Chromatic Cardinalities

[PIER FEDERICO PACCHIAROTTI]

$[\text{UP}: \mathrm{CMon}_m^{(p)}]$: (Harpaz)

$$\begin{tikzcd} \mathrm{CMon}_m^{(p)}(-) : \mathrm{Cat}_\infty^{\mathrm{Ani}_m^{\pi,(p)}\text{-}L} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Cat}_\infty^{\oplus\text{-}m,(p)} : \subseteq \arrow[l, shift left, hook', "\text{f.f.}"] \end{tikzcd}$$

$$(\text{on } \mathrm{Cat}_\infty^{\oplus\text{-}m}) \ \wr\wr$$

$$\mathrm{Fun}^{\mathrm{Ani}_m^{\pi,(p)}\text{-}R}\left(\mathrm{Span}(\mathrm{Ani}_m^{\pi,(p)})^{op}, -\right) \ni M \text{ in s.t. } \forall A \in \mathrm{Ani}_m^{\pi,(p)},$$

$$M|_{(\mathrm{Ani}_m^{\pi,(p)})^{op}} \circ \mathrm{St}\begin{pmatrix} A_{*/} \\ \downarrow \\ A \end{pmatrix} : M(A) \xrightarrow{\simeq} \lim_A M(p)$$

i.e. $\forall \mathcal{C} \in \mathrm{Cat}_\infty^{\oplus\text{-}m,(p)}$, $\quad ev_* : \mathrm{CMon}_m^{(p)}(\mathcal{C}) \xrightarrow{\simeq} \mathcal{C}$

$$\exists!\ x^{(-)} \longmapsto x \in \mathcal{C}$$

$\left[\begin{array}{c}\text{Higher} \\ \text{Cardinality}\end{array}\right]$: $\mathcal{C} \in \mathrm{Cat}_\infty$, $M \in \mathrm{Fun}(\mathrm{Span}(\mathrm{Ani}_m^{\pi,(p)})^{op}, \mathcal{C})$:

$$\forall A \in \mathrm{Ani}_m^{\pi}, \quad |A|_M := M\begin{pmatrix} & A & \\ \swarrow & & \searrow \\ * & & * \end{pmatrix} : M(*) \xrightarrow{A^*} M(A) \xrightarrow{\int_A} M(*)$$

Rk: A $\mathcal{C}$-ambidextrous $\rightsquigarrow$ $\forall x \in \mathcal{C}$,

$$|A|_x := |A|_{x^{(-)}} : \ |A|_x \simeq \int_A id_{A^*(x)}\, d\mu_A \in \pi_0 \mathrm{Map}_{\mathcal{C}}(x, x)$$

assemble into $|A|_{\mathcal{C}} := \underbrace{c_A^!}_{\text{mult}} \circ A_!(-) \circ \underbrace{\mu_A}_{\text{diag.}} : id_{\mathcal{C}} \to id_{\mathcal{C}}$

and retrieves the "A-fold sum" of $id_{\mathcal{C}}$.

PROPERTIES:

- $\forall F : \mathcal{C} \to \mathcal{D}$ in $\mathrm{Cat}_\infty^{\oplus\text{-}m,(p)}$ preserves cardinalities (i.e. F preserves $\lim^{\mathrm{Ani}_m^{\pi,(p)}} \simeq \mathrm{colim}^{\mathrm{Ani}_m^{\pi,(p)}}$)
- $\mathcal{C}^{\otimes} \in \mathrm{CAlg}(\mathrm{Cat}_\infty^L) \rightsquigarrow \forall x \in \mathcal{C}$, $x \otimes (-) : \mathcal{C} \circlearrowleft$ in Cat_∞^L

  $\rightsquigarrow \forall A$ $\mathcal{C}$-ambidextrous,

  $$|A|_x \simeq |A|_{\mathbb{1}} \otimes x \in \pi_0 \mathrm{Map}_{\mathcal{C}}(x, x)$$

Ex: $\forall p$ prime, $\mathrm{Sp}_{K(n)}, \mathrm{Sp}_{T(n)} \in \mathrm{Cat}_\infty^{\oplus\text{-}\infty}$ $\forall n$

The localisations of chromatic filtrations
$$\begin{tikzcd} \mathrm{Sp}_{T(n)} \arrow[r, shift left, "\mathbb{S}_{K(n)} \otimes (-)"] \arrow[r, phantom, "\perp"] & \mathrm{Sp}_{K(n)} \arrow[l, shift left, hook', "\text{ff}"] \end{tikzcd}$$
match cardinalities:

$$\forall A \in \mathrm{Ani}^{\pi}, \quad |A|_{\mathbb{S}_{T(n)}} \longmapsto |A|_{\mathbb{S}_{K(n)}} \in \pi_0 \mathbb{S}_{K(n)} := \pi_0 \mathrm{Map}(\mathbb{S}_{K(n)}, \mathbb{S}_{K(n)})$$

$\rightsquigarrow$ we have a family of el.'s of $\pi_0 \mathbb{S}_{K(n)}$ which lifts to $\pi_0 \mathbb{S}_{T(n)}$ (per se a hard task!)

The main result describes them for $n=1$.

§ The main theorem

Fix p prime: $\quad K := \begin{cases} KU_p^\wedge & , p \neq 2 \\ KO_2^\wedge & , p = 2 \end{cases}$, $\quad \psi^g$ Adams operation corresponding to a top. gen. $g \in \begin{cases} \mathbb{Z}_p^\times \\ \mathbb{Z}_2^\times/\{\pm 1\} \end{cases}$

<u>Def:</u> $\quad \mathbb{S}_{K(1)} \longrightarrow K \xrightarrow{\psi^g} K \quad$ (co) fibre seq. in $Sp_p^\wedge$

$$\rightsquigarrow \quad \mathbb{S}_{K(1)} \otimes (-) : Sp_p^\otimes \underset{ff}{\overset{L}{\rightleftarrows}} Sp_{K(1)}^\otimes \quad \text{symm. mon. localisation}$$

$$\mathbb{S} \longmapsto \mathbb{S}_{K(1)}$$

<u>RK:</u> The induced LES supplies w/ a description: $\quad \pi_0 \mathbb{S}_{K(1)} \overset{\mathrm{CRing}}{\cong} \begin{cases} \mathbb{Z}_p & , p \neq 2 \\ \dfrac{\mathbb{Z}_2[\varepsilon]}{(\varepsilon^2, 2\varepsilon)} & , p = 2 \end{cases}$

[Baez-Dolan]: (Homotopy Cardinality, [CSY21], Ex. (2.2.2))

$$Sp_{\mathbb{Q}} \in \mathcal{C}at_\infty^{\oplus\text{-}\infty} : \quad A \in \mathrm{Ani}^{\pi}, \quad |A|_0 := \sum_{a \in \pi_0(A)} |\pi_n(A,a)|^{(-1)^n} \in \pi_0 \mathbb{S}_{\mathbb{Q}} \cong \mathbb{Q}$$

<u>Main Thm:</u> [CY, Thm A, (5.21)]

Fix p prime:

$$\forall A \in \mathrm{Ani}_{\geq 1}^{\pi,(p)}, \qquad |A|_{\mathbb{S}_{K(1)}} \simeq \begin{cases} 1 & , p \neq 2 \\ 1 + \log_2(|A|_0) \cdot \varepsilon & , p = 2 \end{cases}$$

<u>Strategy:</u> 1) Reduction steps to $A \simeq \mathbb{B}C_p$

2) Compute $|\mathbb{B}C_p|_{\mathbb{S}_{K(1)}}$ for p odd and for $p = 2$

①: <u>Reduction to $A \simeq \mathbb{B}C_p$:</u>

- <u>CLAIM:</u> The identities in the Thm are multiplicative along principal fibre sequences w/ connected fibre-anima,

<u>i.e.</u> $\forall\ F \xrightarrow{\in \mathrm{Ani}_{\geq 1}^{\pi,(p)}} E \longrightarrow B$ principal fibre seq. in $\mathrm{Ani}^{\pi,(p)}$,

it holds: $\quad \bullet \ |E| = |F| \cdot |B| \quad \in \pi_0 \mathbb{S}_{K(1)}$

$\quad \bullet \ 1 + \log_2(|E|_0) \cdot \varepsilon = \big(1 + \log_2(|F|_0) \cdot \varepsilon\big)\big(1 + \log_2(|B|_0) \cdot \varepsilon\big)$ $\quad \pi_0 \mathbb{S}_{\mathbb{Q}} \ni$

<u>PROOF:</u> This is [CSY21], Thm A, iv) to

- $Sp_{K(1)} \in \mathcal{C}at_\infty^{\oplus\text{-}0}$ w/ $\mathrm{Ht}(Sp_{K(1)}) = 1$, $\quad F \in \mathrm{Ani}_{\geq 1}^{\pi,(p)}$
- $Sp_{\mathbb{Q}} \in \mathcal{C}at_\infty^{\oplus\text{-}0}$ w/ $\mathrm{Ht}(Sp_{\mathbb{Q}}) = 0$, $\quad F \in \mathrm{Ani}_{\geq 1}^{\pi,(2)}$ $\quad$ [+ recall that $\varepsilon^2 = 0$]

<u>RK:</u> p-local π-finite spaces are nilpotent (see [CSY21], RK (3.8))

- Reduction to $A \simeq \mathbb{B}^n C_p$:

A nilpotent + connected $\rightsquigarrow$ refine its Postnikov tower to a sequence of principal fibrations: $\qquad (\exists k_i > 0)$

$$A = A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_k = pt \quad \text{w/ fibres } \mathrm{Fib}(A_i \to A_{i-1}) \simeq \mathbb{B}^{k_i} C_p$$

$\overset{\text{CLAIM}}{\Longrightarrow}$ Inductively reduce to compute $|-|_{\mathbb{S}_{K(1)}}$ on fibres.

- Reduction to $A \simeq \mathbb{B}C_p$:

$(k \geq 2) \quad \mathbb{B}^{k-1} C_p \longrightarrow * \longrightarrow \mathbb{B}^k C_p$ principal fibre seq. (b/c $\ell \geq 2 \rightsquigarrow \mathbb{B}^\ell C_p \in \mathrm{Ani}_{\geq 2}^{\pi}$)

$\overset{\text{CLAIM}}{\Longrightarrow}$ Inductively reduce to compute $|-|_{\mathbb{S}_{K(1)}}$ on fibres.

②: <u>Computation of $|\mathbb{B}C_p|$ w/ $p \neq 2$:</u>

<u>Lemma:</u> Fix $p \neq 2$ prime: $\quad \forall A \in \mathrm{Ani}_{\geq 1}^{\pi,(p)}, \quad |A|_{\mathbb{S}_{K(1)}} = 1 \quad \in \pi_0 \mathbb{S}_{K(1)}$

<u>PROOF:</u> $\mathbb{S}_{K(1)} \longrightarrow KU_p^\wedge$ in $\mathrm{Alg}(Sp_{K(1)})$ induces an adjunction

$$(-) \underset{\mathbb{S}_{K(1)}}{\otimes} KU_p^\wedge : Sp_{K(1)} \overset{L}{\rightleftarrows} \mathrm{Mod}_{KU_p^\wedge}(Sp_{K(1)})$$

whose counit gives - by [CSY22], (2.2.4)

$$\pi_0 \mathbb{S}_{K(1)} \xrightarrow{\cong} \pi_0 KU_p^\wedge \cong \mathbb{Z}_p$$

$$|\mathbb{B}C_p|_{KU_p^\wedge} \longmapsto 1$$

§ Computation $|\mathbb{B}C_2|\,\$_{K(1)}$

$Sp_{K(1)} \in \mathcal{C}at_\infty^{\otimes\text{-}\infty} \rightsquigarrow \forall\, \substack{p \text{ prime} \\ 0\le m\le\infty}$, $\mathrm{CMon}_m^{(p)}(Sp_{K(1)}) \xrightarrow[\simeq]{ev_*} Sp_{K(1)}$

$\exists!\, \$^{(-)}_{K(1)} \longmapsto \$_{K(1)}$

GOAL: Find "computable" $M \in \mathrm{CMon}_m^{(p)}(Sp_{K(1)})$

s.t. $M(*) \xrightarrow{\simeq} \$_{K(1)}$ in $Sp_{K(1)}$

[explicit on π_0]

History:
- [Quillen]: $p \neq 2$ prime $\rightsquigarrow \$_{K(1)} \simeq$ $K(1)$-local K-theory of $\mathbb{F}_\ell$ for $\ell \in \mathbb{Z}_p^\times$ top. generator
- [Friedlander]: $\ell \equiv_8 3,5$, $p=2$ primes
  $\rightsquigarrow L_{K(1)}\, GW(\mathbb{F}_\ell) \simeq \$_{K(1)} \in Sp_{K(1)}$
- [Morel]: k field $\rightsquigarrow \$^{0,0} \in SH(k)$
  $\rightsquigarrow \pi_0\,\$^{0,0} \cong GW(k)$

Thm: ([CY], Thm C, (4.0.1))

$p=2$, $\ell \equiv_8 3,5$ primes $\rightsquigarrow L_{K(1)}\, GW^{(-)}(\mathbb{F}_\ell) \simeq \$^{(-)}_{K(1)} \in \mathrm{CMon}_1^{(2)}(Sp_{K(1)})$

Recall: ([CY], § 5.1.1)

- $\frac{1}{2} \in R \in \mathrm{CRing} \rightsquigarrow R^\times \xrightarrow{[-]} \mathcal{QF}_0(R) \in \mathcal{QF}_0(R)^{gp} =: GW_0(R)$ in Ab

  $\mathcal{QF}_0(R) = \{(P, b: P\times P \to R) \mid \cdot\, P \in \mathrm{Proj}^{fg}(R),\ \cdot\, b$ non-deg. symm. bil. form$\}$

  ({quadratic forms} = {symm. bil. forms})

  $r \longmapsto [r] := [(R, x\otimes y \mapsto rxy)]$
- $R \cong \mathbb{F}_q$ w/ $q = \ell^m$ $\exists m$, $\ell \neq 2$ prime:

  $\forall (P,b) \in \mathcal{QF}_0(\mathbb{F}_q)$, b can be diagonalized, i.e. $[b] = \sum_{i=1}^{\dim P} [x_i]$ $\exists x_i \in \mathbb{F}_q^\times / \mathbb{F}_q^{\times 2}$

  Thm: $GW_0(\mathbb{F}_q) \cong \dfrac{\mathbb{Z}[e]}{(e^2, 2e)} \in \mathrm{CRing}$

  $[(P,b)] \longmapsto rk(b) + \det(b)\, e$

Let's enhance $\mathcal{QF}_0(R)$ to $\mathcal{QF}^{(-)}: \mathrm{CAlg}(\mathcal{C}at_\infty^\times) \xrightarrow{\mathrm{Lax}\text{-}\otimes} \mathrm{CMon}_m^{(p)}(\mathrm{Ani})$

§ Higher GW-Theory

<u>CONSTRUCTION:</u> (Lax fixed points f.)

- $(-)^{op} : \mathrm{Cat}_\infty \circlearrowleft$ induces $\underline{\mathrm{Cat}_\infty} \in \mathrm{Cat}_\infty^{\mathbb{B}C_2}$ $\rightsquigarrow$ $\underline{\mathrm{Cat}_\infty}^{hC_2} \xrightarrow{w_*} \mathrm{Cat}_\infty$, $\quad (\mathcal{C}, i : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}^{op}) \longmapsto \mathcal{C}$

  cat.'s w/ anti-involution + coherence

- [Haugseng - Melani - Safronov]: $\mathrm{Tw} := (id_{\Delta^{op}} \star (-)^{rev})^* : s\mathrm{Ani} \to s\mathrm{Ani}$, $\quad \mathcal{C} \mapsto ([n] \mapsto \mathcal{C}([n] \star [n]^{rev}))$

$$\begin{tikzcd} cSeg(s\mathrm{Ani}) \arrow[r, "\mathrm{Tw}"] & cSeg(s\mathrm{Ani}) \\ s\mathrm{Ani} \arrow[u, shift left, "\dashv"] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & s\mathrm{Ani} \arrow[u, shift left, "\dashv"] \end{tikzcd}$$

comes equipped w/ a canonical nat. tr.

$$(s, t) : \mathrm{Tw}(-) \overset{\cdot}{\longrightarrow} (-)^{op} \times id_{\mathrm{Cat}_\infty}$$

<u>Obs:</u> $\forall\, \mathcal{C} \in \mathrm{Cat}_\infty$ w/ $|-|$ & PB's, $G \cong \langle \sigma \rangle \in \mathrm{Grp}$ cyclic,

$$\left\{ \{\sigma\} \times \mathcal{C} \xrightarrow{\alpha|_\sigma} \mathcal{C},\ \alpha^{\# G} \xrightarrow{\simeq} id_{\mathcal{C}} \right\} \hookrightarrow \mathrm{ho}\left(\mathrm{Cat}_\infty^{\mathbb{B}G}\right) \text{ in } \mathrm{Cat}_1$$

<u>Lemma:</u> [CY, 3.1.4] The data above can be enhanced to

$$\underline{(s,t)} : \underline{\mathrm{Tw}(-)} \overset{\cdot}{\longrightarrow} \underline{(-)^{op} \times id_{\mathrm{Cat}_\infty}} \text{ in } \mathrm{Fun}_{\mathrm{Cat}_\infty}\left(\underline{\mathrm{Cat}_\infty}^{hC_2}, \mathrm{Cat}_\infty^{\mathbb{B}C_2}\right)$$

<u>Informally,</u>

- $\underline{\mathrm{Tw}(\mathcal{C}, i)}$:
  - <u>OBJ:</u> $* \longmapsto \mathrm{Tw}(\mathcal{C})$
  - <u>MOR:</u> $C_2 \longmapsto \left\{ id_{\mathrm{Tw}(\mathcal{C})},\ (i.\mathrm{Tw})(\mathcal{C}) : \mathrm{Tw}(\mathcal{C}) \xrightarrow{i.\mathcal{C}(id \star (-)^{rev})} \mathrm{Tw}(\mathcal{C})^{op} \right\}$, $\quad \mathrm{Tw}(\mathcal{C}^{op}) \cong \mathrm{Tw}(\mathcal{C})^{op}$
- $\underline{(-)^{op} \times id_{\mathrm{Cat}_\infty}}$:
  - <u>OBJ:</u> $* \longmapsto \mathcal{C}^{op} \times \mathcal{C}$
  - <u>MOR:</u> $\left\{ C_2 \longmapsto id_{\mathcal{C}^{op} \times \mathcal{C}},\ i \times i : \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C} \times \mathcal{C}^{op} \right\}$ $\quad$ <u>RK:</u> $i \simeq i^{-1}$

- Consider the C_2-equiv. f. $\left[(\mathcal{C}, i) \mapsto \underline{(id, i)} : \overset{\text{trivial}}{(\mathcal{C}, id)} \longrightarrow \underline{\mathcal{C}^{op} \times \mathcal{C}} \right]$ in $\mathrm{Fun}\left(\underline{\mathrm{Cat}_\infty}^{hC_2}, \mathrm{Cat}_\infty^{\mathbb{B}C_2}\right)$.

<u>Def:</u> ([CY, 3.1.5])

- <u>Lax fixed pts:</u> $\Phi^{lax} : \underline{\mathrm{Cat}_\infty}^{hC_2} \longrightarrow \mathrm{Cat}_\infty^{\mathbb{B}C_2}$, $\quad (\mathcal{C}, i) \longmapsto \underline{\mathcal{C}} \times_{\underline{\mathcal{C} \times \mathcal{C}^{op}}} \underline{\mathrm{Tw}(\mathcal{C})}$ $\quad$

  <u>Informally,</u> $\Phi^{lax}(-,-)$ consists of the following data:

  - <u>OBJ:</u> $\alpha : x \to i(x)$ in $\mathcal{C}$
  - <u>MOR:</u> $\mathrm{Map}_{\Phi^{lax}(\mathcal{C}, i)}\left(\begin{smallmatrix} x \\ \downarrow \alpha \\ ix \end{smallmatrix}, \begin{smallmatrix} y \\ \downarrow \beta \\ iy \end{smallmatrix} \right) \simeq \left\{ \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & y \arrow[d, "\beta"] \\ i(x) & i(y) \arrow[l, "i(f)"] \end{tikzcd} \text{'s} \right\}$

- <u>Fixed pts:</u> $\Phi \le \Phi^{lax} : \underline{\mathrm{Cat}_\infty}^{hC_2} \longrightarrow \mathrm{Cat}_\infty^{\mathbb{B}C_2}$ <u>s.t.</u>

  $\Phi(\mathcal{C}, i) \underset{\text{FULL}}{\subseteq} \Phi^{lax}(\mathcal{C}, i)$ spanned by $\{ x \xrightarrow[\alpha]{\simeq} i(x) \text{'s} \}$

<u>Def:</u> ([CY, 3.1.7])

$$\mathrm{QF}(-) := \left(\Phi(-,-)^{hC_2} \right)^{\simeq} : \underline{\mathrm{Cat}_\infty}^{hC_2} \xrightarrow{\Phi} \mathrm{Cat}_\infty^{\mathbb{B}C_2} \xrightarrow{(-)^{hC_2}} \mathrm{Cat}_\infty \xrightarrow{(-)^{\simeq}} \mathrm{Ani}$$

[Heine - Lopez-Avila - Spitzweck]

<u>Note:</u>

$$\begin{array}{ccccccc} \mathrm{CAlg}(\mathrm{Cat}_\infty) & \xrightarrow{(-)^{dual}} & \mathrm{CAlg}(\mathrm{Cat}_\infty)^{rig} & \underset{\text{for}}{\longleftarrow} & \left(\underline{\mathrm{CAlg}(\mathrm{Cat}_\infty)^{rig}} \right)^{hC_2} & \underset{\text{for}}{\longrightarrow} & \underline{\mathrm{Cat}_\infty}^{hC_2} \\ \mathcal{C}^{\otimes} & \longmapsto & (\mathcal{C}^{dual})^{\otimes} & \longmapsto & ((\mathcal{C}^{dual})^{\otimes}, (-)^{\vee}) & \longmapsto & (\mathcal{C}^{dual}, (-)^{\vee}) \end{array}$$

(section)

$$(\mathcal{C}^{dual}, (-)^{\vee}) \xrightarrow{\mathrm{QF}(-)} \left(\Phi(\mathcal{C}^{dual}, (-)^{\vee})^{hC_2} \right)^{\simeq} \in \mathrm{Ani}$$

- <u>OBJ:</u> $b : x \xrightarrow{\simeq} x^{\vee} \in \Phi(\mathcal{C}^{dual}, (-)^{\vee})$
  - <u>s.t.</u> invariant under $x \mapsto x^{\vee}$
  - <u>i.e.</u> $b : x \otimes x \to \mathbb{1}$ non-deg. symm. bilinear form

<u>CONSTRUCTION</u>: Restrict the construction along

$$\begin{tikzcd}
 & \left(\mathcal{C}at_\infty^{\mathrm{Ani}_m^{\pi,(p)}\text{-}L}\right)^{\otimes^{\mathrm{Lurie}}} \arrow[dl, shift left] \arrow[dd, "(-)^{op}", "\simeq"', "\simeq"] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] & \\
\mathcal{C}at_\infty^{\times} \arrow[ur, shift left] \arrow[ur, phantom, "\perp"] \arrow[dr, shift left] \arrow[dr, phantom, "\perp"] \arrow[loop below, "(-)^{op}"] & & \left(\mathcal{C}at_\infty^{\oplus\text{-}m,(p)}\right)^{\otimes^{\mathrm{Lurie}}} \arrow[ul, hook, shift left, "ff"] \arrow[dl, hook, shift left, "ff"] \arrow[loop below, "(-)^{op}"] \\
 & \left(\mathcal{C}at_\infty^{\mathrm{Ani}_m^{\pi,(p)}\text{-}R}\right)^{\otimes^{\mathrm{Lurie}}} \arrow[ul, shift left] \arrow[ur, shift left, "\mathrm{CMon}_m^{(p)}(-)"] \arrow[ur, phantom, "\top"] &
\end{tikzcd}$$

in $\mathrm{CAlg}(\widehat{\mathcal{C}at}_\infty^{\times})^{BC_2}$

to obtain a $\mathrm{CMon}_m^{(p)}$ - enhancement:

$$\begin{tikzcd}
\left(\underline{\mathcal{C}at_\infty}^{\oplus\text{-}m,(p)}\right)^{hC_2} \arrow[rr, "\mathcal{Q}\mathcal{F}(-)"] & \circlearrowleft & \mathrm{Ani} \\
\mathrm{CMon}_m^{(p)}\left(\left(\underline{\mathcal{C}at_\infty}^{\oplus\text{-}m,(p)}\right)^{hC_2}\right) \arrow[u, "w_*", "\simeq"'] \arrow[rr, dashed, "\mathcal{Q}\mathcal{F}^{(-)}(-)"'] & & \mathrm{CMon}_m^{(p)}(\mathrm{Ani}) \arrow[u, "w_*"']
\end{tikzcd}$$

We enhance our main application by the compatibility of $(-)^{dual}$ w/ both $\mathrm{CMon}_m^{(p)}(-)$ and the $(-)^{op}$ C_2-action:

$$\begin{tikzcd}
\mathrm{CAlg}(\mathcal{C}at_\infty) \arrow[r, "(-)^{dual}"] & \mathrm{CAlg}(\mathcal{C}at_\infty)^{rig} \arrow[r, "\text{section}"] & \underline{\mathcal{C}at_\infty}^{hC_2} \\
\mathrm{CAlg}(\mathcal{C}at_\infty^{\oplus\text{-}m,(p)}) \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{CAlg}(\mathcal{C}at_\infty^{\oplus\text{-}m,(p)})^{rig} \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \underline{\mathcal{C}at_\infty}^{\oplus\text{-}m,(p)\, hC_2} \arrow[u, hook]
\end{tikzcd}$$

$\left[(\mathrm{Ani} \to \mathrm{CMon}(\mathrm{Ani}))\otimes(-) = 0\text{-additivization}\right]$

<u>Def</u>: $\left([\mathrm{CY},\ 13.2.9]\right)$ $(m \geq 0)$

$$\begin{tikzcd}
 & & \mathrm{CMon}(\mathrm{Ani}) \otimes \mathrm{CMon}_m^{(p)}(\mathrm{Ani}) \\
\left(\underline{\mathcal{C}at_\infty}^{\oplus\text{-}m,(p)}\right)^{hC_2} \arrow[r, "\mathcal{Q}\mathcal{F}^{(-)}(-)"] \arrow[ddddrr, "\mathrm{GW}^{(-)}(-)"'] & \mathrm{CMon}_m^{(p)}(\mathrm{Ani}) \arrow[ur, "\simeq"] \arrow[r, phantom, "\simeq"] & \mathrm{CMon}_m^{(p)}(\mathrm{CMon}(\mathrm{Ani})) \arrow[u, phantom, "\text{\rotatebox{90}{$\simeq$}}\ [\text{Ben Moshe - Schlank}]"'] \arrow[d, phantom, "\text{\rotatebox{90}{$\subseteq$}}\ \text{FULL}"] \\
 & \text{def}\ \circlearrowleft & \mathrm{Fun}\left(\mathrm{Span}(\mathrm{Ani}_m^{\pi,(p)})^{op}, \mathrm{CMon}(\mathrm{Ani})\right) \arrow[d, "(-)^{gp}_*"] \\
 & & \mathrm{Fun}\left(\mathrm{Span}(\mathrm{Ani}_m^{\pi,(p)})^{op}, \mathrm{CGrp}(\mathrm{Ani})\right) \arrow[d, phantom, "\text{\rotatebox{90}{$\simeq$}}"] \\
 & & \\
 & & \mathrm{Fun}\left(\mathrm{Span}(\mathrm{Ani}_m^{\pi,(p)})^{op}, \mathrm{Sp}\right)
\end{tikzcd}$$

<u>n.t.</u> $\mathrm{GW}^{A}(\mathcal{C}) \simeq \mathcal{Q}\mathcal{F}(\mathcal{C}^{A})^{gp} \in \mathrm{Sp}_{\geq 0}$

Computation of $|\mathbb{B}C_2|$:

Thm: ([CY, (4.0.1)])

$p = 2$, $\ell \equiv_8 3,5$ primes $\rightsquigarrow$ $L_{K(1)} GW^{(-)}(\mathbb{F}_\ell) \xrightarrow{\simeq} \$^{(-)}_{K(1)}$ in $\mathrm{CMon}_1^{(2)}(\mathrm{Sp})$

induces $GW^{(-)}(\mathbb{F}_\ell)^\wedge_2 \simeq \tau_{\geq 0} \$^{(-)}_{K(1)}$

in $\mathrm{Fun}(\mathrm{Span}(\mathrm{Ani}_1^{\pi,(2)})^{op}, \mathrm{Sp})$

Rk: After ev_{pt}, this retrieves [Friedlander]: $L_{K(1)} GW(\mathbb{F}_\ell) \simeq \$_{K(1)} \in \mathrm{Sp}_{K(1)}$

Sketch: By previous work of [Fiedorowicz-Hauschild-May] & [Atiyah-Segal],

- $L_{K(1)} GW^{(-)}(\mathbb{F}_\ell) \simeq L_{K(1)}\left(\underline{GW^{pt}(\bar{\mathbb{F}}_\ell)}\right)^{h\mathbb{Z}}$ Frobenius fixed pts

  $\left[\ell \equiv_8 3,5\right] \simeq \$_{K(1)}$

  [realize this as the fixed pts of an Adams operation on KO]

- $GW^{(-)}(\mathbb{F}_\ell)^\wedge_2 \simeq \tau_{\geq 0}\left(L_{K(1)} GW^{(-)}(\mathbb{F}_\ell)\right)$

Lemma: [CY, (5.1.6)]

$p = 2$, $\ell \equiv_8 3,5$ primes $\rightsquigarrow$

The identification of connected covers above induces

$$\begin{array}{ccc} \pi_0 GW(\mathbb{F}_\ell) & \longrightarrow & \pi_0 \$_{K(1)} \quad \text{in CRing} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}[e]/(e^2, 2e) & \xrightarrow{\ ?\ } & \mathbb{Z}_{(2)}[\varepsilon]/(\varepsilon^2, 2\varepsilon) \\ e & \longmapsto & \varepsilon \end{array}$$

(b/c $\cdot\ e \not\mapsto 0$, since iso after $\tau_{\geq 0}$; $\cdot\ e, \varepsilon$ only nilpotents)

i.e. $\pi_0 GW(\mathbb{F}_\ell) \longrightarrow \pi_0 \$_{K(1)}$

$[(V,q)] \longmapsto [rk(q) + det(q)\varepsilon]$

$\quad = rk(q) + det(q)[e]$

Computation of $|\mathbb{B}C_2|_{\$_{K(1)}}$:

It suffices to show $|\mathbb{B}C_2| = 1 + e \in \pi_0 GW(\mathbb{F}_\ell)$.

By def. $|\mathbb{B}C_2| = \int_{\mathbb{B}C_2} \mathbb{B}C_2^*(1)$ w/ $[1] \in \mathcal{QF}(\mathbb{F}_\ell)$ repres. $1 \in \pi_0 GW(\mathbb{F}_\ell)$:

$$\mathbb{B}C_2^* : \pi_0 GW(\mathbb{F}_\ell) \xrightarrow{res_1^{C_2}} \pi_0 GW^{\mathbb{B}C_2}(\mathbb{F}_\ell) \cong \pi_0 GW(\mathrm{Mod}_{\mathbb{F}_\ell}(\mathrm{Sp}^\omega)^{\mathbb{B}C_2})$$

$$(\mathbb{F}_\ell, [1]) \longmapsto (\overset{\text{trivial}}{\mathbb{F}_\ell}, [1])$$

Hence, $|\mathbb{B}C_2| = \int_{\mathbb{B}C_2} (\overset{\text{triv.}}{\mathbb{F}_\ell}, [1]) \overset{[CY,(5.1.5)]}{=} [\#C_2 \cdot 1] = [2] = [1] + ([2] - [1])$

$([2]-[1]) \in \mathbb{F}_\ell^\times / \mathbb{F}_\ell^{\times 2}$

b/c $\ell \equiv_8 3,5$

$\rightsquigarrow [2] - [1] = e \in \pi_0 GW(\mathbb{F}_\ell)$

$\square$