Oberseminar on

Parametrized semiadditivity

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Various interesting ∞-categories arising throughout mathematics, like any abelian category, the ∞-category of spectra, or the derived ∞-category of a scheme, are *semiadditive*, meaning that they admit both finite coproducts and finite products and that these two constructions agree. This simple *property* provides the ∞-category with useful algebraic *structure*. For example, every semiadditive ∞-category admits an enrichment in the ∞-category of commutative monoids: given two morphisms $f, g\colon X \to Y$, their sum $f + g$ is given as the composite

$$f + g\colon X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} Y \times Y \cong Y \amalg Y \xrightarrow{\nabla} Y, \tag{1}$$

where $\Delta = (\mathrm{id}_X, \mathrm{id}_X)$ denotes the diagonal and $\nabla = (\mathrm{id}_Y, \mathrm{id}_Y)$ denotes the fold map ('codiagonal'). Getting this structure for free is particularly helpful in higher categorical contexts, where specifying a commutative monoids requires an infinite tower of coherence data in addition to the map (1).

The idea of semiadditivity admits a plethora of useful generalizations, for example in equivariant homotopy theory, in chromatic homotopy theory, and in the context of six-functor formalisms. These generalizations are based on the notion of *ambidexterity* by Hopkins and Lurie [HL13], which describes the phenomenon that certain functors f^* admit both a left adjoint $f_!$ and a right adjoint f_* which canonically agree with each other. Also in these generalized settings the *property* of semiadditivity provides additional *structure* in the form of enrichments in certain sheaves with transfers/Mackey functors.

The purpose of this seminar is to provide an introduction to the general theory of parametrized semiadditivity developed in [CLL24], to explore a variety of natural examples of ambidexterity/parametrized semiadditivity, and to study various higher semiadditive analogues of classical concepts.

Program

- **Talk 1: Introduction** (October 15th 2024, Bastiaan Cnossen)
  Give an informal discussion of the main examples of parametrized semiadditivity (ordinary semiadditivity, equivariant semiadditivity, higher semiadditivity and ambidexterity in geometry). Define what it means for a functor $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ to be $\mathcal{Q}$-*semiadditive* for an inductible subcategory $\mathcal{Q} \subseteq \mathcal{A}$. Give the equivalent characterizations from [CLL24, Proposition 3.44].

  Reference: [CLL24, Sections 1 and 3].

- **Talk 2: The universality of spans** (October 22nd 2024, Denis-Charles Cisinski)
  Define the parametrized span category $\underline{\mathrm{Span}}(\mathcal{Q})$, prove that it is $\mathcal{Q}$-semiadditive, and discuss the proof of its universal property.

  Reference: [CLL24, Sections 4 and 5].

- **Talk 3: The universality of commutative monoids** (October 29th 2024, Divya Ghanshani)
  Introduce the parametrized category $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ of $\mathcal{Q}$-commutative monoids in a $\mathcal{B}$-category $\mathcal{C}$, and prove its universal property. Give the alternative description of $\underline{\mathrm{CMon}}^{\mathcal{Q}}(\mathcal{C})$ in terms of sheaves with transfers in case $\mathcal{C}$ comes from a non-parametrized category.

  Reference: [CLL24, Sections 7 and 8, parts of 6].

- **Talk 4: Equivariant semiadditivity** (November 5th 2024, Luca Pol)
  Introduce the notion of a *global category*, and give various examples (Borel G-objects, Kasparov KK-categories, genuine G-spectra, G-global spectra, G-Mackey functors). Formulate the universal properties of the global categories of genuine G-spectra and G-global spectra from [CLL23b, Theorem C] and [CLL23a, Theorem C], respectively. Conclude (using the previous talk) that both admit descriptions in terms of spectral Mackey functors [CLL24, Section 9.3].

  Reference: [CLL23a; CLL23b].

- **Talk 5: Higher semiadditivity in chromatic homotopy theory** (November 12th 2024, Sil Linskens)
  Recall the ∞-categories $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$ of $K(n)$-local and $T(n)$-local spectra, [CSY22, Section 5.1]. Introduce the notion of m-semiadditivity for $-2 \leq m \leq \infty$, and state the main theorems of [HL13; CSY22] about the ∞-semiadditivity of $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{T(n)}$. Discuss the Bootstrap Machine from [CSY22, Theorem 3.4.10]. Explain how one applies the Bootstrap Machine to the case $\mathcal{C} = \mathrm{Sp}_{K(n)}$.

  Reference: [HL13], [CSY22].

- **Talk 6: Chromatic cardinalities** (November 19th 2024, Pier Federico Pacchiarotti)
  Define the *cardinality* $|A|\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ of a $\mathcal{C}$-ambidextrous space A, and formulate the main result of Carmeli–Yuan [CY23]. Define Grothendieck-Witt theory, and discuss how to equip

it with the structure of a higher commutative monoid. Discuss how this structure induces the higher semiadditive structure on $\mathbb{S}_{K(1)}$, and explain why this leads to a computation of the cardinalities $|A|_{\mathbb{S}_{K(1)}}$.

Reference: [CY23], [CSY21, Section 3].

- **Talk 7: Semiadditive height** (November 26th 2024, Tashi Walde)
Define the *semiadditive height* of a (p-typically) m-semiadditive ∞-category $\mathcal{C}$, and more generally of objects $X \in \mathcal{C}$. Show that every p-local n-semiadditive ∞-category of height at most n is even ∞-semiadditive [CSY21, Theorem 3.2.7]. Discuss the Semiadditive Redshift Theorem [CSY21, Theorem 3.3.2]. Discuss the height decomposition [CSY21, Theorem 4.2.7] for stable m-semiadditive ∞-categories.

  Reference: [CSY21].

- **Talk 8: The semiadditive Fourier transform** (December 3rd 2024, Pelle Steffens)
Give a quick recollection of the ordinary Fourier transform $\mathfrak{F}_\omega \colon R[M] \xrightarrow{\sim} R^{M^*}$ for a primitive m-th root of unity $\omega \colon \mathbb{Z}/m \to R^\times$, and formulate its chromatic analogue due to Hopkins–Lurie [Bar+22, Theorem 1.1]. Introduce the n-shifted Brown–Comenetz dual $I_p^{(n)}M$ of a connective spectrum M, and define the *pre-orientations* of a commutative algebra S in $\mathcal{C}$. Given a pre-orientation ω, construct the semiadditive Fourier transform $\mathfrak{F}_\omega \colon \mathbb{1}[-] \to \mathbb{1}^{\lfloor I_p^{(n)}(-) \rfloor}$. Discuss the notion of *orientability* of a pre-orientation ω, and discuss the fact that in the higher semiadditive case there exists a *universal* oriented commutative algebra $\mathbb{1}[\omega^{(n)}]$ called the *cyclotomic extension of height n*.

  Reference: [Bar+22].

- **Talk 9: Global semiadditivity for Kasparov K-theory** (December 10th 2024, Benjamin Dünzinger)
Introduce the stable ∞-categories KK^G via their universal property [BEL21]. Discuss the resulting global category $\underline{\mathrm{KK}}$, and show it is equivariantly semiadditive [CLL23c, Example 5.10]. Discuss the fact from [BEL21, Theorem 1.23] that the two adjoints to the inflation functor $\mathrm{KK} \to \mathrm{KK}^G$ agree, and speculate on the *global* semiadditivity of $\underline{\mathrm{KK}}$.

  Additionally/alternatively: Discuss the parametrized semiadditivity of the functor $X \mapsto \mathrm{Mot}^{\mathcal{E}}_{\mathrm{loc}}(X)$ which sends a space X to the ∞-category of localizing motives for X-indexed $\mathcal{E}$-linear dualizable ∞-categories, $\mathcal{E} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$.

  Reference: [BEL21].

- **Talk 10: Ambidexterity and transchromatic character theory** (January 7th 2024, Sil Linskens)

  Introduce the *HKR character map* $\chi \colon E_n^X \to C_k^{L_p^{n-k}X}$ for a π-finite space X. Give a categorification of this map in terms of tempered local systems, and show that it enhances to a parametrized functor. Use the result from Talk 3 to deduce the naturality of the map χ in X as functors $\mathrm{Span}(\mathrm{Spc}_\pi) \to \mathrm{Sp}$.

Reference: [Ben24].

- *No talk on January 14th.*

- **Talk 11: The universal bi-adjointable functor (January 21th, Bastiaan Cnossen)**
Generalize the notions of parametrized (co)completeness to functors $\mathcal{C}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{E}$ into an arbitrary $(\infty,2)$-category $\mathcal{E}$. Provide a proof sketch that, when $P \subseteq I \subseteq \mathcal{A}$ are both inductible, the universal I-cocomplete P-complete such functor is the inclusion $\mathcal{A}^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(\mathcal{A}, I)_I^P$ into its span 2-category. Mention the more general conjecture and how it relates to six-functor formalisms.

  This result is work in progress by Ben-Moshe–Cnossen–Lenz–Linskens–Yanovski.

- **Talk 12: Higher semiadditivity of (∞, k)-categories (January 28th, Tashi Walde)**
Define what it means for an (∞,k)-category to be m-semiadditive. Discuss the result by Scheimbauer and Walde that the universal m-semiadditive (∞,k)-category is the (∞,k)-category of iterated spans of m-finite spaces. More generally, discuss the free m-semiadditive (∞,k)-category generated by an ∞-category $\mathcal{C}$ in terms of decorated spans.

  This result is work in progress by Scheimbauer–Walde.

- **Talk 13: Lax (semi)additivity (Johannes Glossner, February 4th)**
Recall the notions of lax limits and colimits in an $(\infty,2)$-category $\mathbb{C}$. Define the notion of a *lax matrix* in $\mathbb{C}$, and construct for every diagram $\mathcal{X}\colon S \to \mathbb{C}$ the *identity matrix* $\mathcal{I}^{\mathcal{X}}$. Define what it means for an $(\infty,2)$-category to be *lax (semi)additive*, and give some examples. Discuss some features of the resulting theory, up to the taste of the speaker.

  Reference: [CDW24].

Further questions

Below you may find two further questions that are closely related to the topic of this seminar.

- **Ambidexterity and character theory**
Introduce the *character* $\chi_V\colon LA \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ of an A-indexed collection $V \in (\mathcal{C}^{\mathrm{dbl}})^A$ of dualizable objects in a presentably symmetric monoidal ∞-category $\mathcal{C}$, where A is $\mathcal{C}$-ambidextrous (see [Car+22]). Discuss how the resulting character map $\chi\colon (\mathcal{C}^{\mathrm{dbl}})^A \to \mathrm{End}_{\mathcal{C}}(\mathbb{1})^{LA}$ can be categorified, and use the result from Talk 3 to deduce that for m-semiadditive $\mathcal{C}$ the character map χ is natural in A as functors $\mathrm{Span}(\mathrm{Spc}_m) \to \mathrm{Spc}$. Explain how this provides a coherent form of the *induced character formula* of [Car+22].

  This result is work in progress by Carmeli–Cnossen–Ramzi–Yanovski.

- **Parametrized semiadditivity for six-functor formalisms**
Define the notion of a three-/six-functor formalism $D\colon \mathrm{Span}(C,E) \to \mathrm{Cat}$, [Man22]. Define

what it means for a morphism f in E to be *cohomologically étale* or *cohomologically proper*. Show that the cohomologically étale and proper maps form an inductible subcategory Q of E, and that D is Q-semiadditive for this choice of Q. Provide various examples of six-functor formalisms and explain which maps are cohomologically étale and proper in these cases.

Reference: [Man22], [Sch23].

References

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[HL13] Michael Hopkins and Jacob Lurie. "Ambidexterity in K(n)-local stable homotopy theory". *arXiv preprint arXiv:1510.03304* (2013).

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[Sch23] Peter Scholze. *Six-Functor Formalisms*. 2023.