n) \cong \mathrm{U}(n)/(\mathrm{U}(k) \times \mathrm{U}(n-k))$$

and

$$\mathrm{Gr}_k(\mathbb{H}^n) \cong \mathrm{Sp}(n)/(\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)).$$

We end the section with an important theorem concerning the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. We say that an element $t \in T^n$ is a *generator* if the group $\{t^k \mid k \in \mathbb{Z}\}$ generated by t is dense in T^n.

Theorem 1.4.25 (Kronecker's theorem)**.** *A vector $v \in \mathbb{R}^n$ represents a generator of T^n if and only if the real numbers 1 and the n components $v_1, \ldots, v_n$ of v are linearly independent over the rational numbers $\mathbb{Q}$. In particular, almost every element v is a generator and generators form a dense subset of the torus.*

Proof. During this proof we write all Lie groups additively. Consider the following five statements:

(i) The elements $1, v_1, \ldots, v_n \in \mathbb{R}$ are linearly dependent over $\mathbb{Q}$;

(ii) We have $\sum_{i=1}^n q_i v_i \in \mathbb{Q}$ for some $0 \neq (q_1, \ldots, q_n) \in \mathbb{Q}^n$;

(iii) We have $\sum_{i=1}^n a_i v_i \in \mathbb{Z}$ for some $0 \neq (a_1, \ldots, a_n) \in \mathbb{Z}^n$;

(iv) The element $[v] \in T^n$ is in the kernel of a nontrivial homomorphism $f : T^n \to S^1$;

(v) The element $[v] \in T^n$ is not a generator.

It is clear that (i) $\iff$ (ii) $\iff$ (iii). For (iii) $\implies$ (iv), define the group homomorphism $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n a_i x_i$. Then $\overline{f}$ sends $\mathbb{Z}^n$ into $\mathbb{Z}$ and thus induces a non-trivial group homomorphism $f : T^n = \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}/\mathbb{Z} = S^1$ satisfying $f([v]) = [\overline{f}(v)] = [\sum_{i=1}^n a_i v_i] = 0$. Conversely, assuming (iv), consider the exact sequence

$$0 \to \mathbb{Z}^n \to \mathbb{R}^n \to T^n \to 0$$

defining the torus. This induces a canonical identification $\mathbb{R}^n = \mathrm{L}\mathbb{R}^n \cong \mathrm{L}T^n$, and we may identify the projection map $\mathbb{R}^n \to T^n$ with the exponential map of T^n, cf. Example 1.2.23. The given homomorphism $f\colon T^n \to S^1$ induces a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^n \arrow[r, hook] \arrow[d, dashed, "\mathrm{L}f|_{\mathbb{Z}^n}"'] & \mathbb{R}^n \arrow[r] \arrow[d, "\mathrm{L}f"] & T^n \arrow[d, "f"] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, hook] & \mathbb{R} \arrow[r] & S^1. \end{tikzcd}$$

Define $a_i := \mathrm{L}f(e_i) \in \mathbb{Z}$, where $e_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{Z}^n \subseteq \mathbb{R}^n$ is the i-th basis vector. Since $[v] \in \ker(f)$, we see that $\mathrm{L}f(v) \in \ker(\mathbb{R} \to S^1) = \mathbb{Z}$. But by linearity we have $\mathrm{L}f(v) = \sum_{i=1}^n a_i v_i$, showing (iii).

To show that (iv) $\iff$ (v), assume first that $[v] \in \ker(f)$. If f is non-trivial, then this kernel is not all of T^n and hence is a proper closed subgroup of T^n, so $[v]$ cannot be a generator. Conversely, if $[v]$ is not a generator, then it is contained in a proper closed subgroup $H \subseteq T^n$, and the quotient group T^n/H is a nontrivial compact connected abelian Lie group. By Theorem 1.3.8 this means that T^n/H is isomorphic to a nontrivial torus $T^k = (S^1)^k$ for some k. But then $[v]$ is in the kernel of the nontrivial homomorphism

$$T^n \to T^n/H \cong T^k = (S^1)^k \xrightarrow{\mathrm{pr}_1} S^1. \qquad \square$$

Corollary 1.4.26. *A compact Lie group contains a dense cyclic subgroup if and only if the group is isomorphic to $T^k \times \mathbb{Z}/l$ for some $k \geq 0$, $l \geq 1$.*

Proof. First we show that $T^k \times \mathbb{Z}/l$ contains a dense cyclic subgroup. Let $t \in T^k$ be a generator, and choose $s \in T^k$ such that $l \cdot s = t$ (where we use additive notation in T^k). Then $l \cdot (s,1) = (t,0)$ is a generator of $T^k \times \{0\}$, and since $1 = \mathrm{pr}_2(s,1)$ generates $\mathbb{Z}/l$ we see that $(s,1)$ generates a dense cyclic subgroup of $T^k \times \mathbb{Z}/l$.

Conversely, let $a \in G$ be an element such that $\langle a \rangle$ is dense in G. The preimage of e under the commutator map $G \times G \to G, (g,h) \mapsto ghg^{-1}h^{-1}$ contains $\langle a \rangle \times \langle a \rangle$, and by continuity it follows that G is abelian. By Corollary 1.3.10, we have $G \cong T^k \times B$ for some k and some finite abelian group. But then B must by cyclic as it is generated by $\mathrm{pr}_2(a) \in B$. $\square$

Definition 1.4.27. A Lie group is called *topologically cyclic* if it contains an element, called a *generator*, whose powers are dense.

So the previous corollary may be stated as saying that the only topologically cyclic compact Lie groups are the products $T^k \times \mathbb{Z}/l$.

1.5 Invariant integration

If G is a finite group and $f\colon X \to \mathbb{R}$ is any continuous function defined on some G-space X, we may turn f into a *G-invariant* function by means of averaging over the group G:

$$\overline{f}(x) := \frac{1}{|G|} \sum_{g \in G} f(gx).$$

This function is G-invariant, meaning that $\overline{f}(gx) = \overline{f}(x)$ for all $g \in G$ and $x \in X$. The goal of this section is to introduce a similar technique when G is more generally a *compact Lie group*.

Definition 1.5.1 (Left-invariant integral)**.** Let G be a compact Lie group and let $C(G)$ denote the vector space of continuous functions $G \to \mathbb{R}$. An *integral* on G is an $\mathbb{R}$-linear map

$$I\colon C(G) \to \mathbb{R}, \qquad f \mapsto I(f)$$

which is *monotone*, meaning that $I(f) \le I(f')$ whenever we have $f(g) \le f'(g)$ for all $g \in G$. We say that I is *left-invariant* if for every $h \in G$ we have

$$I(f) = I(f \circ l_h), \qquad \text{where} \qquad l_h\colon G \to G,\ \ g \mapsto hg.$$

We say that I is *normalized* if $I(1) = 1$.

One may construct left-invariant integrals on G using the theory of differential forms on manifolds; the details may be found in Section I.5 of Bröcker and tom Dieck [Bt95]. However, for our purposes the following existence and uniqueness statement will suffice:

Theorem 1.5.2 (Bröcker and tom Dieck [Bt95, Theorem 5.13])**.** *Let G be a compact Lie group. Then there exists a unique left-invariant normalized integral*

$$C(G) \to \mathbb{R}, \qquad f \mapsto \int f(g)dg.$$

The unique integral on G from the theorem is called the *Haar integral*. (There exists in fact a much more general version of Haar integrals for locally compact topological groups.) We have relations

$$\int 1\, dg = 1 \qquad \text{and} \qquad \int f(hg)dg = \int f(g)dg,$$

expressing the conditions that the Haar integral is normalized and left-invariant, respectively.

Remark 1.5.3. When G is a finite group, the Haar integral is given by $f \mapsto \frac{1}{|G|}\sum_{g \in G} f(g)$.

Proposition 1.5.4. *Let G be a compact Lie group. For an element $h \in G$ and each automorphism $\varphi\colon G \to G$, there are relations*

$$\int f(g)dg = \int f(gh)dg = \int f(g^{-1})dg = \int (f \circ \varphi)(g)dg$$

for all $f \in C(G)$.

Proof. By uniqueness of the invariant integral, it suffices to observe that the three formulas on the right each define invariant integrals $C(G) \to \mathbb{R}$. It is clear that each of them are still normalized integrals, hence it remains to show they are still left-invariant.

- The assignment $f \mapsto \int f(gh)dg$ is left invariant: for $k \in G$ we have $\int f(kgh)dg = \int f(gh)dg$ for all $k \in G$ by left-invariance of $\int$ applied to the function $g \mapsto f(gh)$.
- The assignment $f \mapsto \int f(g^{-1})dg$ is left-invariant: we have

$$\int f(hg^{-1})dg = \int f((gh^{-1})^{-1})dg = \int f(g^{-1})dg,$$

  where the last equality is due to the right-invariance of the integral proved in the previous point.

- The assignment $f \mapsto \int f(\varphi(g))dg$ is left-invariant: if we set $k := \varphi^{-1}(h)$, we have

$$\int f(h\varphi(g))dg = \int f(\varphi(kg))dg = \int f(\varphi(g))dg,$$

where the middle equality holds by left-invariance of $\int$.

$\square$

Proposition 1.5.5. *Let $H \subseteq G$ be a closed subgroup and assume that there exists a left-invariant normalized integral $\int : C(G/H) \to \mathbb{R}$ (for example the Haar measure on the compact Lie group G/N if $H = N$ is normal). Then for any continuous function $f: G \to \mathbb{R}$ we have*

$$\int_G f(g)dg = \int_{G/H} \left(\int_H f(gh)dh \right) d(gH).$$

Proof. This is again clear from uniqueness since the right-hand side defines a normalized left-invariant integral on G. $\square$

Remark 1.5.6. The integration of real-valued functions on G directly generalizes to integration of functions $G \to V$ for an arbitrary finite-dimensional real vector space V equipped with an inner product $\langle -, - \rangle$, or more generally a real *Hilbert space*. Indeed, for any $f: G \to V$ and $v \in V$ we may consider the map

$$\langle f, v \rangle : G \to \mathbb{R}, \qquad g \mapsto \langle f(g), v \rangle,$$

which is a continuous real-valued function that can be integrated. The assignment $V \to \mathbb{R}$ given by $v \mapsto \int_G \langle f(g), v \rangle \, dg$ is a continuous linear functional, and must therefore be given by scalar product with some element $\int f(g) \, dg \in V$. In other words, the element $\int f(g) \, dg \in V$ is defined by the property that for every $v \in V$ we have

$$\left\langle \int f(g) \, dg, v \right\rangle = \int_G \langle f(g), v \rangle \, dg.$$

It is clear from this definition that the resulting function $\int : C^0(G, V) \to V$ is $\mathbb{R}$-linear, and that for any function $f: G \to \mathbb{R}$ and any vector $w \in V$ we have $\int_G f(g) w \, dg = (\int_G f(g) \, dg) \cdot w$.

Taking $V = \mathbb{C}$, this in particular allows us to integrate complex-valued functions on G. If V is moreover a complex vector space, then the integral is even $\mathbb{C}$-linear: multiplication by i is an $\mathbb{R}$-linear endomorphism of V, and hence commutes with integration by Lemma 1.5.8. For the same reason, the identity $\int_G f(g) w \, dg = (\int_G f(g) \, dg) \cdot w$ continues to hold for any function $f: G \to \mathbb{C}$ and any vector $w \in V$.

Lemma 1.5.7. *Let V be a finite-dimensional real vector space. Then the integral $\int : C^0(G, V) \to V$ is independent of the choice of inner product.*

Proof. Let $\langle -, - \rangle_2$ be a different choice of inner product, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis with respect to the first inner product $\langle -, - \rangle_1 = \langle -, - \rangle$. Given a function $f: G \to V$, we may write f as

$$f(g) = \sum_{i=1}^{n} f_i(g) \cdot e_i,$$

where $f_i(g) := \langle e_i, f(g)\rangle_1$. We may then compute

$$\begin{aligned}\left\langle \int_G f(g)\,dg, v\right\rangle_2 &= \left\langle \sum_{i=1}^n \int_G f_i(g)e_i\,dg, v\right\rangle_2 = \sum_{i=1}^n \left(\int f_i(g)\,dg\right)\langle e_i, v\rangle_2 \\ &= \int_G \langle \sum_{i=1}^n f_i(g)\,dg e_i, v\rangle_2 = \int_G \langle f(g), v\rangle_2,\end{aligned}$$

where we used that $\int_G f_i(g)e_i\,dg = \left(\int f_i(g)\,dg\right)\cdot e_i$. This verifies the defining property of the integral with respect to the inner product $\langle -,-\rangle_2$, as desired. □

Lemma 1.5.8. *Let $\varphi\colon V \to W$ be an $\mathbb{R}$-linear map of finite-dimensional real vector spaces. Then φ commutes with integration: for a continuous map $f\colon G \to V$ we have $\int_G \varphi(f(g))\,dg = \varphi(\int_G f(g)\,dg)$.*

Proof. Choosing an inner product on W, it will suffice to show that we have an equality

$$\left\langle \int_G \varphi(f(g))\,dg, w\right\rangle = \left\langle \varphi\left(\int_G f(g)\,dg\right), w\right\rangle$$

for every $w \in W$. The left-hand side is by definition equal to $\int_G \langle \varphi(f(g)), w\rangle\,dg$. Letting $U := \mathrm{im}(\varphi) \subseteq W$ be the image of φ, we observe that the equality is clear if $w \in U^\perp$, since then both sides are zero. It thus remains to show the equality for $w \in U$. We may choose some linear section $s\colon U \hookrightarrow V$ of the surjective map $\varphi\colon V \twoheadrightarrow U = \mathrm{im}(\varphi)$, giving a decomposition $V = s(U) \oplus \ker(\varphi)$. Combining the inner product on U with some chosen inner product on $\ker(\varphi)$ we obtain an inner product on $V = s(U) \oplus \ker(\varphi)$ satisfying $\langle \varphi(v), w\rangle = \langle v, s(w)\rangle$ for all $v \in V$ and $w \in U$. Then for $w \in U$ we may compute

$$\begin{aligned}\left\langle \int_G \varphi(f(g))\,dg, w\right\rangle &= \int_G \langle \varphi(f(g)), w\rangle\,dg = \int_G \langle f(g), s(w)\rangle\,dg \\ &= \left\langle \int_G f(g)\,dg, s(w)\right\rangle = \left\langle \varphi\left(\int_G f(g)\,dg\right), w\right\rangle,\end{aligned}$$

which is what we had to show. □

2 Representations of compact Lie groups

This chapter is directly taken from Chapter 2 of Bröcker and tom Dieck [Bt95].

In this chapter, we will study the theory of finite-dimensional representations of compact Lie groups. In Section 2.1, we introduce and discuss the notion of a complex representation and show that every representation is the direct sum of irreducible representations. Uniqueness of the direct sum decomposition is shown in Section 2.2 using the notion of a 'semisimple module'. In Section 2.3 we introduce various constructions of representations, like tensor products, symmetric powers and exterior powers.

In Section 2.4 we introduce the notion of a *character* of a representation, which determines the representation up to isomorphism. We will also prove the orthogonality theorem for characters and for entries of matrix representations.

In Section 2.5 we give an elementary construction of the irreducible representations of SU(2), SO(3), U(2) and O(3). In Section 2.6 we turn to representations on real and quaternionic vector spaces and the relation between these and complex representations. In Section 2.7 we introduce the character ring of a compact Lie group and introduce the exterior power and Adams operations on it.

2.1 Representations

We start by introducing the notion of a (complex) representation of a Lie group.

Definition 2.1.1. A *(complex) representation* of a Lie group G is a finite-dimensional complex vector space V equipped with a continuous action

$$G \times V \to V, \qquad (g, v) \mapsto gv$$

of G on V such that for each $g \in G$ the translation $l_g \colon V \to V, v \mapsto gv$ is a linear map. The *dimension* of the representation is defined as the dimension of V as a complex vector space.

Remark 2.1.2. Recall that the map $(g, v) \mapsto gv$ being an action means that $ev = v$ and $(gh)v = g(hv)$, or in other words:

$$l_e = \mathrm{id}_V \qquad \text{and} \qquad l_g \circ l_h = l_{gh}.$$

It follows that $l_{g^{-1}}$ is a linear inverse of l_g, and so the assignment $g \mapsto l_g$ determines a continuous group homomorphism

$$l \colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$$

(which is hence a morphism of Lie groups by Proposition 1.3.18). Conversely, any such homomorphism determines a continuous linear action

$$G \times V \to V, \qquad (g,v) \mapsto l_g(v).$$

It follows that a complex representation is the same thing as a pair (V,l) where $l\colon G \to \mathrm{Aut}(V)$ is a morphism of Lie groups.

Definition 2.1.3. A *matrix representation* of G is a continuous homomorphism $l\colon G \to \mathrm{GL}_n(\mathbb{C})$.

Example 2.1.4 (Adjoint representation)**.** As we will discuss in Section 2.6, one can analogously define *real* G-representations. We have encountered an important example of such in Definition 1.2.25: the *adjoint representation*

$$\mathrm{Ad}\colon G \to \mathrm{Aut}_{\mathbb{R}}(\mathrm{L}G), \qquad g \mapsto \mathrm{L}c_g,$$

where $c_g\colon G \to G, h \mapsto ghg^{-1}$ is the conjugation map. This representation is of great importance to the structure theory of G.

Definition 2.1.5. A representation V is called *faithful* if the associated homomorphism $G \to \mathrm{Aut}(V)$ is injective.

Later we will show that every compact Lie group admits a faithful representation and is therefore isomorphic to a closed subgroup of a matrix group. The desire to represent abstract groups in the concrete numerical terms of matrices and matrix multiplications is one of the origins of representation theory.

Remark 2.1.6. Assume that G is a finite group, regarded as a Lie group with the discrete topology. Then we may consider what is called the *group ring* $\mathbb{C}[G]$ of G over $\mathbb{C}$. The underlying vector space of $\mathbb{C}[G]$ is the complex vector space with a basis given by the elements of G, so that elements of $\mathbb{C}[G]$ are formal linear combinations $\sum_{g\in G}\lambda_g \cdot g$ with $\lambda_g \in \mathbb{C}$. Multiplication in $\mathbb{C}[G]$ is given by

$$\left(\sum_g \lambda_g g\right) * \left(\sum_h \mu_h h\right) = \sum_{g,h} \lambda_g \mu_h gh = \sum_g \left(\sum_h \lambda_{gh^{-1}} \mu_h\right) g.$$

Exercise 2.1.7. Every complex G-representation V determines a left $C[G]$-module by means of the map

$$\mathbb{C}[G] \otimes V \to V, \qquad (\sum_g \lambda_g g) \otimes v \mapsto \sum_g \lambda_g g v.$$

Show that this is well-defined and determines a one-to-one correspondence between complex G-representations and finite-dimensional left $C[G]$-modules.

This module interpretation of representations can be generalized to the case for compact Lie groups. Here the situation is more subtle, since we need to take into account continuity in the definition of "group ring" and "module". For this reason, we will not provide all the formal details. Nevertheless, it is useful to have at least a heuristic understanding of the concepts involved, because they will come back in some of the proofs in later sections.

Note that an element of $\mathbb{C}[G]$ defines a function $G \to \mathbb{C}$ given by assigning the coefficient λ_g to g. For a general Lie group, we will therefore consider the space $C^0(G,\mathbb{C})$ of continuous functions $f\colon G \to \mathbb{C}$. In terms of functions, the multiplication rule on $\mathbb{C}[G]$ takes the form

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(gh^{-1}) \cdot f_2(h).$$

If G is a compact Lie group, this finite sum has to be replaced by integrals, leading to the *convolution product*

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h) dh$$

of two continuous functions f_1 and f_2 on G. It turns out that the space $C^0(G,\mathbb{C})$, equipped with the supremum norm and the convolution product, is what is known as a 'Banach algebra', and that G-representations are suitable modules over this algebra. We will not define these notions, but the approach will occasionally be used as a guiding perspective.

Definition 2.1.8. A *morphism* $f\colon V \to W$ between G-representations is a linear map which is G-equivariant, i.e. $f(gv) = gf(v)$ for $g \in G$ and $v \in V$. Morphisms are sometimes also called *intertwining operators*. We let $\mathrm{Hom}_G(V,W)$ denote the set of all morphisms. This defines a category, and in particular we have a notion of *isomorphism* of representations: a morphism which admits an inverse.

Example 2.1.9. Let α and β be two matrix representations $G \to \mathrm{GL}_n(\mathbb{C})$, corresponding to two G-representations $V_\alpha = (\mathbb{C}^n,\alpha)$ and $V_\beta = (\mathbb{C}^n,\beta)$. Using the correspondence between linear maps $V_\alpha \to V_\beta$ and complex $(n\times n)$-matrices, we see that V_α and V_β are isomorphic if and only if there is an invertible matrix A such that

$$A\alpha(g)A^{-1} = \beta(g) \qquad \text{for all} \qquad g \in G.$$

If two homomorphisms α and β are related in this way, we say they are *conjugate* to each other. (This should not be confused with complex conjugation.)

Definition 2.1.10. Let V be a complex G-representation. An (Hermitian) inner product $\langle -,-\rangle\colon V\times V \to \mathbb{C}$ is called *G-invariant* if $\langle gv, gw\rangle = \langle v,w\rangle$ for all $g \in G$ and $v,w \in V$. A representation together with a G-invariant inner product is called a *unitary representation*.

If we choose an orthonormal basis for a unitary representation V, then the associated matrix representation is a homomorphism $G \to \mathrm{U}(n)$. Conversely, any such continuous homomorphism defines a unitary representation on $\mathbb{C}^n$ with its standard inner product.

The following result is in fact more generally true for compact topological groups; we will state it only for compact Lie groups.

Theorem 2.1.11. *Let V be a representation of a compact Lie group G. Then V admits a G-invariant inner product.*

Proof. Let $\langle -,-\rangle\colon V\times V \to \mathbb{C}$ be an arbitrary inner product. Using the unique left-invariant normalized integral on G, we define

$$\langle v,w\rangle_G := \int_G \langle gv, gw\rangle dg.$$

Then $\langle -,-\rangle_G$ is easily seen to be linear in v, conjugate linear in w, G-invariant since the integral is left-invariant, and positive definite since the integral of a positive continuous function is positive. Thus $\langle -,-\rangle_G$ is the desired G-invariant inner product. □

Definition 2.1.12. Let V be a G-representation. A G-invariant linear subspace $U \subseteq V$ (i.e. $gu \in U$ for $g \in G$ and $u \in U$) is called a *subrepresentation* of V. A nonzero representation V is called *irreducible* if it has no subrepresentations other than $\{0\}$ and V. A representation which is not irreducible is called *reducible*.

Proposition 2.1.13. *Let G be a compact Lie group. If V is a subrepresentation of a G-representation U, then there is a complementary subrepresentation $W \subseteq U$ such that $U = V \oplus W$, in the sense that every element of U may uniquely be written as $v + w$ for $v \in V$ and $w \in W$. Every G-representation can be written as a direct sum $\bigoplus_{i=1}^n V_i$ of irreducible subrepresentations $V_i \subseteq V$.*

Proof. By Theorem 2.1.11, we may choose a G-invariant inner product on U. We define W as the orthogonal complement of V in U. If $w \in W$, then also $gw \in W$ for all $g \in G$ since for $u \in U$ we have $\langle gw, u\rangle = \langle w, g^{-1}u\rangle = 0$. So W is a subrepresentation satisfying $U = V \oplus W$.

The second statement now follows by induction on the dimension of U: if $U \neq \{0\}$, there exists some subrepresentation $V \subseteq U$ with $0 < \dim(V) < \dim(U)$ and we may write $U = V \oplus W$. □

The next theorem is an extremely useful tool in the theory of representations:

Theorem 2.1.14 (Schur's lemma)**.** *Let G be a Lie group and let V and W be irreducible G-representations. Then:*

(i) A morphism $V \longrightarrow W$ is either zero or an isomorphism;

(ii) Every morphism $f\colon V \longrightarrow V$ has the form $f(v) = \lambda v$ for some $\lambda \in \mathbb{C}$;

(iii) We have

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(V,W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W. \end{cases}$$

Proof. Since the kernel $\ker(f) \subseteq V$ of f is a subrepresentation and V is irreducible, the kernel is either $\{0\}$ or V. In the latter case f is zero, and in the former f is injective. If f is injective, then its image $\mathrm{im}(f) \subseteq W$ is a nonzero subrepresentation, and by irreducibility of W it must then be all of W. We conclude that f is an isomorphism, showing (i).

To prove (ii), assume that f is nontrivial and let λ be any eigenvalue of f. Let $W = \{w \in V \mid f(w) = \lambda w\}$ be the corresponding eigenspace. It is easy to see that W is a subrepresentation, and thus $W = V$, showing (ii). Part (iii) follows from (i) and (ii). □

Definition 2.1.15. The actions of the groups $\mathrm{SU}(n)$, $\mathrm{U}(n)$ and $\mathrm{GL}_n(\mathbb{C})$ on $\mathbb{C}^n$ given by matrix multiplication give rise to representations of these groups, which are known as the *standard representations*.

A representation is called *trivial* if each group element $g \in G$ acts as the identity: $gv = v$ for all $v \in V$.

Definition 2.1.16. Let V and W be G-representations. We define their *direct sum* as the vector space $V \oplus W$ with the G-action given by $g(v,w) = (gv, gw)$.

In terms of matrices, this corresponds to the following construction: given $G \to \mathrm{GL}_m(\mathbb{C}), g \mapsto A(g)$ and $G \to \mathrm{GL}_n(\mathbb{C}), g \mapsto B(g)$, we obtain their direct sum representation $G \to \mathrm{GL}_{m+n}(\mathbb{C})$ by forming block matrices:

$$g \mapsto \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}.$$

Proposition 2.1.17. *An irreducible representation of an abelian Lie group G is one-dimensional.*

Proof. Since G is abelian, the translation function $l_g \colon V \to V, v \mapsto gv$ is a morphism of G-representations for each $g \in G$: we have $l_g(hv) = g(hv) = h(gv) = hl_g(v)$. By Theorem 2.1.14, it follows that l_g is multiplication by some $\lambda(g) \in \mathbb{C}$. But this implies that any linear subspace $W \subseteq V$ is a subrepresentation: for $w \in W$ and $g \in G$ we have $gw = \lambda(g)w \in W$. The result now follows: if $\dim(V) > 1$, then V would have some one-dimensional subspace W, but then W is a subrepresentation with $0 < \dim(W) < \dim(V)$, which contradicts the irreducibility of V. □

Let G be a compact Lie group. We denote by $\mathrm{Irr}_G(\mathbb{C})$ a complete set of pairwise non-isomorphic irreducible complex G-representations, i.e. every irreducible G-representation is isomorphic to exactly one element of $\mathrm{Irr}_G(\mathbb{C})$. For arbitrary representations of G we use the following terminology:

Definition 2.1.18. If U is isomorphic to a subrepresentation of V, we say that U is *contained* in V. If W is irreducible, we call the dimension $\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V)$ the *multiplicity* of W in V.

The significance of the multiplicity is the following. Assume that we are given a decomposition $V = \bigoplus_i V_i$ of V into irreducible subrepresentations (such a decomposition exists by Proposition 2.1.13). Then we have

$$\mathrm{Hom}_G(W, V) = \bigoplus_i \mathrm{Hom}_G(W, V_i),$$

and thus by Schur's lemma we see that $\dim_{\mathbb{C}} \mathrm{Hom}_G(W, V)$ is simply the number of V_i that are isomorphic to W. In particular, this multiplicity is nonzero if and only if W is contained in V, and this happens for only those finitely many $W \in \mathrm{Irr}_G(\mathbb{C})$ isomorphic to some V_i.

We will now describe a *canonical decomposition* of representations which makes the multiplicities even more transparent. Given an irreducible representation W, consider the map

$$d_W \colon \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \to V, \qquad \varphi \otimes w \mapsto \varphi(w).$$

The group G acts on the domain of this may by $g(\varphi \otimes w) = \varphi \otimes gw$, and it is easy to see that d_W is G-equivariant with respect to this action. We can assemble all the maps d_W into a map

$$d = (d_W) \colon \bigoplus_W \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \to V,$$

where W ranges over $\mathrm{Irr}_G(\mathbb{C})$. Note that d is again a morphism of G-representations.

Proposition 2.1.19. *For every G-representation V, the above map $d \colon \bigoplus_W \mathrm{Hom}_G(W, V) \otimes_{\mathbb{C}} W \to V$ is an isomorphism.*

Proof. Notice that the map d is compatible with isomorphisms $V \cong V'$ of G-representations and with direct sum decompositions $V = V_1 \oplus V_2$. By writing V as a direct sum of irreducible subrepresentations, it will thus suffice to consider the case where $V \in \mathrm{Irr}_G(\mathbb{C})$. In this case, Schur's lemma

(Theorem 2.1.14) tells us that the only nonzero summand in the domain is $\mathrm{Hom}_G(V,V)\otimes_{\mathbb{C}} V$, which is isomorphic to $\mathbb{C}\otimes_{\mathbb{C}} V$. The map $d\colon C\otimes_{\mathbb{C}} V\to V$ is given by $\lambda\otimes v\mapsto \lambda v$ and this is indeed an isomorphism. This finishes the proof. $\square$

We let $V(W)\subseteq V$ denote the image of the map $d_W\colon \mathrm{Hom}_G(W,V)\otimes_{\mathbb{C}} W\to V$, and refer to it as the *W-isotypical summand* of V. We call $\dim_{\mathbb{C}}\mathrm{Hom}_G(W,V)$ the *multiplicity* of $V(W)$.

Exercise 2.1.20. Show that $V(W_1)=V(W_2)$ whenever there exists an isomorphism $W_1\cong W_2$.

In particular, the subrepresentation $V(W)$ only depends on the isomorphism class of W. It satisfies the following property:

Proposition 2.1.21. *The subrepresentation $V(W)$ is generated by the irreducible subrepresentations of V that are isomorphic to W.*

Proof. Given a subrepresentation $W'\subseteq V$, write $i\colon W'\hookrightarrow V$ for the inclusion of W' into V, so that $i\in\mathrm{Hom}_G(W',V)$. Then we have $d_{W'}(i\otimes w)=i(w)=w$, and thus $W'\subseteq V(W')$. Now if $W'\cong W$, then we have $W'\subseteq V(W')=V(W)$, and thus $V(W)$ contains all subrepresentations of V isomorphic to W.

Conversely, let $U\subseteq V$ denote the subrepresentation of V generated by the irreducible subrepresentations of V that are isomorphic to W. Given $\varphi\in\mathrm{Hom}_G(W,V)$, the image of φ lies inside U: either φ is zero, in which case it is clear, or φ is injective, in which case we have $W\cong\varphi(W)\subseteq V$. It follows that also the image of d_W is contained in U. $\square$

Definition 2.1.22 (Restriction)**.** Let $\alpha\colon H\to G$ be a homomorphism of Lie groups. For a G-representation V, we denote by $\alpha^*(V)$ the H-representation which has the same underlying complex vector space, but with H-action given by

$$H\times V\to V,\qquad (h,v)\mapsto\alpha(h)v.$$

If $\alpha\colon H\hookrightarrow G$ is the inclusion of a subgroup H of G, we refer to $\alpha^*(V)$ as the *restriction of V to H* and denote it by $\mathrm{res}^G_H(V)$.

2.2 Semisimple modules

We have seen that any complex representation of a complex Lie group is a direct sum of irreducible representations. We have also seen that there is always a ‘canonical’ choice of such a decomposition. In this section, we will briefly record a completely analogous discussion of *semisimple modules* in a more general setting that can be applied to other situations we will encounter later.

Definition 2.2.1. Let Ω be any set. We say that an abelian group M has the structure of a (left) Ω*-module* if for every element $\alpha\in\Omega$ we are given an endomorphism $M\to M, x\mapsto\alpha x$ of M. An Ω*-homomorphism* between two Ω-modules M and N is a group homomorphism $f\colon M\to N$ such that $f(\alpha x)=\alpha f(x)$ for all $\alpha\in\Omega$. A subgroup N of an Ω-module M is called a *submodule* if $\alpha x\in N$ for all $x\in N$ and $\alpha\in\Omega$. A non-zero Ω-module M is called *simple* or *irreducible* if its only submodules are $\{0\}$ and M.

Observe that kernels, images and cokernels of Ω-homomorphisms are again Ω-modules. The arguments used in Theorem 2.1.14 directly give the following result:

Lemma 2.2.2 (Schur's lemma)**.** *Let $f\colon M \to N$ be an Ω-homomorphism. If M is simple, then f is either injective or zero; if N is simple then f is either surjective or zero; if both are simple then f is either an isomorphism or zero.*

Definition 2.2.3 (Semisimple module)**.** The *direct sum* of a family of Ω-modules $\{M_i\}_{i\in I}$ is the group-theoretic direct sum $\bigoplus_{i\in I} M_i$ equipped with the componentwise Ω-operation. An Ω-module is called *semisimple* if it is a direct sum of simple modules.

Theorem 2.2.4. *Let M be an Ω-module that is generated by a family $\{N_j\}_{j\in J}$ of simple submodules. Let E be any submodule of M. Then there is a submodule F of M such that $M = E \oplus F$ and F is the direct sum of some subfamily $\{N_j\}_{j\in I}$ for some $I \subseteq J$.*

Furthermore, there is an index set $I' \subseteq J$ with $I \cap I' = \emptyset$ such that M is the direct sum of the simple submodules $\{N_j\}_{j\in I\cup I'}$ and E is isomorphic to the direct sum of the $\{N_j\}_{j\in I'}$. In particular, every submodule of M is semisimple.

Proof. Consider those subsets $K \subseteq J$ such that the submodule $N(K) \subseteq M$ generated by $\{N_j\}_{j\in K}$ is a direct sum $\bigoplus_{j\in K} N_j$ (in the sense that every element of $N(K)$ can uniquely be written as a sum of elements $x_j \in N_j$ of which only finitely many are nonzero), and such that $N(K) \cap E = \{0\}$. Then Zorn's lemma applies to this collection of subsets, hence we may let $I \subseteq J$ be maximal with respect to these properties. We claim that $F = N(I)$ satisfies $E \oplus F = M$. If not, there must be some $j \in J$ such that N_j is not contained in $E \oplus N(I)$, since M is generated by the N_j. Since N_j is simple, this means that $N_j \cap E = N_j \cap N(I) = \{0\}$. But then we see that $N(I \cup \{j\}) = N(I) \oplus N_j$ is a direct sum and that $E \cup N(I \cup \{j\}) = \{0\}$, contradicting the maximality of I. Setting $F = N(I)$ this proves the first statement.

For the second statement, we simply apply the first statement to the submodule $N(I)$ in place of E. We then obtain $I' \subseteq J$ with $I \cap I' = \emptyset$ and with $M = N(I) \oplus N(I')$. Since also $M = E \oplus N(I)$, we thus get the desired isomorphism $E \cong M/N(I) \cong N(I')$. □

A semisimple Ω-module M is called *isotypical* if it is a direct sum of simple submodules each of which is isomorphic to a given simple module S. We also say that M is *S-isotypical* in this case. If M is S-isotypical, Theorem 2.2.4 implies that each simple submodule of M is isomorphic to S and hence each submodule of M is itself S-isotypical.

Proposition 2.2.5. *Let M be a semisimple Ω-module and S a simple Ω-module. Let $M(S)$ be the submodule of M generated by all simple submodules of M isomorphic to S. Then $M(S)$ is S-isotypical and the S-isotypical submodules of M are submodules of $M(S)$. If N is a submodule of M, then $N(S) = N \cap M(S)$. Finally, M is the direct sum of the submodules $M(S)$, where S ranges over a complete set of pairwise nonisomorphic simple Ω-modules contained in M.*

Proof. The submodule $M(S)$ is semisimple and S-isotypical by Theorem 2.2.4, and contains every S-isotypical submodule of M by definition. If N is a submodule of M, then $N(S)$ is S-isotypical and thus $N(S) \subseteq M(S) \cap N$. Conversely $M(S) \cap N$ is S-isotypical so $M(S) \cap N \subseteq N(S)$, showing equality. To prove the final statement, note that by collecting those simple submodules isomorphic to the same S we may write $M = \bigoplus_S M_S$ with M_S S-isotypical. Note also that if M has no submodule isomorphic to S, we have $M_S = M(S) = \{0\}$, so we may take S to run over the index set stated in the proposition. Fixing S, let N be a submodule of M isomorphic to S. Then by Schur's lemma the map $N \hookrightarrow M \to \bigoplus_{T \not\cong S} M_T$ is the zero map, and hence $N \subseteq M_S$. We conclude that $M_S = M(S)$, finishing the proof of the proposition. □

We refer to $M(S)$ as the *S-isotypical part* of M.

Example 2.2.6 (Representations of Lie groups)**.** We may apply all of the above to the situations of representations of a compact Lie group G. Consider the set $\Omega = G \sqcup \mathbb{C}$. If V is a G-representation, we have an action map $g\colon V \to V$ for every $g \in G \subseteq \Omega$, and we let every scalar $\lambda \in \mathbb{C} \subseteq \Omega$ act on V via scalar multiplication. This lets us regard V as an Ω-module. It is clear that an Ω-submodule is the same thing as a subrepresentation, and the simple Ω-submodules of V are precisely the irreducible subrepresentations. In particular, Proposition 2.1.13 says that V is semisimple as an Ω-module. All in all, we recover the decomposition of V into its S-isotypical parts, where S runs through a complete set of pairwise nonisomorphic irreducible G-representations.

2.3 Linear algebra and representations

Let us now discuss various algebraic constructions that can be used to obtain new representations from old ones. We have already seen how to form *direct sums* $V \oplus W$ in Definition 2.1.16, and now we consider tensor products, exterior powers and homomorphisms. Throughout, we fix a (not necessarily compact) Lie group G.

Definition 2.3.1. Let V and W be G-representations. Recall that the *tensor product* $V \otimes W$ comes equipped with a bilinear map $V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$ which is *universal*: any other bilinear map $V \times W \to U$ factors uniquely through a linear map $V \otimes W \to U$. We obtain a linear G-action on $V \otimes W$ by

$$g(v \otimes w) := gv \otimes gw,$$

which is well-defined since the right-hand expression is linear in both v and w.

If $\{v_1, \ldots, v_n\}$ is a basis of V and $\{w_1, \ldots, w_m\}$ is a basis of W, then $\{v_i \otimes w_k\}$ is a basis of $V \otimes W$. If (r_{ij}) is the corresponding matrix representation of V and (s_{kl}) is that of W, then the matrix representation of $V \otimes W$ is given by the matrix $(r_{ij}s_{kl})$ whose entry in the (i,k)-th row and (j,l)-th column is $r_{ij}s_{kl}$. Indeed, if $gv_j = \sum_i r_{ij} v_i$ and $gw_l = \sum_k s_{kl} w_k$, then by bilinearity we get

$$g(v_j \otimes w_l) = \sum_{i,k} r_{ij} s_{kl} (v_i \otimes w_k).$$

The matrix $(r_{ij}s_{kl})$ is sometimes called the *Kronecker product* of (r_{ij}) and (s_{kl}).

Definition 2.3.2. Let V and W be G-representations. Then the vector space $\mathrm{Hom}(V, W)$ of $\mathbb{C}$-linear maps $V \to W$ is again a G-representation with action given by

$$(g \cdot f)(v) := g f(g^{-1} v)$$

for $f \in \mathrm{Hom}(V, W)$, $g \in G$ and $v \in V$. In other words, $g \cdot f$ fits in the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, dotted, "g \cdot f"] & W. \end{tikzcd}$$

Definition 2.3.3. Given a G-representation V, we define the *dual representation* V^* as

$$V^* := \mathrm{Hom}(V, \mathbb{C}),$$

where $\mathbb{C}$ is the trivial representation. Given a basis $\{v_1, \dots, v_n\}$ of V, we obtain a dual basis $v_1^*, \dots, v_n^*$ of V^*, and thus we may write $g \cdot v_j^* = \sum_i s_{ij}(g) v_i^*$ for some matrix $(s_{ij}(g))$. Explicitly, we have:

$$s_{ij}(g) = (gv_j^*)(v_i) = v_j^*(g^{-1}v_i) = v_j^*\left(\sum_k r_{ki}(g^{-1})v_k\right) = r_{ji}(g^{-1}),$$

so that g acts on V^* via the transpose matrix of the action of g^{-1} on V.

Definition 2.3.4. Given a complex vector space V, we define its *conjugate space* $\overline{V}$ as the complex vector space with the same underlying real vector space as V, but with scalar multiplication defined by

$$\mathbb{C} \times V \to V, \qquad (z, v) \mapsto \overline{z}v.$$

If V is a G-representation, then also $\overline{V}$ is a G-representation, called the *conjugate representation*. If we choose an invariant inner product on V, then the assignment

$$\overline{V} \to V^*, \qquad v \mapsto \langle -, v \rangle$$

is an isomorphism of G-representations.

Lemma 2.3.5. *Let V be a unitary G-representation with G-invariant inner product $\langle -, - \rangle \colon V \times V \to \mathbb{C}$, and let $V_1, V_2 \subseteq V$ be two irreducible subrepresentations of V. If V_1 and V_2 are nonisomorphic, then V_1 and V_2 are orthogonal with respect to the inner product, in the sense that $\langle v_1, v_2 \rangle = 0$ for all $v_1 \in V_1$ and $v_2 \in V_2$.*

Proof. The inner product induces a bilinear map $\langle -, - \rangle \colon V_1 \times \overline{V_2} \to \mathbb{C}$, hence a $\mathbb{C}$-linear map $f \colon V_1 \to \overline{V_2}^* \cong V_2$, determined by the property that $\langle f(v_1), v_2 \rangle = \langle v_1, v_2 \rangle$ for all $v_1 \in V_1$ and $v_2 \in V_2$. By the G-invariance of the inner product, this map is G-equivariant: we have $\langle f(gv_1), v_2 \rangle = \langle gv_1, v_2 \rangle = \langle v_1, g^{-1}v_2 \rangle = \langle f(v_1), g^{-1}v_2 \rangle = \langle gf(v_1), v_2 \rangle$ for all $v_2 \in V_2$, implying that $f(gv_1) = gf(v_1)$. By Schur's lemma, this implies $f = 0$, and hence $\langle v_1, v_2 \rangle = 0$ for all $v_1 \in V_1$ and $v_2 \in V_2$, as desired. □

Definition 2.3.6. For a G-representation V, we obtain the *i-th symmetric power* $S^i(V)$ and the *i-th exterior power* $\Lambda^i(V)$ as quotients of $V^{\otimes i}$ by forcing the assignment $(v_1, \dots, v_i) \mapsto v_1 \otimes \cdots \otimes v_i$ to become *symmetric* or *alternating*, respectively. The image in $\Lambda^i(V)$ is typically denoted $v_1 \wedge \cdots \wedge v_i$. If $\{v_1, \dots, v_n\}$ is a basis of V, then a basis of $\Lambda^i(V)$ is given by the elements $v_{k_1} \wedge \cdots \wedge v_{k_i}$ for $k_1 < \cdots < k_i$. In particular, $\Lambda^n(V)$ is a one-dimensional vector space. The action of $g \in G$ on $\Lambda^n(G)$ is via scalar multiplication by the determinant of $l_g \colon V \to V$.

The algebraic constructions introduced above come with many canonical isomorphisms, which result in canonical isomorphisms between the corresponding representations:

$$\begin{aligned}(U\otimes V)\otimes W &\cong U\otimes(V\otimes W)\\ U\otimes V&\cong V\otimes U\\ U\otimes(V\oplus W)&\cong(U\otimes V)\oplus(U\otimes W)\\ \Lambda^k(V\oplus W)&\cong\bigoplus_{i=0}^{k}\Lambda^i(V)\otimes\Lambda^{k-i}(W)\\ S^k(V\oplus W)&\cong\bigoplus_{i=0}^{k}S^i(V)\otimes S^{k-i}(W)\\ V\otimes V&\cong S^2(V)\oplus\Lambda^2(V).\end{aligned}$$

Of particular importance for representation theory is the isomorphism

$$\theta\colon V^*\otimes W\xrightarrow{\cong}\mathrm{Hom}(V,W),\qquad v^*\otimes w\mapsto(u\mapsto v^*(u)w).$$

If $\{v_1,\ldots,v_n\}$ and $\{w_1,\ldots,w_m\}$ are bases for V and W, and $f\colon V\to W$ is given by $fv_j=\sum_i r_{ij}w_i$, then an inverse of θ is given by $\theta^{-1}(f)=\sum_{i,j}r_{ij}v_j^*\otimes w_i$.
(This canonical map exists even when V and W are infinite-dimensional, but it is only an isomorphism if V or W is finite-dimensional.)

Definition 2.3.7. For $V=W$, the map

$$\mathrm{Hom}(V,V)\cong V^*\otimes V\to\mathbb{C},\qquad v^*\otimes u\mapsto v^*(u)$$

associates to $f\in\mathrm{Hom}(V,V)$ its *trace* $\mathrm{tr}(f)\in\mathbb{C}$. If we pick a basis $\{v_1,\ldots,v_n\}$, then we have $\theta^{-1}(f)=\sum_{i,j}r_{ij}v_j^*\otimes v_i$, and thus we see that

$$\mathrm{tr}(f)=\sum_{i,j}r_{ij}v_j^*(v_i)=\sum_i r_{ii},$$

which is indeed the usual definition of the trace of a matrix.

Proposition 2.3.8. *The trace satisfies the following properties:*

(i) The map $\mathrm{tr}\colon\mathrm{Hom}(V,V)\to\mathbb{C}$ *is* $\mathbb{C}$*-linear;*

(ii) We have $\mathrm{tr}(\varphi f\varphi^{-1})=\mathrm{tr}(f)$ *for every* $\mathbb{C}$*-linear automorphism of* V*;*

(iii) For $f\colon V\to W$ *and* $h\colon W\to V$ *we have* $\mathrm{tr}(fh)=\mathrm{tr}(hf)$*;*

(iv) We have $\mathrm{tr}(f\oplus h)=\mathrm{tr}(f)+\mathrm{tr}(h)$*;*

(v) We have $\mathrm{tr}(f\otimes h)=\mathrm{tr}(f)\cdot\mathrm{tr}(h)$*;*

(vi) A morphism $f\colon V\to V$ *induces a dual morphism* $f^*\colon V^*\to V^*$ *and we have* $\mathrm{tr}(f)=\mathrm{tr}(f^*)$*;*

(vii) If $f\colon V\to V$ *is idempotent, i.e.* $f^2=f$*, then* $\mathrm{tr}(f)$ *is equal to the dimension of the image of* f*;*

(viii) A morphism $f\colon V\to V$ *induces a morphism* $\overline{f}\colon\overline{V}\to\overline{V}$ *and we have* $\mathrm{tr}(\overline{f})=\overline{\mathrm{tr}(f)}$*.*

Proof. The proof is left as an exercise. □

2.4 Characters and orthogonality relations

In this section, we will introduce the *character* of a representation, and show that it completely determines the representation. We will also study the orthogonality relations satisfied by characters. Throughout this section, we will assume that all Lie groups are compact, so that we have access to the invariant integration from Section 1.5. We will now use the invariant integration to gain some deeper insights into the structure of representations.

Construction 2.4.1. Given a (complex) representation V of G, its fixed point set

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$

is a linear subspace of V. We may project onto this subspace by defining

$$p(v) := \int_G gv\, dg$$

for $v \in V$. By left-invariance of integration, we have $hp(v) = h\int_G gv\,dg = \int_G hgv\,dg = p(v)$, so that $p(v) \in V^G$. Furthermore, if $v \in V^G$ then $\int_G gv\,dg = \int_G v\,dg = v$. So the resulting map $p\colon V \to V^G$ is a projection onto V^G.

Example 2.4.2. Given another representation W, we may apply the above construction to $\mathrm{Hom}(V,W)$. Recall that G acts on $\mathrm{Hom}(V,W)$ via $(g\cdot f)(v) = gf(g^{-1}v)$, and in particular that we have $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$, the subspace of G-equivariant linear maps $V \to W$. The projection operator thus takes the form

$$p\colon \mathrm{Hom}(V,W) \to \mathrm{Hom}_G(V,W), \qquad f \mapsto \int_G (g\cdot f)\,dg.$$

If V is irreducible, then Schur's lemma tells us that any G-equivariant endomorphism $V \to V$ is a scalar multiple of the identity map. The following result identifies this scalar multiple in the case of $\int_G (g\cdot f)\,dg$.

Proposition 2.4.3. *Let V be an irreducible G-representation. Then for $f \in \mathrm{Hom}(V,V)$ we have*

$$\int_G (g\cdot f)\,dg = \frac{1}{|V|}\operatorname{tr}(f)\,\mathrm{id}_V,$$

where $|V| = \dim_{\mathbb{C}} V$ and $\operatorname{tr}(f)$ is the trace of f.

Proof. By Schur's lemma we know that the integral is $c\cdot \mathrm{id}_V$ for some $c \in \mathbb{C}$. To compute c, we apply the linear map $\operatorname{tr}\colon \mathrm{Hom}(V,V) \to \mathbb{C}$, using that it commutes with integration (as does any linear map):

$$|V|\cdot c = \operatorname{tr}(c\cdot \mathrm{id}_V) = \int_G \operatorname{tr}(g\cdot f) = \int_G \operatorname{tr}(l_g \circ f \circ l_g^{-1}) = \int_G \operatorname{tr}(f) = \operatorname{tr}(f).$$

In the second-to-last step we used the relation $\operatorname{tr}(fh) = \operatorname{tr}(hf)$. □

We will next investigate the matrix coefficients of G-representations. Given a basis $\{v_1,\dots,v_n\}$ of V, the corresponding matrix representation $G \to \mathrm{GL}_n(\mathbb{C})$, $g \mapsto (r_{ij}(g))$ is determined by $gv_j = \sum_i r_{ij}(g)v_i$. So if $v_1^*,\dots,v_n^*$ is the dual basis of V^*, we see that

$$r_{ij}(g) = v_i^*(gv_j).$$

More generally, we will consider functions of the form

$$G \to \mathbb{C}, \qquad g \mapsto \varphi(gv),$$

where $\varphi \in V^*$ and $v \in V$. Such functions are called *representative functions* on G.

Proposition 2.4.4. *Let V be an irreducible G-representation. Then for $\varphi \in V^*$, $v \in V$ and $f \in$ $\mathrm{Hom}(V,V)$ we have*

$$\int_G \varphi(gf(g^{-1}v))\,dg = \frac{1}{|V|}\operatorname{tr}(f)\varphi(v).$$

Proof. This is an immediate consequence of Proposition 2.4.3 by plugging in v on both sides and applying $\varphi\colon V \to \mathbb{C}$. □

Corollary 2.4.5. *For $v,w \in V$ and $\varphi,\psi \in V^*$ we have*

$$\int_G \varphi(g^{-1}w)\psi(gv)\,dg = \frac{1}{|V|}\psi(w)\varphi(v).$$

Proof. We apply the previous result using the linear map

$$f\colon V \to V, \qquad u \mapsto \psi(u)w.$$

We have

$$\operatorname{tr}(f) = \operatorname{tr}(V \xrightarrow{\psi} \mathbb{C} \xrightarrow{w} V) = \operatorname{tr}(\mathbb{C} \xrightarrow{w} V \xrightarrow{\psi} \mathbb{C}) = \psi(w),$$

and

$$\varphi(g^{-1}f(gv)) = \varphi(g^{-1}\psi(gv)w) = \varphi(g^{-1}w)\psi(gv),$$

and the result follows. □

It will be useful to give a formulation of these results in terms of a chosen Hermitian inner product $\langle -,- \rangle$ on V:

Theorem 2.4.6 (Orthogonality relations)**.** *Let V be irreducible. Then:*

(i) For $f \in \mathrm{Hom}(V,V)$ and $v,w \in V$ we have

$$\int_G \langle gf(g^{-1}v), w\rangle\,dg = \frac{1}{|V|}\operatorname{tr}(f)\langle v,w\rangle; \qquad \textit{and}$$

(ii) For $v,w,\alpha,\beta \in V$ we have

$$\int_G \langle g^{-1}v,\alpha\rangle\langle g\beta,w\rangle\,dg = \frac{1}{|V|}\langle \beta,\alpha\rangle\langle v,w\rangle.$$

If V happens to be a unitary representation, we may also write (ii) as

$$\int_G \overline{\langle g\alpha, v\rangle}\langle g\beta,w\rangle\,dg = \frac{1}{|V|}\overline{\langle \alpha,\beta\rangle}\langle v,w\rangle.$$

Proof. For part (i) we take $\varphi\colon V \to \mathbb{C}, v \mapsto \langle v,w\rangle$ and apply Proposition 2.4.4. For part (ii) we further take $\psi(\beta) = \langle \beta,\alpha\rangle$ and apply Corollary 2.4.5 (up to changing some variable names). □

Theorem 2.4.7 (Orthogonality relations)**.** *Let V and W be nonisomorphic irreducible G-representations. Then for G-invariant inner products $\langle -,-\rangle$ on V and W and any $\alpha, v \in V$ and $\beta, w \in W$ we have*

$$\int_G \overline{\langle g\alpha, v\rangle}\langle g\beta, w\rangle\, dg = 0.$$

Proof. Fix α and β. From $\overline{\langle g\alpha, w\rangle} = \langle v, g\alpha\rangle$ it follows that the integral is linear in v and conjugate linear in w, hence defines a bilinear form $b\colon V \times \overline{W} \to \mathbb{C}$. It is G-invariant:

$$b(hv, hw) = \int_G \overline{\langle g\alpha, hv\rangle}\langle g\beta, hw\rangle\, dg = \int_G \overline{\langle h^{-1}g\alpha, v\rangle}\langle h^{-1}g\beta, w\rangle\, dg = b(v,w),$$

hence it determines a G-equivariant map $b'\colon V \to \mathrm{Hom}(\overline{W}, \mathbb{C}) = \overline{W}^* \cong W$. But then by Schur's lemma, b' must be the zero map, finishing the proof. □

Remark 2.4.8. Let us explain why the previous two theorems are called 'Orthogonality relations'. The space $C^0(G,\mathbb{C})$ of continuous functions $G \to \mathbb{C}$ admits an inner product given by

$$\langle \varphi, \psi\rangle = \int_G \varphi(g)\overline{\psi(g)}\, dg.$$

Theorems 2.4.6 and 2.4.7 concern the behavior of representative functions under this inner product. If $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ are orthonormal bases of the irreducible unitary representations V and W, then we have matrix representations

$$\begin{aligned} r_{ij}(g,V) &= \langle gv_j, v_i\rangle \\ r_{kl}(g,W) &= \langle gw_l, w_k\rangle, \end{aligned}$$

and Theorems 2.4.6 and 2.4.7 yield

$$\begin{aligned} &\int_G r_{ij}(g,V)\overline{r_{kl}}(g,V)\, dg = \frac{1}{|V|}\delta_{ik}\delta_{jl} \\ &\int_G r_{ij}(g,V)\overline{r_{kl}}(g,W)\, dg = 0, \qquad\qquad \text{if } V \not\cong W. \end{aligned}$$

So we see that each of the $|V|^2$ matrix coefficients of an irreducible representation V are all orthogonal to each other in $C^0(G,\mathbb{C})$, and all matrix coefficients of two nonisomorphic representations are also orthogonal to each other. The linear subspace they generate is known as the *ring of representative functions*. We will see in Chapter 3 that this subring is in fact a dense subspace of $C^0(G,\mathbb{C})$, so that certain questions about continuous functions $G \to \mathbb{C}$ may be reduced to representative functions.

We now introduce characters of representations. They are important because they are functions on G which determine representations uniquely up to isomorphism, and it is easier to handle functions than matrices.

Definition 2.4.9. Let V be a (complex) G-representation. The *character* of V is the function

$$\chi_V\colon G \to \mathbb{C}, \qquad g \mapsto \mathrm{tr}(l_g),$$

where $\mathrm{tr}(l_g)$ is the trace of the linear map $l_g\colon V \to V$, $v \mapsto gv$. The character of an irreducible representation is called an *irreducible character*.

The properties of the trace given in Proposition 2.3.8 directly provide properties of characters:

Proposition 2.4.10. *(i) The character $\chi_V\colon G \to \mathbb{C}$ is a smooth function;*

(ii) If V and W are isomorphic, then $\chi_V = \chi_W$;

(iii) $\chi_V(ghg^{-1}) = \chi_V(h)$;

(iv) $\chi_{V\oplus W} = \chi_V + \chi_W$;

(v) $\chi_{V\otimes W} = \chi_V \cdot \chi_W$;

(vi) $\chi_{V^*}(g) = \chi_V(g^{-1})$;

(vii) $\chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$;

(viii) $\chi_V(e) = \dim_{\mathbb{C}}(V)$.

Proof. For (i), we may choose a basis of V to obtain a matrix representation $G \to \mathrm{GL}_n(\mathbb{C})$, $g \mapsto (r_{ij}(g))$, which is continuous and thus smooth by Proposition 1.3.18. It then follows that also $\chi_V(g) = \sum_{i=1}^n r_{ii}(g)$ is smooth.
For (ii), let $f\colon V \to W$ be an isomorphism. Then we have $l_g^W f = f l_g^V$, and thus

$$\chi_V(g) = \mathrm{tr}(l_g^V) = \mathrm{tr}(f^{-1} l_g^W f) = \mathrm{tr}(l_g^W) = \chi_W(g).$$

Properties (iii)-(v) follow directly from the analogous properties of the trace from Proposition 2.3.8. For (vi), we apply 2.3.8(vi), using that the left translation by g on V^* is given by the map $(l_{g^{-1}})^*\colon V^* \to V^*$. Finally, (vii) follows from (ii) and the isomorphism $\overline{V} \cong V^*$, and (viii) is clear. □

Definition 2.4.11. Given a Lie group G, a *class function on G* is a continuous function $f\colon G \to \mathbb{C}$ which is constant on conjugacy classes of G: we have $f(ghg^{-1}) = f(h)$ for all $g, h \in G$. Part (iii) of the proposition says that the character χ_V is a class function for all V.

Example 2.4.12. Let V be the trivial representation. Then we have $\chi_V(g) = \dim_{\mathbb{C}}(V)$ for all $g \in G$.

Example 2.4.13. Let $V = \mathbb{C}^2$ equipped with the 'flip-action' by $\mathbb{Z}/2 = \{e, \sigma\}$, i.e. σ acts via $\sigma(x, y) = (-x, y)$. Then we have

$$\chi_V(e) = 1 + 1 = 2, \qquad \chi_V(\sigma) = -1 + 1 = 0.$$

Example 2.4.14. Let $V = \mathbb{C}^2$ equipped with the 'rotation action' by $\mathbb{Z}/2 = \{e, \sigma\}$, i.e. σ acts via $\sigma(x, y) = (-x, -y)$. Then we have

$$\chi_V(e) = 1 + 1 = 2, \qquad \chi_V(\sigma) = -1 + (-1) = -2.$$

The next theorem expresses the orthogonality relations for characters. In Chapter 3 we will show that the characters of irreducible representations form a complete orthogonal system in the space of class functions.

Theorem 2.4.15 (Orthogonality relations for characters)**.** *Let G be a compact Lie group and let V and W be G-representations.*

(i) $\int_G \chi_V(g)\,dg = \dim V^G$*;*

(ii) $\langle \chi_W, \chi_V \rangle = \int_G \overline{\chi_V}(g)\chi_W(g)\,dg = \dim \mathrm{Hom}_G(V,W)$*;*

(iii) If V and W are irreducible, then

$$\langle \chi_W, \chi_V \rangle = \int_G \overline{\chi_V}(g)\chi_W(g)\,dg = \begin{cases} 1 & \text{if } V \cong W; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The projection operator p from Construction 2.4.1 has trace $\dim V^G$. Since the trace is linear and thus commutes with integration by Lemma 1.5.8, we get

$$\dim(V^G) = \mathrm{tr}(p) = \mathrm{tr}(\int_G l_g\,dg) = \int \mathrm{tr}(l_g)\,dg = \int \chi_V(g)\,dg.$$

This proves (i). To prove (ii), recall that $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$, so using (i) we get

$$\dim \mathrm{Hom}_G(V,W) = \int_G \chi_{\mathrm{Hom}(V,W)}(g)\,dg,$$

but we have

$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_{V^*\otimes W}(g) = \chi_{V^*}(g)\cdot\chi_W(g) = \overline{\chi_V(g)}\chi_W(g).$$

The last statement (iii) is an immediate consequence of (ii) and Schur's lemma. □

Theorem 2.4.16. *A representation of a compact Lie group is determined up to isomorphism by its character.*

Proof. Let V_i be a complete set of pairwise nonisomorphic irreducible representations. For a representation V, there exists an isotypical decomposition $V \cong \bigoplus_i n_i V_i$, where n_i is the multiplicity of V_i in V. We then have $\chi_V = \sum_i n_i \chi_{V_i}$ by Proposition 2.4.10, and thus $n_i = \langle \chi_V, \chi_{V_i} \rangle$. It follows that the n_i may be recovered from the character of V, and these numbers determine V up to isomorphism. □

Proposition 2.4.17. *Suppose that* $\langle \chi_V, \chi_V \rangle = 1$*. Then V is irreducible.*

Proof. If we write $V = \oplus_i n_i V_i$, then $\langle \chi_V, \chi_V \rangle = \sum_{i,j} n_i n_j \langle \chi_{V_i}, \chi_{V_j} \rangle = \sum_i n_i^2$, hence it follows that $n_i = 0$ for all but one i, for which $n_i = 1$. □

We may use Proposition 2.4.17 to identify the irreducible representations of a product $G \times H$ of compact Lie groups G and H. Note that for representations V of G and W of H we have the tensor product $V \otimes W$ with action given by $(g,h)(v \otimes w) = gv \otimes hw$.

Proposition 2.4.18. *Let G and H be compact Lie groups. If V is an irreducible G-representation and W is an irreducible H-representation, then* $V \otimes W$ *is an irreducible* $(G \times H)$*-representation. Furthermore, any irreducible* $(G \times H)$*-representation is of this form.*

Proof. The first half follows from Proposition 2.4.17:

$$\begin{aligned}\int_{G\times H}\chi_{V\otimes W}(g,h)\overline{\chi_{V\otimes W}(g,h)}\,d(g,h) &= \int_{G\times H}\chi_V(g)\chi_W(h)\overline{\chi_V(g)\chi_W(h)}\,d(g,h)\\ &= \int_G \chi_V(g)\overline{\chi_V(g)}\,dg\cdot\int_H \chi_W(h)\overline{\chi_W(h)}\,dh = 1\cdot 1 = 1.\end{aligned}$$

For the second half, suppose that U is a $(G\times H)$-representation. We may consider the underlying H-representation of U (restricting along the inclusion $H\hookrightarrow G\times H$) and obtain an isotypical decomposition

$$\varphi\colon \bigoplus_j \mathrm{Hom}_H(W_j,U)\otimes W_j \xrightarrow{\cong} U, \qquad f\otimes w\mapsto f(w)$$

of U. Observe that G acts on $f\in\mathrm{Hom}_H(W_j,U)$ by $(gf)(w) = g\cdot f(w)$: since the G- and H-actions on U commute, we again have $gf\in\mathrm{Hom}_H(W_j,U)$. Furthermore, the isomorphism φ is G-equivariant, and hence it is even an isomorphism of $(G\times H)$-representations.

For every j, we may now find a decomposition $\mathrm{Hom}_H(W_j,U) = \bigoplus_i n_{ij}V_i$ as G-representations. This gives us an isomorphism $U\cong\bigoplus_{i,j} n_{ij}V_i\otimes W_j$. In particular, if U was irreducible it must have been of the form $V\otimes W$. □

Remark 2.4.19. From the proof of Proposition 2.4.18, we obtain a bijection

$$\begin{aligned}\mathrm{Irr}_G(\mathbb{C})\times\mathrm{Irr}_H(\mathbb{C}) &\xrightarrow{\cong} \mathrm{Irr}_{G\times H}(\mathbb{C})\\ (V,W) &\mapsto V\otimes W.\end{aligned}$$

If G is an abelian compact Lie group, then all its irreducible representations are one-dimensional (see Proposition 2.1.17). Since we may take the representations to be unitary, it follows that the irreducible characters are precisely the homomorphisms of Lie groups $G\to\mathrm{U}(1)$.

Definition 2.4.20. For an abelian compact Lie group, we define the *character group* $\hat{G}$ to be the set of irreducible characters $G\to\mathrm{U}(1)$, with group structure given by the pointwise product in $\mathrm{U}(1)$.

Let us close the section by recording for later reference a few identities concerning the behavior of representative functions and characters with respect to the convolution product on $C^0(G;\mathbb{C})$. All of the identities follow from the orthogonality relations.

Proposition 2.4.21. *Let $g\mapsto(u_{ij}(g))$ and $g\mapsto(v_{kl}(g))$ be nonisomorphic irreducible unitary matrix representations of the compact Lie group G, with characters χ_U and χ_V. Let $|U|$ and $|V|$ denote the dimensions of the representations. Then:*

(i) We have

$$u_{ij}*u_{kl} = \frac{1}{|U|}\delta_{jk}u_{il} \qquad \textit{and} \qquad u_{ij}*v_{kl} = 0.$$

(ii) We have

$$\chi_V * v_{ij} = v_{ij}*\chi_V = \frac{1}{|V|}v_{ij} \qquad \textit{and} \qquad \chi_U * v_{kl} = v_{kl}*\chi_U = 0.$$

(iii) We have

$$\chi_V*\chi_V = \frac{1}{|V|}\chi_V \qquad \textit{and} \qquad \chi_V*\chi_U = 0.$$

Proof. For (i), first recall the definition of the convolution product:

$$(u_{ij} * u_{kl})(g) = \int_G u_{ij}(gh^{-1})u_{kl}(h)\,dh.$$

Now, the matrix coefficients $u_{ij}(gh^{-1})$ are the matrix coefficients of a product of matrices:

$$u_{ij}(gh^{-1}) = \sum_m u_{im}(g)u_{mj}(h^{-1}).$$

Since U is unitary, we have $u_{mj}(h^{-1}) = \overline{u_{jm}(h)}$. Substituting this expression into the integral, we get

$$(u_{ij} * u_{kl})(g) = \sum_m u_{im}(g) \int_G \overline{u_{jm}(h)}u_{kl}(h)\,dh.$$

But by the orthogonality relation from Remark 2.4.8, the integral reduces to $\frac{1}{|U|}\delta_{jk}\delta_{ml}$, hence we end up with $\frac{1}{|U|}\delta_{jk}u_{il}$ as claimed. The second relation is proved similarly.

Parts (ii) and (iii) are an immediate consequence of (i), using that $\chi_V = \sum_{i=1}^n v_{ii}(g)$. □

2.5 Representations of $\mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathrm{U}(2)$ and $\mathrm{O}(3)$

Our goal in this section is to give elementary descriptions of the irreducible representations of the groups $\mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathrm{U}(2)$ and $\mathrm{O}(3)$.

Before we start, let us briefly address the lower dimensional groups.

Example 2.5.1. The groups $\mathrm{SU}(1)$ and $\mathrm{SO}(1)$ are trivial, hence have no nontrivial representations.

Example 2.5.2. Since $\mathrm{O}(1) = \{\pm 1\}$ is abelian, the irreducible representations are given by $\mathrm{Hom}(\{\pm 1\}, \mathrm{U}(1)) = \{\pm 1\}$, corresponding to the trivial action and the flip action on $\mathbb{C}$.

Example 2.5.3. Also $\mathrm{SO}(2) = S^1$ is abelian, so again the irreducible representations correspond to irreducible characters $S^1 \to U(1) = S^1$. But by Exercise 4.4, all continuous group homomorphisms $S^1 \to S^1$ are of the form $\lambda \mapsto \lambda^n$ for some $n \in \mathbb{Z}$. The corresponding one-dimensional representations are

$$S^1 \times \mathbb{C} \to \mathbb{C} \colon (\lambda, x) \mapsto \lambda^n x.$$

We will sometimes also denote this representation by A_n.

Example 2.5.4. Let us identify all the irreducible representations of $\mathrm{O}(2) = \mathrm{SO}(2) \ltimes \mathbb{Z}/2$. Let V be an irreducible $\mathrm{O}(2)$-representation. Then its underlying $\mathrm{SO}(2)$-representation might no longer be irreducible, but it contains an irreducible subrepresentation $U \subseteq V$. By the previous example, U is one-dimensional, and $R_\theta = \left(\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right) \in \mathrm{SO}(2)$ acts on U via $R_\theta u = e^{in\theta}u$ for some integer $n \in \mathbb{Z}$. Now consider the generator $\tau \in \mathbb{Z}/2 \subseteq \mathrm{O}(2)$, given by the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the linear subspace $\tau U \subseteq V$ is closed under the $\mathrm{SO}(2)$-action: we have

$$R_\theta \tau u = \tau R_{-\theta} u = \tau e^{-in\theta} u \in \tau U.$$

It follows that the sum $U + \tau U \subseteq V$ is an $\mathrm{O}(2)$-subrepresentation of V, hence by irreducibility must equal V. If $\tau U = U$, then this forces $n = 0$, and V must be obtained from one of the two irreducible $\mathbb{Z}/2$-representations (the trivial one or the one with the flip action.) If $\tau U \neq U$, then V is two-dimensional, and any element $u \in U$ provides a basis $\{u, \tau u\}$ of V. In this basis, the matrices corresponding to R_θ and $R_\theta \tau$ are

$$R_\theta = \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} \qquad \text{and} \qquad R_\theta \tau = \begin{pmatrix} 0 & e^{in\theta} \\ e^{-in\theta} & 0 \end{pmatrix}.$$

With that out of the way, let us move to the irreducible representations of $\mathrm{SU}(2)$.

Definition 2.5.5. Let $V_0 = \mathbb{C}$ denote the trivial representation of $\mathrm{SU}(2)$. Let $V_1 = \mathbb{C}^2$ denote the standard representation. For every $n \geq 2$, we define

$$V_n := S^n V_1,$$

the n-th symmetric power of V_1.

Our goal is to show that each of the V_n is irreducible and that they in fact cover *all* irreducible $\mathrm{SU}(2)$-representations.

We start by giving a more explicit description of V_n. If we denote the two basis vectors of $\mathbb{C}^2$ by z_1 and z_2, then a basis of V_n is given by the monomials $\{z_1^n, z_1^{n-1} z_2, \ldots, z_1 z_2^{n-1}, z_2^n\}$, and we may identify V_n with the vector space of homogeneous polynomials of degree n in the variables z_1 and z_2. (In particular V_n has dimension $n+1$.) If we view polynomials as functions $\mathbb{C}^2 \to \mathbb{C}$, then we obtain a natural left action of $\mathrm{GL}_2(\mathbb{C})$ and hence $\mathrm{SU}(2)$ on the polynomials by letting

$$(gP)z = P(zg),$$

where

$$P \in \mathbb{C}[z_1, z_2], \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad z = (z_1, z_2)$$

and

$$zg = (az_1 + cz_2, bz_1 + dz_2).$$

Since g acts as a homogeneous linear transformation, the subspace $V_n \subseteq \mathbb{C}[z_1, z_2]$ is $\mathrm{SU}(2)$-invariant. Note that with this description, $n = 0$ results in the trivial representation and for $n = 1$ we get the standard representation.

In the following result, we write $P_k(z_1, z_2) = z_1^k z_2^{n-k}$, $0 \leq k \leq n$, for the basis of V_n.

Proposition 2.5.6. *The representations V_n are irreducible.*

Proof. It suffices to show that each $\mathrm{SU}(2)$-equivariant endomorphism A of V_n is a multiple of the identity. For $a \in \mathrm{U}(1)$, define

$$g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \in \quad \mathrm{SU}(2).$$

Then we have

$$g_a P_k = a^{2k-n} P_k \qquad \text{and} \qquad g_a A P_k = A g_a P_k = A a^{2k-n} P_k = a^{2k-n} A P_k.$$

Now choose a such that all powers a^{2k-n} for $0 \leq k \leq n$ are distinct. It then follows that the a^{2k-n}-eigenspace of g_a in V_n is generated by P_k, and in particular we have $AP_k = \lambda_k P_k$ for some scalars $\lambda_k \in \mathbb{C}$. Our goal is to show that all these scalars are the same.

To do this, we consider again the real rotations

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathrm{SU}(2), \qquad \theta \in \mathbb{R},$$

and we compute

$$\begin{aligned} AR_\theta P_n &= A(z_1 \cos(\theta) + z_2 \sin(\theta))^n \\ &= \sum_k \binom{n}{k} \cos(\theta)^k \cdot \sin(\theta)^{n-k} \cdot AP_k \\ &= \sum_k \binom{n}{k} \cos(\theta)^k \cdot \sin(\theta)^{n-k} \cdot c_k \cdot P_k. \end{aligned}$$

Similarly, we obtain

$$R_\theta A P_n = R_\theta c_n P_n = \sum_k \binom{n}{k} \cos(\theta)^k \cdot \sin(\theta)^{n-k} \cdot c_n \cdot P_k.$$

Since A is $\mathrm{SU}(2)$-equivariant, these two expressions must agree, but then comparing coefficients we see that $c_k = c_n$. This implies that $A = c_n \cdot \mathrm{id}$ as desired. □

To show that the V_n comprise *all* the irreducible representations, we will analyze the possible class functions on $\mathrm{SU}(2)$.

Observation 2.5.7. The space of class functions on $\mathrm{SU}(2)$ agrees with the space of even 2π-periodic continuous functions $\mathbb{R} \to \mathbb{C}$.

Proof. Consider now for $\theta \in \mathbb{R}$ the matrix

$$e(\theta) := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in \mathrm{SU}(2).$$

Any class function $f\colon \mathrm{SU}(2) \to \mathbb{C}$ may be restricted to a continuous function $fe\colon \mathbb{R} \to \mathbb{C}$ in this way. Since any element in $\mathrm{SU}(2)$ is conjugate to a diagonal matrix, which must have diagonal entries λ and λ^{-1} with $|\lambda| = 1$, we see that any element in $\mathrm{SU}(2)$ is conjugate to some $e(\theta)$. It follows that f is completely determined by fe. Furthermore, the matrices $e(\theta)$ and $e(\tau)$ are conjugate if and only if $\theta \equiv \pm\tau \mod 2\pi$, so fe is even and 2π-periodic. This shows that we obtain a bijection of sets. Since the topologies on both sides are induced by the supremum norm, this is even a homeomorphism of topological spaces. □

Proposition 2.5.8. *Let $\chi_n\colon \mathrm{SU}(2) \to \mathbb{C}$ denote the character of V_n. Then the functions χ_n generate a uniformly dense subspace of $C^0(\mathrm{SU}(2);\mathbb{C})$ (i.e. dense with respect to the supremum norm).*

Proof. By Observation 2.5.7 we may restrict attention to even 2π-periodic functions $\mathbb{R} \to \mathbb{C}$. Since $e(\theta)P_k = (e^{i\theta}x_1)^k (e^{-i\theta}x_2)^{n-k} = e^{i(2k-n)\theta} P_k$, the character χ_n of V_n corresponds to the function

$$\kappa_n\colon \mathbb{R} \to \mathbb{C}, \qquad \theta \mapsto \sum_{k=0}^{n} e^{i(2k-n)\theta}.$$

Now, using the expression $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, and the formula $1 + x + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$, we may rewrite this expression as

$$\begin{aligned}\sum_{k=0}^{n} e^{i(2k-n)\theta} = e^{-in\theta} \sum_{k=0}^{n} (e^{2i\theta})^k &= \frac{e^{-i\theta(n+1)}}{e^{-i\theta}} \cdot \frac{(e^{2i\theta})^{n+1} - 1}{e^{2i\theta} - 1} \\ &= \frac{e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)}.\end{aligned}$$

Let us write $\kappa_n(\theta)$ for this function. Using the addition theorem for sin, we obtain

$$\kappa_n(\theta) = \cos(n\theta) + \kappa_{n-1}(\theta)\cos(\theta),$$

so that $\kappa_0 = 1$, $\kappa_1 = 2\cos(\theta)$, $\kappa_2 = 2(\cos(\theta))^2 + \cos(2\theta) = 1 + 2\cos(2\theta)$, etcetera. It follows in particular that the subspace of continuous functions $\mathbb{R} \to \mathbb{C}$ generated by the κ_n agrees with the subspace generated by the functions $\theta \mapsto \cos(n\theta)$ for $n \in \mathbb{N}$. But it is a well-known theorem from elementary Fourier analysis that the space generated by the functions $\cos(n\theta)$ is uniformly dense in the space of even 2π-periodic continuous functions $\mathbb{R} \to \mathbb{C}$. This finishes the proof. $\square$

Proposition 2.5.9. *Every irreducible unitary representation of* $\mathrm{SU}(2)$ *is isomorphic to one of the* V_n.

Proof. Let W be an irreducible representation with character χ and assume it is different from all the V_n. Then by orthogonality, Theorem 2.4.15(iii), we have $\langle \chi, \chi_n \rangle = 0$ for all $n \in \mathbb{N}$ and $\langle \chi, \chi \rangle = 1$. But since the χ_n generate a dense subspace this gives a contradiction. $\square$

We now turn to the irreducible representations of SO(3). For this, we use the following relation between SU(2) and SO(3):

Proposition 2.5.10. *There exists a surjective homomorphism* $\pi\colon \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ *whose kernel is* $\{\pm I\}$.

Proof. The proof will make heavy use of the quaternions $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$. First recall that we may identify the Lie group SU(2) with the group $\mathrm{Sp}(1) = \{x \in \mathbb{H} \mid |x| = 1\}$ of *unit quaternions*. We will further consider the linear subspace of the quaternions given by the *pure quaternions*:

$$\mathbb{H}_{\mathrm{pure}} := \langle i, j, k \rangle = \{bi + cj + dk \mid b, c, d \in \mathbb{R}\} = \{z \in \mathbb{H} \mid \mathrm{Re}(z) = 0\}.$$

Note that there is a preferred isomorphism of $\mathbb{R}$-vector spaces $\mathbb{R}^3 \cong \mathbb{H}_{\mathrm{pure}}$ given by $(b, c, d) \mapsto bi + cj + dk$. Hence to construct the map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ we may equivalently specify a linear action of $\mathrm{Sp}(1)$ on $\mathbb{H}_{\mathrm{pure}}$ which has determinant 1 and preserves the symplectic inner product. We define this as follows:

$$\mathrm{Sp}(1) \times \mathbb{H}_{\mathrm{pure}} \to \mathbb{H}_{\mathrm{pure}}, \qquad (\lambda, v) \mapsto \lambda v \lambda^{-1},$$

where we use that $\mathbb{H}$ is a (non-commutative) ring. We need to show that this is well-defined, i.e. that $\lambda v \lambda^{-1}$ is indeed again a pure quaternion if v is. To this end, we first observe the following fact: for all $z, w \in \mathbb{H}$ we have $\mathrm{Re}(zw) = \mathrm{Re}(wz)$. Indeed, it will suffice to show this when $z, w \in \{1, i, j, k\}$ are basis vectors, where it is easily verified. But then

$$\mathrm{Re}(\lambda v \lambda^{-1}) = \mathrm{Re}(\lambda \lambda^{-1} v) = \mathrm{Re}(v) = 0,$$

so the action of $\mathrm{Sp}(1)$ restricts as desired. It is norm-preserving since $N(\lambda v \lambda^{-1}) = N(\lambda)N(v)N(\lambda)^{-1} = N(v)$, and thus defines a Lie group homomorphism $\mathrm{Sp}(1) \to \mathrm{O}(\mathbb{H}_{\mathrm{pure}}) \cong \mathrm{O}(3)$. Since $\mathrm{Sp}(1)$ is connected, we see that it in fact must factor through $\mathrm{SO}(3)$, giving the homomorphism $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$.

Next we will show that π is surjective, so consider a matrix $A \in \mathrm{SO}(3)$. We claim that A has 1 as an eigenvalue. Indeed, we have

$$\det(A - I) = \det(A^T)\det(A - I) = \det(A^T A - A^T) = \det(I - A^T) = \det(I - A) = -\det(A - I),$$

so that $\det(A - I) = 0$. We conclude that there exists some vector $u \in \mathbb{H}_{\mathrm{pure}}$ such that $Au = u$. Without loss of generality we assume $|u| = 1$. We call v the *rotation axis* of A. Since A is an orthogonal matrix, it follows that A preserves the orthogonal complement of $\mathbb{R}u \subseteq \mathbb{H}_{\mathrm{pure}}$. The restriction of A to this orthogonal complement is an element of $\mathrm{SO}(2) \cong S^1$, i.e. it is given by some rotation θ. It follows that the matrix A is the given by a rotation of angle θ around the axis u (according to the right-hand rule).

Now consider the quaternion

$$\lambda := \cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2}) \cdot u \in \mathbb{H},$$

which is easily seen to be a unit quaternion (using that $|u| = 1$). We claim that for any $v \in \mathbb{H}_{\mathrm{pure}}$, rotation of v by angle θ around axis u is precisely given by $v \mapsto \lambda v \lambda^{-1}$. To show this, write $v = v_{||} + v_{\perp}$, where $v_{||}$ is parallel to u while $v_{\perp}$ is orthogonal to u. In this way, we can reduce to the cases where v is either parallel to u or orthogonal to u:

- If v is parallel to u, then one can prove that $uv = vu$, and consequently also $\lambda v = v\lambda$. So $\lambda v \lambda^{-1} = v\lambda\lambda^{-1} = v$. This is what we want, since rotation around axis u leaves v fixed.

- If v is perpendicular to u, then one can prove that instead we have $uv = -vu$. Consequently we get

  $$\lambda v \lambda^{-1} = \lambda v(\cos(\theta/2) - u\sin(\theta/2)) = \lambda(\cos(\theta/2) + u\sin(\theta/2))v = \lambda^2 v = (\cos(\theta) + u\sin(\theta))v.$$

  And again this is precisely the rotation with angle θ around axis u.

Finally, we need to show that the kernel of $\mathrm{Sp}(1) \to \mathrm{SO}(3)$ is $\{\pm 1\}$, i.e. if $\lambda v \lambda^{-1} = v$ for all v then $\lambda \in \{\pm 1\}$. But if we write $\lambda = \lambda_1 + \lambda_2$ with $\lambda_1 \in \mathbb{R}$ and λ_2 pure, then by choosing v perpendicular to λ_2 we see that $\lambda v \lambda^{-1} = \lambda^2 v$ hence this is only v if $\lambda^2 = 1$, and thus $\lambda = \pm 1$. This finishes the proof. □

Corollary 2.5.11. *The irreducible representations of* $\mathrm{SO}(3)$ *are in bijective correspondence with the irreducible representations* V_n *of* $\mathrm{SU}(2)$ *on which* $-I$ *acts as the identity.*

Proof. It is immediate from Proposition 2.5.10 that the representations of $\mathrm{SO}(3)$ are in bijective correspondence with representations of $\mathrm{SU}(2)$ on which $-I$ acts as the identity. Since $\pi\colon \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ is surjective, such a representation is irreducible as an $\mathrm{SO}(3)$-representation if and only if it is irreducible as an $\mathrm{SU}(2)$-representation. □

Corollary 2.5.12. *The irreducible representations of* $\mathrm{SO}(3)$ *are the representations* $W_k := V_{2k}$ *for* $k \in \mathbb{N}$.

Proof. We know that $-I$ acts as multiplication by $(-1)^n$ on V_n, which acts as the identity if and only if $n = 2k$ is even. $\square$

Remark 2.5.13. The dimension of W_k is $2k+1$. We will show below that W_k admits an alternative presentation in terms of harmonic polynomials. The homomorphism $\pi\colon \mathrm{SU}(2) \twoheadrightarrow \mathrm{SO}(3)$ sends the matrix $e(\theta)$ to

$$R(2\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta), \end{pmatrix}$$

and hence by Observation 2.5.7 the class functions $\mathrm{SO}(3) \to \mathbb{C}$ are determined by their values on the matrices $R(\theta)$.

The next result is often referred to as the *Clebsh-Gordan formula*. It plays an important role in physics, specifically in quantum mechanics: it comes up in the context of adding angular momenta of quantum states.

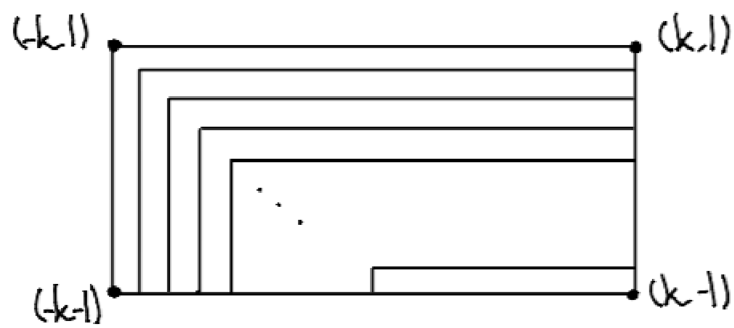
Proposition 2.5.14. *There is an isomorphism*

$$V_k \otimes V_l \cong \bigoplus_{j=0}^{q} V_{k+l-2j}, \qquad \textit{where} \qquad q = \min\{k, l\}.$$

Proof. Since representations are determined by their characters, it suffices to prove the corresponding result for characters. By Observation 2.5.7 the characters are determined by their values on the elements $e(\theta)$, and thus it suffices to verify the purely combinatorial identity

$$\left(\sum_{i_0=0}^{k} x^{k-2i_0}\right)\left(\sum_{i_1=0}^{l} x^{l-2i_1}\right) = \sum_{j=0}^{q}\left(\sum_{m=0}^{k+l-2j} x^{k+l-2j-2m}\right)$$

and replace x by $e^{i\theta}$. We may assume that $l \leq k$. Now we may arrange the pairs of indices $(k-2i_0, l-2i_1)$ in a rectangular scheme; to each such pair there corresponds a term $x^{k-2i_0}x^{l-2i_1} = x^{k+l-2(i_0+i_1)}$ in the left-hand product. The right-hand sum then comes about by summing first over the pairs of indices on the separate lines indicated in the figure below, and then adding up all the resulting sums. (It might be easiest to see this by doing an induction on $k+l$, and observing that when going from $k+l-2$ to $k+l$, the additional terms on both sides precisely correspond to the additional line of terms along the top-left of the diagram.)



$\square$

Remark 2.5.15. The representations V_k are sometimes enumerated by half integers in the literature ,say $V_k = V(k/2)$. The Clebsch-Gordan formula then reads

$$V(a) \otimes V(b) \cong V(|a-b|) \oplus V(|a-b|+1) \oplus \cdots \oplus V(a+b).$$

The irreducible representations of $U(2)$ may be expressed in terms of those of S^1 and $\mathrm{SU}(2)$ using the following:

Proposition 2.5.16. *There is a surjective group homomorphism*

$$S^1 \times \mathrm{SU}(2) \twoheadrightarrow \mathrm{U}(2), \qquad (\lambda, A) \mapsto \lambda \cdot A$$

whose kernel is $\{(1,I), (-1,-I)\}$.

Proof. It is clear that the given assignment determines a homomorphism of Lie groups. It is also clearly surjective, since any $B \in \mathrm{U}(2)$ may be written as $B = \lambda \cdot A$ where we take $\lambda \in S^1$ to be a square root of $\det(B) \in S^1$ and we take $A := \lambda^{-1} \cdot B$. To compute the kernel, assume that $\lambda \cdot A = I$. Then $A = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. But this is only in $\mathrm{SU}(2)$ if $\lambda^{-2} = 1$, and thus $\lambda = \pm 1$. □

Recall from Example 2.5.3 that we write A_m for the one-dimensional S^1-representations $S^1 \times \mathbb{C} \to \mathbb{C}$, $(\lambda, x) \mapsto \lambda^m x$.

Corollary 2.5.17. *The irreducible representations of* $\mathrm{U}(2)$ *are of the form* $A_m \otimes V_n$ *for* $n, m \in \mathbb{Z}$ *with* $n+m$ *even.*

Proof. By Proposition 2.5.16, the irreducible representations of $\mathrm{U}(2)$ may be identified with the irreducible representations of the product $S^1 \times \mathrm{SU}(2)$ on which the pair $(-1,-I)$ acts trivially; see the proof of Corollary 2.5.12 for a similar reduction step. By Proposition 2.4.18, we see that the irreducible representations of $S^1 \times \mathrm{SU}(2)$ are given by the tensor products $A_m \otimes V_n$, where the A_m are the irreducible S^1-representations from Example 2.5.3 and the V_n are the irreducible $\mathrm{SU}(2)$-representations from Proposition 2.5.9. The element $(-1,-I)$ acts on $A_m \otimes V_n$ via multiplication by $(-1)^{n+m}$, hence acts trivially if and only if $n+m$ is even. □

Let us now explain the connection between the representations W_k of $\mathrm{SO}(3)$ and 'spherical harmonic functions'.

Construction 2.5.18. For a natural number $l \geq 0$, let P_l be the complex vector space of homogeneous complex polynomials in three variables of degree l. Viewing these polynomials as functions $\mathbb{R}^3 \to \mathbb{C}$, we obtain an action of $\mathrm{SO}(3)$ on this space:

$$(Af)(x) = f(xA), \quad A \in \mathrm{SO}(3), \quad x \in \mathbb{R}^3, \quad f \in P_l.$$

This $\mathrm{SO}(3)$-representation is not irreducible: for example, P_2 contains the invariant subspace generated by the polynomial $x_1^2 + x_2^2 + x_3^2$.

Consider the *Laplace operator* Δ on $\mathbb{R}^3$: for a function $f \colon \mathbb{R}^3 \to \mathbb{R}$ we define $\Delta f := \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \colon \mathbb{R}^3 \to \mathbb{R}$. For a complex-valued function $f = f_1 + i f_2$ we define $\Delta f = \Delta f_1 + i \Delta f_2$. We may then consider the subspace

$$\mathcal{H}_l := \{ f \in P_l \mid \Delta f = 0 \},$$

called the space of *harmonic polynomials* of degree l. Restricting functions in $\mathcal{H}_l$ to $S^2 \subseteq \mathbb{R}^3$ yields the *spherical harmonics* of degree l. Note that a homogeneous function on $\mathbb{R}^3$ is uniquely determined by its restriction to S^2.

Lemma 2.5.19. *We have*

$$\dim(P_l) = \frac{1}{2}(l+1)(l+2) \qquad \textit{and} \qquad \dim(\mathcal{H}_l) = 2l+1.$$

Proof. A basis of P_l is given by the monomials $x_1^p x_2^q x_3^r$ with $p+q+r=l$. For every $0 \le r \le l$, there are $l-r+1$ ways to find nonnegative p and q such that $p+q=l-r$, and thus the dimension of P_l is

$$\sum_{r=0}^{l}(l-r+1) = \sum_{r=0}^{l}(l+1) = \frac{1}{2}(l+1)(l+2).$$

For the second part, we may write every polynomial $f \in P_l$ in the form

$$f(x_1,x_2,x_3) = \sum_{k=0}^{l} \frac{x_1^k}{k!} f_k(x_2,x_3),$$

where f_k is a homogeneous polynomial of degree $l-k$ in variables x_2,x_3. We then have

$$\Delta f = \sum_{k=0}^{l-2} \frac{x_1^k}{k!} f_{k+2} + \sum_{k=0}^{l} \frac{x_1^k}{k!}\left(\frac{\partial^2 f_k}{\partial x_2^2} + \frac{\partial^2 f_k}{\partial x_3^2}\right).$$

So we see that $\Delta f = 0$ if and only if

$$f_{k+2} = -\left(\frac{\partial^2 f_k}{\partial x_2^2} + \frac{\partial^2 f_k}{\partial x_3^2}\right), \qquad 0 \le k \le l-2$$

In particular, an element of $\mathcal{H}_l$ is uniquely determined by the polynomials f_0 and f_1: the higher f_k are computed by the previous formula. We conclude that the dimension of $\mathcal{H}_l$ is the sum of the dimensions of the spaces of homogeneous polynomials in two variables of degrees l and $l-1$, i.e.

$$\dim \mathcal{H}_l = (l+1) + l = 2l+1.$$

□

Lemma 2.5.20. *The subspace* $\mathcal{H}_l \subseteq P_l$ *is* $\mathrm{SO}(3)$*-invariant.*

Proof. It suffices to show that the Laplace operator is rotation invariant, i.e. commutes with the action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$. So see this, consider the *gradient*

$$\mathrm{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_3}, \frac{\partial f}{\partial x_3}\right)$$

of a smooth map $f\colon \mathbb{R}^3 \to \mathbb{R}$, and the *divergence*

$$\mathrm{div}(g) = \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}$$

for a smooth map $g\colon \mathbb{R}^3 \to \mathbb{R}^3$. It follows from the chain rule that for $A \in \mathrm{SO}(3) \subseteq \mathrm{End}(\mathbb{R}^3)$ we have

$$\mathrm{grad}(Af) = A\mathrm{grad}(f) \qquad \text{and} \qquad \mathrm{div}(Ag) = A\mathrm{div}(g).$$

Since we have $\Delta(f) = \mathrm{div}(\mathrm{grad}(f))$, this finishes the proof. □

So we have obtained an SO(3)-representation $\mathcal{H}_l$, and we may ask how it decomposes into irreducible SO(3)-representations. Since we have $\dim(\mathcal{H}_l) = 2l+1 = \dim(W_l)$, the following result is not too surprising.

Proposition 2.5.21. *The space $\mathcal{H}_l$ of harmonic polynomials of degree l is an irreducible* SO(3)*-representation isomorphic to W_l.*

Proof. Since the irreducible SO(3)-representations are of the form W_k, we may decompose $\mathcal{H}_l$ as

$$\mathcal{H}_l \cong \bigoplus_i W_{k_i}$$

for certain natural numbers k_i. Because of dimension reasons it will suffice that $k_i \geq l$ for some i. To do this, we consider the action of the rotation matrix $R(\theta)$ on this direct sum. This matrix corresponds to the complex matrix $e(\theta/2) \in \mathrm{SU}(2)$ (see Remark 2.5.13), and thus the eigenvalues of the action of $R(\theta)$ on W_{k_i} are the eigenvalues of $e(\theta/2)$ on V_{2k_i}, which as we saw before are $e^{im\theta}$ for $|m| \leq k_i$. In particular, the eigenvalue $e^{-il\theta}$ can only appears if $k_i \geq l$ for some i, and thus it will suffice to show that $\mathcal{H}_l$ admits an eigenvector with eigenvalue $e^{-il\theta}$. For this, consider $f_l(x_1,x_2,x_3) = (x_2+ix_3)^l$. Then $f_l \in \mathcal{H}_l$, and we have

$$\begin{aligned}
R(\theta)f_l(x_1,x_2,x_3) &= (x_2\cos(\theta)+x_3\sin(\theta)+i(-x_2\sin(\theta)+x_3\cos(\theta)))^l \\
&= (e^{-i\theta}x_2+ie^{-i\theta}x_3)^l \\
&= e^{-il\theta}f_l(x_1,x_2,x_3).
\end{aligned}$$

This finishes the proof. $\square$

2.6 Real and quaternionic representations

So far, we have restricted our attention to *complex* representations. In this section, we will introduce real and quaternionic representations and see how they may be expressed in terms of complex representations.

Definition 2.6.1. Let G be a Lie group. A *real* (resp. *quaternionic*) *G-representation* is a real (resp. quaternionic) vector space U with a continuous action $G \times U \to U$ such that the left translations are $\mathbb{R}$-linear (resp. $\mathbb{H}$-linear). If U carries a G-invariant inner product we call U an *orthogonal* (resp. *symplectic*) representation.

Choosing a basis for U yields corresponding matrix representations $G \to \mathrm{GL}_n(\mathbb{R})$ (resp. $G \to \mathrm{GL}_n(\mathbb{H})$). If U is orthogonal (resp. symplectic) it has the form $G \to \mathrm{O}(n)$ (resp. $G \to \mathrm{Sp}(n)$).

As in the case of complex representations, invariant inner products always exist when G is compact. This leads to the conclusion that every real (resp. quaternionic) representation is a direct sum of irreducible subrepresentations, i.e. real and quaternionic representations of compact groups are semisimple in the sense of Definition 2.2.3.

For the remainder of the section, we assume that G is compact.

We want to analyze the interaction between real, complex, and quaternionic representations. This requires a certain amount of bookkeeping, but is not mathematically deep. In order to start the bookkeeping, we will do two things: We will introduce notation for various categories of representations and functors between them, and we will treat the complex representations as the basic objects, viewing both real and quaternionic representations as complex representations with additional structure.

Definition 2.6.2. Let V be a complex G-representation. A *real structure* on V is a conjugate-linear G-equivariant map $J\colon V \to V$ such that $J^2 = \mathrm{id}$. A *quaternionic structure* on V is a conjugate-linear G-equivariant map $J\colon V \to V$ such that $J^2 = -\mathrm{id}$. In both cases, we refer to J as the *structure map*. A complex representation is said to be of *real* (resp. *quaternionic*) *type* if it admits a real (resp. quaternionic) structure.

Remark 2.6.3. A representation can be simultaneously of real and quaternionic type.

Next we introduce some categories of representations and functors between them. We write

$$K = \mathbb{R}, \mathbb{C}, \text{ or } \mathbb{H},$$

and we denote by

$$\mathrm{Rep}_K(G)$$

the category[1] whose objects are the G-representations on K-vector spaces and whose morphisms are K-linear G-equivariant maps. We denote by

$$\mathrm{Rep}^+_{\mathbb{C}}(G) \qquad \text{and} \qquad \mathrm{Rep}^-_{\mathbb{C}}(G)$$

the categories of complex G-representations with real and quaternionic structure. Morphisms in these categories are $\mathbb{C}$-linear G-equivariant maps which commute with the structure maps.

Construction 2.6.4. Let U be a real G-representation. We define the complex G-representation $e_+(U)$ as

$$e_+(U) := \mathbb{C} \otimes_{\mathbb{R}} U.$$

It comes equipped with a real structure given by $J(z \otimes u) := \overline{z} \otimes u$. Given a morphism $f\colon U \to U'$ of real representations, we obtain a morphism $e_+(f)\colon e_+(U) \to e_+(U'), z \otimes u \mapsto z \otimes f(u)$. This is compatible with identity maps and compositions, hence this construction defines a functor

$$e_+\colon \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Rep}^+_{\mathbb{C}}(G).$$

Conversely, let (V, J) be a complex representation with a real structure. We denote by V_+ and V_- the $(+1)$- and (-1)-eigenspaces of J. Since morphisms in $\mathrm{Rep}^+_{\mathbb{C}}(G)$ commute with J, they need to preserve these eigenspaces. In particular, sending V to V_+ determines another functor

$$s_+\colon \mathrm{Rep}^+_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{R}}(G).$$

For $v \in V$, we may write $2v = (v + Jv) + (v - Jv)$, and we conclude that $V = V_+ \oplus V_-$. Notice that multiplication by $i \in \mathbb{C}$ induces isomorphisms $V_+ \to V_-$ and $V_- \to V_+$.

Lemma 2.6.5. *The functor* $e_+\colon \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Rep}^+_{\mathbb{C}}(G)$ *is an equivalence with inverse* $s_+\colon \mathrm{Rep}^+_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{R}}(G)$.

[1]The word ‘category’ indicates that we have a class of mathematical objects together with morphisms between them that may be composed.

Proof. Given $(V,J) \in \mathrm{Rep}^+_{\mathbb{C}}(G)$, a natural isomorphism $e_+s_+(V,J) \cong (V,J)$ is given by $\mathbb{C}\otimes_{\mathbb{R}}V_+ \to V, z\otimes v \mapsto zv$. A natural isomorphism $s_+e_+U \cong U$ is given by $U \to (\mathbb{C}\otimes_{\mathbb{R}}U)_+, u \mapsto 1\otimes u$. We leave the details to the reader. □

Construction 2.6.6. If W is a quaternionic representation, we let $e_-(W)$ denote its underlying complex representation. The action of $j \in \mathbb{H}$ determines a conjugate-linear map $J\colon W \to W$ satisfying $J^2 = -\mathrm{id}$, hence this provides W with a quaternionic structure. This is compatible with morphisms of representations, and thus determines a functor

$$e_-\colon \mathrm{Rep}_{\mathbb{H}}(G) \to \mathrm{Rep}^-_{\mathbb{C}}(G).$$

Conversely, if (V,J) is a complex representation with quaternionic structure, we may define a quaternionic representation $s_-(V,J)$ by equipping V the $\mathbb{H}$-linear structure defined by

$$(z_1+z_2j)v := z_1v + z_2J(v), \qquad z_1,z_2 \in \mathbb{C}, v \in V.$$

This defines a functor $s_-\colon \mathrm{Rep}^-_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{H}}(G)$.

Lemma 2.6.7. *The functor e_- is an equivalence with inverse s_-.*

Proof. It is straightforward to see that $e_-s_-(V,J) = (V,J)$ and $s_-e_-(U) = U$. (Actual equalities this time, not just natural isomorphisms.) □

We will now consider relations (i.e., functors) between different types of representations coming from restriction, extension and conjugation.

First, we have various forgetful functors that for example view a complex representation as real, or forget about the structure map J, and so on. Explicitly, given a complex G-representation V we define $r^{\mathbb{C}}_{\mathbb{R}}(V)$ as the underlying real representation of V, and we denote by $r_+(V,J) := V$ the underlying complex representation with no additional structure. This gives functors

$$\begin{aligned}
r^{\mathbb{C}}_{\mathbb{R}}&\colon \mathrm{Rep}_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{R}}(G)\\
r^{\mathbb{H}}_{\mathbb{C}}&\colon \mathrm{Rep}_{\mathbb{H}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G)\\
r^{\mathbb{H}}_{\mathbb{R}}&\colon \mathrm{Rep}_{\mathbb{H}}(G) \to \mathrm{Rep}_{\mathbb{R}}(G), \qquad\qquad r^{\mathbb{H}}_{\mathbb{R}} = r^{\mathbb{C}}_{\mathbb{R}}\circ r^{\mathbb{H}}_{\mathbb{C}}\\
r_+&\colon \mathrm{Rep}^+_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G)\\
r_-&\colon \mathrm{Rep}^-_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G),
\end{aligned}$$

which are sometimes also called *restriction functors*. Second, we have *extension functors*

$$\begin{aligned}
e^{\mathbb{C}}_{\mathbb{R}}&\colon \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G),\\
e^{\mathbb{H}}_{\mathbb{C}}&\colon \mathrm{Rep}_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{H}}(G),\\
e^{\mathbb{H}}_{\mathbb{R}}&\colon \mathrm{Rep}_{\mathbb{R}}(G) \to \mathrm{Rep}_{\mathbb{H}}(G).
\end{aligned}$$

The first is defined via $e^{\mathbb{C}}_{\mathbb{R}}(U) = \mathbb{C}\otimes_{\mathbb{R}}U$, while the second is $e^{\mathbb{H}}_{\mathbb{C}}(V) = \mathbb{H}\otimes_{\mathbb{C}} V$, where we view $\mathbb{H}$ as a right $\mathbb{C}$-module via right multiplication: $\mathbb{H}\times\mathbb{C} \to \mathbb{H}\,(w,v) \mapsto wv$. Finally, we let $e^{\mathbb{H}}_{\mathbb{R}} = e^{\mathbb{H}}_{\mathbb{C}}e^{\mathbb{C}}_{\mathbb{R}}$, i.e. $e^{\mathbb{H}}_{\mathbb{R}}(V) = \mathbb{H}\otimes_{\mathbb{R}} V$.

Third, we have the *conjugation functor*

$$c\colon \mathrm{Rep}_{\mathbb{C}}(G) \to \mathrm{Rep}_{\mathbb{C}}(G),$$

which sends V to the conjugate representation $\overline{V}$.

Proposition 2.6.8. *The above functors satisfy the following relations*

$$\begin{aligned} r_{\mathbb{R}}^{\mathbb{C}} e_{\mathbb{R}}^{\mathbb{C}} &= 2 & c e_{\mathbb{R}}^{\mathbb{C}} &= e_{\mathbb{R}}^{\mathbb{C}} \\ e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} &= 1 + c & r_{\mathbb{R}}^{\mathbb{C}} c &= r_{\mathbb{R}}^{\mathbb{C}} \\ e_{\mathbb{C}}^{\mathbb{H}} r_{\mathbb{C}}^{\mathbb{H}} &= 2 & c r_{\mathbb{C}}^{\mathbb{H}} &= r_{\mathbb{C}}^{\mathbb{H}} \\ r_{\mathbb{C}}^{\mathbb{H}} e_{\mathbb{C}}^{\mathbb{H}} &= 1 + c & e_{\mathbb{C}}^{\mathbb{H}} c &= e_{\mathbb{C}}^{\mathbb{H}} \\ e_{\mathbb{R}}^{\mathbb{C}} &= r_+ e_+ & r_{\mathbb{C}}^{\mathbb{H}} &= r_- e_- \\ c^2 &= \mathrm{id}\,. \end{aligned}$$

Here the first equality should be interpreted as saying that there is a natural isomorphism $r_{\mathbb{R}}^{\mathbb{C}} e_{\mathbb{R}}^{\mathbb{C}}(V) \cong V \oplus V$, the second one that there is a natural isomorphism $e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}}(V) \cong V \oplus \overline{V}$, etcetera.

Proof. We will not spell out each of the individual cases; most of them are fairly straightforward. As an example, let us prove that $e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} = 1 + c$. For a complex G-representation V, we consider the map $\alpha\colon V \oplus V \to \mathbb{C} \otimes_{\mathbb{R}} V$ defined by

$$\alpha(v,w) := \frac{1}{2}(1 \otimes v - i \otimes iv) + \frac{1}{2}(1 \otimes w + i \otimes iw).$$

This is $\mathbb{R}$-linear by construction, and it satisfies the relation $\alpha(iv, -iw) = i\alpha(v,w)$. In particular, we may regard α as a $\mathbb{C}$-linear map $V \oplus \overline{V} \to \mathbb{C} \otimes_{\mathbb{R}} V$. An inverse to α is given by the map

$$\beta\colon\ \mathbb{C} \otimes_{\mathbb{R}} V \to V \oplus \overline{V}, \qquad z \otimes v \mapsto (zv, \bar{z}v).$$

As another example, the relation $r_{\mathbb{C}}^{\mathbb{H}} e_{\mathbb{C}}^{\mathbb{H}} = 1 + c$ comes from the $\mathbb{C}$-linear isomorphism

$$\mathbb{H} \otimes_{\mathbb{C}} V \to V \oplus \overline{V}, \qquad (z_1 + j z_2) \otimes v \mapsto (z_1 v, z_2 v).$$

Here we view $\{1, j\}$ as a $\mathbb{C}$-linear basis of $\mathbb{H}$ with respect to right multiplication and $z_1, z_2 \in \mathbb{C}$. The other relations are proved in a similar way. ◻

We now come to the main goal of this section: To relate the irreducible real and quaternionic representations to the irreducible complex representations. To this end, consider the sets

$$\mathrm{Irr}_G(K), \qquad K = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

of isomorphism classes of irreducible K-linear G-representations. We will partition each of them into three disjoint (possibly empty) parts, thus ending up with nine sets. To simplify the exposition, we will use the following notational convention:

Convention 2.6.9. The letters U, V, and W will, respectively, denote elements of $\mathrm{Irr}_G(\mathbb{R})$, $\mathrm{Irr}_G(\mathbb{C})$ and $\mathrm{Irr}_G(\mathbb{H})$.

Definition 2.6.10. We define the following nine subsets $\mathrm{Irr}_G(K)_L \subseteq \mathrm{Irr}_G(K)$ for $K, L = \mathbb{R}, \mathbb{C}, \mathbb{H}$:

(1) Consider an irreducible real G-representation $U \in \mathrm{Irr}_G(\mathbb{R})$.

- We say that U is *of real type* if the complex representation $V = e_{\mathbb{R}}^{\mathbb{C}} U$ is irreducible;
- We say that U is *of complex type* if it is of the form $U = r_{\mathbb{R}}^{\mathbb{C}} V$, where $V \in \mathrm{Irr}_G(\mathbb{C})$ satisfying $V \not\cong \overline{V}$;

- We say that U is *of quaternionic type* if it is of the form $U = r_{\mathbb{R}}^{\mathbb{C}} V$, where $V \in \mathrm{Irr}_G(\mathbb{C})$ satisfies $V \cong \overline{V}$.

(2) Consider an irreducible complex G-representation $V \in \mathrm{Irr}_G(\mathbb{C})$.

- We already defined in Definition 2.6.2 what it means for V to be of real or of quaternionic type;
- We say that V is *of complex type* if $V \not\cong \overline{V}$.

(3) Consider an irreducible quaternionic G-representation $W \in \mathrm{Irr}_G(\mathbb{H})$.

- We say W is *of real type* if $W = e_{\mathbb{C}}^{\mathbb{H}} V$ where $V \in \mathrm{Irr}_G(\mathbb{C})$ is of real type.
- We say W is *of complex type* if $W = e_{\mathbb{C}}^{\mathbb{H}} V$ where $V \in \mathrm{Irr}_G(\mathbb{C})$ with $V \not\cong \overline{V}$;
- We say W is *of quaternionic type* if $V = r_{\mathbb{C}}^{\mathbb{H}}(W)$ is irreducible.

We let $\mathrm{Rep}_G(K)_L \subseteq \mathrm{Rep}_G(K)$ denote the subset of representations of type L.

Our next goal will be to prove the following theorem:

Theorem 2.6.11. *For $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, the set $\mathrm{Irr}_G(K)$ is a disjoint union of the three subsets $\mathrm{Irr}_G(K)_{\mathbb{R}}$, $\mathrm{Irr}_G(K)_{\mathbb{C}}$ and $\mathrm{Irr}_G(K)_{\mathbb{H}}$.*

As a preparation for the proof of the theorem, we will prove the following criterion for complex representations to be of real/quaternionic type:

Proposition 2.6.12. *A complex representation V is of real (resp. quaternionic) type if and only if there exists a nondegenerate symmetric (resp. skew-symmetric) G-invariant bilinear form $B\colon V \times V \to \mathbb{C}$.*

Proof. First assume that such a form B is given. To treat both cases at the same time, define $\varepsilon = 1$ in the real case and $\varepsilon = -1$ in the quaternionic case, so that we have

$$B(v,w) = \varepsilon B(w,v)$$

for all $v, w \in V$. We may choose a G-invariant Hermitian inner product $\langle -,- \rangle$ on V and define the map $f\colon V \to V$ by demanding that $B(v,w) = \langle v, fw \rangle$ for all $v, w \in V$. Then f is conjugate linear and G-equivariant, and it is an isomorphism since B is nondegenerate. We have the relation

$$\langle v, fw \rangle = B(v,w) = \varepsilon B(w,v) = \varepsilon \langle w, fv \rangle = \varepsilon \overline{\langle fv, w \rangle},$$

and applying this twice we obtain

$$\langle v, f^2 w \rangle = \varepsilon \overline{\langle fv, fw \rangle} = \langle f^2 v, w \rangle,$$

showing that f^2 is Hermitian. It follows that also εf^2 is Hermitian, so that it has real eigenvalues. We also see that we have $\langle v, \varepsilon f^2 v \rangle \geq 0$ for all $v \in V$, so that εf^2 has *positive* real eigenvalues. We thus obtain a decomposition

$$V = \bigoplus_{i=1}^{n} V_{\lambda_i}$$

where $V_{\lambda_i} \subseteq V$ is the eigenspace of $\lambda_i \in \mathbb{R}_{>0}$. Notice that every V_{λ_i} is G-invariant, so that this is a decomposition of V as a G-representation. We may now define the G-automorphism $h\colon V \to V$ as the direct sum of the morphisms

$$\sqrt{\lambda_i} \cdot \mathrm{id}_{V_{\lambda_i}} : V_{\lambda_i} \to V_{\lambda_i}$$

for $i = 1, \ldots, n$. Since the V_{λ_i} are invariant under f we see that $hf = fh$, and by construction we have that $h^2 = \varepsilon f^2$. If we now define

$$J := hf^{-1} \colon V \to V,$$

then we see that $J^2 = hf^{-1}hf^{-1} = h^2 f^{-2} = \varepsilon \cdot \mathrm{id}_V$, as desired.

Conversely, suppose that V has a structure map J satisfying $J^2 = \varepsilon \cdot \mathrm{id}_V$. We consider the two cases separately:

- If $\varepsilon = 1$, then the map $\mathbb{C} \otimes_{\mathbb{R}} V_+ \to V, z \otimes v \mapsto z \cdot v$ is an isomorphism by Lemma 2.6.5. Consider now any nondegenerate G-invariant symmetric $\mathbb{R}$-bilinear form $\langle -,- \rangle \colon V_+ \times V_+ \to \mathbb{R}$. Then this uniquely extends to a $\mathbb{C}$-bilinear form $B\colon V \times V \to \mathbb{C}$ on $V = \mathbb{C} \otimes_{\mathbb{R}} V_+$, and B is again nondegenerate, G-invariant and symmetric.

- If $\varepsilon = -1$, then we may regard V as a quaternionic representation by Construction 2.6.6 by letting $j \in \mathbb{H}$ act as $J\colon V \to V$. We may now choose a G-invariant symplectic inner product $\langle -,- \rangle \colon V \times V \to \mathbb{H}$ on V. We may write this inner product as

  $$\langle u, v \rangle = H(u,v) + B(u,v)j$$

  where $H(u,v)$ and $B(u,v)$ are complex numbers. Using the relations $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ and $\langle u, \lambda v \rangle = \langle u, v \rangle \overline{\lambda}$ for $u, v \in V$ and $\lambda \in \mathbb{H}$, it is easy to verify that B is $\mathbb{C}$-bilinear. Moreover, since $\langle u, v \rangle = \overline{\langle v, u \rangle}$, we have

  $$H(u,v) + B(u,v)j = \overline{H(v,u) + B(v,u)j} = \overline{H(v,u)} - j\overline{B(v,u)} = \overline{H(v,u)} - B(v,u)j,$$

  and it follows that B is skew-symmetric. It remains to show that B is nondegenerate. To see this, suppose that $B(u, v_0) = 0$ for all $u \in V$. Then we have $\langle u, v_0 \rangle \in \mathbb{C}$ for all $u \in V$, but since $\langle ju, v_0 \rangle = j \langle u, v_0 \rangle$ this implies that $\langle u, v_0 \rangle = 0$ for all $u \in V$. But since symplectic inner products are nondegenerate, this means that $v_0 = 0$, as desired.

□

Definition 2.6.13. A complex representation V is called *self-conjugate* if there exists an isomorphism $V \cong \overline{V}$.

Proposition 2.6.14. *Let V be an irreducible self-conjugate complex representation. Then V is of real type or of quaternionic type, and not of both.*

Proof. We will use Proposition 2.6.12, and thus we will examine the space $\mathrm{Hom}(V \otimes V, \mathbb{C}) \cong V^* \otimes V^*$ of bilinear forms on V. We may decompose this space as a direct sum

$$\mathrm{Hom}(V \otimes V, \mathbb{C}) = S \oplus A,$$

where S denotes the subspace of symmetric bilinear forms and A denotes the subspace of skew-symmetric bilinear forms. The G-invariant forms are the elements of the fixed-point set $(S \oplus A)^G =$

$S^G \oplus A^G$. Since V is self-conjugate, we have isomorphisms $V \cong \overline{V} \cong V^*$, and since V is irreducible it thus follows from Schur's lemma that the space

$$\mathrm{Hom}(V \otimes V, \mathbb{C})^G \cong \mathrm{Hom}_G(V, V^*)^G = \mathrm{Hom}_G(V, V^*)$$

is one-dimensional. So we are in one of the following two cases:

- If $\dim(S^G) = 1$ and $\dim(A^G) = 0$, then V admits a symmetric G-invariant bilinear form inducing an isomorphism $V \cong V^*$, so that this bilinear form is nondegenerate. Furthermore, V does *not* admit a nondegenerate skew-symmetric G-invariant bilinear form.
- Otherwise we have $\dim(S^G) = 0$ and $\dim(A^G) = 1$, and here the situation is reversed.

It then follows from Proposition 2.6.12 that V is either of real or of quaternionic type, but not of both. □

We may now prove Theorem 2.6.11:

Proof of Theorem 2.6.11. We divide the proof into three parts, corresponding to the three cases $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We will always use the letters U, V, W (with or without indices) to denote irreducible representations over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, respectively.

First part: Let $V \in \mathrm{Irr}_G(\mathbb{C})$. If V is either of real or quaternionic type, then the structure map $J\colon V \to V$ is a conjugate-linear isomorphism, giving an isomorphism $V \cong \overline{V}$. So either $V \not\cong \overline{V}$, in which case V is of complex type, or $V \cong \overline{V}$, in which case V is either of real or quaternionic type by Proposition 2.6.14.

Second part: Let $U \in \mathrm{Irr}_G(\mathbb{R})$, and let us simply write e and r for $e_{\mathbb{R}}^{\mathbb{C}}$ and $r_{\mathbb{R}}^{\mathbb{C}}$. If eU is irreducible, then by definition U is of real type. Observe that then $V = eU$ is a complex representation of real type, with real structure constructed in Construction 2.6.4. If eU is not irreducible, we may find a decomposition

$$eU = V_1 \oplus \cdots \oplus V_n$$

into irreducible complex G-representations, with $n \geq 2$. Since we have $reV \cong U \oplus U$ by Proposition 2.6.8, we get an $\mathbb{R}$-linear isomorphism $U \oplus U \cong V_1 \oplus \cdots \oplus V_n$, and by irreducibility of U we see that $n \leq 2$, hence $n = 2$ and $rV_1 = rV_2 = U$. We then see that $eU \cong erV_1 \cong V_1 \oplus \overline{V_1}$, implying that $V_2 \cong \overline{V_1}$. There are now two possibilities:

- If $V_1 \not\cong \overline{V_1}$, then $U = rV_1 \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{C}}$.
- If $V_1 \cong \overline{V_1}$, it must be either of real or quaternionic type. But we can show that V_1 is not of real type: if it were, then we would have $rV_1 \cong (V_1)_+ \oplus (V_1)_-$, contradicting the irreducibility of $U = rV_1$. We conclude that $V_1 \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{H}}$, hence $U = rV_1 \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{H}}$.

This finishes the case $K = \mathbb{R}$.

Third part: Let $W \in \mathrm{Irr}_G(\mathbb{H})$, and write r and e for $r_{\mathbb{C}}^{\mathbb{H}}$ and $e_{\mathbb{C}}^{\mathbb{H}}$. Note that rW always carries a quaternionic structure, so if rW is irreducible we have $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{H}}$. If rW is not irreducible, we may decompose it as a direct sum of irreducible subrepresentations

$$rW = V_1 \oplus \cdots \oplus V_n,$$

with $n \geq 2$. As in the previous case, we may then use the relations $er = 2$ and $re = 1 + c$ from Proposition 2.6.8 to see that $n = 2$ and that $V_2 \cong \overline{V_1}$. This time we have $eV_1 = W$. If $V_1 \not\cong \overline{V_1}$, then $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{C}}$ by definition. If $V_1 \cong \overline{V_1}$, it only remains to show that it is not of quaternionic type. But if it were, then we could write $V_1 = rX$ for some $X \in \mathrm{Rep}_{\mathbb{H}}(G)$, and then $W = eV_1 = erX \cong X \oplus X$, which contradicts the irreducibility of W. This finishes the case $K = \mathbb{H}$, and thus the theorem is proved. □

We may summarize this entire discussion in the following table:

L=	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$U \in \mathrm{Irr}_G(\mathbb{R})_L$	$V{:=}e_{\mathbb{R}}^{\mathbb{C}}U$ V of real type	$U{=}r_{\mathbb{R}}^{\mathbb{C}}V$ $V{\not\cong}\overline{V}$	$U{=}r_{\mathbb{R}}^{\mathbb{C}}V$ V of quaternionic type
$V \in \mathrm{Irr}_G(\mathbb{C})_L$	V of real type	$V \not\cong \overline{V}$	V of quaternionic type
$W \in \mathrm{Irr}_G(\mathbb{H})_L$	$W{=}e_{\mathbb{C}}^{\mathbb{H}}V$ V of real type	$W{=}e_{\mathbb{C}}^{\mathbb{H}}V$ $V{\not\cong}\overline{V}$	$V{:=}r_{\mathbb{C}}^{\mathbb{H}}W$ V of quaternionic type

In the course of proving Theorem 2.6.11, we have established the first six of the following implications. Analogous reasoning leads to the remaining ones, which we leave as an exercise to the reader.

Proposition 2.6.15. *(1) Let $U \in \mathrm{Irr}_G(\mathbb{R})$.*

- *If $U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{R}}$, then $e_{\mathbb{R}}^{\mathbb{C}}U \cong V$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{R}}$;*
- *If $U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{C}}$, then $e_{\mathbb{R}}^{\mathbb{C}}U \cong V \oplus \overline{V}$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{C}}$;*
- *If $U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{H}}$, then $e_{\mathbb{R}}^{\mathbb{C}}U \cong V \oplus V$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{H}}$.*

(2) Let $W \in \mathrm{Irr}_G(\mathbb{H})$.

- *If $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{R}}$, then $r_{\mathbb{C}}^{\mathbb{H}}W \cong V \oplus V$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{R}}$;*
- *If $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{C}}$, then $r_{\mathbb{C}}^{\mathbb{H}}W \cong V \oplus \overline{V}$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{C}}$;*
- *If $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{H}}$, then $r_{\mathbb{C}}^{\mathbb{H}}W \cong V$ for $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{H}}$.*

(3) Let $V \in \mathrm{Irr}_G(\mathbb{C})$.

- *If $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{R}}$, then* $\begin{cases} r_{\mathbb{R}}^{\mathbb{C}}V \cong U \oplus U & \textit{for } U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{R}}; \\ e_{\mathbb{C}}^{\mathbb{H}}V \cong W & \textit{for } W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{R}}; \end{cases}$
- *If $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{C}}$, then* $\begin{cases} r_{\mathbb{R}}^{\mathbb{C}}V \cong U \cong r_{\mathbb{R}}^{\mathbb{C}}\overline{V} & \textit{for } U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{C}}; \\ e_{\mathbb{C}}^{\mathbb{H}}V \cong W \cong e_{\mathbb{C}}^{\mathbb{H}}\overline{V} & \textit{for } W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{C}}; \end{cases}$
- *If $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{H}}$, then* $\begin{cases} r_{\mathbb{R}}^{\mathbb{C}}V \cong U & \textit{for } U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{H}}; \\ e_{\mathbb{C}}^{\mathbb{H}}V \cong W \oplus W & \textit{for } W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{H}}. \end{cases}$

The next result is an important and easily remembered characterization of the sets $\mathrm{Irr}_G(\mathbb{R})_K$:

Theorem 2.6.16. *Given an irreducible real G-representation $U \in \mathrm{Irr}_G(\mathbb{R})$, we have $U \in \mathrm{Irr}_G(\mathbb{R})_K$ if and only if the endomorphism algebra $\mathrm{Hom}_G(U, U)$ of U is isomorphic to K.*

Proof. We may regard $\mathrm{Hom}_G(U,U)$ as an algebra over $\mathbb{R}$. By Schur's lemma for simple modules, Lemma 2.2.2, every nonzero endomorphism $U \to U$ is invertible, and it follows that $\mathrm{Hom}_G(U,U)$ is in fact a division algebra over $\mathbb{R}$. Recall that there are only three possibilities for such, namely $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (this is known as the Frobenius Theorem).

- First suppose that $\mathrm{Hom}_G(U,U) \cong \mathbb{C}$. This implies that $\mathbb{C}$ acts on U, and thus $U = r^{\mathbb{C}}_{\mathbb{R}} V$ for some irreducible V. If V were of real type, then $r^{\mathbb{C}}_{\mathbb{R}} V = V_+ \oplus V_-$ would not be irreducible, and if V were of quaternionic type, then $\mathbb{H}$ would be contained in $\mathrm{Hom}_G(U,U)$. So $V \in \mathrm{Irr}_G(\mathbb{C})_{\mathbb{C}}$ and thus $U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{C}}$.
- Next suppose that $\mathrm{Hom}_G(U,U) \cong \mathbb{H}$. Then $U = r^{\mathbb{H}}_{\mathbb{R}} W$ for some irreducible W. From Proposition 2.6.15 we see that $W \in \mathrm{Irr}_G(\mathbb{H})_{\mathbb{H}}$ and thus $U \in \mathrm{Irr}_G(\mathbb{R})_{\mathbb{H}}$.
- Finally, suppose that $\mathrm{Hom}_G(U,U) \cong \mathbb{R}$. Then U cannot be of the form $r^{\mathbb{C}}_{\mathbb{R}} V$ for any irreducible V of complex or quaternionic type, since in either case $\mathrm{Hom}_G(U,U)$ would contain $\mathbb{C}$. So U must be of real type.

This completes the proof. $\square$

We have just seen that the type of an irreducible real representation is determined by its endomorphism algebra. The next proposition shows how the type of an irreducible *complex* representation is determined by its character.

Proposition 2.6.17. *Let V be an irreducible complex representation of G with character $\chi_V : G \to \mathbb{C}$. Then:*

$$\int_G \chi_V(g^2)\, dg \;=\; \begin{cases} 1 & V \text{ is of real type,} \\ 0 & V \text{ is of complex type,} \\ -1 & V \text{ is of quaternionic type.} \end{cases}$$

Proof. We may decompose the tensor product $V \otimes V$ as a direct sum $V \otimes V = S \oplus A$, where S denotes the subspace of symmetric tensors and A denotes the subspace of antisymmetric tensors. Notice that both are G-invariant subspaces. We claim that we have

$$\chi_V(g^2) = \chi_S(g) - \chi_A(g)$$

for every $g \in G$. To see this, we fix some $g \in G$ and let $H \subseteq G$ be the closed subgroup generated by g. Since H is automatically abelian, we may thus find a decomposition $V = \bigoplus_i V(i)$, where each of the $V(i)$ are irreducible H-representations and hence one-dimensional. We thus get $A = \Lambda^2(V) = \Lambda^2(\bigoplus_i V(i))$. Now recall the general fact that $\Lambda^k(B \oplus C) = \bigoplus_j \Lambda^j(B) \otimes \Lambda^{k-j}(C)$ and observe that $\Lambda^2(V(i)) = 0$ for all i since $V(i)$ is one-dimensional. It follows that we obtain a decomposition

$$A \cong \bigoplus_{i<j} V(i) \otimes V(j).$$

We may now calculate

$$\begin{aligned}
\chi_V(g^2) &= \sum_i \chi_{V(i)}(g^2) = \sum_i (\chi_{V(i)}(g))^2 \\
&= \left(\sum_i \chi_{V(i)}(g)\right)^2 - 2\sum_{i<j} \chi_{V(i)}(g)\chi_{V(j)}(g) \\
&= \chi_V(g)\cdot\chi_V(g) - 2\chi_A(g) \\
&= (\chi_{V\otimes V}(g) - \chi_A(g)) - \chi_A(g) \\
&= \chi_S(g) - \chi_A(g),
\end{aligned}$$

establishing the claim. Now we may integrate both sides to obtain

$$\int \chi_V(g^2)\,dg = \int \chi_S(g)\,dg - \int \chi_A(g)\,dg = \dim S^G - \dim A^G,$$

using part (i) of Theorem 2.4.15. But we have $\dim S^G = \dim(S^*)^G$, which is the dimension of the space of symmetric G-invariant bilinear forms on V. The same is true for A^G with respect to antisymmetric forms. The proposition is now a direct consequence of Proposition 2.6.12. □

Similar to the situation for complex representations discussed at the end of Section 2.1, there is also a *canonical decomposition* of real G-representations X. For $U \in \mathrm{Irr}_G(\mathbb{R})$, we denote by $D(U) = \mathrm{Hom}_G(U,U)$ the endomorphism ring of U. Then $\mathrm{Hom}_G(U,X)$ is a right $D(U)$-module via evaluation. Therefore the tensor product $\mathrm{Hom}_G(U,X)\otimes_{D(U)} U$ is defined and the map $\mathrm{Hom}_G(U,X)\otimes U \to X, \varphi\otimes u \mapsto \varphi(u)$ factors through this tensor product. Thus we obtain a homomorphism

$$d\colon \bigoplus_{U\in\mathrm{Irr}_G(\mathbb{R})} \mathrm{Hom}_G(U,X)\otimes_{D(U)} U \to X.$$

Proposition 2.6.18. *The map d is an isomorphism.*

Proof. The proof is analogous to that of Proposition 2.1.19 and will be omitted. □

We refer to the image of the map $\mathrm{Hom}_G(U,X)\otimes_{D(U)} U \to X$ as the *U-isotypical part* of X.

2.7 The character ring and the representation ring

Let G be a compact Lie group. The relations $\chi_{V\oplus W} = \chi_V + \chi_W$ and $\chi_{V\otimes W} = \chi_V\chi_W$ suggest that characters might generate an interesting ring of functions on G. Therefore we let

$$R(G) = R(G,\mathbb{C})$$

denote the additive group of functions $G\to\mathbb{C}$ generated by characters of complex G-representations. Since every character may be written as a sum of irreducible characters (Proposition 2.1.13) and since the irreducible characters are orthogonal to each other (Theorem 2.4.15), we see that the irreducible characters are linearly independent and generate $R(G)$. As a consequence, $R(G)$ is the free abelian group generated by these characters. Since $\chi_V\cdot\chi_W = \chi_{V\otimes W}$, the group $R(G)$ is closed under multiplication in $C^0(G,\mathbb{C})$, and thus is a commutative subring of $C^0(G,\mathbb{C})$.

Definition 2.7.1. We refer to the ring $R(G)$ as the *character ring* of (complex representations of) G. Elements of $R(G)$ are called *virtual characters*.

Warning 2.7.2. Although a G-representation is completely determined by its character, the ring $R(G)$ is a weaker invariant of G than the set of representations itself. This is because we cannot distinguish which elements of $R(G)$ are *positive*, i.e. are actual characters. Moreover, we cannot necessarily recover the irreducible characters from $R(G)$.

The character ring has an alternative description directly in terms of representations which exhibits a useful universal property.

Definition 2.7.3. We let $R^+(G)$ denote the set of isomorphism classes of complex G-representations. We equip $R^+(G)$ with an addition and multiplication operation given by $\oplus$ and $\otimes$. Then $R^+(G)$ is a *commutative semiring*: it satisfies all the usual axioms of a commutative ring, except that there are no additive inverses.

We obtain a function $\chi\colon R^+(G) \to R(G)$ by sending a representation V to its character χ_V. Notice that this is a morphism of semirings, in that it preserves both addition and multiplication. It has the following universal property:

Proposition 2.7.4. *Let A be an abelian group and let $\varphi\colon R^+(G) \to A$ be a function satisfying $\varphi(V \oplus W) = \varphi(V) + \varphi(W)$ and $\varphi(0) = 0$. Then there exists a unique group homomorphism $\Phi\colon R(G) \to A$ such that $\varphi = \Phi \circ \chi$:*

$$\begin{tikzcd} R^+(G) \arrow[r, "\varphi"] \arrow[d, "\chi"'] & A \\ R(G). \arrow[ur, dashed, "\exists!\Phi"'] & \end{tikzcd}$$

If $A = R$ is in fact a commutative ring and φ additionally satisfies the conditions $\varphi(V \otimes W) = \varphi(V)\varphi(W)$ and $\varphi(1) = 1$, then the map Φ is even a ring homomorphism.

Proof. The group $R(G)$ is a free abelian group generated by the irreducible characters, so the group homomorphism Φ is uniquely determined by the specification

$$\Phi(\chi_V) := \varphi(V)$$

for every irreducible G-representation V. The relation $\Phi \circ \chi = \varphi$ then holds true on irreducible characters, and the assumptions on φ then guarantee that it holds true on every representation. For the assertion that Φ is even a ring map, we may check this on the generators of $R(G)$: for irreducible G-representations V and W we have

$$\Phi(\chi_V \cdot \chi_W) = \Phi(\chi_{V\otimes W}) = \varphi(V \otimes W) = \varphi(V)\varphi(W) = \Phi(\chi_V)\Phi(\chi_W),$$

as desired. $\square$

Given any abelian monoid, there is a formal construction of an abelian group satisfying the above universal property that does not refer to characters, called the *Grothendieck construction*. Hence the previous proposition says that $R(G)$ is the Grothendieck construction of $R^+(G)$. For this reason, $R(G)$ is also known as the (complex) *representation ring* of G. Given a representation V, we will sometimes also abusively write $V \in R(G)$, identifying a representation with its image under the inclusion $R^+(G) \hookrightarrow R(G)$.

We may use the universal property of $R(G)$ to equip it with some additional structure of *λ-operations* and *Adams operations*.

Construction 2.7.5 (λ-operations)**.** We will construct functions $\lambda^i \colon R(G) \to R(G)$ for every natural number $i \geq 0$.

Let $R(G)[[t]]$ denote the commutative ring of formal power series with coefficients in $R(G)$. For G-representation V, we define the formal power series $\lambda_t(V)$ by

$$\lambda_t(V) := \Lambda^0(V) + \Lambda^1(V)t + \Lambda^2(V)t^2 + \dots \quad \in \quad \mathbb{R}(G)[[t]].$$

Recall that the k-th exterior power $\Lambda^k(V)$ satisfies the following fundamental property:

$$\Lambda^k(V \oplus W) = \sum_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W).$$

We may express these relations for all k simultaneously via the relation

$$\lambda_t(V \oplus W) = \lambda_t(V) \cdot \lambda_t(W)$$

Notice that we have $\Lambda^0(V) = \mathbb{C}$ with the trivial action, which corresponds to the unit $1 \in R(G)$, and thus $\lambda_t(V)$ is an element of the multiplicative group

$$1 + tR(G)[[t]] \subseteq R(G)[[t]]$$

of formal power series over $R(G)$ with constant term 1. (The required multiplicative inverses can be constructed by inductively solving for the coefficients of the inverse power series.) We conclude that the assignment $V \mapsto \lambda_t(V)$ determines a homomorphism $R^+(G) \to 1 + tR(G)[[t]]$, and thus the universal property Proposition 2.7.4 provides a group homomorphism

$$\lambda_t \colon R(G) \to 1 + tR(G)[[t]].$$

Now, for $x \in R(G)$ we define

$$\lambda_t(X) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \dots,$$

thereby obtaining maps $\lambda^i \colon R(G) \to R(G)$ satisfying the relations

$$\lambda^0(x) = 1, \qquad \lambda^1(x) = x, \qquad \lambda^k(x+y) = \sum_i \lambda^i(x) \cdot \lambda^{k-i}(y).$$

Remark 2.7.6. A ring R together with maps $\lambda^i \colon R \to R$ is called a *λ-ring*, hence the above construction shows that $R(G)$ admits the structure of a λ-ring.

The behavior of the operations λ^i with respect to composition and products is quite complicated, and is given by certain universal polynomials with integer coefficients (independent of G):

$$\begin{aligned} \lambda^i\lambda^j(x) &= P_{i,j}(\lambda^1 x, \lambda^2(x), \dots, \lambda^{ij}(x)), \\ \lambda^k(xy) &= P_k(\lambda^1(x), \dots, \lambda^k(x); \lambda^1(y), \dots, \lambda^k(y)). \end{aligned}$$

We will not go into details about how to construct the polynomials $P_{i,j}$ and P_k.

Construction 2.7.7 (Adams operations)**.** We will now construct the *Adams operations*

$$\Psi^k \colon R(G) \to R(G), \qquad k \in \mathbb{N},$$

which are ring homomorphisms satisfying $\Psi^k \Psi^l = \Psi^{kl}$. We will define them as certain polynomials in the λ^i-maps. To do this, consider an integer $m \geq k$ and consider the polynomial

$$x_1^k + \cdots + x_m^k$$

in the variables $x_1, \ldots, x_m$. Since this is a symmetric polynomial, we may write it as a polynomial $Q_k(\sigma_1, \ldots, \sigma_k)$ in the elementary symmetric polynomials σ_i in the variables $x_1, \ldots, x_m$. Notice that the polynomial Q_k is independent of m for $m \geq k$: setting $x_{m+1} = 0$ we see that $\sigma_k(x_1, \ldots, x_m, 0) = \sigma_k(x_1, \ldots, x_m)$. We may then define

$$\Psi^k(x) := Q_k(\lambda^1(x), \ldots, \lambda^k(x)) \quad \in \quad R(G).$$

The Adams operations may be described very concretely in terms of operations on the ring $C^0(G; \mathbb{C})$:

Proposition 2.7.8. *If we view x and $\Psi^k(x)$ as virtual characters $G \to \mathbb{C}$, then we have*

$$\Psi^k(x)(g) = x(g^k)$$

for all $g \in G$.

Proof. We start by proving the claim when $x = \chi_V$ is the character of a G-representation. Fixing $g \in G$, let H be the closed abelian subgroup of G generated by g. We may regard V as an H-representation, and as such find a decomposition $V = \bigoplus_{i=1}^n V_i$ into one-dimensional irreducible H-representations V_i. Let $v_i \in V_i$ be a generator and write $gv_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. Then a basis for $\Lambda^m(V)$ is given by the elements $v_{i_1} \wedge \cdots \wedge v_{i_m}$ for $i_1 < \cdots < i_m$, and hence we see that the trace of $l_g \colon \Lambda^m(V) \to \Lambda^m(V)$ is given by $\sigma_m(\lambda_1, \ldots, \lambda_n)$, where σ_m denotes the m-th elementary symmetric polynomial in the λ_i. In other words, we have

$$\chi_{\Lambda^m(V)} = \sigma_m(\lambda_1, \ldots, \lambda_n).$$

Now by definition of Ψ^k we have

$$\Psi^k(V)(g) = Q_k(\chi_{\Lambda^1(V)}, \ldots, \chi_{\Lambda^k(V)}) = Q_k(\sigma_1, \ldots, \sigma_k) = \lambda_1^k + \cdots + \lambda_n^k.$$

But since $g^k v_i = \lambda_i^k v_i$, notice that also the trace of g^k on V is $\lambda_1^k + \cdots + \lambda_n^k$. This proves the claim for $x = \chi_V$.

In the case of an arbitrary virtual character x, we consider a formal power series $\Psi_t(x) = \sum_{n \geq 1} \Psi(x, n) t^n$ defined by the following relation:

$$\Psi_{-t}(x) = -t \left(\frac{d}{dt} \lambda_t(x) \right) / \lambda_t(x) = -t \frac{d}{dt} \log \lambda_t(x).$$

It can be shown (see the exercises) that we have

$$\Psi(x, n) = Q_n(\lambda^1(x), \ldots, \lambda^n(x)) = \Psi^n(x)$$

for every x. From the definition of the $\Psi(x, n)$ it is clear that they are additive in x, and thus so are the $\Psi^n(x)$. Since we have already proved the proposition for positive virtual characters χ_V, the claim for arbitrary virtual characters now follows immediately. □

Corollary 2.7.9. *The Adams operation Ψ^k is a ring homomorphism for $k \geq 0$ and we have $\Psi^k\Psi^l = \Psi^{kl}$.*

Proof. The ring structure on $C^0(G;\mathbb{C})$ is given by pointwise addition and multiplication in $\mathbb{C}$, and it is clear that precomposition with the function $G \to G$, $g \mapsto g^k$ preserves the pointwise addition and multiplication. The second statement also follows since $x(g^{kl}) = x((g^k)^l)$. $\square$

Proposition 2.7.10. *For compact Lie groups G and H, there is a canonical isomorphism of rings*

$$R(G) \otimes R(H) \xrightarrow{\cong} R(G \times H).$$

Proof. By the universal properties of $R(G)$ and $R(H)$, it suffices to say what the map does on elementary tensors of the form $V \otimes W \in R(G) \otimes R(H)$, which we send to the $(G \times H)$-representation $V \otimes W$. Notice that the LHS is the free abelian group generated by $V \otimes W$ for $V \in \mathrm{Irr}_G(\mathbb{R})$ and $W \in \mathrm{Irr}_H(\mathbb{R})$, while the RHS is the free abelian group generated by the irreducible $(G \times H)$-representations, and thus it follows from Proposition 2.4.18 that this map is an isomorphism of abelian groups. Since the ring structure on both sides is given by tensor product of representations, we see that it is even an isomorphism of rings. $\square$

Definition 2.7.11. We let $R(G,\mathbb{R})$ denote the character ring of *real characters*, i.e. the subring of $C^0(G;\mathbb{R})$ generated by characters χ_V of real G-representations V. Other notations for this ring are $\mathrm{RO}(G)$ and $K_{\mathbb{R}}(G)$.

The ring $R(G,\mathbb{R})$ admits a similar universal property as $R(G) = R(G,\mathbb{C})$, and it also comes with λ^i and Ψ^i operations.

If we try to do the same thing with quaternionic representations, we see that these only form an abelian group $R(G,\mathbb{H})$: the tensor product of two quaternionic representations is only a real representation. Each of the maps appearing in Proposition 2.6.8 preserves direct sums, and therefore induce maps of abelian groups between the corresponding representation rings (/groups), so that we get maps like

$$r^{\mathbb{C}}_{\mathbb{R}} \colon R(G,\mathbb{C}) \to R(G,\mathbb{R}), \qquad e^{\mathbb{C}}_{\mathbb{R}} \colon R(G,\mathbb{R}) \to R(G,\mathbb{C}), \qquad \text{etcetera...}$$

Each of the relations from Proposition 2.6.8 remains valid. From the relations $r^{\mathbb{C}}_{\mathbb{R}} e^{\mathbb{C}}_{\mathbb{R}} = 2$ and $e^{\mathbb{H}}_{\mathbb{C}} r^{\mathbb{H}}_{\mathbb{C}} = 2$ we conclude:

Lemma 2.7.12. *The maps*

$$e^{\mathbb{C}}_{\mathbb{R}} \colon R(G,\mathbb{R}) \to R(G,\mathbb{C}) \qquad \text{and} \qquad r^{\mathbb{H}}_{\mathbb{C}} \colon R(G,\mathbb{H}) \to R(G,\mathbb{C})$$

are injective. $\square$

As an immediate corollary, we see:

Corollary 2.7.13. *Let U_1 and U_2 be real G-representations such that $U_1 \otimes_{\mathbb{R}} \mathbb{C} \cong U_2 \otimes_{\mathbb{R}} \mathbb{C}$. Then $U_1 \cong U_2$.* $\square$.

The assignments $G \mapsto R(G) = R(G,\mathbb{C})$ and $G \mapsto \mathrm{RO}(G) = R(G,\mathbb{R})$ may be considered as functors: if we are given a group homomorphism $\alpha \colon H \to G$, then the assignment $V \mapsto \alpha^* V$ from Definition 2.1.22 induces ring homomorphisms

$$\alpha^* \colon R(G) \to R(H) \qquad \text{and} \qquad \alpha^* \colon \mathrm{RO}(G) \to \mathrm{RO}(H).$$

Note that $(\mathrm{id})^* = \mathrm{id}$ and $(\beta\alpha)^* = \alpha^*\beta^*$ so that this indeed defines a functor from compact Lie groups to commutative rings.

3 The Peter-Weyl Theorem

The content of this chapter is based on Bröcker and tom Dieck [Bt95, Chapter III] and Bump [Bum13, Part I].

The individual entries of a matrix representation of a compact Lie group G are continuous functions on G. They generate the ring of representative functions. The celebrated theorem of Peter and Weyl says that the representative functions are dense in the space of continuous functions. This result will be proved in Section 3.3 with the help of some functional analysis that will be reviewed in Section 3.2. We start in Section 3.1 with a useful alternative characterization of representative functions in terms of G-invariant finite-dimensional subspaces of the ring of continuous functions on G.

In Section 3.4 we will discuss various applications of the Peter-Weyl theorem, including the fact that every compact Lie group is a closed subgroup of some unitary group $U(n)$.

Throughout the chapter, G will be a compact Lie group.

3.1 Representative functions

Let us recall the definition of a representative function, which we already briefly encountered in Section 2.4 during the discussion of orthogonality relations.

Definition 3.1.1. A function $f\colon G \to \mathbb{C}$ is called a *representative function* if it is of the form

$$f(g) = \varphi(gv),$$

where V is a (finite-dimensional complex) G-representation, $v \in V$ is a vector and $\varphi\colon V \to \mathbb{C}$ is a dual vector.

We denote by $\mathcal{F}(G) = \mathcal{F}(G,\mathbb{C})$ the subvectorspace of $C(G) = C^0(G;\mathbb{C})$ consisting of the representative functions.

Example 3.1.2. Given a matrix representation $(r_{ij}(g))$ of G, each of the matrix coefficients r_{ij} is a representative function: taking $v = e_j \in \mathbb{C}^n$ and $\varphi = e_i^* \in (\mathbb{C}^n)^*$, we have

$$r_{ij}(g) = e_i^*(ge_j) = \varphi(gv).$$

Lemma 3.1.3. *Representative functions are closed under (pointwise) addition, multiplication and conjugation in $C(G)$. In particular, they form a subring of $C(G)$.*

Proof. Consider two representative functions $f(g) = \varphi(gv)$ and $f'(g) = \varphi'(gv')$, based on two G-representations V and V'. Then:

- The sum $f+f'$ is represented by the direct sum $V\oplus V'$, with vector (v,v') and dual vector $\varphi+\varphi'\colon V\oplus V'\to\mathbb{C}$:

$$(\varphi+\varphi')(g(v,v')) = (\varphi+\varphi')(gv,gv') = \varphi(gv)+\varphi'(gv') = f(g)+f'(g) = (f+f')(g).$$

- The product $f\cdot f'$ is represented by the tensor product $V\otimes V'$, with vector $v\otimes v'$ and dual vector $\varphi\cdot\varphi'\colon V\otimes V'\to\mathbb{C}$:

$$(\varphi\cdot\varphi')(g(v\otimes v')) = (\varphi\cdot\varphi')(gv\otimes gv') = \varphi(gv)\cdot\varphi'(gv') = f(g)\cdot f'(g) = (f\cdot f')(g).$$

- The conjugate function $\overline{f}$ is represented by the conjugate representation $\overline{V}$, with vector $v\in\overline{V}$ and with dual vector $\overline{\varphi}\colon\overline{V}\to\mathbb{C}$.

Note that also the constant functions $0\colon G\to\mathbb{C}$ and $1\colon G\to\mathbb{C}$ are representative functions by taking $V=\mathbb{C}$ with the trivial action, taking $v=1\in\mathbb{C}$ and taking $\varphi\colon\mathbb{C}\to\mathbb{C}$ to be the zero map or the identity map, respectively. $\square$

Corollary 3.1.4. *Sending a pair (φ,v) to the representative function $g\mapsto\varphi(gv)$ determines an injection*

$$s_V\colon V^*\otimes V\hookrightarrow\mathcal{F}(G).$$

Proof. The given assignment determines a map $V^*\times V\to\mathcal{F}(G)$ which is linear in both φ and v, so extends uniquely to a map $s_V\colon V^*\otimes V\to\mathcal{F}(G)$. By choosing a basis of V, it follows from the orthogonality relations of matrix coefficients, Remark 2.4.8, that this map is injective. $\square$

Definition 3.1.5. We denote the image of s_V by $\mathcal{F}_V(G)\subseteq\mathcal{F}(G)$, and refer to its elements as *representative functions coming from V*.

By the orthogonality relations from Remark 2.4.8 we see that $\mathcal{F}_V(G)$ is orthogonal in $\mathcal{F}(G)$ to $\mathcal{F}_W(G)$ whenever V and W are non-isomorphic irreducible representations.

Example 3.1.6. The character χ_V of a G-representation V is a representative function in $\mathcal{F}_V(G)$: choosing a basis of V we may write it as $\chi_V=\sum_{i=1}^n r_{ii}$.

Definition 3.1.7. The group G acts from the left and the right on itself. This induces two left actions of G on the space $C(G)$: given $f\in C(G)$ and $h\in G$ we define two new functions $\lambda(h)f$ and $\rho(h)f$ by

$$(\lambda(h)f)(g) := f(h^{-1}g), \qquad (\rho(h)f)(g) := f(gh).$$

Here we use h^{-1} in the definition of λ to guarantee that λ is still a left action and not a right action.

Lemma 3.1.8. *Let $f\colon G\to\mathbb{C}$ be a representative function coming from a G-representation V. Then for every $h\in G$ also the left-translation $\lambda(h)f$ and right-translation $\rho(h)f$ are representative functions coming from V.*

Proof. By picking a basis, we may assume V is given by a matrix representation $(r_{ij}(g))$. By matrix multiplication, we have the relation

$$r_{ij}(gh) = \sum_k r_{ik}(g)r_{kj}(h).$$

It follows that if f is a linear combination of the matrix coefficients $r_{ij}(-)$, then the assignment $g \mapsto f(gh)$ is still a linear combination of the matrix coefficients $r_{ik}(-)$, and thus again a representative function coming from V. A similar reasoning works for $f(h^{-1}g)$ as well. $\square$

Theorem 3.1.9. *Consider a continuous function $f: G \to \mathbb{C}$. The following are equivalent:*

(1) The left-translations $\lambda(h)f$ for $h \in G$ span a finite-dimensional subspace of $C(G)$;

(2) The right-translations $\rho(h)f$ for $h \in G$ span a finite-dimensional subspace of $C(G)$;

(3) The function f is a representative function.

Proof. Assume first that (3) holds, so that $f \in \mathcal{F}_V(G)$ for a certain G-representation V. Then also $\lambda(h)f \in \mathcal{F}_V(G)$ and $\rho(h)f \in \mathcal{F}_V(G)$ by Lemma 3.1.8. It follows that the subspaces they generate are contained in $\mathcal{F}_V(G)$. But the dimension of $\mathcal{F}_V(G)$ is $\dim(V)^2$, and in particular is finite. This proves that (3) implies (1) and (2).

Conversely, if (2) holds, we may define V as the finite-dimensional subspace of $C(G)$ spanned by the functions $\rho(h)f$ for $h \in G$. We define the dual vector $\varphi: V \to \mathbb{C}$ by $\varphi(f) := f(e)$. Then $f(g) = f(eg) = \varphi(\rho(g)f)$, so f is a representative function, showing that (2) implies (3). Similarly (1) implies (3) by applying the argument to $\tilde{f}(g) := f(g^{-1})$. $\square$

3.2 Some analysis on compact groups

We are interested in the space $C(G) = C^0(G;\mathbb{C})$ of continuous functions on G. We may equip $C(G)$ with the supremum norm

$$|u|_\infty = \sup\{|u(g)| \mid g \in G\}.$$

We may also consider the norm coming from the inner product on $C(G)$:

$$\langle u, v\rangle := \int_G u(g)\overline{v(g)}\,dg, \qquad |u|_2 := \sqrt{\langle u, u\rangle}.$$

While $C(G)$ is complete with respect to $|-|_\infty$, it is not complete with respect to $|-|_2$: its completion is known as the space $L^2(G)$ of *square-integrable functions on* G:

$$L^2(G) := \{f: G \to \mathbb{C} \mid \int_G |f|^2\,dg < \infty\}.$$

The inner product on $C(G)$ extends to an inner product on $L^2(G)$ given by the same formula, making $L^2(G)$ into a Hilbert space. Similarly, the left-actions λ and ρ of G on $C(G)$ defined in Definition 3.1.7 directly extend to $L^2(G)$. Since $L^2(G)$ is the completion of $C(G)$, it follows that $C(G)$ is dense in $L^2(G)$.

We will now recall some notions from functional analysis, just enough to state the *spectral theorem for compact operators*.

Definition 3.2.1. Recall that a *Hilbert space* is a complex vector space $\mathcal{H}$ equipped with a Hermitian inner product $\langle -,-\rangle: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $\mathcal{H}$ is complete with respect to the induced metric $d(x,y) := |x-y|$ on $\mathcal{H}$, meaning that any Cauchy sequence in $\mathcal{H}$ converges.

Given another Hilbert space $\mathcal{H}'$, a linear map (or 'linear operator') $T: \mathcal{H} \to \mathcal{H}'$ is called *bounded* if there exists a constant A such that $|Tx| \leq A \cdot |x|$ for all $v \in \mathcal{H}$.

One can prove that a linear operator T is continuous with respect to the metric topologies on $\mathcal{H}$ and $\mathcal{H}'$ if and only if it is bounded. For a linear operator $T\colon \mathcal{H} \to \mathcal{H}'$, we define its *operator norm* $|T|_\infty$ as

$$|T|_\infty := \sup\{|Tx| \mid x \in \mathcal{H}, |x| = 1\}.$$

Note that T is bounded if and only if $|T|_\infty < \infty$.

Definition 3.2.2. We say a bounded operator T is *compact* if for every bounded sequence $(x_1, x_2, x_3, \dots)$ in $\mathcal{H}$, the sequence $(Tx_1, Tx_2, \dots)$ in $\mathcal{H}'$ has a convergent subsequence.

A linear operator $T\colon \mathcal{H} \to \mathcal{H}$ is called *self-adjoint* if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{H}$.

Lemma 3.2.3. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator. Then all eigenvalues of T are real and all eigenspaces of T are orthogonal to each other.*

Proof. If λ is an eigenvalue of T with eigenvector T, we observe that

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle$$

so that $\lambda = \overline{\lambda}$ and thus $\lambda \in \mathbb{R}$. Furthermore, for two distinct eigenvalues λ and μ with eigenvectors x and y, respectively, we see that

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \mu\langle x, y\rangle,$$

which implies that $\langle x, y\rangle = 0$ and thus x and y are orthogonal to each other. □

In the context of Hilbert spaces, we have the following notion of direct sums of subspaces:

Definition 3.2.4. Let $\mathcal{H}$ be a Hilbert space and let $(\mathcal{H}_i)_{i\in I}$ a collection of subspaces of $\mathcal{H}$ such that $\mathcal{H}_i \perp \mathcal{H}_j$ for all $i \neq j$. We define the *direct sum* $\widehat{\bigoplus}_{i\in I}\mathcal{H}_i \subseteq \mathcal{H}$ as

$$\widehat{\bigoplus}_{i\in I}\mathcal{H}_i := \left\{ \sum_{i\in I} x_i \in \mathcal{H} \;\middle|\; x_i \in \mathcal{H}_i, \sum_{i\in I} |h_i| < \infty \right\}.$$

Note that $\widehat{\bigoplus}_{i\in I}\mathcal{H}_i$ is the closure in $\mathcal{H}$ of the group-theoretic direct sum $\bigoplus_{i\in I}\mathcal{H}_i$.

The following theorem will be treated as a black box:

Theorem 3.2.5 (Spectral theorem for compact operators, Bump [Bum13, Theorem 3.1], Bröcker and tom Dieck [Bt95, Theorem 2.6])**.** *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a compact self-adjoint operator with eigenspaces $\mathcal{H}_\lambda$ for $\lambda \in \mathbb{C}$. Then for every $\varepsilon > 0$ the subspace*

$$\bigoplus_{|\lambda|\geq\varepsilon} \mathcal{H}_\lambda$$

is finite-dimensional. Furthermore, $\mathcal{H}$ is the direct sum of all the subspaces $\mathcal{H}_\lambda$:

$$\mathcal{H} = \widehat{\bigoplus}_{\lambda\in\mathbb{C}}\mathcal{H}_\lambda.$$

Note in particular that for every $\lambda \neq 0$ the eigenspace $\mathcal{H}_\lambda$ is finite-dimensional. Also note that for a fixed size $\varepsilon > 0$ there are only finitely many eigenvalues λ of T such that $|\lambda| > \varepsilon$. Consequently, T only has a countable number of eigenvalues.

In the next section, we will need to apply the spectral theorem to convolution operators:

Definition 3.2.6. For a function $\varphi \in C(G)$, we define the *convolution operator* $T_\varphi\colon C(G) \to C(G)$ as follows:

$$T_\varphi f(h) = \int_G \varphi(hg^{-1}) f(g) dg.$$

Observe that this operator is continuous with respect to the $|-|_2$-norm, so that it uniquely extends to an operator $T_\varphi\colon L^2(G) \to L^2(G)$.

Proposition 3.2.7 (Bröcker and tom Dieck [Bt95, Section III.2]). *For a function $\varphi \in C(G)$, the operator $T_\varphi\colon L^2(G) \to L^2(G)$ is a bounded G-equivariant compact operator on $L^2(G)$. It satisfies*

$$|T_\varphi f|_\infty \leq |\varphi|_\infty |f|_2.$$

If $\varphi(g) = \overline{\varphi(g^{-1})}$, then T_φ is self-adjoint, and in particular the Spectral Theorem applies.

Sketch. We will also take this proposition as a black box, and refer to the given reference for a proof. But let us at least sketch the easy parts of the proposition. The fact that T_φ is equivariant follows directly from the invariance of the integral. For the stated inequality, we compute

$$|T_\varphi f|_\infty \leq \int_G |\varphi(hg^{-1})||f(g)|\, dg \leq |\varphi|_\infty \cdot \int_G |f(g)|\, dg \leq |\varphi|_\infty |f|_2.$$

Since $|T_\varphi f|_2 \leq |T_\varphi f|_\infty$, this proves in particular that T_φ is bounded with operator norm at most $|\varphi|_\infty$. Self-adjointness of T_φ in case that $\varphi(g) = \overline{\varphi(g^{-1})}$ is also easy to verify. Only the compactness of T_φ remains. This uses a theorem of Ascoli that we will not go into. □

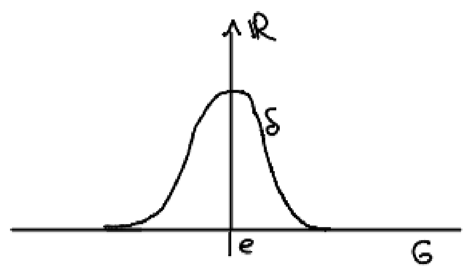
3.3 The Peter-Weyl theorem

We will now state and prove the Peter-Weyl theorem.

Theorem 3.3.1 (Peter-Weyl, part I). *The subspace $\mathcal{F}(G)$ of representative functions is dense in $C(G)$ (and hence also dense in $L^2(G)$).*

Proof. Let $f \in C(G)$. We will prove that for every $\varepsilon > 0$ there exists a representative function f' with $|f - f'|_\infty < \varepsilon$. Since G is compact, f is uniformly continuous, which means that there exists an open neighborhood $U \subseteq G$ of the identity such that for $g \in U$ we have $|\lambda(g)f - f| < \varepsilon$.

Let $\delta\colon G \to \mathbb{R}_{\geq 0}$ be a nonnegative function supported in U such that $\int_G \delta(g)dg = 1$. We may choose δ such that $\delta(g) = \delta(g^{-1})$.



Consider the convolution operator $T_\delta\colon C(G) \to C(G)$ which sends f to $T_\delta f$, defined by

$$T_\delta f(h) = \int_G \delta(g) f(gh)\, dg = \int_G \delta(hg^{-1}) f(g)\, dg.$$

We claim that $|T_\delta f - f|_\infty < \varepsilon$: for $h \in G$ we have

$$\begin{aligned} |T_\delta f(h) - f(h)| &= \left| \int_g (\delta(g) f(gh) - \delta(g) f(h))\, dg \right| \\ &\leq \int_U \delta(g) |(f(gh) - f(h))| dg \\ &\leq \int_U \delta(g) |\lambda(g) f - f|_\infty dg \\ &\leq \int_U \delta(g) \varepsilon dg = \varepsilon. \end{aligned}$$

So it will suffice to approximate $T_\delta f$ by a representative function.

By Proposition 3.2.7, the Spectral Theorem applies to the operator $T_\delta\colon L^2(G) \to L^2(G)$. If λ is an eigenvalue of T_δ, let $V(\lambda) \subseteq L^2(G)$ denote the λ-eigenspace. These subspaces are mutually orthogonal, and $L^2(G)$ is the direct sum of the $V(\lambda)$ as a Hilbert space. They are closed under the G-action, as T_δ is G-equivariant. Furthermore, $V(\lambda)$ is finite-dimensional when $\lambda \neq 0$.

Let f_λ be the projection of f onto $V(\lambda)$, so that $f = \sum_\lambda f_\lambda$. As all the f_λ are orthogonal, we have

$$\sum_\lambda |f_\lambda|_2^2 = |f|_2^2 < \infty,$$

and thus we may choose $q > 0$ such that

$$\left| \sum_{0<|\lambda|<q} f_\lambda \right|_2 \leq \sqrt{\sum_{0<|\lambda|<q} |f_\lambda|_2^2} < \frac{\varepsilon}{|\delta|_\infty}.$$

So if we define $f' := \sum_{|\lambda|>q} T_\delta(f_\lambda)$, we get

$$|T_\delta(f) - f'|_\infty = \left| \sum_{0<|\lambda|<q} T_\delta f_\lambda \right|_\infty = \left| T_\delta \left(\sum_{0<|\lambda|<q} f_\lambda \right) \right|_\infty \leq |\delta|_\infty \left| \sum_{0<|\lambda|<q} f_\lambda \right|_2 < \varepsilon.$$

It thus remains to show that $f' = \sum_{|\lambda|>q} T_\delta(f_\lambda)$ is a representative function. But since representative functions are closed under sums, it suffices to show that each $T_\delta(f_\lambda)$ is a representative function. For this we use the characterization of Theorem 3.1.9: we need to show that the left-translations of $T_\delta(f_\lambda)$ span a finite-dimensional subspace of $L^2(G)$. As T_δ is G-equivariant, we may also prove this for the left-translations of f_λ. But these are all contained in the finite-dimensional eigenspace $V(\lambda)$, hence we are done. □

Corollary 3.3.2. *The characters $\chi_V\colon G \to \mathbb{C}$ of irreducible representations V generate a dense subspace of the space of continuous class functions $G \to \mathbb{C}$.*

Proof. Let $\varphi\colon G \to \mathbb{C}$ be a class function, and let $\varepsilon > 0$. By the Peter-Weyl theorem, we may find a representative function f with $|\varphi - f| < \varepsilon$. Defining $\psi(g) := \int_G f(hgh^{-1}) dh$, we see that also

$|\varphi - \psi| < \varepsilon$. So it suffices to show that ψ is a linear combination of irreducible characters. By assumption, we may write $f(g) = \sum_{i=1}^{n} f_i(gv_i)$ for some irreducible representations V_i, $v_i \in V_i$ and $f_i \colon V_i \to \mathbb{C}$. But then we have $\psi(g) = \sum_i \int_G f_i(hgh^{-1}v_i)dh$, and by Proposition 2.4.4 we see that $\int_G f_i(hgh^{-1}v_i)dh = \frac{1}{|V_i|} f_i(v_i)\chi_{V_i}(g)$. So ψ is a linear combination of irreducible characters as desired. □

3.4 Applications of the Peter-Weyl theorem

We will now discuss various applications of the Peter-Weyl theorem. We start with the following useful lemma:

Lemma 3.4.1. *Let G be a compact Lie group. Then for every $1 \neq g \in G$ there exists an irreducible G-representation V such that g acts non-trivially on V.*

Proof. Let $f \colon G \to \mathbb{C}$ be a continuous function with $f(g) \neq f(1)$. By the Peter-Weyl theorem we can find a representative function $u \colon G \to \mathbb{C}$ with $u(g) \neq u(1)$. Since u is a representative function, there exists a representation V of G together with a vector $v \in V$ and a dual vector $\varphi \in V^*$ such that $u(h) = \varphi(hv)$ for all $h \in G$. Since $u(g) \neq u(1)$, it in particular follows that $gv \neq 1v = v$, so that g acts non-trivially on V. □

Abelian compact Lie groups

Lemma 3.4.2. *A compact Lie group G is abelian if and only if every irreducible representation of G is one-dimensional.*

Proof. We showed in Proposition 2.1.17 that every irreducible representation of an abelian Lie group is one-dimensional. Conversely, assume for contradiction that G is not abelian, so that there exists a non-trivial commutator $x = ghg^{-1}h^{-1} \neq 1$ in G. By the lemma, there exists an irreducible representation V on which x acts nontrivially. In particular, x is not contained in the kernel of the map $G \to \mathrm{GL}(V) \cong \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. But this is a contradiction: this is a group homomorphism to the abelian group $\mathbb{C}^\times$, and hence sends the commutator $x = ghg^{-1}h^{-1}$ to 1. □

Faithful representations

Definition 3.4.3. We say that a G-representation V is *faithful* if the associated map $G \to \mathrm{Aut}(V)$ is injective.

To show that faithful representations always exist, we will make use of the following *descending chain property*:

Lemma 3.4.4. *Let G be a compact Lie group and let*

$$\cdots \subseteq K_3 \subseteq K_2 \subseteq K_1 \subseteq G$$

be a nested sequence of closed subgroups of G. Then this sequence is eventually constant, meaning that there is some $N \in \mathbb{N}$ such that $K_i = K_N$ for all $i \geq N$.

Proof. We have seen in Theorem 1.3.17 that every K_i is a closed submanifold of G. If $K_{i+1} \neq K_i$, we claim that either $\dim K_{i+1} < \dim K_i$ or that K_{i+1} has strictly fewer components than K_i. Indeed, if $\dim K_{i+1} = \dim K_i$, then K_{i+1} contains an open neighborhood of $e \in K_i$, and hence by Lemma 1.3.5 contains the entire connected component of $e \in K_i$. But this implies that K_{i+1} is simply a disjoint union of components of K_i, so since it is not equal to K_i it must have strictly fewer components.

The result now follows: since G has finite dimension there can only be finitely many inclusions $K_{i+1} \subseteq K_i$ in which the dimension goes down, and in every dimension there can be only finitely many inclusions $K_{i+1} \subseteq K_i$ in which the number of components goes down, so after some finite number of steps the process needs to stabilize. □

Proposition 3.4.5. *Every compact Lie group admits a (finite-dimensional) faithful representation.*

Proof. The statement is clear if G is the trivial group, so assume it is not and let $g \in G \setminus \{1\}$. By Lemma 3.4.1, there exists a representation V_1 of G on which g acts non-trivially. In particular, the kernel K_1 of the map $G \to \mathrm{Aut}(V_1)$ is a proper subgroup of G.

If K_1 is not the trivial group, we may choose $k_1 \in K_1 \setminus \{1\}$ and similarly obtain a G-representation V_2 such that k_1 is not in the kernel K_2 of $G \to \mathrm{Aut}(V_2)$. Since $k_1 \in K_1 \setminus K_2$, we see that the intersection $K_1 \cap K_2$ is strictly contained in K_1. If $K_1 \cap K_2$ is not the trivial group, we may choose $k_2 \in (K_1 \cap K_2) \setminus \{1\}$, etcetera. Continuing in this fashion, we get a nested sequence

$$\bigcap_{i=1}^{n} K_i \subsetneq \cdots \subsetneq K_1 \cap K_2 \subsetneq K_1 \subsetneq G$$

of strict inclusions of closed subgroups of G. By the descending chain property, this process must stabilize after a finite number of steps, meaning that we must reach $\bigcap_{i=1}^{n} K_i = \{1\}$ for some $n \in \mathbb{N}$. If we now consider the direct sum representation $V := \bigoplus_{i=1}^{n} V_i$, we see that the kernel of $G \to \mathrm{Aut}(V)$ is contained in $\bigcap_{i=1}^{n} K_i = \{1\}$. It follows that V is a faithful G-representation. □

Corollary 3.4.6. *Every compact Lie group is a closed subgroup of a matrix group* $\mathrm{U}(n) \subseteq \mathrm{O}(2n)$.

Proof. By the previous proposition we may pick a faithful G-representation V. By choosing a G-invariant inner product on V and choosing an orthonormal basis with respect to this inner product, we obtain the desired inclusion $G \hookrightarrow \mathrm{U}(n)$ for $n = \dim(V)$. □

Unitary representations on Hilbert spaces

Definition 3.4.7. If $\mathcal{H}$ is a Hilbert space, then we say a representation $\pi\colon G \to \mathrm{End}(\mathcal{H})$ is *unitary* if $\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle$ for all $v, w \in \mathcal{H}$ and $g \in G$.

Theorem 3.4.8 (Peter-Weyl, part II)**.** *Let $\mathcal{H}$ be a Hilbert space and G be a compact group. Let $\pi\colon G \to \mathrm{End}(\mathcal{H})$ be a unitary representation. Then $\mathcal{H}$ is a direct sum of finite-dimensional irreducible subrepresentations.*

Proof. Step 1: We start by showing that $\mathcal{H}$ has a finite-dimensional invariant subspace. Since $\mathcal{H}$ is non-zero, we may pick a non-zero vector $v \in \mathcal{H}$. Using compactness of G, we may find a neighborhood N of the identity of G such that if $g \in N$ then $|gv - v| \leq |v|/2$. Let $\varphi\colon G \to \mathbb{R}_{\geq 0}$ be a non-negative continuous function on G supported in N and satisfying $\int_G \varphi(g)dg = 1$. We claim that $\int_G \varphi(g)gv\,dg \neq 0$. To see this, we will show that its inner product with v is non-zero. Note that

$$\left\langle \int_G \varphi(g)gv\,dg, v \right\rangle = \langle v, v\rangle + \left\langle \int_N \varphi(g)(gv - v)\,dg, v \right\rangle.$$

This is non-zero, as the size of $\langle v, v\rangle$ is $|v|^2$, while the size of the second term is

$$\left|\left\langle \int_N \varphi(g)(gv - v)\,dg, v \right\rangle\right| \leq \int_N |\varphi(g)||gv - v||v|\,dg \leq \int_N \varphi(g)|v|^2/2\,dg = |v|^2/2.$$

Next, using the Peter–Weyl theorem, we may find a representative function f such that $|f - \varphi|_\infty < \varepsilon$, where ε can be chosen arbitrarily. Since we have

$$\left|\int_G (f - \varphi)(g) \cdot gv\,dg\right| \leq \varepsilon|v|,$$

we may pick ε sufficiently small to guarantee that $\int_G f(g)gv\,dg \neq 0$.

Since f is a representative function, also the function $g \mapsto f(g^{-1})$ is a representative function, and hence there is a finite-dimensional G-representation V with a vector $v \in V$ and a dual vector $\varphi\colon V \to \mathbb{C}$ such that $f(g^{-1}) = \varphi(gv)$. Define a map $T\colon V \to \mathcal{H}$ by

$$T(x) := \int_G \varphi(g^{-1}x)gv\,dg.$$

It follows from G-invariance of the integral that T is G-equivariant. It is non-zero, since $T(v) = \int_G f(g)gv\,dg \neq 0$. Since V is finite-dimensional, the image of T is a finite-dimensional subrepresentation of $\mathcal{H}$, as desired.

Step 2: We have proven that every nonzero unitary representation of G has a nonzero finite-dimensional invariant subspace, and by passing to a smaller subspace if necessary we may assume it to be irreducible. From this we deduce the stated result. Let $(\mathcal{H}, \pi)$ be a unitary representation of G, and let Σ be the set of all sets of orthogonal finite-dimensional irreducible invariant subspaces of $\mathcal{H}$, ordered by inclusion. More explicitly, if $S \in \Sigma$ and $U, V \in S$, then U and V are finite-dimensional irreducible invariant subspaces, and if $U \neq V$ then $U \perp V$. By Zorn's lemma, Σ has a maximal element S and we are done if S spans $\mathcal{H}$ as a Hilbert space. Otherwise, let $\mathcal{H}'$ be the orthogonal complement of the span of S. By what we have shown, $\mathcal{H}'$ contains an invariant irreducible subspace. We may append this subspace to S, contradicting its maximality. $\square$

Theorem 3.4.9 (Peter-Weyl, part III)**.** *Every irreducible unitary representation of G is contained in the regular representation $L^2(G)$ of G. More precisely, $L^2(G)$ is the direct sum $\widehat{\bigoplus}_{V \in \mathrm{Irr}_G(\mathbb{C})} V^{\dim V}$, where $\mathrm{Irr}_G(\mathbb{C})$ is the set of isomorphism classes of irreducible G-representations.*

Proof. For an irreducible unitary G-representation V, recall that $\mathcal{F}_V(G) \subseteq C(G) \subseteq L^2(G)$ denotes the subspace of representative functions of V. It follows from the Peter Weyl theorem, Part I, that

$$L^2(G) \cong \widehat{\bigoplus}_{V \in \mathrm{Irr}_G(\mathbb{C})} \mathcal{F}_V(G).$$

So it remains to show that $\mathcal{F}_V(G) \cong V^{\dim V}$. For every $w \in V$, consider the map

$$V \to \mathcal{F}_V(G); \quad v \mapsto (g \mapsto \langle gv, w\rangle).$$

which sends a vector v to the corresponding representative function. One can check that this is G-equivariant. If $w \neq 0$, it is not the zero-map, and thus necessarily injective since V is irreducible. Thus every non-zero vector $w \in V$ gives an embedding of V into $\mathcal{F}_V(G)$. If $w_1 \perp w_2$, these two embeddings can be seen to be orthogonal in $\mathcal{F}_V(G)$. By choosing an orthogonal basis of V we may thus find $\dim(V)$ copies of V inside $\mathcal{F}_V(G)$. By dimension reasons we must have $\mathcal{F}_V(G) \cong V^{\dim V}$. $\square$

4 The maximal torus of a compact Lie group

The content of this chapter is based on Bröcker and tom Dieck [Bt95, Chapter IV] and Bump [Bum13, Chapter 16].

In this chapter, we show that every connected compact Lie group G contains a 'maximal torus' T. This maximal torus is unique up to conjugation, and every element of G is contained in some conjugate of G. If we let N denote the normalizer of T in G, then the Weyl group $W := N/T$ of T is a finite group that acts transitively on T. It follows that there is a one-to-one correspondence between class functions on G and functions on T which are invariant under the action of W. In particular, the characters of G are W-invariant characters of T. For this reason, understanding the action of the Weyl group on the torus is important to representation theory. The maximal torus also plays an important role in the structure theory of connected compact Lie groups in Chapter 5.

4.1 Maximal tori

Throughout this section, we let G be a compact connected Lie group.

Definition 4.1.1. Recall that a Lie group is called a *torus* if it is isomorphic to $T^k = \mathbb{R}^k/\mathbb{Z}^k$ for some $k \geq 0$. A closed subgroup $T \subseteq G$ is called a *maximal torus* if it is a torus and if there is no other torus $T' \subseteq G$ such that $T \subsetneq T'$.

Remark 4.1.2. A maximal torus exists for every G. To see this, let T and T' be two tori in G such that $T \subsetneq T'$. Since T' is connected, it follows that $\dim(T) < \dim(T')$. But then if we pick a torus T in G with maximal possible dimension, we see that it must be maximal.

The same reasoning shows that in fact any torus in G is contained in a maximal torus.

Remark 4.1.3. A maximal torus is the same thing as a maximal connected abelian subgroup, since the closure of such a subgroup is a connected abelian compact Lie group and hence a torus by Theorem 1.3.8.

Recall that an automorphism $\varphi\colon T^k \to T^k$ induces a commutative diagram of homomorphisms

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^k \arrow[r, hook] \arrow[d, dashed, "\mathrm{L}\varphi|_{\mathbb{Z}^k}"'] & \mathbb{R}^k \arrow[r] \arrow[d, "\mathrm{L}\varphi"] & T^k \arrow[d, "\varphi"] \\ 0 \arrow[r] & \mathbb{Z}^k \arrow[r, hook] & \mathbb{R}^k \arrow[r] & T^k, \end{tikzcd}$$

where we identify T^k with $\mathbb{R}^k/\mathbb{Z}^k$ and $\mathrm{L}T^k$ with $\mathbb{R}^k$. Therefore, the automorphism φ corresponds to an invertible $(n\times n)$-matrix $\mathrm{L}\varphi$ with integral entries: we have an isomorphism

$$\mathrm{L}\colon \operatorname{Aut}(T^k) \cong \operatorname{Aut}(\mathbb{Z}^k) = \mathrm{GL}_k(\mathbb{Z}), \qquad \varphi \mapsto L\varphi|_{\mathbb{Z}^k}.$$

Definition 4.1.4. Let T be a maximal torus in G and let $N = N(T)$ be the normalizer of T in G, i.e.

$$N = \{g \in G \mid gTg^{-1} = T\}.$$

Then we define the *Weyl group of* G as the quotient group $W = N/T$.

Remark 4.1.5. Note that the definition of the Weyl group depends on the choice of a maximal torus T. We will see in Theorem 4.1.7 below that all maximal tori are conjugate, so that different choices of T yield isomorphic Weyl groups. However, when we talk about *the* maximal torus in G and *the* Weyl group of G, we have implicitly chosen a particular T which is to remain fixed throughout the discussion.

The normalizer N of T acts on T by conjugation:

$$N\times T \to T, \qquad (n,t)\mapsto ntn^{-1}.$$

Since T is abelian, the conjugation action of T on itself is trivial, and thus we obtain an induced action of the Weyl group on T:

$$W\times T \to T, \qquad (nT,t)\mapsto ntn^{-1}.$$

Theorem 4.1.6. *The Weyl group W is finite.*

Proof. Let N_0 be the connected component of the identity in N. We will show that $N_0 = T$. Since N is compact, it will follow that also the quotient $W = N/N_0$ is both compact and discrete, and hence finite.

We may view the action of N on T as a continuous map

$$N \to \operatorname{Aut}(T) \overset{\mathrm{L}}{\cong} \mathrm{GL}_k(\mathbb{Z}), \qquad n \mapsto \operatorname{Ad}(n)|_{\mathrm{L}T}.$$

Since the group $\mathrm{GL}_k(\mathbb{Z})$ is discrete and the subgroup $N_0 \subseteq N$ is connected, it follows that the image of N_0 in $\mathrm{GL}_k(\mathbb{Z})$ is just the identity matrix. In particular N_0 acts trivially on T, meaning that the elements of N_0 commute with the elements of T. Now, let $\alpha\colon \mathbb{R}\to N_0$ be a one-parameter group. Then the product $\alpha(\mathbb{R})\cdot T = \{\alpha(r)\cdot t \mid r\in\mathbb{R}, t\in T\} \subseteq G$ is a connected abelian Lie group containing T, hence by maximality we have $\alpha(\mathbb{R})\cdot T = T$ and in particular $\alpha(\mathbb{R})\subseteq T$. But the groups $\alpha(\mathbb{R})$ cover an open neighborhood of the identity in N_0 and since N_0 is connected they must therefore generate N_0 by Lemma 1.3.5. We conclude that $N_0 = T$. $\square$

The main goal of this chapter is to prove the following theorem due to Cartan:

Theorem 4.1.7 (Conjugation Theorem)**.** *Let G be a compact connected Lie group and let T be a maximal torus.*

(1) Any other maximal torus T' in G is conjugate to T: there exists $k\in G$ such that $T' = kTk^{-1}$.

(2) Any element g of G is contained in kTk^{-1} for some $k\in G$.

4.2 Geodesics on Riemannian manifolds

For the proof of the Conjugation Theorem, we will need a digression into *geodesics*.

Definition 4.2.1. A smooth manifold M is called a *Riemannian manifold* if it comes equipped with a *Riemannian structure*: a smooth family of inner products $\langle -,-\rangle\colon T_xM \times T_xM \to \mathbb{R}$ on the tangent spaces T_xM for all $x \in M$.

The smoothness condition on the inner products may be formulated most concretely by picking a local charts $\mathbb{R}^n \cong U \subseteq M$, so that $TM|_U \cong T\mathbb{R}^k = \mathbb{R}^n \times \mathbb{R}^n$. Denoting the components of $\mathbb{R}^n$ by $x_1, \dots, x_n$, we obtain for every point $x \in U$ a basis of $T_x(M) \cong T_x(\mathbb{R}^n)$ that we denote by $\partial/\partial x_1, \dots, \partial/\partial x_n$. We may then define the local functions

$$g_{ij} := \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \colon \qquad U \to \mathbb{R}$$

for $1 \leq i, j \leq n$. Smoothness of the inner product then means that the g_{ij} are smooth functions.

Definition 4.2.2. Let M be a Riemannian manifold. A path $p\colon [0,1] \to M$ is called *admissible* if it is smooth and the velocity vector $\dot{p}(t) := p_*(\frac{d}{dt})$, where t is the coordinate function on $[0,1]$, is never zero. In this case, we define the *length* or *arclength* of p as

$$|p| := \int_0^1 |\dot{p}(t)|, dt,$$

where the length of the velocity vector is defined in terms of the inner product on $T_{p(t)}$. If we choose local coordinates $x_1, \dots, x_n$ on M and write $p_i(t) = x_i(p(t))$, then the integrand is given by

$$|\dot{p}(t)| = \sqrt{\sum_{i,j} g_{ij} \frac{\partial p_i}{\partial t} \frac{\partial p_j}{\partial t}}.$$

We say that the path p is *well-paced* if we have

$$\int_0^a |\dot{p}(t)|\, dt = |p| a$$

for all $0 \leq a \leq 1$. Intuitively, this means that the point $p(t)$ moves along the path at a constant "velocity".

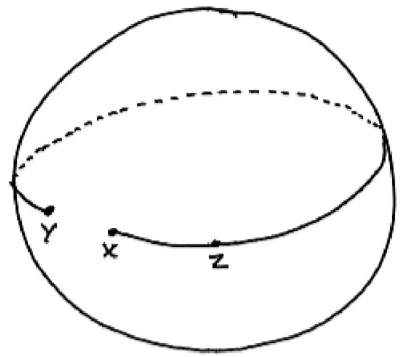
As a consequence of the chain rule, one observes that the length of p is unchanged under reparametrization. Moreover, every path has a unique reparametrization that is well-paced.

Definition 4.2.3. If M is a Riemannian manifold, we may equip it with a metric defined by

$$d(x,y) := \inf\{|p| \mid p\colon [0,1] \to M \text{ is an admissible path from } x \text{ to } y\}.$$

One can check that this indeed satisfies the triangle identity, and that it induces the standard topology on M. In general there might not be a *shortest* path between x and y.

Definition 4.2.4. If $p\colon [0,1] \to M$ is a smooth path such that for any other smooth path $q\colon [0,1] \to M$ with the same endpoints of p we have $|p| \leq |q|$, then we say that p is a *path of shortest length*.



We would now like to generalize the notion of paths of shortest length to that of a *geodesic*: paths that are not necessarily of shortest length, but which are at least "locally of shortest length", in the sense that if we restrict it to small enough intervals it defines a path of shortest length.

Let us illustrate this with an example. In the following figure, the path from x to y is clearly not of shortest length. However, it is still a geodesic: if we break it into small segments, then each of the individual segments is indeed of shortest length. For example, the segment from x to z is a path of shortest length.

Although the heuristic "locally of shortest length" is certainly valid, we will not take it as our formal definition. Instead, we use the following:

Definition 4.2.5. Consider an admissible path $p\colon [0,1] \to M$ from x to y. We say that p is a *geodesic* if the following condition is satisfied:

> Consider a smooth deformation of p, given by a family of paths $p_u\colon [0,1] \to M$ for $u \in (-\varepsilon, \varepsilon)$ with $p_0 = p$ and such that the map $(-\varepsilon,\varepsilon) \times [0,1] \to M$ is smooth. Assume that we have $p_u(0) = x$ and $p_u(1) = y$ for all $u \in (-\varepsilon,\varepsilon)$. Define the smooth function $f\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ by $f(u) := |p_u|$. Then we have $f'(0) = 0$.

In other words, a path is geodesic if its length does not change up to first order if we slightly deform the path.

Example 4.2.6. Every path of shortest length is in particular a geodesic: the function $f(u)$ will have a minimum at $u = 0$ by assumption on $p_0 = p$, and thus the derivative $f'(0)$ must be zero.

In particular, the geodesics in $\mathbb{R}^n$ with the standard Euclidean metric are just the straight lines.

Example 4.2.7. The path from x to y in the figure is not of shortest length, but it is still a geodesic. Even though we may deform the path by raising it up above the equator and simultaneously shrinking it, such a deformation will still satisfy $f'(0) = 0$.

The one main thing we will need about geodesics is the following powerful existence statement:

Theorem 4.2.8 ([Bum13, Theorem 16.1])**.** *Let M be a compact connected Riemannian manifold, and let x and y be points of M. Then there exists a geodesic $p\colon [0,1] \to M$ with $p(0) = x$ and $p(1) = y$.*

Proof sketch. Since the proof is fairly intricate, we will merely give a sketch and refer to [Bum13, Theorem 16.1] for the details.

Step 1: One starts by showing that for every point $x \in M$ and for every tangent vector $X \in T_x(M)$, there exists for sufficiently small ε a unique geodesic $p\colon (-\varepsilon, \varepsilon) \to M$ such that $p(0) = x$ and such that $\dot{p}(0) = X$. For this, one needs to use the theory of first-order differential equations: the property for a path p to be a geodesic can be phrased in local coordinates in terms of a differential equation, and then the general theory of first-order differential equations provides the existence and uniqueness of the solution to these equations.

Step 2: Using these locally defined geodesics, one may select specific choices of local coordinates known as *geodesic coordinates*. The special property about geodesic coordinates is that in these coordinates the geodesics out of the origin look like straight lines.

Step 3: One can then show that 'short' geodesics are paths of shortest length:

(i) If $p\colon [0,1] \to M$ is a geodesic, then there exists some $\varepsilon > 0$ such that the restriction of p to $[0, \varepsilon]$ is the unique path of shortest length from $p(0)$ to $p(\varepsilon)$.

(ii) For a point $x \in M$, there exists a neighborhood $N$of x in M such that for all $y \in N$ there exists a unique path of shortest distance from x to y, and that path is a geodesic.

Step 4: To finish the proof of the theorem, we now consider two points x and y of M. Since the distance $d(x, y)$ is defined as the infimum over all lengths of paths from x to y, we may find a sequence of paths p_i for $i \in \mathbb{N}$ such that the lengths $|p_i|$ converge to $d(x, y)$. Invoking some theory of equicontinuity and convergence, one can show that there is a subsequence that converges uniformly to some path p, which is a priori not known to be smooth again but will always be continuous. On sufficiently short intervals $0 \leq a < b \leq 1$, the points $p(b)$ and $p(a)$ are near enough so that the unique path of shortest distance between them is geodesic by Step 3. But since p minimizes the distance, it follows that p is locally a geodesic, and hence it is in fact a geodesic. □

4.3 Proof of the Conjugation Theorem

We will now use the above digression on geodesics to give a proof of the Conjugation Theorem, Theorem 4.1.7.

Theorem 4.3.1. *Let G be a compact Lie group. There exists a Riemannian metric on G that is invariant under both left and right translation. In this metric, a geodesic is a translate (either left or right) of a map $t \mapsto \exp(tX)$ for some $X \in \mathrm{L}G$.*

Proof. Recall the *adjoint representation* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{L}G)$, which provides a smooth action of G on its tangent space $\mathrm{L}G$. Since G is compact, we may choose some G-invariant inner product $\langle -, - \rangle$ on $\mathrm{L}G$. We saw in Observation 1.2.10 that for every $g \in G$ the map $l_g\colon G \to G$ induces an isomorphism $\mathrm{L}G \xrightarrow{\sim} T_g G$, and we may use this isomorphism to equip all the tangent spaces $T_g G$ with an inner product. The resulting inner products are smooth in G because we even have an isomorphism $G \times \mathrm{L}G = T_e G \xrightarrow{\cong} TG$ of vector bundles over G, and this equips G with the structure of a Riemannian manifold which is invariant under left translation.

Right translation by g induces a different isomorphism $\mathrm{L}G = T_e G \to T_g G$, but these two isomorphisms differ by the map $\mathrm{Ad}(g)\colon \mathrm{L}G \to \mathrm{L}G$ and since the original inner product on $\mathrm{L}G$ is G-invariant under $\mathrm{Ad}(g)$ we see that the Riemannian structure we have obtained on G is invariant under both left and right translation.

It remains to show that a geodesic is a translate of the exponential map. This is essentially a local statement: if we know that any short segment of a geodesic is of the form $t \mapsto g \cdot \exp(tX)$, then in fact the whole geodesic must be of this form (since on the intersections of the segments the g and X must agree). Furthermore, by translating the problems to the origin we may assume that the geodesic starts at $e \in G$ and show it is of the form $t \mapsto \exp(tX)$.

First, we consider the case where $G \cong T^k = \mathbb{R}^k/\mathbb{Z}^k$ is a torus. By a linear change of variables, we may assume that the inner product on $\mathbb{R}^n = T_e(G)$ corresponding to the Riemannian structure is the standard Euclidean inner product, at the cost of having to replace the subgroup $\mathbb{Z}^k$ by an arbitrary lattice $\Lambda \subseteq \mathbb{R}^k$. But in this case the geodesics in $\mathbb{R}^k$, and hence in the quotient $\mathbb{R}^k/\Lambda$, are simply the straight lines, see Example 4.2.6. In particular the geodesics are translates of the exponential map.

We now turn to the general case. Consider $X \in \mathrm{L}G$, and let $E_X \colon (-\varepsilon, \varepsilon) \to G$ denote the geodesic through the origin which is tangent to $X \in \mathrm{L}G$. It is defined for sufficiently small ε (depending on X). If $\lambda \in \mathbb{R}$, then the map $t \mapsto E_X(\lambda t)$ is also a geodesic through the origin which is tangent to λX, and so by uniqueness we have $E_X(\lambda t) = E_{\lambda X}(t)$. It follows that there is a neighborhood U of the origin in $\mathrm{L}G$ and a map $E \colon U \to G$ such that $E_X(t) = E(tX)$ for all $X, tX \in U$. We must show that E coincides with the exponential map.

If $g \in G$, then translating $t \mapsto E(tX)$ on the left by g and on the right by g^{-1} gives another geodesic, which is tangent to $\mathrm{Ad}(g)X$. So, if $tX \in U$, we have

$$gE(tX)g^{-1} = E(t\,\mathrm{Ad}(g)X).$$

We now fix $X \in \mathrm{L}G$. Let T be a maximal torus containing the one-parameter subgroup $\{e^{tX} \mid t \in \mathbb{R}\}$, using Remark 4.1.2. In particular, we have $\mathrm{Ad}(g)X = X$ for all $g \in T$, and hence it follows from the above equation that g commutes with $E(tX)$ for all t. So we see that the path $t \mapsto E(tX)$ runs through the centralizer group $C(T)$, and thus in particular through the normalizer group $N(T)$. Since the quotient $N(T)/T$ is finite by Theorem 4.1.6, it follows that in fact $E(tX) \in T$ for all $t \in \mathbb{R}$.

Now, the translation-invariant Riemannian structure on G induces a translation-invariant Riemannian structure on T, and since the geodesic path $t \mapsto E(tX)$ of G is contained in T, it is also a geodesic path in T. The result therefore follows from the special case of the torus, which we treated before. $\square$

Theorem 4.3.2. *Let G be a compact connected Lie group. Then the exponential map* $\exp \colon \mathrm{L}G \to G$ *is surjective.*

Proof. Choose a Riemannian structure on G as in Theorem 4.3.1. By Theorem 4.2.8, given a group element $g \in G$, there exists a geodesic path from the identity to g. By Theorem 4.3.1, this path is of the form $t \mapsto \exp(tX)$ for some $X \in \mathrm{L}G$. It follows that $g = \exp(X)$ is in the image of the exponential map, as desired. $\square$

Remark 4.3.3. We have already seen in Exercise 3.1 that Theorem 4.3.2 is false for the non-compact Lie group $\mathrm{SL}_2(\mathbb{R})$, so compactness is necessary.

To proceed, we need the following lemma:

Lemma 4.3.4. *Let G be a Lie group and let V be a G-representation over K equipped with a G-invariant bilinear form $B\colon V \times V \to K$. Let $\pi\colon \mathrm{L}G \to \mathrm{L}\,\mathrm{Aut}(V) = \mathrm{End}(V)$ denote the derivative of the G-action on V. Then for all $X \in \mathrm{L}G$ and $v, w \in V$ we have*

$$B(\pi(X)v, w) + B(v, \pi(X)w) = 0.$$

Proof. G-invariance of B means that we have

$$B(\exp(tX)v, \exp(tX)w) = B(v, w).$$

In particular, the derivative of this expression with respect to t is zero. By applying the chain rule, we may write this derivative as the sum

$$\left.\frac{\partial}{\partial t_1}\right|_0 B(\exp(t_1X)v, w) + \left.\frac{\partial}{\partial t_2}\right|_0 B(v, \exp(t_2X)w) = B(\pi(X)v, w) + B(v, \pi(X)w),$$

finishing the proof. □

Theorem 4.3.5. *Let G be a compact connected Lie group, and let T be a maximal torus. For every $g \in G$, there exists $k \in G$ such that $g \in kTk^{-1}$.*

Proof. Let t_0 be a generator of T, in the sense of Definition 1.4.27. Using Theorem 4.3.2, we may find tangent vectors $X \in \mathrm{L}G$ and $H_0 \in \mathrm{L}T$ such that $g = \exp(X)$ and $t_0 = \exp(H_0)$.

Consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{L}G)$ from Definition 1.2.25. Since G is compact, we may choose a G-invariant inner product $\langle -, - \rangle$ on $\mathrm{L}G$. (Here we are talking about a real G-representation since $\mathrm{L}G$ is a real vector space.) We may then choose $k \in G$ such that the real value $\langle X, \mathrm{Ad}(k)H_0 \rangle$ is maximal, and define $H := \mathrm{Ad}(k)H_0 \in \mathrm{L}G$. In particular, the element $\exp(H) = kt_0k^{-1}$ generates a torus kTk^{-1}.

For arbitrary $Y \in \mathrm{L}G$, the inner product $\langle X, \mathrm{Ad}(\exp(tY))H \rangle$ has a maximum when $t = 0$, and thus we get

$$0 = \frac{d}{dt} \langle X, \mathrm{Ad}(\exp(tY))H \rangle|_{t=0} = \langle X, \mathrm{ad}(Y)H \rangle.$$

Using Proposition 1.2.26, it follows that

$$0 = \langle X, \mathrm{ad}(Y)H \rangle = \langle X, [Y, H] \rangle = -\langle X, [H, Y] \rangle = -\langle X, \mathrm{ad}(H)(Y) \rangle,$$

and thus by Lemma 4.3.4, it follows that also

$$0 = \langle \mathrm{ad}(H)(X), Y \rangle = \langle [H, X], Y \rangle.$$

Since this holds for arbitrary Y and the inner product is by definition positive definite, we conclude that $[H, X] = 0$. It follows from Exercise 4.5 that $\exp(H)$ commutes with $\exp(tX)$ for all $t \in \mathbb{R}$. Since $\exp(H)$ generates the maximal torus kTk^{-1}, we deduce that in fact all elements of kTk^{-1} commute with $\exp(tX)$. Since kTk^{-1} is a maximal torus it follows that $\exp(tX) \in kTk^{-1}$ for all t, and in particular $g = \exp(X) \in kTk^{-1}$. □

We are now ready to prove the Conjugation Theorem 4.1.7:

Proof of the Conjugation Theorem 4.1.7. Part (2) is contained in Theorem 4.3.5. For (1), let t' be a generator of T'. By Theorem 4.3.5, t' is contained in kTk^{-1} for some k, and thus $T' \subseteq kTk^{-1}$. Since both are maximal tori, they are equal. □

4.4 Consequences of the Conjugation Theorem

We will now discuss a few consequences of the Conjugation Theorem. Throughout, we let G be a compact connected Lie group and let T be a fixed choice of maximal torus in G with normalizer N and Weyl group W.

Remark 4.4.1. The torus T, together with the Weyl group action, is up to isomorphism uniquely determined by G. To see this, suppose that T' is another maximal torus with Weyl group W'. By Theorem 4.1.7, there exists an inner automorphism φ of G such that $\varphi(T) = T'$. It follows that $\varphi(N) = N'$, where N and N' are the normalizers of T and T'. Thus φ induces an isomorphism

$$\tilde{\varphi}\colon W = N/T \xrightarrow{\cong} N'/T' = W'$$

and the following diagram commutes:

$$\begin{array}{ccc} W \times T & \longrightarrow & T \\ {\scriptstyle \tilde{\varphi}\times\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ W' \times T' & \longrightarrow & T'. \end{array}$$

Definition 4.4.2. The dimension of a maximal torus in G is called the *rank* of G and is denoted by $\operatorname{rank}(G)$.

For a subgroup $H \subseteq G$, recall that the *centralizer* $C(H)$ of H is the subgroup

$$C(H) := \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

Theorem 4.4.3. *Let G be a compact connected Lie group and let T be its maximal torus.*

(i) We have $C(T) = T$, so that T is a maximal abelian subgroup of G;

(ii) If $S \subseteq G$ is a connected abelian subgroup, then $C(S)$ is the union of those (maximal) tori T containing S;

(iii) The center $Z(G)$ of G is the intersection of all maximal tori in G.

Proof. Note that (i) follows from (ii) by taking $T = S$, so we directly go to the proof of (ii).

For (ii), note that the closure $\overline{S}$ of S is a compact connected abelian subgroup of G, and hence a torus. Furthermore, if $g \in C(S)$ then also $g \in C(\overline{S})$ and hence $C(S) = C(\overline{S})$. So we may assume that $S = \overline{S}$ is a torus. Now, observe first that every maximal torus T containing S is certainly contained in $C(S)$, giving one inclusion. For the other inclusion, let $x \in C(S)$ and let B be the closure of the subgroup generated by x and S. Then B is a compact and abelian subgroup of G. If we let B_0 denote its connected component of the unit, then B_0 is a connected compact abelian Lie group and hence a torus. Since $S \subseteq B_0$, the element xB_0 generates the finite group B/B_0, so we see that $B/B_0 \cong \mathbb{Z}/m$ for some integer m. By Corollary 1.3.10 we have $B \cong B_0 \times \mathbb{Z}/m$ and thus Corollary 1.4.26 B admits a generator $g \in B$, i.e. B is the closure of $\{g^n \mid n \in \mathbb{Z}\}$. Now, if we let T be a maximal torus containing g, then it follows that $S \cup \{x\} \subseteq B \subseteq T$. In particular, x is contained in a maximal torus containing S, as desired.

For (iii), notice that (i) immediately implies that an element $x \in C(G)$ in the center of G is contained in $C(T) = T$ for every maximal torus T. Conversely, if $x \in G$ is contained in every maximal torus, then it must commute with every element $g \in G$, since g is contained in some maximal torus. $\square$

Warning 4.4.4. Although every maximal torus is a maximal abelian subgroup, not every maximal abelian subgroup is a maximal torus: see Exercise 11.4 for a counterexample.

The fact that $C(T) = T$ may be stated as follows:

Corollary 4.4.5. *The Weyl group W acts faithfully on the maximal torus, in the sense that the group homomorphism $W \to \mathrm{Aut}(T)$ defined by the action of W on T is injective.*

Proof. An element $g \in N$ is in the kernel of $N \to \mathrm{Aut}(T)$ if and only if it commutes with all elements of T, i.e. $g \in C(T) = T$. □

As a consequence, we may interpret W as a group of automorphisms of T. The maximal torus decomposes into orbits under the action of W.

Lemma 4.4.6. *Two elements of the maximal torus are conjugate in G if and only if they lie in the same orbit under the action of the Weyl group.*

Proof. It is clear that two elements that are in the same orbit of the Weyl group action are conjugate in G. Conversely, consider $x, y \in T$ and $g \in G$ such that $gxg^{-1} = y$. Let $C(x)$ and $C(y)$ denote the centralizers of x and y. Conjugation by g then induces a map $c(g)\colon C(x) \to C(y)$, and since $T \subseteq C(x)$ we have $c(g)T \subseteq C(y)$. Since also $T \subseteq C(y)$, it follows that both T and $c(g)T$ are maximal tori in the connected component of the unit $C(y)_0$ of $C(y)$. By the Conjugation Theorem, there exists $h \in C(y)_0$ such that $T = c(h)(c(g)T) = c(hg)T$. But this means that $hg \in N$ and that $c(hg)x = c(h)c(g)x = c(h)y = y$. So $w = hgT \in W$ is an element satisfying $wx = y$. □

Let $\mathrm{Con}(G)$ denote the space of conjugacy classes of G. More precisely, we define $\mathrm{Con}(G)$ as the space of orbits under the action of G on itself via conjugation $\tilde{q}\colon G \times G \to G, (g,x) \mapsto gxg^{-1}$. The topology on $\mathrm{Con}(G)$ is the quotient topology coming from the projection map $G \to \mathrm{Con}(G)$.

Proposition 4.4.7. *There is a canonical homeomorphism*

$$\kappa\colon T/W \xrightarrow{\cong} \mathrm{Con}(G), \qquad Wt \mapsto c(G)t$$

taking the orbit of t under the operation of W to the conjugacy class of t.

Proof. The map κ is well-defined and continuous. It is surjective because every element of G is conjugate to an element of the torus T by the Conjugation theorem. It is injective by Lemma 4.4.6. Since both are quotients of a compact Hausdorff space by an action of a compact Lie group, the are both compact Hausdorff spaces, and so κ is a homeomorphism. □

Let $C^0(X) = C^0(X, \mathbb{C})$ denote the space of continuous complex-valued functions on a space X. Conjugation induces an action of W on $C^0(T)$ and an action of G on $C^0(G)$:

$$\begin{aligned} W \times C^0(T) \to C^0(T), &\qquad (w, f) \mapsto f \circ w^{-1}; \\ G \times C^0(G) \to C^0(G), &\qquad (g, f) \mapsto f \circ c(g^{-1}). \end{aligned}$$

Note that we have

$$C^0(T/W) = C^0(T)^W, \qquad \text{and} \qquad C^0(\mathrm{Con}(G)) = C^0(G)^G.$$

Corollary 4.4.8. *There is a canonical isomorphism of complex vector spaces*

$$C^0(\mathrm{Con}(G)) \xrightarrow{\cong} C^0(T)^W, \qquad f \mapsto f|_T$$

between the space of class functions on G and the space of continuous functions on the maximal torus invariant under the action of the Weyl group. □

The representation ring $R(G)$ may be viewed as a subring of $C^0(\mathrm{Con}(G))$, and we obtain:

Corollary 4.4.9. *Restriction to the maximal torus induces an injective homomorphism*

$$R(G) \to R(T)^W, \qquad \chi \mapsto \chi|_T.$$

In particular, $R(G)$ has no zero-divisors (i.e. it is an integral domain).

Proof. The first statement is immediate from Corollary 4.4.8. For the second statement, notice that for an abelian compact Lie group H we have an isomorphism of rings $R(H) \cong \mathbb{Z}[\hat{H}]$ since both sides are freely generated by the irreducible characters of H. In particular, for $H = T^k = \mathbb{R}^k/\mathbb{Z}^k$ we have $R(T^k) \cong \mathbb{Z}[\mathbb{Z}^k]$, which has no zero-divisors. □

Remark 4.4.10. The map $R(G) \to R(T)^W$ turns out to even be an isomorphism; however, we will not have the time to prove this in this course.

4.5 The maximal tori and Weyl groups of the classical groups

We will now discuss the maximal tori and Weyl groups of various of the matrix groups, starting with the unitary group $\mathrm{U}(n)$ and the special unitary group $\mathrm{SU}(n)$.

Definition 4.5.1. Let $\Delta(n) \subseteq \mathrm{U}(n)$ denote the subgroup of diagonal matrices

$$\Delta(n) = \left\{ D = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \;\middle|\; z_i \in S^1 \right\}.$$

We let $\mathrm{S}\Delta(n) = \Delta(n) \cap \mathrm{SU}(n)$ be the subgroup of diagonal matrices satisfying $z_1 \cdot \dots \cdot z_n = 1$.

Proposition 4.5.2. *The subgroups $\Delta(n)$ and $\mathrm{S}\Delta(n)$ are maximal tori in $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.*

Proof. The inclusions $T \subseteq \mathrm{U}(n)$ and $T \subseteq \mathrm{SU}(n)$ of any torus T are unitary representations of abelian groups, hence by Proposition 2.1.17 are direct sums of one-dimensional representations. Equivalently, this means that these inclusions are conjugate to homomorphisms with image in $\Delta(n)$. (See also Exercise 5.2.) We may always achieve this through conjugation by an element of $\mathrm{SU}(n)$, and this sends $\mathrm{SU}(n)$ into itself. □

We will represent a diagonal matrix D as above by the n-tuple $(\theta_1, \dots, \theta_n) \in (\mathbb{R}/\mathbb{Z})^n$, with $z_j = e^{2\pi i \theta_j}$. This diagonal matrix generates the torus $\Delta(n)$ if and only if the real numbers $1, \theta_1, \dots, \theta_n$ are linearly independent over the rationals, see Theorem 1.4.25. In particular, the values of the $\theta_j \in \mathbb{R}/\mathbb{Z}$ are distinct for a generating element. Since the eigenvalues of a matrix stay fixed under conjugation, and hence also under the action of the Weyl group W of $\mathrm{U}(n)$, W acts on a generating element

$D = (\theta_1, \dots, \theta_n)$ by permuting the θ_j. This determines the action of W on all of $\Delta(n)$, and it follows that we may identify W with a subgroup of S_n, the group of permutations of $\{1, \dots, n\}$. An element $\alpha \in S_n$ acts on $\Delta(n)$ by

$$\alpha^{-1}(\theta_1, \dots, \theta_n) = (\theta_{\alpha(1)}, \dots, \theta_{\alpha(n)}).$$

Theorem 4.5.3. *The Weyl group of the unitary group* $\mathrm{U}(n)$ *is the full symmetric group* S_n*.*

Proof. Since S_n is generated by the two-cycles, it suffices to show that any two-cycle is in W. But this follows from the computation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \eta & 0 \\ 0 & \zeta \end{pmatrix}. \qquad \square$$

A diagonal matrix $D = (\theta_1, \dots, \theta_n)$ is in $\mathrm{S}\Delta(n)$ if and only if we have

$$\theta_1 + \cdots + \theta_{n-1} = -\theta_n \mod \mathbb{Z}.$$

Thus the torus $\mathrm{S}\Delta(n)$ has dimension $n-1$, but the θ_j of a generating element are still pairwise distinct. Since the transformation used in the proof of Theorem 4.5.3 has determinant 1, we immediately get

Theorem 4.5.4. *The Weyl group of* $\mathrm{SU}(n)$ *is also the symmetric group* S_n*.*

Next we consider the special orthogonal group $\mathrm{SO}(n)$. Recall that

$$\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1 \cong \left\{ \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R}/\mathbb{Z} \right\}.$$

The decompositions

$$\begin{aligned} \mathbb{R}^{2n} &= \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 && \text{into } n \text{ summands, and} \\ \mathbb{R}^{2n+1} &= \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2 \oplus \mathbb{R} && \text{into } n+1 \text{ summands} \end{aligned}$$

give rise to inclusions

$$T(n) := \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2) \subseteq \mathrm{SO}(2n) \subseteq \mathrm{SO}(2n+1).$$

Theorem 4.5.5. *The subgroup* $T(n)$ *is a maximal torus in* $\mathrm{SO}(2n)$ *and* $\mathrm{SO}(2n+1)$*.*

Proof. A torus $T \subseteq \mathrm{SO}(2n)$ (or $\mathrm{SO}(2n+1)$) corresponds to a real representation V of T with a T-invariant inner product. We may find an orthogonal decomposition $V \cong V_1 \oplus \cdots \oplus V_k$ of V into irreducible real T-representations. The classification we had of all the irreducible *complex* T^k-representations (combining Example 2.5.3 and Proposition 2.4.18) immediately imply a classification of all the irreducible *real* T^k-representations:

- The trivial complex representation $\mathbb{C}$ is of real type and corresponds to the one-dimensional trivial real representation $\mathbb{R}$;
- The non-trivial complex representations $T^k \times \mathbb{C} \to \mathbb{C}$, $(\lambda_1, \dots, \lambda_k, v) \mapsto \lambda_1^{n_1} \dots \lambda_k^{n_k} v$ are of complex type and correspond to two-dimensional real representations on $\mathbb{R}^2$.

Choosing orthonormal bases for each of the V_i then shows that the inclusion $T \hookrightarrow \mathrm{SO}(2n)$ (resp. $T \hookrightarrow \mathrm{SO}(2n+1)$) is conjugate to a subgroup of $T(n)$. □

The tori $\Delta(n)$ and $T(n)$ correspond to each other under the canonical inclusions $\mathrm{U}(n) \subseteq \mathrm{SO}(2n) \subseteq \mathrm{SO}(2n+1)$. With this in mind, we also represent the elements of $T(n)$ by n-tuples $(\theta_1, \dots, \theta_n)$, $\theta_i \in \mathbb{R}/\mathbb{Z}$. The element represented by such an n-tuple has the matrix

$$\begin{pmatrix} \cos 2\pi\theta_i & -\sin 2\pi\theta_i \\ \sin 2\pi\theta_i & \cos 2\pi\theta_i \end{pmatrix} \in \mathrm{SO}(2)$$

as its i-th component. The following will help us describe the Weyl group of $\mathrm{SO}(n)$:

Notation 4.5.6. We use the notation

$$\theta_{-i} := -\theta_i$$

for $i = 1, \dots, n$, and we let G_n denote the group of permutations φ of the set $\{-n, \dots, -1, 1, \dots, n\}$ that satisfy $\varphi(-i) = -\varphi(i)$. There is a split exact sequence

$$1 \to (\mathbb{Z}/2)^n \to \mathrm{G}_n \to S_n \to 1,$$

where S_n acts on $(\mathbb{Z}/2)^n$ by permuting the factors. (This construction is also known as the *wreath product* of $\mathbb{Z}/2$ by S_n.) The group G_n acts on the torus $T(n)$ with action given by

$$\varphi^{-1}(\theta_1, \dots, \theta_n) = (\theta_{\varphi(1)}, \dots, \theta_{\varphi(n)}).$$

Theorem 4.5.7. *The Weyl group of* $\mathrm{SO}(2n+1)$ *is* G_n*. The Weyl group of* $\mathrm{SO}(2n)$ *is* $\mathrm{S}\,\mathrm{G}_n$*, the subgroup of* G_n *consisting of even permutations.*

Proof. We begin with $\mathrm{SO}(2n+1)$. The eigenvalues of the matrix associated to the generating element $(\theta_1, \dots, \theta_n)$ are the numbers $e^{2\pi i(\pm\theta_j)}$ (this is a familiar observation from linear algebra but can also be deduced from the general equation $e_{\mathbb{R}}^{\mathbb{C}} r_{\mathbb{R}}^{\mathbb{C}} V \cong V \oplus \overline{V}$). Now, these eigenvalues remain fixed under conjugation, and hence the action of the Weyl group maps the generating element above to an element $(\lambda_1, \dots, \lambda_n)$ with

$$(-\lambda_n, \dots, -\lambda_1, \lambda_1, \dots, \lambda_n) = (\theta_{\varphi(-n)}, \dots, \theta_{\varphi(-1)}, \theta_{\varphi(1)}, \dots, \theta_{\varphi(n)})$$

for some permutation φ of $\{-n, \dots, -1, 1, \dots, n\}$. For a generating element, all the θ_i may be chosen distinct, and it follows that $\varphi \in \mathrm{G}_n$. Since the action of the Weyl group is determined by what it does on this generating element, it follows that $W \subseteq \mathrm{G}_n$. We thus need to show the converse: every permutation $\varphi \in \mathrm{G}_n$ comes from an element of the Weyl group.

Case 1. Assume first that we have $\varphi(\{1, \dots, n\}) = \{1, \dots, n\}$. That is to say, the element $\varphi \in \mathrm{G}_n$ lies in the subgroup $\mathrm{S}_n \subseteq \mathrm{G}_n$ in the description of G_n as a semidirect product $(\mathbb{Z}/2)^n \ltimes \mathrm{S}_n$. In this case, we may use an equation like the one we had earlier, namely

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}, \qquad 1, A, B \in \mathrm{SO}(2).$$

Case 2. Let $\varphi \in \mathrm{G}_n$ be the element given by $\varphi(-1) = 1$, $\varphi(1) = -1$ and $\varphi(i) = i$ for $|i| > 1$. This is the element $(-1, 1, \dots, 1) \in (\mathbb{Z}/2)^n \subseteq \mathrm{G}_n$ in the description of G_n as a direct product. Then conjugation by the matrix

$$h = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

yields φ. (Indeed, conjugating the (2×2)-matrix $\left(\begin{smallmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ gives $\left(\begin{smallmatrix} \cos 2\pi\theta_i & \sin 2\pi\theta \\ -\sin 2\pi\theta_i & \cos 2\pi\theta \end{smallmatrix}\right)$.) Now, since $\mathrm{G}_n = (\mathbb{Z}/2)^n \ltimes \mathrm{S}_n$ is generated by the elements considered in Cases 1 and 2, it follows that $\mathrm{G}(n) = W$.

Now we turn to the group $\mathrm{SO}(2n)$. The subgroup $\mathrm{S\,G}_n$ of G_n acts via even permutations and, as in Case 1 above, lies in the Weyl group. The parity of a permutation in $\mathrm{G}_n = (\mathbb{Z}/2)^n \ltimes \mathrm{S}_n$ depends only on the factor in $(\mathbb{Z}/2)^n$, and even permutations in $(\mathbb{Z}/2)^n$ may be obtained via conjugation by a product of an even number of matrices of the type in Case 2. Since such a product is in $\mathrm{SO}(2n)$, we conclude that $\mathrm{S\,G}_n \subseteq W$.

We now have to show that W is not bigger than $\mathrm{S\,G}_n$. Since the index of $\mathrm{S\,G}_n$ in G_n is 2, it suffices to show that the permutation φ with $\varphi(1) = -1$ and $\varphi(j) = j$ for $|j| > 1$ is not in W. But if conjugation by some $g \in \mathrm{SO}(2n)$ gave rise to this automorphism of $T(n)$, we would have $g^{-1}h$ acting as the identity on $T(n)$, where h is the matrix in Case 2 above. This leads to $g^{-1}h \in T(n) \subseteq \mathrm{SO}(2n)$ by Corollary 4.4.5, and this, in turn, says that $h \in \mathrm{SO}(2n)$, which is false. □

5 Miscellaneous topics

In this last chapter, we will discuss various miscellaneous topics. They would each deserve their own extensive treatment, but due to time constraints we will only give an overview.

5.1 Lie theory

The exposition in this section is based on Lee [Lee02, Chapters 14 and 15] and on lecture notes by Alexander Kirillov.

In Section 1.2, we saw that the tangent space $\mathrm{L}G = T_eG$ of a Lie group G comes equipped with the structure of a *Lie algebra*: a real vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying $[X,X]=0$ for all $X\in\mathfrak{g}$ and satisfying the Jacobi identity:

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

The Lie algebra $\mathrm{L}G$ fully determines some of the behavior of the Lie group G. For example, we saw that every morphism of Lie groups $f\colon H\to G$ induces a morphism of Lie groups $\mathrm{L}f\colon \mathrm{L}G\to\mathrm{L}H$ and that when G is connected the morphism f is in fact uniquely determined by $\mathrm{L}f$, see Lemma 1.3.4. If $H\hookrightarrow G$ is a Lie subgroup, we obtain a Lie subalgebra $\mathrm{L}H\hookrightarrow\mathrm{L}G$.

The following are some natural questions about the interaction between Lie groups and Lie algebras:

(1) Is every Lie subalgebra $\mathfrak{h}\subseteq\mathrm{L}G$ of the form $\mathrm{L}H$ for some Lie subgroup $H\hookrightarrow G$?

(2) Can every morphism $\mathrm{L}G\to\mathrm{L}H$ of Lie algebras be lifted to a morphism $G\to H$ of Lie groups?

(3) Is every Lie algebra $\mathfrak{g}$ of the form $\mathrm{L}G$ for some Lie group G?

For (1), the answer is negative if we restrict to *embedded* Lie subgroups $H\subseteq G$, in the sense that H is a (necessarily closed) submanifold of G:

Example 5.1.1. Consider the Lie group $G=T^2=\mathbb{R}^2/\mathbb{Z}^2$, so that $\mathrm{L}G=\mathbb{R}^2$ with zero Lie bracket. Choose an irrational number α, and consider the subspace

$$\mathfrak{h}:=\{(t,\alpha t)\mid t\in\mathbb{R}\}\subseteq\mathrm{L}G.$$

Since the Lie bracket on $\mathrm{L}G$ is zero, $\mathfrak{h}$ is automatically a one-dimensional Lie subalgebra. However, there is no closed subgroup corresponding to $\mathfrak{h}$: the image of $\mathfrak{h}$ under the exponential map $\exp\colon \mathbb{R}^2\to\mathbb{R}^2/\mathbb{Z}^2$ is everywhere dense on the torus, hence the closure is all of T^2.

Also the answer to (2) is negative in general:

Example 5.1.2. Consider the Lie groups $G = S^1 = \mathbb{R}/\mathbb{Z}$ and $H = \mathbb{R}$. Then we have canonical identifications $\mathrm{L}G = \mathbb{R} = \mathrm{L}H$, where the Lie brackets are identically zero (since G and H are abelian). The identity map $\mathrm{L}G \to \mathrm{L}H$ is then a morphism of Lie algebras. But this cannot be lifted to a morphism $f: S^1 \to \mathbb{R}$ of Lie groups: since the exponential map $\exp: \mathbb{R} \to S^1$ is surjective, $f: S^1 \to \mathbb{R}$ must be given by $\exp(\theta) \mapsto \theta$, but this is not a well-defined map on S^1.

It turns out, however, that it is possible to fix these problems by slightly reformulating the questions. For (1), we need to allow for arbitrary *immersed* Lie subgroups $H \hookrightarrow G$ as defined in Definition 1.3.13, and not only the embedded ones; see Theorem 5.1.10 below. For (2) we need to make sure that the source G does not admit any nontrivial 'loops', see Theorem 5.1.13 below. The answer to question (3) is always positive, see Theorem 5.1.17 below.

Our goal in this section is to prove these three fundamental results in Lie theory. We start with some preparations.

Definition 5.1.3. A *distribution* on a smooth manifold M is a smooth subbundle $D \subseteq TM$ of the tangent bundle of M. This means that for every $p \in M$ we are given a linear subspace $D_p \subseteq T_pM$ such that their union $D = \bigcup_{p \in M} D_p \subseteq TM$ is a smooth submanifold and the resulting map $D \to M$ is a vector bundle.

Definition 5.1.4. Given a smooth manifold M, an *immersed submanifold* is a smooth manifold N together with an injective immersion $i: N \hookrightarrow M$. We will mostly identify N with its image in M and write $N \subseteq M$, but it is important to note that N does not need to carry the subspace topology of M! (If it did, we call N an *embedded submanifold*; this notion agrees with that of Definition 1.3.11.) Since any immersion is at least locally an embedding, the smooth structure on N is still determined by the smooth structure on M.

Definition 5.1.5. Given a distribution D on M, an *integral manifold* for D is an immersed submanifold $H \subseteq M$ such that for every point $p \in X$ we have $T_pH = D_p \subseteq T_pM$. We say that a distribution D is *completely integrable* if for every $p \in M$ there exists an integral manifold for D containing p.

Remark 5.1.6. One can show that a union of connected integral manifolds containing p is still a connected integral manifold containing p, so every complete point p is contained in a maximal connected integral manifold N_p. For two points p and q, if $N_p \cap N_q$ contains a point x, then N_p and N_q are both connected integral manifolds containing x, hence so is their union $N_p \cup N_q$. By maximality it follows that $N_p = N_q$, showing that all the N_p are either disjoint or equal. In other words, we have decomposed M as a disjoint union of immersed submanifolds. This is known as a *foliation*, and the individual submanifolds N_p are known as the *leaves of the foliation*.

Example 5.1.7. Note that every vector field X on M defines a one-dimensional distribution by letting $D_p \subseteq T_pM$ be the subspace spanned by the vector X_p. Note that in this case every integral curve for X is in particular an integral manifold for D.

Although integral curves for vector fields locally always exist and are unique, this is not the case for integral manifolds of distributions. We will use as a black box the following characterizations for which distributions are completely integrable:

Theorem 5.1.8 (Frobenius integrability criterion,[Lee02, Theorem 14.5])**.** *Let D be a distribution on M, and say that a vector field X is tangent to D if $X_p \in D_p$ for all $p \in M$. Then D is completely* *integrable if and only if the space of vector fields tangent to D is closed under the Lie bracket: if X and Y are vector fields tangent to D, then also their Lie bracket $[X,Y]$ is tangent to D.* $\square$

We also record the following fact:

Proposition 5.1.9 ([Lee02, Theorem 14.7]). *Let $H \subseteq M$ be an integral manifold for a completely integrable distribution D on M. If $f: N \to M$ is a smooth map such that $f(N) \subseteq H$, then f is smooth as a map from N to H.* $\square$

Using these two results, we may now prove our first fundamental result about Lie groups and Lie algebras:

Theorem 5.1.10 (Integral Subgroup Theorem). *Let G be a Lie group. If $\mathfrak{h} \subseteq \mathrm{L}G$ is a Lie subalgebra, there is a unique connected Lie subgroup $H \hookrightarrow G$ such that $\mathfrak{h} = \mathrm{L}H \subseteq \mathrm{L}G$.*

Proof. We start by proving the existence of H. Recall from the discussion after Definition 1.2.17 that we may identify $\mathrm{L}G$ with the space of left-invariant vector fields on G. With this in mind, we define a distribution $D \subseteq TG$ by setting

$$D_g := \{X_g \in T_g G \mid X \in \mathfrak{h}\}.$$

If $(X_1, \ldots, X_k)$ is a basis for $\mathfrak{h}$, then the subspace D_g is spanned by the vectors $X_1|_g, \ldots, X_k|_g$ at any $g \in G$, and so D is spanned by smooth vector fields and it follows that it is a smooth subbundle of TG.

By construction, a vector field X on G is tangent to D if and only if $X_g \in l_g \mathfrak{h} \subseteq T_g G$ for all $g \in G$. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathrm{L}G$ and in particular closed under the Lie bracket, it follows that if X and Y are vector fields that are tangent to D then also $[X,Y]$ is tangent to D. So by the Frobenius integrability criterion, D is completely integrable.

By the discussion in Remark 5.1.6, we obtain for every $g \in G$ a maximal connected integral manifold $\mathcal{H}_g \subseteq G$. Note that for all $g, g' \in G$ the isomorphism $l_g : T_{g'} G \xrightarrow{\cong} T_{gg'} G$ identifies $D_{g'}$ with $D_{gg'}$, and it follows that if $M \subseteq G$ is a connected integral manifold then so is $l_g(M)$. If M is maximal, then so is $l_g(M)$, and it follows that left multiplication takes leaves to leaves: $L_g(\mathcal{H}_{g'}) = \mathcal{H}_{gg'}$.

Define $H := \mathcal{H}_e \subseteq G$, the leaf of the unit, which is an immersed submanifold of G. We claim that H is a Lie subgroup of G. We have $e \in H$ by construction. For $h, h' \in H$, we have

$$hh' = l_h(h') \in l_h(H) = L_h(\mathcal{H}_e) = \mathcal{H}_h = H,$$

where the last equality holds since by uniqueness of maximal connected integral manifolds. Similarly, we have

$$h^{-1} = h^{-1} e \in l_{h^{-1}}(\mathcal{H}_e) = l_{h^{-1}}(\mathcal{H}_h) = \mathcal{H}_{h^{-1}h} = H.$$

So the groups structure on G restrict to a group structure on H. It follows from Proposition 5.1.9 that this group structure is smooth, so that H is an (immersed) Lie subgroup of G. Furthermore, by definition of H as an integral manifold we have that $\mathrm{L}H = T_e H = \mathfrak{h}$ as desired.

It is now clear that H is the *unique* connected subgroup with Lie algebra $\mathfrak{h}$: if $\tilde{H}$ is another connected subgroup of G with the same Lie algebra, then it follows from Lemma 1.3.5 that $\tilde{H}$ is generated by the image of the exponential map $\mathfrak{h} \hookrightarrow \mathrm{L}G \xrightarrow{\exp} G$. But also H is generated by this image, hence the subgroups agree. $\square$

We now move on to the question of lifting morphisms of Lie algebra to morphisms of Lie groups.

Lemma 5.1.11. *Let $f\colon G \to H$ be a surjective morphism of Lie groups. Then f is a covering map if and only if the induced map $\mathrm{L}f\colon \mathrm{L}G \to \mathrm{L}H$ is an isomorphism.*

Proof. If f is a covering map, then f is in particular a local diffeomorphism, hence induces an isomorphism $\mathrm{L}f\colon \mathrm{L}G \cong \mathrm{L}H$ on tangent spaces. Conversely, assume that $\mathrm{L}f$ is an isomorphism, and let $K \subseteq G$ denote the kernel of f. Since K is a closed subgroup of G, it is automatically a Lie group. The inclusion $\mathrm{L}K \subseteq \mathrm{L}G$ induces an isomorphism $\mathrm{L}G/\mathrm{L}K \xrightarrow{\cong} \mathrm{L}H$, and the assumption on f then implies that $\mathrm{L}K = 0$, showing that K is a discrete subgroup. Note that K is a normal subgroup of G and by surjectivity of f we may identify H with the quotient group G/K. It then follows from Theorem 1.4.11 that the map $f\colon G \to H$ is a K-principal bundle. In particular, it is locally of the form $\mathrm{pr}_1\colon U \times K \to U$, hence by discreteness of K it follows that f is a covering map. □

Lemma 5.1.12. *Let G be a connected Lie group. Then the universal cover $\tilde{G}$ of G admits a canonical structure of a Lie group such that the covering map $p\colon \tilde{G} \to G$ is a morphism of Lie groups.*

Proof. Using the theory of covering spaces, G admits a universal covering $\tilde{G}$, and the covering map $p\colon \tilde{G} \to G$ is a local homeomorphism. In particular, $\tilde{G}$ is naturally a smooth manifold. If we fix a base point $e \in \tilde{G}$ satisfying $p(e) = e$, then the lifting property of covering spaces implies that the maps $m\colon G\times G \to G$ and $(-)^{-1}\colon G \to G$ admit unique lifts to maps $\tilde{G}\times\tilde{G} \to \tilde{G}$ and $(-)^{-1}\colon \tilde{G} \to \tilde{G}$ satisfying $e \cdot e = e$ and $e^{-1} = e$. This equips $\tilde{G}$ with the structure of a Lie group and the covering map $p\colon \tilde{G} \to G$ is a morphism of Lie groups. □

Theorem 5.1.13. *Let G and H be Lie groups and assume that G is simply-connected.*[1] *Then every Lie algebra homomorphism $\varphi\colon \mathrm{L}G \to \mathrm{L}H$ lifts uniquely to a Lie group homomorphism $f\colon G \to H$, in the sense that $\varphi = \mathrm{L}f$.*

In particular, sending f to $\mathrm{L}f$ defines a bijection $\mathrm{Hom}(G,H) \cong \mathrm{Hom}(\mathrm{L}G,\mathrm{L}H)$.

Proof. For the last statement, we already know that for connected G the map $\mathrm{Hom}(G,H) \to \mathrm{Hom}(\mathrm{L}G,\mathrm{L}H)$ is injective by Lemma 1.3.4. So it remains to show that it is surjective, i.e. that every morphism of Lie algebras $\varphi\colon \mathrm{L}G \to \mathrm{L}H$ can be lifted to a morphism of Lie groups $f\colon G \to H$ with $\mathrm{L}f = \varphi$.

Consider the product group $G \times H$. The Lie algebra of this group is the product is $\mathrm{L}G \times \mathrm{L}H$. We may now consider the linear subspace

$$\mathfrak{k} := \{(x,\varphi(x)) \in \mathrm{L}(G\times H) \mid x \in \mathrm{L}G\} \subseteq \mathrm{L}(G\times H).$$

This is a Lie subalgebra: since φ is a morphism of Lie algebras we have $[(x,\varphi(x)),(y,\varphi(y))] = ([x,y],[\varphi(x),\varphi(y)]) = ([x,y],\varphi([x,y]))$. As a consequence of Theorem 5.1.10, there is a corresponding connected Lie subgroup $K \hookrightarrow G\times H$ satisfying $\mathrm{L}K = \mathfrak{k}$. Composing this immersion with the projection $\mathrm{pr}_1\colon G\times H \to G$, we get a morphism of Lie groups $\pi\colon K \to G$, and the induced map of Lie algebras $\mathrm{L}\pi\colon \mathfrak{h} = \mathrm{L}\tilde{K} \to \mathrm{L}G'$ is an isomorphism. By Lemma 5.1.11, it follows that π is a covering map. Since G is simply-connected, the theory of covering maps says that G has up to isomorphism a unique connected covering map given by the identity. Since K is connected, it follows that π must be an isomorphism.

[1]This means that G is connected and that every continuous map $f\colon S^1 \to X$ extends to a continuous map $H\colon D^2 \to X$.

We may now define the map $s\colon G \to G \times H$ as the composite $G \xrightarrow{\pi^{-1}} K \hookrightarrow G \times H$. Passing to Lie algebras, s induces the map

$$\mathrm{L}G \to \mathrm{L}G \times \mathrm{L}H, \qquad X \mapsto (X, \varphi(X)).$$

So if we take f to be the composite $G \xrightarrow{s} G \times H \xrightarrow{\mathrm{pr}_2} H$, we see that $\mathrm{L}f = \varphi\colon G \to H$. This finishes the proof. □

Corollary 5.1.14. *Let G and H be simply-connected Lie groups with isomorphic Lie algebras. Then G and H are isomorphic as Lie groups.*

Proof. Given an isomorphism $\varphi\colon \mathrm{L}G \xrightarrow{\cong} \mathrm{L}H$ of Lie algebras, Theorem 5.1.13 provides morphisms of Lie groups $f\colon G \to H$ and $f'\colon H \to G$ satisfying $\mathrm{L}f = \varphi$ and $\mathrm{L}f' = \varphi^{-1}$. It follows that $\mathrm{L}(f' \circ f) = \mathrm{id}$ and $\mathrm{L}(f \circ f') = \mathrm{id}$, and hence $f' \circ f = \mathrm{id}$ and $f \circ f' = \mathrm{id}$. So f is an isomorphism between G and H. □

Corollary 5.1.15. *Two connected Lie groups G and H have isomorphic Lie algebras if and only if their universal covers $\tilde{G}$ and $\tilde{H}$ are isomorphic Lie groups.*

Proof. For the "if"-direction, note that $\tilde{G} \cong \tilde{H}$ implies

$$\mathrm{L}G \cong \mathrm{L}\tilde{G} \cong \mathrm{L}\tilde{H} \cong \mathrm{L}H.$$

Conversely, if $\mathrm{L}G \cong \mathrm{L}H$, then also $\mathrm{L}\tilde{G} \cong \mathrm{L}\tilde{H}$. Since $\tilde{G}$ and $\tilde{H}$ are both simply-connected, this implies the result by Corollary 5.1.14. □

We now move on to the third fundamental theorem, addressing question (3) above about realizing every Lie algebra as coming from a Lie group. For this, we need another black box:

Theorem 5.1.16 (Ando's theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ admits a faithful representation, i.e. an injective Lie algebra homomorphism $\rho\colon \mathfrak{g} \to \mathrm{gl}_n(\mathbb{R})$ for some n.*

The proof of this theorem is long, hard and very algebraic.

Theorem 5.1.17 (Lie's third theorem)**.** *Any finite-dimensional Lie algebra is isomorphic to a Lie algebra of a (connected) Lie group.*

Proof. By Ando's theorem, every finite-dimensional Lie algebra is isomorphic to some Lie subalgebra $\mathfrak{g} \subseteq \mathrm{gl}_n(\mathbb{R})$ for some n. Since $\mathrm{gl}_n(\mathbb{R}) = \mathrm{L}\,\mathrm{GL}_n(\mathbb{R})$, it follows from Theorem 5.1.10 that there exists an immersed connected Lie subgroup $H \subseteq G$ such that $\mathrm{L}H = \mathfrak{h}$. This finishes the proof. □

Combining the three fundamental results from Theorem 5.1.10, Theorem 5.1.13 and Theorem 5.1.17, we obtain the following tight connection between Lie groups and Lie algebras:

Theorem 5.1.18 (Lie Group - Lie Algebra Correspondence)**.** *The assignment $G \mapsto \mathrm{L}G$ defines a one-to-one correspondence between isomorphism classes of simply-connected Lie groups and isomorphism classes of finite-dimensional Lie algebras.*

Proof. Injectivity was shown in Corollary 5.1.14. For surjectivity, let $\mathfrak{g}$ be a Lie algebra. By Theorem 5.1.17, $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathrm{L}G$ of some connected Lie group G. By Lemma 5.1.12, the universal cover $\tilde{G}$ of G is again a Lie group, and $\tilde{G}$ is simply-connected. By Lemma 5.1.11 the covering map $p\colon \tilde{G} \to G$ induces an isomorphism of Lie algebras, so that $\tilde{G}$ is the desired simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. □

Remark 5.1.19. In terms of category theory, one may say that sending G to $\mathrm{L}G$ defines an equivalence of categories between the category of simply-connected Lie groups and the category of finite-dimensional Lie algebras.

A question that remains is: what happens if we allow non-simply-connected Lie groups? Since every Lie group has a simply-connected covering group, the answer in the connected case is easy to describe:

Proposition 5.1.20. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The connected Lie groups whose Lie algebras are isomorphic to $\mathfrak{g}$ are (up to isomorphism) precisely those of the form G/Γ, where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and where Γ is a discrete normal subgroup of G.*

Proof. Given $\mathfrak{g}$, let G be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. Given any other Lie group H whose Lie algebra is isomorphic to $\mathfrak{g}$, let $\varphi\colon \mathrm{L}G \to \mathrm{L}H$ be a Lie algebra isomorphism. By Theorem 5.1.13, we may lift φ to a Lie group homomorphism $f\colon G \to H$ satisfying $\mathrm{L}f = \varphi$. Because φ is an isomorphism, f is a local diffeomorphism, and hence the kernel $K = \ker(f)$ is a discrete normal subgroup of G. Since f is a surjective Lie group homomorphism with kernel K, it follows that H is isomorphic to G/Γ. □

Lemma 5.1.21. *Let $\Gamma \subseteq G$ be a discrete normal subgroup of a connected Lie group G. Then Γ is contained in the center of G.*

Proof. Consider an element $\gamma \in \Gamma$, and consider the map $G \to \Gamma$, $g \mapsto g\gamma g^{-1}$. This is a smooth map which sends e to γ, and since G is connected and Γ is discrete this means that this map is constant. But this means that γ commutes with all elements of G, showing that $\gamma \in Z(G)$ as desired. □

5.2 Classification of compact connected Lie groups

The content of this section is not part of the exam material. It is meant as an overview of results that exist in the literature, hence various results will be stated without proof.

In this section, we will address the question: how many different compact Lie groups are there? Would it be possible to give some kind of classification of all compact Lie groups?

Note that every *finite* group is in particular a compact Lie group. So a classification of all compact Lie groups should in particular entail a classification of all finite groups. This is generally considered impossible (though a classification of all finite *simple* groups exists).

For this reason, we will restrict attention to compact *connected* Lie groups. In a sense, this class is precisely orthogonal to the class of finite groups: an arbitrary compact Lie group G is an extension of a finite group G/G_e by a compact connected Lie group G_e:

$$1 \to G_e \hookrightarrow G \twoheadrightarrow G/G_e \to 1.$$

For the classification of compact connected Lie groups, we want to use covering theory to reduce to the simply-connected case. We have seen in the previous section that every connected Lie group is of the form G/Γ, where G is a simply-connected Lie group and where $\Gamma \subseteq G$ is a discrete central subgroup. However, the Lie group G has no reason to be compact again; for example, the (compact) torus T^k is of the form $\mathbb{R}^k/\mathbb{Z}^k$ and $\mathbb{R}^k$ is not compact. It turns out that the case of tori is in some sense the only thing that can go wrong:

Theorem 5.2.1 ([Bt95, Theorem V.8.1])**.** *Every compact connected Lie group G admits a finite cover which is isomorphic to the direct product of a simply-connected Lie group $\tilde{G}$ and a torus S. In particular, $\tilde{G}$ is also compact.* □

Corollary 5.2.2. *Every compact connected Lie group G is of the form $(\tilde{G}\times T)/\Gamma$ for a torus T, a simply-connected compact Lie group G and a finite central subgroup $\Gamma \subseteq \tilde{G}\times T$.*

Proof. If $p\colon \tilde{G}\times T \twoheadrightarrow G$ is a finite cover as in Theorem 5.2.1, and $\Gamma := \ker(p)$, then Γ is a finite normal subgroup, and hence a central subgroup by Lemma 5.1.21. So $G = (\tilde{G}\times T)/\Gamma$. □

We conclude that in order to classify all the compact connected Lie groups, it will suffice to:

(1) Classify all the *simply-connected* Lie groups;

(2) Given a simply-connected Lie group $\tilde{G}$ and a torus T, classify all the finite discrete subgroups of $Z(\tilde{G})\times T$.

Let us start with part (1). By the Correspondence from Theorem 5.1.18, this is equivalent to classifying all the finite-dimensional Lie algebras. This task is still somewhat hard, but we can break it up by looking only at the *simple* Lie algebras:

Definition 5.2.3. A Lie algebra $\mathfrak{g}$ is called *simple* if it satisfies:

- It is non-abelian, i.e. the Lie bracket $[-,-]\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is not identically zero;
- It does not have non-trivial ideals, i.e. every subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[x,y] \in \mathfrak{h}$ for $x \in \mathfrak{h}$ and $y \in \mathfrak{g}$ is either $\mathfrak{h} = 0$ or $\mathfrak{h} = \mathfrak{g}$.

A Lie algebra is called *semisimple* if it is a direct sum of simple Lie algebras.

Definition 5.2.4. A Lie group G is called *simple* if it does not have a nontrivial connected normal subgroup (i.e. different from $\{e\}$ and G).

Lemma 5.2.5. *A connected Lie group is simple if and only if its Lie algebra* LG *is simple.*

Proof. If $H \subseteq G$ is a connected normal subgroup, then this induces a Lie subalgebra L$H \subseteq$ LG. The fact that H is normal in LG implies that LH is an ideal in LG. So if LG is simple, then LH is either 0 or LG, meaning that H is either $\{e\}$ or G.

Conversely, if $\mathfrak{h} \subseteq$ LG is an ideal, then by Theorem 5.1.10 it is of the form LH for some connected Lie subgroup $H \subseteq G$. To see that H is normal, we have to show that $gHg^{-1} \subseteq H$ for all $g \in G$. If $U \subseteq H$ is an open neighborhood of $e \in H$, then U generates H by Lemma 1.3.5, hence it will suffice to show that $gUg^{-1} \subseteq H$. Choosing U small enough, it lies in the image of the exponential map

$\exp\colon \mathfrak{h} = \mathrm{L}H \to H$, and hence it will suffice to show that the map $\mathrm{Ad}(g)\colon \mathrm{L}G \to \mathrm{L}G$ preserves the subspace $\mathfrak{h} \subseteq \mathrm{L}G$. Since G is connected this may again be tested on an open neighborhood, and there it follows from the condition that the map $[-,-]\colon \mathfrak{h} \times \mathrm{L}G \to \mathrm{L}G$ lands in $\mathfrak{h}$, i.e. that $\mathfrak{h}$ is an ideal. This shows that H is a connected normal subgroup, hence if G is simple we have $H = \{e\}$ or $H = G$ and thus $\mathfrak{h} = 0$ or $\mathfrak{h} = \mathrm{L}G$. □

The following should be true, but I have so far been unable to find a clean reference:

Theorem 5.2.6. *Let G be a compact connected Lie group with finite fundamental group. Then its Lie algebra* $\mathrm{L}G$ *is semisimple.*

Corollary 5.2.7. *Every compact simply-connected Lie group G is isomorphic to a product $G_1 \times \cdots \times G_n$ of compact simply-connected* simple *Lie groups.*

Proof. Since G is simply-connected, its fundamental group is trivial, so that by Theorem 5.2.6 its Lie algebra $\mathrm{L}G$ is semisimple. We may thus write $\mathrm{L}G = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ as a direct sum of Lie subalgebras $\mathfrak{g}_i \subseteq \mathrm{L}G$ such that each $\mathfrak{g}_i$ is simple. For every i, we may find a simply-connected Lie group G_i satisfying $\mathrm{L}G_i \cong \mathfrak{g}_i$. It then follows that

$$\mathrm{L}(G_1 \times \cdots \times G_n) \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathrm{L}G,$$

and thus by Corollary 5.1.14 we get that $G \cong G_1 \times \cdots \times G_n$. Since G is compact, it follows that each of the factors G_i is compact, and since each $\mathfrak{g}_i$ is simple also each G_i is simple by Lemma 5.2.5. This finishes the proof. □

Corollary 5.2.8. *Every compact connected Lie group is of the form*

$$G \cong (G_1 \times \cdots \times G_n \times T)/\Gamma,$$

where $G_1, \ldots, G_n$ are compact simply-connected simple Lie groups, T is a torus and $\Gamma \subseteq Z(G_1) \times \cdots \times Z(G_n) \times T$ is a finite central subgroup.

So, we have basically reduced the problem of classifying all the compact simply-connected *simple* Lie groups, or equivalently the problem of classifying all the finite-dimensional simple Lie algebras It turns out that problem has a very concrete solution, giving rise to just a small list of simply-connected Lie groups.

Definition 5.2.9. For $n \geq 1$, we define the *spin group* $\mathrm{Spin}(n)$ as the connected cover of the special orthogonal group $\mathrm{SO}(n)$. Since the fundamental group of $\mathrm{SO}(n)$ is $\mathbb{Z}/2$, we obtain a short exact sequence

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

Theorem 5.2.10. *Every compact simply-connected simple Lie group G is isomorphic to precisely one from the following list:*

(A_n) $\mathrm{SU}(n+1)$ *for $n \geq 1$, of dimension $n^2 + 2n$;*

(B_n) $\mathrm{Spin}(2n+1)$ *for $n \geq 2$, of dimension $2n(2n+1)/2$;*

(C_n) $\mathrm{Sp}(n)$ *for $n \geq 3$, of dimension $2n^2 + n$;*

(D_n) $\mathrm{Spin}(2n)$ *for* $n \geq 4$*, of dimension* $2n(2n-1)/2$*;*

(E_n, F_n, G_n) *The so-called 'exceptional groups'* G_2*,* F_4*,* E_6*,* E_7 *and* E_8*, of dimensions* 14*,* 52*,* 78*,* 133 *and* 248*, respectively.*

The restrictions on the natural numbers n that are allowed are just there to make sure the four families do not have any overlap. The proof of the theorem is combinatorial in nature: one classifies finite-dimensional simple Lie algebras by means of their *root systems*, discussed in Section 5.3, and these can in turn be described in terms of certain *Dynkin diagrams*, see Section 5.4.

Now that we have classified all the simply-connected simple Lie groups, it remains to compute their centers:

(A_n) The center of $\mathrm{SU}(n+1)$ is cyclic of order n;

(B_n) The center of $\mathrm{Spin}(2n+1)$ is $\mathbb{Z}/2$;

(C_n) The center of $\mathrm{Sp}(n)$ is $\mathbb{Z}/2$;

(D_n) The center of $\mathrm{Spin}(2n)$ is $\mathbb{Z}/2\times\mathbb{Z}/2$ for n even, and $\mathbb{Z}/4$ for n odd.

(E_n, F_n, G_n) The centers of the exceptional groups are

$$Z(G_2) = \{1\}, \qquad Z(F_4) = \{1\}, \qquad Z(E_6) = \mathbb{Z}/3, \qquad Z(E_7) = \mathbb{Z}/2, \qquad Z(E_8) = \{1\}.$$

Combining these results, we thus obtain a complete classification of all the compact connected Lie groups.

5.3 Root systems

The content of this section is not part of the exam material.

In this section, we will discuss the notion of a *root system* on the Lie algebra of a Lie group. The notion of a root system plays a central role in the classification of simple Lie algebras discussed in the previous section. Root systems also play a crucial role in the representation theory of Lie algebras and Lie groups, as they help determine the 'weights' of representations and the characters of irreducible representations.

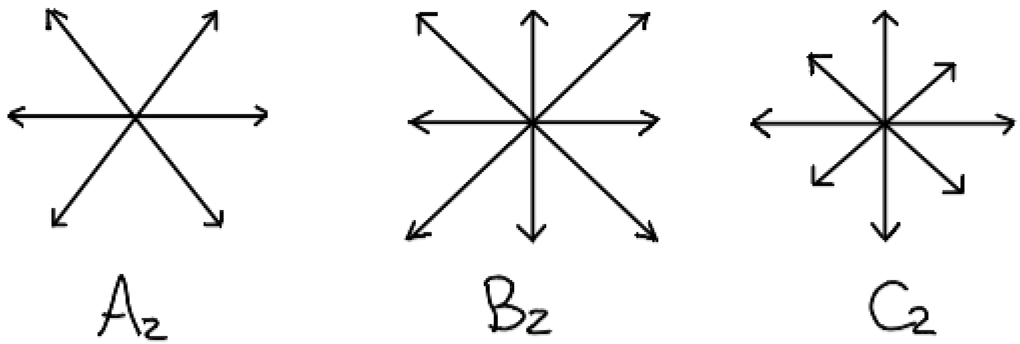
First, an informal definition: a *root system* in a finite-dimensional real vector space V is a "very symmetrical" set of vectors in V. Examples of root systems in $\mathbb{R}^2$ are the following sets of vectors:

Let us now make precise what "very symmetrical" means.

Definition 5.3.1. A *symmetry* associated to a non-zero vector $\alpha \in V$ is an automorphism $s\colon V \to V$ that pointwise fixes some hyperplane $\mathcal{H} \subseteq V$ and sends α to $-\alpha$.

Lemma 5.3.2. *Let $R \subseteq V$ be a finite set which spans V. Given a non-zero vector $\alpha \in V$, there is at most one symmetry associated to α which maps R into itself.*

Proof. Let S be the group of automorphisms of V which maps R into itself. Observe that S is finite since R is finite. We choose an S-invariant inner product on V, so that any automorphism of V mapping R into itself is orthogonal with respect to this metric. It thus follows that the only possible symmetry associated to α is the one given by reflection in the hyperplane orthogonal to α. $\square$



Definition 5.3.3. A finite subset $R \subseteq V$ is called a *(reduced) root system* if it satisfies the following conditions:

(i) The set R spans V and $0 \notin R$;

(ii) If $\alpha, \beta \in R$ are proportional, then $\alpha = \beta$ or $\alpha = -\beta$;

(iii) For each vector $\alpha \in R$ there is an associated symmetry $s_\alpha \colon V \to V$ mapping R into itself;

(iv) For all $\alpha, \beta \in R$ the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The dimension of V is called the *rank* of the root system and the elements of R are called *roots*. For every root α we let $\mathcal{H}_\alpha \subseteq V$ denote the hyperplane fixed by the (uniquely determined) symmetry s_α. We obtain a unique dual vector $\alpha^* \in V^*$ satisfying

$$\alpha^*|_{\mathcal{H}_\alpha} = 0 \qquad \text{and} \qquad \alpha^*(\alpha) = 2,$$

and we may describe s_α in terms of this dual vector:

$$s_\alpha(v) = v - \alpha^*(v) \cdot \alpha.$$

The element α^* is called the *inverse root* of α. Condition (iv) is now equivalent to:

(iv') We have $\alpha^*(\beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

From the symmetry of this formulation, we see that the system $R^* \subseteq V^*$ of inverse roots is a root system in V^*. This is called the *inverse root system* of R.

Definition 5.3.4. The *Weyl group* of a root system R in V is the subgroup W of $\mathrm{Aut}(V)$ generated by the symmetries s_α for $\alpha \in R$.

Since elements of W send R into itself and R generates V, it follows that the Weyl group W is a finite group.

The reason we are interested in root systems is the following theorem:

Theorem 5.3.5 (Cartan, Killing)**.** *Every finite-dimensional semisimple Lie algebra over $\mathbb{C}$ has an associated root system, and the root system determines the Lie algebra up to isomorphism.*

Root systems from Lie groups

Consider a connected compact Lie group G with maximal torus T, taken to be fixed throughout this subsection. We are interested in studying its Lie algebra $\mathrm{L}G$, and in particular in understanding its associated root system. As we will see, the underlying vector space V of this root system is the dual space LT^* of the Lie algebra of T. To obtain this root system, we consider the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathrm{L}G)$. The roots of G will be obtained from $\mathrm{L}G$, regarded as a T-representation by restricting the action.

Before specializing to the case $V = \mathrm{L}G$, let us start by analyzing T-representations in more detail. First recall the description of the irreducible representations of T:

Proposition 5.3.6. *The characters of the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ are of the form*

$$\theta\colon \mathbb{R}^n/\mathbb{Z}^n \to S^1, \qquad [x] \mapsto \exp(2\pi i\alpha(x)),$$

where $\alpha(x) = \langle a, x\rangle = \sum_i a_i x_i$ for some $a = (a_1,\dots,a_n) \in \mathbb{Z}^n$.

Proof. Given a character $\theta\colon T^n \to S^1$, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^n \arrow[r, hook] \arrow[d, dashed, "\alpha|_{\mathbb{Z}^n}"] & \mathrm{L}T^n \arrow[r, two heads] \arrow[d, "\alpha:=\mathrm{L}\theta"] & T^n \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook] & \mathbb{R} \arrow[r, "\exp"] & S^1 \arrow[r] & 1,
\end{tikzcd}$$

determining the map α. □

Definition 5.3.7. Let V be a complex T-representation. An irreducible character $\theta\colon T \to S^1$ is called a *weight* of V if the corresponding isotypical summand

$$V(\theta) := \{v \in V \mid xv = \theta(x)\cdot v \text{ for all } x \in T\}$$

is nonzero. In this case, $V(\theta)$ is called the *weight space* of θ.

It follows from Proposition 2.1.19 that every complex T-representation is the direct sum of its weight spaces.

We will also need an ‘infinitesimal’ form of the notion of a weight.

Definition 5.3.8. Let $\mathfrak{g}$ be a Lie algebra. A *representation of* $\mathfrak{g}$ is a finite-dimensional complex vector space V equipped with a morphism of Lie algebras $\mathfrak{g} \to \mathrm{End}(V) = \mathrm{L}\,\mathrm{Aut}(V)$.

Spelling out the definition, this means that V comes equipped with a map $\mathfrak{g}\times V \to V$, $(X, v) \mapsto Xv$ satisfying the property that $X(Yv) - Y(Xv) = [X,Y]v$ for all $X, Y \in \mathfrak{g}$ and $v \in V$.

Example 5.3.9. If V is a representation of a Lie group G, then the derivative of the Lie group morphism $\varphi\colon G \to \mathrm{Aut}(V)$ determines a representation $\mathrm{L}\varphi\colon \mathrm{L}G \to \mathrm{L}\,\mathrm{Aut}(V)$ of the Lie algebra of G. Given $X \in \mathrm{L}G$, the induced action on V is given by the *Lie derivative*:

$$\mathrm{L}_X\colon V \to V, \qquad v \mapsto \lim_{t\to 0}\frac{1}{t}(\exp(tX)\cdot v - v).$$

The example has a partial converse:

Lemma 5.3.10. *Let G be a simply-connected Lie group. Then every representation of the Lie algebra* $\mathrm{L}G$ *comes from a representation of* G.

Proof. By Theorem 5.1.13 every morphism of Lie groups $\mathrm{L}G \to \mathrm{End}(V) = \mathrm{L}\,\mathrm{Aut}(V)$ lifts uniquely to a morphism of Lie groups $G \to \mathrm{Aut}(V)$. □

We now apply this to the situation of roots $\theta\colon T \to \mathrm{U}(1)$ of the Lie group G. Since such θ is a morphism of Lie groups, we may pass to associated Lie algebras and obtain a map

$$\Theta := \mathrm{L}\theta\colon \mathrm{L}T \to \mathrm{L}\,\mathrm{U}(1).$$

Recall from Example 1.2.34 that $\mathrm{L}\,\mathrm{U}(1) = \{a \in \mathbb{C} \mid a = -\overline{a}\} = i\mathbb{R}$.

Definition 5.3.11. Let V be a complex T-representation. An $\mathbb{R}$-linear map $\Theta\colon \mathrm{L}T \to \mathrm{L}\,\mathrm{U}(1) = i\mathbb{R} \subseteq \mathbb{C}$ is called an *infinitesimal weight* of V if the corresponding *weight space*

$$V(\Theta) = \{v \in V \mid L_X v = \Theta(X)\cdot v \text{ for all } X \in \mathrm{L}T\}$$

is nonzero.

Proposition 5.3.12. *The assignment $\theta \mapsto \mathrm{L}\theta$ defines a bijection between weights and infinitesimal weights of V. For every weight θ we have $V(\theta) = V(\mathrm{L}\theta)$.*

Proof sketch. Let us merely sketch how to produce the inverse. Given an infinitesimal weight $\Theta\colon \mathrm{L}T \to \mathrm{L}\,\mathrm{U}(1)$, we may pick some $0 \neq v \in V(\Theta)$ and compute that we have $\exp(X)\cdot v = e^{\Theta(X)}\cdot v$ for all $X \in \mathrm{L}T$. It follows that the assignment $\mathrm{L}T \to \mathrm{U}(1)$ given by $X \mapsto X \mapsto e^{\Theta(X)}$ factors through the map $\exp\colon \mathrm{L}T \to T$, and hence we obtain a map $\theta_\Theta\colon T \to \mathrm{U}(1)$. One may show that this is a weight of V and that the assignment $\Theta \mapsto \theta_\Theta$ is an inverse of $\theta \mapsto \mathrm{L}\theta$. □

Since V is a complex vector space, every representation $\mathrm{L}T \times V \to V$ of the Lie algebra $\mathrm{L}T$ on V extends to the complexified Lie algebra $\mathrm{L}T_{\mathbb{C}} = \mathbb{C} \times_{\mathbb{R}} \mathrm{L}T$. The extended action $\mathrm{L}T_{\mathbb{C}} \times V \to V$ is also denoted by $(X, v) \mapsto \mathrm{L}_X v$.

Definition 5.3.13. A $\mathbb{C}$-linear form $\Phi\colon \mathrm{L}T_{\mathbb{C}} \to \mathbb{C}$ is called a *complex infinitesimal weight* of V if the corresponding *weight space*

$$V(\Phi) = \{v \in V \mid \mathrm{L}_H v = \Phi(H)\cdot v \text{ for all } H \in \mathrm{L}T_{\mathbb{C}}\}$$

is nonzero.

Note that the map $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{L}\,\mathrm{U}(1) \to \mathbb{C}, z \otimes u \mapsto zu$ is an isomorphism. An $\mathbb{R}$-linear form $\Theta\colon \mathrm{L}T \to \mathrm{L}\,\mathrm{U}(1)$ therefore induces a $\mathbb{C}$-linear form

$$\Phi_\Theta\colon\ \mathbb{C} \otimes \mathrm{L}T \to \mathbb{C} \otimes \mathrm{L}\,\mathrm{U}(1) \cong \mathbb{C}$$

sending $X + iY$ to $\Theta(X) + i\Theta(Y)$ for $X, Y \in \mathrm{L}T$.

Proposition 5.3.14. *The assignment $\Theta \mapsto \Phi_\Theta$ defines a bijection between infinitesimal and complex infinitesimal weights of the complex T-representation V. Furthermore, $V(\Theta) = V(\Phi_\Theta)$.*

Proof. Omitted, see Bröcker and tom Dieck [Bt95, Proposition 9.6]. □

Definition 5.3.15. An $\mathbb{R}$-linear form $\alpha\colon \mathrm{L}T \to \mathbb{R}$ is called a *real (infinitesimal) weight* of the complex T-representation V if the map $2\pi i\alpha\colon \mathrm{L}T \to i\mathbb{R}$ is an infinitesimal weight of V. For $V = \mathrm{L}G$ we call α a *real root* of G.

All in all, we obtain for every complex T-representation bijections

$$\begin{tikzcd}
 & \{\text{ Complex infinitesimal weights } \Phi\colon \mathrm{L}T_{\mathbb{C}} \to \mathbb{C} \text{ of } V\ \} \\
\{\text{ Weights } \theta\colon T \to \mathrm{U}(1) \text{ of } V\ \} \arrow[r, "\cong"] & \{\text{ Infinitesimal weights } \Theta\colon \mathrm{L}T \to \mathrm{L}\,\mathrm{U}(1) = i\mathbb{R} \text{ of } V\ \} \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
 & \{\text{ Real weights } \alpha\colon \mathrm{L}T \to \mathbb{R} \text{ of } V\ \}.
\end{tikzcd}$$

We will now specialize to the case $V = \mathrm{L}G$.

Definition 5.3.16. The *roots* of G are the weights of the T-representation $\mathrm{L}G$. More specifically:

- The *global roots* are the weights $\theta\colon T \to \mathrm{U}(1)$ of $\mathrm{L}G$;
- The *infinitesimal roots* are the corresponding linear forms $\Theta = \mathrm{L}\theta\colon \mathrm{L}T \to i\mathbb{R}$;
- The *complex roots* are the induced linear forms $\Phi\colon \mathrm{L}T_{\mathbb{C}} \to \mathbb{C}$,
- The *real roots* are the induced linear forms $\alpha = \Theta/2\pi i\colon \mathrm{L}T \to \mathbb{R}$.

We let $R\langle\mathbb{C}\rangle$ denote the set of complex roots and $R = R\langle\mathbb{R}\rangle$ the set of real roots of G.

The decomposition into weight spaces now takes the form

$$\mathrm{L}G_{\mathbb{C}} = L_0 \oplus \bigoplus_{\alpha} L_\alpha, \qquad \alpha \in R\langle\mathbb{C}\rangle,$$

with weight spaces given by

$$L_\alpha = \{X \in \mathrm{L}G_{\mathbb{C}} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathrm{L}T_{\mathbb{C}}\}.$$

Theorem 5.3.17. *Let G be a compact connected Lie group with maximal torus T and let $V \subseteq LT^*$ denote the subspace generated by the set R of real roots of G. Then R is a root system in V and the Weyl group W of G, viewed as a subgroup of* $\mathrm{Aut}(V)$*, is the same as the Weyl group of this root system as defined in Definition 5.3.4.*

The root system on $V \subseteq LT^*$ from the theorem is precisely the root system associated to the Lie algebra $\mathrm{L}G$ via Theorem 5.3.5.

5.4 Weyl chambers and Dynkin diagrams

The content of this section is not part of the exam material.

In this final section we will discuss Dynkin diagrams and their relation to root systems.

Before we start, let us recall some of the motivation. In Theorem 5.2.10, we stated without proof a full classification of all the compact simply-connected simple Lie groups. Roughly speaking, this classification proceeds as follows:

(1) Via the correspondence from Theorem 5.1.18, the compact simply-connected simple Lie groups correspond to certain finite-dimensional Lie algebras called the *compact* simple Lie algebras;

(2) By sending a Lie algebra $\mathfrak{g}$ to its complexification $\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, one obtains a one-to-one correspondence between real compact simple Lie algebras and complex compact simple Lie algebras.

(3) Via Theorem 5.3.5, the latter are classified by their root systems.

To complete the picture of Theorem 5.2.10, we will now proceed with the following two steps:

(4) Root systems are completely determined by a choice of *base* for the root system together with a collection of *Cartan numbers*;

(5) Bases of root systems may be encoded in terms of certain *Dynkin diagrams*.

Bases and Weyl chambers

In the next couple of results and definitions, we let V be a finite-dimensional real vector space and let R be a root system in V with Weyl group W.

To every root α we associated the hyperplane $\mathcal{H}_\alpha \subseteq V$ which is fixed under the corresponding symmetry $s_\alpha \in W$. Since W is finite, we may choose a W-invariant inner product on V, and with respect to this metric $\mathcal{H}_\alpha$ is the hyperplane orthogonal to α and s_α is the reflection in $\mathcal{H}_\alpha$. The hyperplanes $\mathcal{H}_\alpha$ divide V into finitely many convex regions called the *Weyl chambers* of the root system:

Definition 5.4.1. The *Weyl chambers* of the root system (V, R) are the connected components of $V \setminus \bigcup_{\alpha\in R} \mathcal{H}_\alpha$. The Weyl chambers are open subsets of V. We denote the closure of a Weyl chamber K by $\overline{K}$. The *walls* of a Weyl chamber K are those subsets of the form $\overline{K} \cap \mathcal{H}_\alpha$ which have dimension $k-1$, where $k = \dim(V)$.

Since the Weyl group sends roots to roots and consists of orthogonal transformations, it also preserves the hyperplanes $\mathcal{H}_\alpha$, and hence it induces an action on the set of all Weyl chambers. We have the following theorem:

Theorem 5.4.2 (Bröcker and tom Dieck [Bt95, Theorem 4.1])**.** *Given a point $p \in V$, let W_p be the isotropy group of p in the Weyl group W of a root system R in V.*

(i) The group W_p acts freely and transitively on the set of Weyl chambers K with $p \in \overline{K}$.

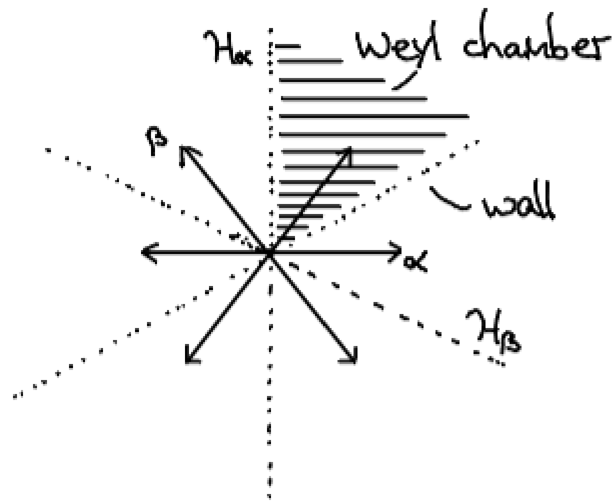


Figure 5.1: Example of a Weyl chamber

(ii) If $p \in \overline{K}$, then W_p is generated by the reflections in the walls of K which contain p.

(iii) The orbit of p meets every closed Weyl chamber in exactly one point.

Note that for $p = 0$ we have $W_p = W$ and we have $p \in \overline{K}$ for all K, so that the statement reduces to the following:

Lemma 5.4.3 (Bröcker and tom Dieck [Bt95, Lemma 4.2])**.** *The Weyl group W acts freely and transitively on the set of all Weyl chambers. Given any Weyl chamber K, W is generated by the reflections in the walls of K.*

I don't want to go into the proofs of these results, even though they are not that difficult. But let us at least see that the statements make sense in the example of a root system given in Figure 5.1:

- For $p = 0$, we see that W indeed acts freely and transitively on the Weyl chambers, and that W is generated by the two walls of the Weyl chamber highlighted in the above figure.
- For a point p in the interior of a Weyl chamber K, we see that $W_p = \{1\}$, and indeed there is only one closed Weyl chamber containing p.
- For a point p contained in a wall $\overline{K} \cap \mathcal{H}_\alpha$, we see that $W_p = \{1, s_\alpha\}$ is of order two, and indeed there are precisely two closed Weyl chambers containing p.

Note that the result in particular says that every closed Weyl chamber is a fundamental domain of the Weyl group action on V, and that every closed Weyl chamber is mapped homeomorphically onto the orbit space of the operation of W by the canonical projection.

Note that a given Weyl chamber is completely described by specifying its walls and specifying on which side of these walls the chamber lies. But specifying a wall $\mathcal{H}_\alpha$ and a side of that wall is the same as specifying a root α. So the Weyl chambers correspond to certain subsets of R which we now describe more algebraically.

Definition 5.4.4. A subset S of the root system R in V is called a *basis* or a *system of simple roots* if S is linearly independent in V and every root $\beta \in R$ may be written as

$$\beta = \sum_{\alpha \in S} m_\alpha \cdot \alpha,$$

where the m_α are integers such that either all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$. The elements of S are called *simple roots* with respect to S.

For example, in the root system from Figure 5.1 we see that the subset $S = \{\alpha, \beta\}$ forms a basis. We will soon see that bases actually exist for every root system. So suppose that S is a fixed basis for R. We may then split up R into two disjoint parts

$$R = R_+ \cup R_-, \qquad R_- = -R_+,$$

where R_+ consists of those roots β whose coefficients m_α in Definition 5.4.4 are all non-negative. These are called the *positive roots*, and the elements of R_- are called the *negative roots*. Note that R_+ and S determine each other: R_+ is obtained from S by taking finite sums of elements in S, and conversely S consists of the 'indecomposable' elements of R_+, i.e. those that cannot be written as a sum of two elements of S.

The next theorem says that bases exist and are in one-to-one correspondence with Weyl chambers:

Proposition 5.4.5. *The following assignment defines a bijection between the set of Weyl chambers and the set of bases in the root system R in V.*

We assign to every Weyl chamber K the set of positive roots $R_+(K) = \{\alpha \in R \mid \langle \alpha, t\rangle > 0$ for one and therefore all $t \in K\}$, and the basis $S(K)$ consists of the elements which are indecomposable in $R_+(K)$.

Conversely, we assign to each basis S the Weyl chamber

$$K(S) = \{t \in V \mid \langle \alpha, t\rangle > 0 \text{ for all } \alpha \in S\}.$$

It has walls $\mathcal{H}_\alpha \cap \overline{K}(S)$, $\alpha \in S$, where $\mathcal{H}_\alpha$ is the hyperplane orthogonal to α. We call $K(S)$ the fundamental *Weyl chamber corresponding to S.*

For example, the Weyl chamber corresponding to the basis $S = \{\alpha, \beta\}$ in Figure 5.1 is the highlighted Weyl chamber.

Dynkin diagrams

We have seen that a root system is completely determined by a choice of basis. We will now discuss a quick and easy diagrammatic description of the bases.

Let R be a root system in V with basis S and Weyl group W. We assume that V is equipped with a W-invariant Euclidean inner product $\langle -, -\rangle$, which allows us to identify V with its dual via the isomorphism

$$\kappa \colon V \to V^*, \qquad v \mapsto \langle v, -\rangle.$$

Recall that to every root α we assigned an *inverse root* $\alpha^* \in V^*$ with the property that $\alpha^*|_{\mathcal{H}_\alpha} = 0$ and $\alpha^*(\alpha) = 2$. Under the isomorphism κ, we may identify α^* with a vector in V, and as such it is given by

$$\alpha^* = 2/|\alpha|^2 \cdot \alpha.$$

Now recall that condition (iv) of a root system demands that the inner products $\langle \alpha^*, \beta \rangle$ are integers for all $\alpha, \beta \in R$.

Definition 5.4.6. The integers

$$n_{\alpha\beta} := \langle \alpha^*, \beta \rangle = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}, \qquad \alpha, \beta \in R$$

are called the *Cartan numbers* of the root system R. We obtain an associated matrix

$$(n_{\alpha\beta}), \qquad \alpha, \beta \in S,$$

called the *Cartan matrix*.

Up to reordering the rows and columns, the Cartan matrix is independent of the choice of basis and is just an invariant of the root system. Furthermore, the Cartan matrix actually completely determines the root system.

These numbers are subject to rather strong conditions, since we have

$$\langle \alpha, \beta \rangle = |\alpha| \cdot |\beta| \cdot \cos(\alpha, \beta),$$

where $\cos(\alpha, \beta)$ is the cosine of the angle between α and β. Thus we get

$$\begin{aligned} n_{\alpha\beta} &= 2\frac{|\beta|}{|\alpha|}\cos(\alpha, \beta) \\ n_{\alpha\beta} n_{\beta\alpha} &= 4\cos^2(\alpha, \beta). \end{aligned}$$

Using these relations, we find that, possibly after switching α and β, there are only the following seven possibilities for non-proportional roots:

	$n_{\alpha\beta}$	$n_{\beta\alpha}$	angle(α, β)	$\lvert\alpha\rvert^2/\lvert\beta\rvert^2$
(i)	0	0	$\pi/2$	?
(ii)	1	1	$\pi/3$	1
(iii)	-1	-1	$2\pi/3$	1
(iv)	1	2	$\pi/4$	2
(v)	-1	-2	$3\pi/4$	2
(vi)	1	3	$\pi/6$	3
(vii)	-1	-3	$5\pi/6$	3

Using this observation, one can show that there are, up to isomorphism, only four root systems of rank 2:

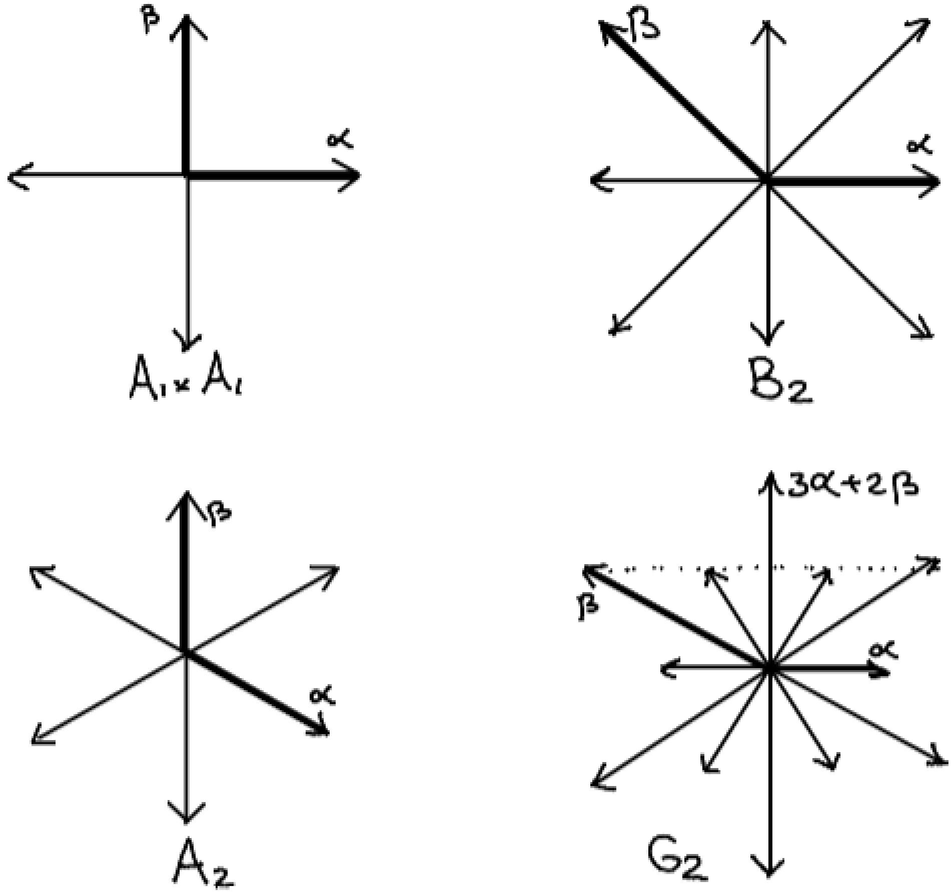


Figure 5.2: Root systems of rank 2

Here the names of the root systems are the standard ones in the literature. In each case we have indicated a choice of basis $S = \{\alpha, \beta\}$ of the root system. It turns out that elements of the basis S always form obtuse angles:

Lemma 5.4.7. *Consider a basis S and roots $\alpha, \beta \in S$ with $\alpha \neq \beta$. Then α and β form an obtuse angle, i.e. $\langle \alpha, \beta \rangle \leq 0$.*

Proof. Assume $\langle \alpha, \beta \rangle > 0$, or equivalently $n_{\alpha\beta} > 0$. Then also $n_{\beta\alpha} > 0$, and since we have $n_{\alpha\beta} \cdot n_{\beta\alpha} < 4$ at least one of these integers must be one. One may deduce from this that then also $\alpha - \beta$ and $\beta - \alpha$ are roots. But at least one of them, say $\alpha - \beta$, then lies in R_+, which gives a contradiction since then $\alpha = (\alpha - \beta) + \beta$ is decomposable. $\square$

All in all, we see that the Cartan matrix of a root system is very restricted:

$$n_{\alpha\alpha} = 2, \qquad n_{\alpha\beta} \in \{0, -1, -2, -3\} \quad \text{for } \alpha \neq \beta.$$

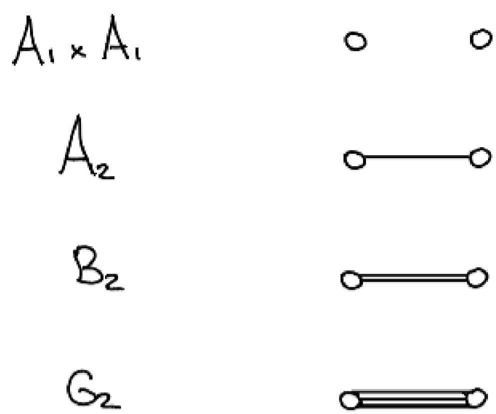
For example, the root systems $A_1 \times A_1$, B_2, A_2 and G_2 gives rise to the matrices

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Definition 5.4.8. Let R be a root system in V with basis S. The associated *Coxeter graph* has one vertex for each element of S, and every pair α, β of distinct vertices is connected by $n_{\alpha\beta} \cdot n_{\beta\alpha}$ edges.

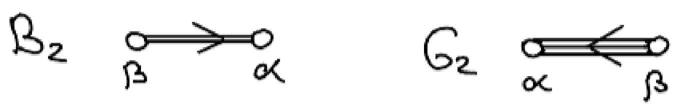
Note that this product can only be 0, 1, 2 or 3. Up to isomorphism of graphs, the Coxeter graph only depends on the root system and not on the choice of basis S.

The Coxeter graphs for the root systems of rank two from Figure 5.2 are given as follows:



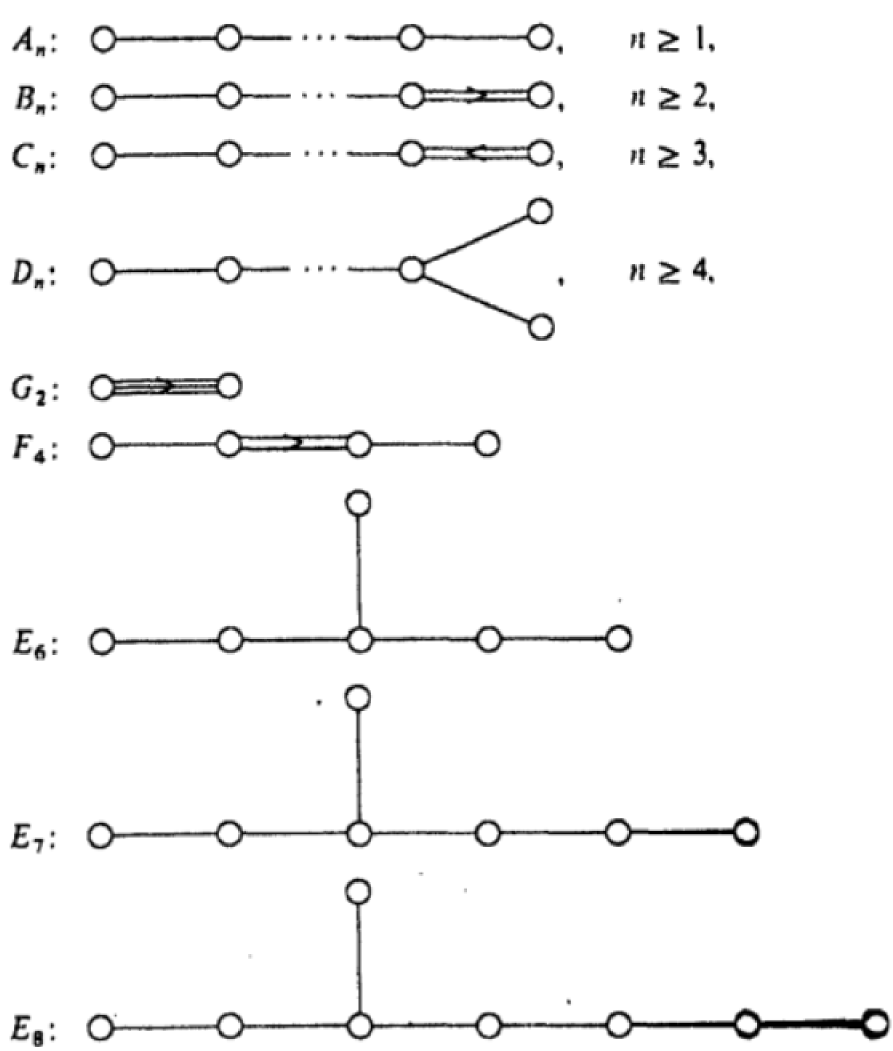
The Coxeter graph characterizes the root system in cases where all roots have the same length. If, however, two different lengths occur, then the graph only gives the angle between the two roots α and β. In this case one requires the additional information of which root is longer.

Definition 5.4.9. The *Dynkin diagram* consists of the Coxeter graph together with the symbol $<$ or $>$ attached to each doubled or tripled edge pointing to the shorter root.



Proposition 5.4.10. *A root system is determined up to isomorphism by its Dynkin diagram.*

Theorem 5.4.11. *The Dynkin diagram of an irreducible root system is isomorphic to exactly one of the diagrams in the following list having n vertices:*


A_n:
$n \geq 1$,
B_n:
$n \geq 2$,
C_n:
$n \geq 3$,
D_n:
$n \geq 4$,
G_2:
F_4:
E_6:
E_7:
E_8:

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