AMBIDEXTERITY IN CHROMATIC HOMOTOPY THEORY

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1. Integration

Suppose that $\mathcal{C}$ is a presentable n-semiadditive category, where $n \geq 1$.

Definition 1.1. Recall that $\mathcal{C}$ is enriched over $\mathrm{CMon_n}(\mathcal{S})$. In particular given a n-finite space B and $X, Y \in \mathcal{C}^B$, the mapping space $\mathrm{Hom}_{\mathcal{C}^B}$ enhances to a functor

$$\mathrm{Hom}_{\mathcal{C}^B}(X, Y)\colon \mathrm{Span}((\mathcal{S}_n)_{/B}) \to \mathcal{S}.$$

Given a map $p\colon A \to B$ of n-finite spaces, we write

$$\int_p \colon \mathrm{Hom}_{\mathcal{C}^A}(p^*X, p^*Y) \to \mathrm{Hom}_{\mathcal{C}^B}(X, Y)$$

for the value of the functor above on the map $A = A \to B$.

There are a ton of relations that these integration maps satisfy, all of which are encoded by the enrichment in n-commutative monoids. We highlight one:

Proposition 1.2. *Let $p\colon A \to B$ be a map of n-finite spaces and consider $f\colon p^*X \to p^*Y$ and $g\colon Y \to Z$. Then*

$$\int_p (p^*g \circ f) \simeq g \circ \int_p f.$$

Proof. $g \circ (-)\colon \mathrm{Hom}_{\mathcal{C}^B}(X, Y) \to \mathrm{Hom}_{\mathcal{C}^B}(Y, Z)$ is a map of n-commutative monoids. ∎

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2. Amenability (or how to climb up the truncation tower)

Definition 2.1. Consider a map $p\colon A \to B$ of n-finite spaces. We define the natural transformation $|p|\colon \mathrm{Id}_{\mathcal{C}^B} \Rightarrow \mathrm{Id}_{\mathcal{C}^B}$ whose component at X equals $|p|_X = \int_f \mathrm{id}_{p^*X}$. In the case of $p\colon A \to *$ we write $|p| = |A|$.

We say a map f (or a space A) is $\mathcal{C}$-amenable if $|f|$ (or $|A|$) is a natural equivalence.

Remark 2.2. One can unwind the definition of $|f|$ to show that its component at X is given by the composite

$$X \xrightarrow{u} f_*f^*X \xrightarrow{\mu} X.$$

Example 2.3. Suppose $A = \{n\}$. Then $|n|$ is the transformation

$$Y \xrightarrow{\Delta} Y^{\times n} \xrightarrow{m} Y.$$

In other words, it is multiplication by n. In particular $\{n\}$ is amenable if and only if multiplication by n is an equivalence.

In general we think of $|A|$ as integration of the identity along A (this gives the “size” of A).

Proposition 2.4. *Amenability can by checked fiberwise. $f\colon A \to B$ is $\mathcal{C}$-amenable if and only if A_b is $\mathcal{C}$-amenable for all $b \in \pi_0(B)$.*

What is the key use of amenability?

Theorem 2.5. *(Higher Maschke's theorem) Suppose $p\colon A \to B$ is $\mathcal{C}$-amenable. Then for every $X \in \mathcal{C}^B$ the counit map $c_!\colon p_!p^*X \to X$ admits a section. In particular every object in $\mathcal{C}^B$ is a retract of an object in the essential image of $p_!$.*

Proof. Consider $X \in \mathcal{C}^B$. The composite

$$p^*X \xrightarrow{u} p^*p_!p^*X \xrightarrow{c} p^*X$$

is the identity by the triangle identity. We may apply $\int_p$ and compute

$$|p| = \int_p \mathrm{id} \simeq \int_p p^*(c) \circ u \simeq c \circ \int_p u.$$

$|p|$ is an equivalence by assumption, and so we find that c admits a section. $\square$

Proposition 2.6. *Suppose $A \xrightarrow{p} B \xrightarrow{q} C$ such that p is $\mathcal{C}$-amenable and q is weakly $\mathcal{C}$-semiadditive. Then pq is $\mathcal{C}$-semiadditive if and only if q is $\mathcal{C}$-semiadditive.*

Proof. $\mathrm{Nm}_p q$ is equal to the composite

$$q_! p_! \xrightarrow{Nm_q} q_* p_! \xrightarrow{Nm_p} q_* p_*.$$

From this the "if" is clear.

Converselt suppose pq is $\mathcal{C}$-semiadditive. Then by 2-out-of-3 $\mathrm{Nm}_q \colon q_! p_! \to q_* p_!$ is an equivalence. In otherwards, Nm_q is an equivalence for objects in the image of $p_!$. That is all objects up to retracts, proving the result. $\square$

Let us use this to simplify our problem slightly.

Proposition 2.7. *Let $\mathcal{C}$ be p-local and stable. Then $\mathcal{C}$ is $n+1$-semiadditive if and only if for $q \colon B^{n+1}C_p \to *$,*

$$\mathrm{Nm}_q \colon q_! \to q_*$$

is an equivalence.

Proof. Given a fiber sequence $A \to B \to C$, B is $\mathcal{C}$-semiadditive if and only if A and C are. Therefore it suffices to prove that $q \colon B^{n+1}C_l \to *$ is $\mathcal{C}$-semiadditive for all primes l. One can show that $q^* \colon \mathcal{C} \to \mathrm{Fun}(BC_l, \mathcal{C})$ is an equivalence if $l \neq p$, and so clearly $\mathcal{C}$-semiadditive. $\square$

Proposition 2.8. *Let $\mathcal{C}$ be a stable, p-local, n-semiadditive category. Suppose that there is a connected n-finite p-space A such that $\pi_n(A) \neq 0$ and $|A|$ is an equivalence. Then $\mathcal{C}$ is $n+1$-semiadditive.*

Proof. n-finite p-spaces are always nilpotent, and so admit a principal refinement of there Postnikov tower. Because $\pi_n(A) \neq 0$ we find that there is a fiber sequence

$$A \to B \to B^{n+1}C_p$$

with B an n-finite space. Because A is amenable and $B^{n+1}C_p$ is connected, we conclude that the map $B \to B^{n+1}C_p$ is amenable. So $B^{n+1}C_p$ is $\mathcal{C}$-semiadditive if and only if B is. But this is because $\mathcal{C}$ is n-semiadditive. $\square$

3. Duality and cardinalities

We have seen that it suffices to compute the cardinalities of the B^mC_p. We will now discuss a very useful technique for accessing the cardinality of n-finite spaces.

Proposition 3.1. *Let A be a n-finite space and let $X \in \mathcal{C}$ be an object. Then*

$$|A|_X \simeq X \otimes |A|_{\mathbb{1}}.$$

In particular $|A|$ is amenable if and only if $|A|_{\mathbb{1}} \in \mathcal{R}_{\mathcal{C}}$ is a unit.

Proof. $X \otimes -$ is an n-semiadditive functor. So $X \otimes |A|_{\mathbb{1}} \simeq |A|_{X \otimes \mathbb{1}} \simeq |A|_X$. $\square$

The n-semiadditive category $\mathcal{C}$ admits a symmetric monoidal universal property. There exists a unique symmetric monoidal n-semiadditive functor

$$\mathrm{Span}(\mathcal{S}_n) \to \mathcal{C}$$

determined by the fact that it sends the point to the unit of $\mathcal{C}$.

Proposition 3.2. *Let $q\colon A \to *$ be an n-finite space, then $\mathbb{1}_A = q_!q^*\mathbb{1}$ is self dual in $\mathcal{C}$.*

Proof. This follows immediately from the fact that A is self dual in $\mathrm{Span}(\mathcal{S}_n)$. The duality data there is given by:

$$\begin{tikzcd} & A \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ A \times A & & A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "\Delta"] & \\ A & & A \times A \end{tikzcd}$$

□

Remark 3.3. You can make this into an if and only if (Basically $\mathrm{Span}(\mathcal{S}_n)$ has a second symmetric monoidal universal property as the free category on $\mathcal{S}_n$ such that every object of $\mathcal{S}_n$ is self-dual.

Definition 3.4. Recall that given a dualizable object X in a symmetric monoidal category $\mathcal{C}$ we write $\dim(X)$ for the composite

$$\mathbb{1} \to X \otimes X^\vee \simeq X^\vee \otimes X \to \mathbb{1}.$$

Proposition 3.5. *Let A be a n-finite space. Then* $\dim(A) = (* \leftarrow A^{S^1} \to *) = |A^{S^1}|_{\mathbb{1}}$. *In particular the same holds in any presentably symmetric monoidal n-semiadditive $\mathcal{C}$.*

Proof. We simply compute

$$\begin{tikzcd}
& & A^{S^1} \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & & \\
& A \arrow[dl, equal] \arrow[dr, "\Delta"] & & A \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\
A & & A \times A & & A
\end{tikzcd}$$

□

Corollary 3.6. *Suppose A is a loop space. Then* $\dim(A) = |A||\Omega A|$.

Proof. A^{S^1} splits as $A \times \Omega A$, and a simple computation shows that $|\Omega A \times A| = |A||\Omega A|$. □

4. Higher Semiadditivity and additive derivations

Let $\mathcal{C}$ be presentably symmetric monoidal, stable and 1-semiadditive. We write $\mathcal{R}_{\mathcal{C}}$ for the endomorphism spectrum of the unit.

Example 4.1. $\mathrm{Sp}_{T(n)}$ is an example, by a theorem of Kuhn. See also Clausen-Mathew, who simplify the proof drastically.

Definition 4.2. We define the functor

$$\Theta^p\colon \mathcal{C} \to \mathcal{C}^{BC_p}, \quad X \mapsto X^{\otimes C_p}.$$

Definition 4.3. We define an operation

$$\alpha\colon \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{C}}, \quad f \mapsto \int_{BC_p} \Theta^p(f).$$

We also define

$$\delta\colon \pi_0\mathcal{R}_{\mathcal{C}} \to \pi_0\mathcal{R}_{\mathcal{C}}, \quad f \mapsto |BC_p|f - \alpha(f)$$

Remark 4.4. One can think of $|BC_p|$ as a stand in for $\frac{1}{p}$. So $\alpha(f) = \frac{f^p}{p}$ and $\delta(f) = \frac{f-f^p}{p}$.

Definition 4.5. Let R be a commutative ring. An additive p-derivation on R is a function $\delta\colon R \to R$ that satisfies

(1) (additivity) $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p+y^p-(x+y)^p}{p}$ for all x, y in R.
(2) (normalization) $\delta(0) = \delta(1) = 0$.

We call a pair (R, δ) a semi-δ-ring.

Proposition 4.6. *$\delta\colon \pi_0\mathcal{R}_{\mathcal{C}} \to \pi_0\mathcal{R}_{\mathcal{C}}$ is an additive p-derivation.*

Example 4.7. Suppose R is a subring of $\mathbb{Q}$. Then the fermat quotient

$$\delta(x) = \frac{x - x^p}{p}$$

is an additive p-derivation. In fact it is the only such p-derivation.

Proposition 4.8. *The key property of the Fermat quotient is as follows: for every $x \in \mathbb{Q}$, if $0 < \nu_p(x) < \infty$ then*

$$\nu_p(\delta(x)) = \nu_p(x) - 1.$$

We will crucially use the following:

Proposition 4.9. *Suppose S is a p-local semi-δ-ring. The map $\mathbb{Z}_{(p)} \to S$ is injective and a semi-δ-ring homomorphism.*

Definition 4.10. We say $x \in S$ is rational if it is in the image of the map $\mathbb{Z}_{(p)} \to S$.

and the following:

Proposition 4.11. *Let (R, δ) be a p-local semi-δ-ring. Write R^{tf} for the quotient of R by the ideal of torsion elements. Then there is a unique p-derivation on R^{tf} such that the quotient $g\colon R \to R^{tf}$ is a homomorphism of p-local semi-δ-rings. Moreover, g detects invertibility.*

Proof. The first statement follows from the fact that $\delta(I_{\mathrm{tor}}) \subset I_{\mathrm{tor}}$. The second follows from the fact that torsion elements are nilpotent in p-local semi-δ-rings. □

5. Constructing amenable spaces

Definition 5.1. Given a space A we write $A \wr C_p = A^{\times p}_{hC_p}$.

Lemma 5.2. *Let A be a n-finite space. Then $\alpha(|A|) = |A \wr C_p|$ in $\pi_0\mathcal{R}_{\mathcal{C}}$.*

Proof. We compute

$$\alpha(|A|) = \int_{BC_p} \Theta^p(|A|) = \int_{BC_p} \Theta^p(\int_A \mathrm{id}_{\mathbb{1}}) = \int_{BC_p} \int_{A\wr C_p \to BC_p} \mathrm{id}_{\mathbb{1}} = \int_{A\wr C_p} \mathrm{id}_{\mathbb{1}} = |A \wr C_p|.$$

The third equivalence is an explicit calculation (or alternatively one uses the fact that Θ^p is a n-semiadditive functor, however note that the target of Θ^p is a $\mathcal{S}$-parametrized category, i.e. does not satisfy descent) □

Proposition 5.3. *Let $n \geq 1$ and let $\mathcal{C}$ be a n-semiadditive, presentably symmetric monoidal, stable, p-local category. Let $g\colon \pi_0(R) \to S$ be a semi-δ-ring homomorphism that detects invertibility, and such that $h(|BC_p|)$ and $h(|B^m C_p|)$ are rational and non-zero. Then $\mathcal{C}$ is $(n+1)$-semiadditive.*

Proof. Call a space h-good if

(1) A is a connected, n-finite p-space such that $\pi_n(A) \neq 0$.
(2) $h(|A|)$ is rational.

As we have seen it is enough to show that there exists an h-good space such that $h(|A|)$ is invertible in S.

By assumption $h(|B^mC_p|)$ is rational and therefore B^mC_p is h-good. If $p \in S^\times$ then all rational elements are invertible, and so we are done. So let us assume $p \notin S^\times$. In this case a rational element $x \in S$ is invertible if and only if $\nu_p(x) = 0$. Writing $\nu(A) = \nu_p(h(|A|))$, it suffices to find h-good A such that $\nu(A) = 0$.

Note $h(|B^mC_p|)$ is non-zero and p is not invertible, so $0 < \nu(B^mC_p) < \infty$. It suffices to show that given an h-good space A with $0 < \nu(A) < \infty$, there exists another h-good space A' with $\nu(A') - \nu(A) - 1$.

We compute:

$$\delta(|A|) = |BC_p||A| - \alpha(|A|) = |BC_p||A| - |A \wr C_p|.$$

So

$$h(|A \wr C_p|) = h(|BC_p|)h(|A|) - \delta(h(|A|)).$$

This shows that $h(|A \wr C_p|)$ is again rational. Moreover

$$\nu_p(h(|A \wr C_p|)) = \min(\nu(BC_p)\nu(A), \nu_p(\delta(h(|A|))) = \min(\nu(A), \nu(A) - 1) = \nu(A) - 1.$$

$\square$

We can combine this with the connection between cardinalities and dimensions to obtain the following master theorem:

Theorem 5.4 (The bootstrapping machine). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal, stable, p-local categories. Assume that*

1. *$\mathcal{C}$ and $\mathcal{D}$ are 1-semadditive.*
2. *The induces map $\mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility.*
3. *For every $0 \le k < n$, the space B^kC_p is dualizable in $\mathcal{D}$ and $\dim(B^kC_p)$ is rational and non-zero.*

Then $\mathcal{C}$ is n-semadditive.

Proof. It suffices to show that $\mathcal{C}$ is n-semiadditive. We argue by induction on k that $\mathcal{C}$ is k-semiadditive and that $\phi(|B^k C_p|)$ is rational and non-zero for all $0 \leq i < k$.

The base case $k = 1$ holds from assumption (1) and the fact that $p = |C_p|_{\mathcal{D}}$ is rational and non-zero (because $\mathbb{Z}_{(p)} \to R_{\mathcal{D}}^{tf}$ is injective).

Now let us assume the inductive hypothesis. First we prove $|B^k C_p|_{\mathcal{D}}$ is rational and non-zero in $R_{\mathcal{D}}^{\mathrm{tf}}$. By the above

$$\dim(B^k C_p) = |B^k C_p||B^{k-1} C_p|$$

in $R_{\mathcal{D}}^{\mathrm{tf}}$. The first and third are rational and non-zero, and so the second is too (note there is no torsion in $R_{\mathcal{D}}^{\mathrm{tf}}$).

To see that $\mathcal{C}$ is $(k+1)$-semiadditive we apply Proposition 5.3. The functor F induces a map $\phi\colon R_{\mathcal{C}} \to R_{\mathcal{D}} \to R_{\mathcal{D}}^{\mathrm{tf}}$. This detects invertibility, and $|B^k C_p|_{\mathcal{D}}$ and $|BC_p|_{\mathcal{D}}$ are rational and non-zero by before, and induction. □

6. Applications to Chromatic Homotopy Theory

Definition 6.1. A colimit preserving functor $\mathcal{C} \to \mathcal{D}$ between stable presentably symmetric monoidal categories is nil-conservative if for every ring R, $F(R) = 0$ implies $R = 0$.

Proposition 6.2. *A nil-conservative functor $F\colon \mathcal{C} \to \mathcal{D}$ is conservative on nilpotent objects.*

Proof. Suppose X is compact and $F(X) = 0$. Then $X \otimes X^\vee$ is a ring and $F(X \otimes X^\vee) = F(X) \otimes F(X)^\vee = 0$. Therefore $X \otimes X^\vee = 0$, and so $X = 0$. □

Corollary 6.3. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a nil-conservative functor. Then the induced ring homomorphism $\mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility.*

Proposition 6.4. *The functor $\hat{E}[-]\colon \mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(n)} \to \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$ is nil-conservative.*

Proof. Given any p-local ring R in spectra. The Nilpotence theorem tells us that

$$\mathrm{Sp}_{T(n)} \to \prod_n \mathrm{Sp}_{K(n)}$$

is nil-conservative. However $\mathrm{Sp}_{T(n)} \to \mathrm{Sp}_{K(m)}$ is zero for $n \neq m$.

$\mathrm{Sp}_{K(n)} \to \mathrm{Mod}_{E_n}(\mathrm{Sp}_{K(n)})$ is even conservative (Galois extension). □

Theorem 6.5. $\mathrm{Sp}_{T(n)}$ *in* ∞*-semiadditive.*

Proof. We apply the bootstrap theorem to the functor $\hat{E}[-]$. It is nil-conservative and so detects invertibility. $\mathrm{Sp}_{T(n)}$ is 1-semiadditive by a theorem of Kuhn. Therefore it suffices to show that $E_n \otimes B^n C_p$ is dualizable, and has rational non-zero dimension. This is a famous theorem of Ravenel-Wilson, who completely compute $K(n)_*(B^k C_p)$. In particular it is concentrated in even degrees, and finite dimensional of rank $p^{\binom{n}{k}}$. This in turn implies that $E_n \otimes B^n C_p$ is also free of rank $p^{\binom{n}{k}}$. So $E_n \otimes B^n C_p$ is dualizable and has dimension $p^{\binom{n}{k}}$. □

Theorem 6.6. $\mathrm{Sp}_{K(n)}$ *is* ∞*-semiadditive.*

Proof. Exactly as above. □

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