Semiadditive height

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November 26, 2024

This are notes for a talk in the Regensburg seminar on parameterized ambidexterity, held on 26.11.2024. All the material is based on [CSY20]; all mistakes are mine.

1 Basic notions from previous talks

Throughout, $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are ∞-categories. We use the letters A, B, C for spaces (usually π-finite) and the letters X, Y, Z either just for objects of $\mathcal{C}, \mathcal{D}$, or for local systems such as $X\colon A \to \mathcal{C}$ wich we also write as $(X_a : \mathcal{C})_{a:A}$.

1.1 Integration of maps

For every $\mathcal{C}$-ambidextrous space A we can integrate/sum over an A-indexed family of maps

$$\int_A : \mathrm{Map}(X,Y)^A = \mathrm{Map}(A^*X, A^*Y) \longrightarrow \mathrm{Map}(X,Y)$$
$$(f_a\colon X \to Y)_{a:A} \longmapsto \int_{a:A_b} f_a$$

(for $X, Y : \mathcal{C}$). Similarly, we can define $\int_q$ for $q\colon A \to B$ by integrating fiberwise.

This integration satisfies

- (Fubini) For each $f\colon A \times A' \to \mathrm{Map}(X,Y)$:
$$\int_{A\times A'} f = \int_{a:A}\int_{a':A'} f(a,a')$$
- (Partial integration) For each $q\colon A \to B$ and $f\colon A \to \mathrm{Map}(X,Y)$:
$$\int_A f = \int_{b:B}\int_{a:A_b} f_a$$
- (Homogeneity) For each $f\colon A \to \mathrm{Map}(X,Y)$ and $g\colon Y \to Z$ and $h\colon Z \to Y$, we have
$$g \circ \int_A f = \int_{a:A}(g\circ f_a) \quad \text{or} \quad \left(\int_A f\right)\circ h = \int_{a:A}(f_a \circ h).$$

- (Base change) For $q\colon A \to B$ and $s\colon B' \to B$ we have

$$s^*\left(\int_{a:A_b} f_a\right)_{b:B} = \left(\int_{a:(A\times_B B')_{b'}} f_a\right)_{b':B'}.$$

  (Note that this equation looks tautological with the "pointwise" notation, because $(A\times_B B')_{b'}$ is just $A_{s(b')}$. In a sense, it is this stability under base change that justifies using that notation to begin with.)

1.2 Cardinality

Definition 1.1. The *cardinality* of a $\mathcal{C}$-ambidextrous space A is the natural transformation $|A|\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ given by

$$|A|_X \coloneqq \int_A \mathrm{id}_X \quad \text{for } X : \mathcal{C}.$$

(And fiberwise $|q| = (|A_b|)_b\colon \mathrm{id}_{\mathcal{C}^B} \to \mathrm{id}_{\mathcal{C}^B}$ for $q\colon A \to B$.) The space A is called $\mathcal{C}$-*amenable* if $|A|$ is invertible.

Theorem 1.2 ("Higher Maschke's Theorem")**.** *Let $X\colon B \to \mathcal{C}$. Let $q\colon A \to B$ be a map such that all $|A_b|_X$ are invertible. Then for each $X : B \to \mathcal{C}$ the counit map $c_!X\colon q_!q^*X \to X$ has a section.*

Proof. Without loss of generality assume $B = *$. The composite

$$X \xrightarrow{\Delta} X^A \xleftarrow[\simeq]{\mathrm{Nm}_A} X[A] \xrightarrow{\nabla} X$$

is exactly the cardinality $|A|_X$. If it is invertible, we thus have a section of $\nabla = c_!X$. $\square$

Theorem 1.3 (Cancellation Theorem)**.** *Let $A \xrightarrow{p} B \xrightarrow{q} C$ be weakly $\mathcal{C}$-ambidextrous such that p is amenable. Then q is $\mathcal{C}$-ambidextrous if and only if qp is $\mathcal{C}$-ambidextrous.*

Proof of Theorem 1.3(Cancellation Theorem). "Only if" is a general fact about ambidexterity, so we only show "if". Assume that Nm_p and Nm_{qp} are invertible. Since we have $q_*\mathrm{Nm}_p \circ \mathrm{Nm}_q p_! = \mathrm{Nm}_{qp}$, it follows that $\mathrm{Nm}_q p_!$ is invertible. By Theorem 1.2, each object X of $\mathcal{C}^B$ is a retract of $p_!p^*X$; and since we have just shown that Nm_q is invertible on $p_!p^*X$, it is thus also invertible on X. $\square$

Corollary 1.4. *Assume that B is connected and weakly $\mathcal{C}$-ambidextrous. If ΩB is amenable, then B is ambidextrous.*

Proof. Apply cancellation theorem to $A = * = C$. $\square$

Recall that a map $q\colon A \to B$ is principal if we have a fiber sequence

$$A \xrightarrow{q} B \to E.$$

with connected E.

Proposition 1.5. *Let $q\colon A \to B$ a principal map with fiber $F = \Omega E$. Assume A,B and F (hence q) are $\mathcal{C}$-ambidextrous. Assume that $|F|_X$ is invertible. Then for each $X : \mathcal{C}$ we have*

$$(|A_b|_X)_{b:B} = (|F|_X)_{b:B}\colon B \to \mathcal{C}(X,X) \quad \textit{and} \quad |A|_X = |B|_X|F|_X.$$

Proof. Fix a basepoint $e : E$ and note that for $b : B$ we have $A_b = E(b,e)$ and $F = E(e,e)$. Thus we have

$$|A_b|_X|F|_X = |E(b,e) \times E(e,e)|_X \cong |E(b,e) \times E(b,e)|_X = |A_b|_X^2 \tag{1}$$

where in the second step we use that $E(b,e)$ is a torsor over $F = E(e,e)$, yielding an isomorphism

$$E(b,e) \times E(e,e) \cong E(b,e) \times E(b,e), \quad (\eta,\mu) \mapsto (\eta,\mu\eta).$$

Since we assume that $|F|_X$ is invertible, so is each $|A_b|_X$ (because E is connected, so that $E(b,e)$ and $E(e,e)$ are abstractly isomorphic.) Thus we may cancel $(|A_b|_X)_{b:B}$ from (1) and obtain $(|F|_X)_b = (|A_b|_X)_b$, as desired. By integrating along B we get

$$|B|_X|F|_X = \int_{b:B} |F|_X = \int_{b:B} |A_b|_X = |A|_X,$$

as desired. □

Corollary 1.6. *Assume that B is connected and $\mathcal{C}$-ambidextrous. If $|\Omega B|_X$ is invertible, then we have*

$$|\Omega B|_X|B|_X = \mathrm{id};$$

in particular $|B|_X$ is also invertible.

Proof. Apply Proposition 1.5 to $B = E$, hence $F = \Omega B$. □

Corollary 1.7. *Given $A \xrightarrow{f} B \xrightarrow{g} C$ with f, g amenable and f principal; then gf is amenable.*

Proof. Without loss of generality, assume $C = *$ and write F for the fiber of f. Then Proposition 1.5 yields $|A| = |B||F|$. If f, g are amenable, then $|F|$ and $|B|$ are invertible; hence also $|A|$. □

2 Acyclic maps

Definition 2.1. Let $q\colon A \to B$ and assume that $\mathcal{C}$ admits q-limits and q-colimits We say that q is

- $\mathcal{C}$-*acyclic*, if $q^*\colon \mathcal{C}^B \to \mathcal{C}^A$ is fully faithful;
- $\mathcal{C}$-*trivial*, if $q^*\colon \mathcal{C}^B \to \mathcal{C}^A$ is an equivalence.

Lemma 2.2. *The conditions of acyclicity and triviality are fiberwise and preserved under composition.*

Lemma 2.3. *Assuming that $\mathcal{C}$ has A-limits and A-colimits, the following are equivalent:*

- *The space A is $\mathcal{C}$-acyclic.*
- *For all X, the fold map $\nabla\colon X[A] \to X$ is an equivalence.*
- *For all X, the diagonal map $\Delta\colon X \to X^A$ is an equivalence.*

Proof. The fold mag and diagonal map are the counit and unit of the adjunctions $A_! \dashv A^*$ and $A^* \dashv A_*$, respectively. Hence we have criterion for fully faithful adjoints. $\square$

Proposition 2.4. *Let A be a connected $\mathcal{C}$-ambidextrous space. For each $X \in \mathcal{C}$, the following are equivalent:*

1. *The fold map $\nabla\colon X[A] \to X$ is an equivalence.*
2. *The diagonal $\Delta\colon X \to X^A$ is an equivalence.*
3. *There exists a homotopy $|A(-,e)|_X \simeq |\Omega A|_X\colon A \to \mathrm{Map}(X,X)$.*
4. *$|\Omega A|_X$ is invertible.*

If this is the case, $|A|_X$ is the inverse of $|\Omega A|_X$

Proof. Choose a basepoint $e\colon * \to A$.

We start with $2 \implies 3$; the implication $1 \implies 3$ is dual.

Note that the composite

$$\mathcal{C}(X,X) \xrightarrow{\Delta\circ-} \mathcal{C}(X,X^A) = \mathcal{C}^A(A^*X, A^*X) \xrightarrow{e^*} \mathcal{C}(X,X)$$

is the identity and the first map is an equivalence because Δ is invertible. Thus the second map is an equivalence as well. This means that two families $f,g\colon A \to \mathrm{Map}(X,X)$ are homotopic if and only if $f(e) = g(e)$. In particular, this yields a homotopy $|A(-,e)|_X \simeq |\Omega A|_X\colon A \to \mathrm{Map}(X,X)$.

To prove $3 \implies 4$, we simply integrate over A and obtain

$$\mathrm{id}_X = |*|_X = \int_{a:A} |A(a,e)|_X \simeq \int_{a:A} |\Omega A|_X = |A|_X |\Omega A|_X,$$

making $|\Omega A|_X$ inverse to $|A|_X$, hence invertible.

Next we prove $4 \implies 1\&2$. Since A is connected, all fibers of $e\colon * \to A$ have cardinality merely equivalent to $|\Omega A|_X$, hence invertible. Thus, for each $a : A$ the composite

$$|A(a,e)|_X\colon X \xrightarrow{\Delta_a} X[A(a,e)] \xrightarrow{\nabla_a} X$$

is invertible. Taking the limit/colimit over $a : A$ we obtain the following diagram

$$\begin{tikzcd}
\bigoplus_{a:A} X \arrow[r, "\bigoplus_{a:A} \Delta_a"] \arrow[d, "\bigoplus_{a:A} |A(a,e)|_X"', "\simeq"] & \bigoplus_{a:A} X[A(a,e)] \arrow[r, "\bigoplus_{a:A} \nabla_a"] & \bigoplus_{a:A} X \\
X[A] \arrow[r, "\nabla"] & X \arrow[u, "(e,\mathrm{id}_e)", "\simeq"'] \arrow[r, "\Delta"] & X[A] \arrow[u, "\simeq", "\bigoplus_{a:A} |A(a,e)|_X"']
\end{tikzcd} \tag{2}$$

where the upper composite and the outer vertical maps are invertible. The diagram (2) commutes because we have

$$\bigoplus_{a:A} \Delta_a = \left(\int_{A(a,a')} \mathrm{id}_X \right)_{a',\mu,a} \in \lim_{a':A} \lim_{\mu:A(a',e)} \lim_{a:A} \mathrm{Map}(X,X)$$

which at $(a',\mu) = (e, \mathrm{id}_e)$ evaluates to $(|A(a,e)|_X)_{a:A}$ (and dually for the right diagram). (One can also check that the outer diagram commutes by directly evaluating the lower and upper composites to

$$|A(a'',e) \times A(a,e)|_X \quad \text{and} \quad |A(a'',a) \times A(a,e)|_X \quad \text{as functions of } a'', a : A$$

respectively; these are equivalent via $(\eta,\mu) \mapsto (\mu^{-1}\eta, \mu)$.)

The other composite

$$X \xrightarrow{\Delta} X[A] \xrightarrow{\nabla} X$$

is just the cardinality $|A|_X$, which is invertible by Corollary 1.6. Hence both Δ and ∇ are invertible. $\square$

Corollary 2.5. *Let A be a connected space. The following are equivalent:*

- *A is $\mathcal{C}$-acyclic and $\mathcal{C}$-ambidextrous.*
- *ΩA is $\mathcal{C}$-amenable and $\mathcal{C}$ admits A-colimits.*

In that case $|A| = |\Omega A|^{-1}$.

Proposition 2.6. *Let A be a connected space and assume that ΩA-colimits exist in $\mathcal{C}$. Then A is $\mathcal{C}$-trivial if and only if ΩA is $\mathcal{C}$-acyclic.*

Proof. Choose a base point $e \colon * \to A$. The map $A^* \colon \mathcal{C} \to \mathcal{C}^A$ is an equivalence

if and only if the composite $e^* \colon \mathcal{C}^A \to \mathcal{C}$ is an equivalence (because the composite e^*A^* is the identity)

if and only if e^* is fully faithful (because e^* is essentialy surjective)

if and only if e^* is $\mathcal{C}$-acyclic (by definition)

if and only if ΩA is $\mathcal{C}$-acyclic (because acyclicity is fiberwise).

$\square$

3 Recollements

Definition 3.1. A recollement of stable ∞-categories is a diagram of the form

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook, "i" description] & \mathcal{C} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \arrow[r, "P" description] & \mathcal{D} \arrow[l, bend right, hook', "j'"'] \arrow[l, bend left, hook', "j"] \end{tikzcd} \tag{3}$$

where i, j, j' are fully faithful and such that $j = \ker R$ and $j' = \ker L$ (as full subcategories of $\mathcal{C}$).

Such a recollement provides a lot of useful structure:

- We have canonical fiber sequences
$$iR \to \mathrm{id}_{\mathcal{C}} \to jP \quad \text{and} \quad iL \to \mathrm{id}_{\mathcal{C}} \to j'P$$
assembled from the various units and counits.
- Via j, the category $\mathcal{D}$ is identified with the right orthogonal complement of $\mathcal{B}$, i.e. $j = \mathcal{B}^{\perp} := \{X : \mathcal{C} \mid \mathcal{C}(i, X) = 0\}$.
- Similarly, $j' = {}^{\perp}\mathcal{B}$.

(Note that while $\mathcal{B}^{\perp}$ and ${}^{\perp}\mathcal{B}$ are both identified with $\mathcal{D}$ as categories, they are in general *not* the same subcategory of $\mathcal{C}$.)

It turns out that one gets most of the data in a recollement for free:

Proposition 3.2. *The following are equivalent pieces of data:*

- *A recollement* (3)*.*
- *A category $\mathcal{C}$ with a full subcategory $\mathcal{B} \subseteq \mathcal{C}$ such that the inclusion admits a left and a right adjoint.*
- *Two categories $\mathcal{B}$ and $\mathcal{D}$ and a functor $F\colon \mathcal{D} \to \mathcal{B}$ (called the* gluing functor*).*

Sketch.

- Given $\mathcal{B} \hookrightarrow \mathcal{C}$ with adjoint L and R, one defines $\mathcal{D} := \ker R$ (fully faithful by definition) and obtains P and j' as its (iterated) left adjoint.
- From the recollement (3) one obtains the functor $F := L \circ j$ (which up to shift is equivalent to $R \circ j'$).
- Given just the functor $F\colon \mathcal{D} \to \mathcal{B}$, one recovers $\mathcal{C}$ as the oplax colimit
$$\mathcal{C} = \mathrm{oplaxlim}(\Delta^1 \xrightarrow{[F]} \mathcal{S}\mathrm{t})$$

of the diagram classifying F. This is the (stable) ∞-category of sections of the cartesian unstraightening of $[F]$ over Δ^1; in other words, it consists of triples

$$(X : \mathcal{B}, Z : \mathcal{D}, f \colon X \to FZ).$$

It is identified with $\mathcal{C}$ via

$$Y \mapsto (LY, PY, L(Y \to jPY)),$$

where $Y \to jPY$ is the counit of the adjunction $P \dashv j$.

$\square$

We call a recollement *trivial* (“split” in [CSY20]) if the following equivalent conditions hold:

- For all $X : \mathcal{B}$ and $Z : \mathcal{D}$ we have $\mathcal{C}(Z, X) = 0$.
- The adjoints L and R are equivalent.
- The gluing functor $F \colon \mathcal{D} \to \mathcal{B}$ is zero.
- The full subcategories ${}^\perp\mathcal{B}$ and $\mathcal{B}^\perp$ agree.
- The functor $\mathcal{C} \xrightarrow{(L,P)} \mathcal{B} \times \mathcal{D}$ is an equivalence.

3.1 Chains of recollements

Consider a descending chain of full subcategories

$$\mathcal{C}_\infty \coloneqq \bigcup_n \mathcal{C}_n \subseteq \cdots \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_0 \subseteq \mathcal{C}$$

such that each inclusion $\mathcal{C}_n \hookrightarrow \mathcal{C}$ has a left right adjoints L_n and R_n, respectively.

Then we have an induced tower of left adjoints

$$\dots \xrightarrow{P_2} \mathcal{C}_2^\perp \xrightarrow{P_1} \mathcal{C}_1^\perp \xrightarrow{P_0} \mathcal{C}_0^\perp$$

which assemble to a functor $P_\infty \colon C \to \lim_n \mathcal{C}_n^\perp$.

Proposition 3.3. *Assuming that $\mathcal{C}$ has sequential limits and colimits, we have a recollement*

$$\begin{tikzcd} \mathcal{C}_\infty \arrow[r, hook] & \mathcal{C} \arrow[l, bend right=50, "L_\infty"'] \arrow[l, bend left=50, "R_\infty"] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \arrow[r, "P_\infty"] & \lim_n \mathcal{C}_n^\perp \arrow[l, bend right=50] \arrow[l, bend left=50] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"] \end{tikzcd}$$

with

$$L_\infty X \coloneqq \operatorname{colim}(X \xrightarrow{\eta_1} L_1 X \xrightarrow{\eta_2} L_2 X \xrightarrow{\eta_3} \dots) \quad \textit{and} \quad R_\infty X \coloneqq \lim(\dots \xrightarrow{\epsilon_3} R_2 X \xrightarrow{\epsilon_2} R_1 X \xrightarrow{\epsilon_1} X)$$

(where, for instance, $\eta_n \colon L_{n-1} \xrightarrow{u_n L_{n-1}} L_n L_{n-1} = L_n$ is induced by the unit u_n of the adjunction $L \colon \mathcal{C} \leftrightarrows \mathcal{C}_0$).

4 Divisibility and Completeness

Assume that $\mathcal{C}$ is pointed. Fix a natural endomorphism $\alpha\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ of the identity.

Definition 4.1.
- An object X of $\mathcal{C}$ is called *α-divisible* if $\alpha_X : X \to X$ is invertible. We denote by $\mathcal{C}[\alpha^{-1}] \subseteq \mathcal{C}$ the full subcategory of α-divisible objects.
- We denote by

$$\widehat{\mathcal{C}_\alpha} := \mathcal{C}[\alpha^{-1}]^{\perp} := \{X : \mathcal{C} \mid \mathrm{Map}(\mathcal{C}[\alpha^{-1}], X) = 0\}$$

  the right orthogonal complement; its objects are called *α-complete*.

Assume now that $\mathcal{C}$ is stable and admits sequential limits and colimits. Then we have a recollement as follows.

$$\begin{tikzcd}
\mathcal{C}[\alpha^{-1}] \arrow[r, hook] & \mathcal{C} \arrow[l, bend right=50, "L"'] \arrow[l, bend left=50, "R"] \arrow[l, phantom, bend right=25, "\perp"] \arrow[l, phantom, bend left=25, "\perp"] \arrow[r, "\widehat{(-)}_\alpha"] & \widehat{\mathcal{C}_\alpha} \arrow[l, hook, bend left=50] \arrow[l, phantom, bend left=25, "\perp"]
\end{tikzcd} \tag{4}$$

We have the following explicit descriptions of the various adjoints

- The left adjoint L "inverts α" on X:

$$LX := \mathrm{colim}(X \xrightarrow{\alpha} X \xrightarrow{\alpha} X \xrightarrow{\alpha} \dots)$$

- The right adjoint R takes the "α-divisible part" of X:

$$RX := \lim(\dots \xrightarrow{\alpha} X \xrightarrow{\alpha} X \xrightarrow{\alpha} X)$$

- The completion functor takes X to

$$\widehat{X_\alpha} := \lim(\dots \to X/\alpha^3 \to X/\alpha^2 \to X/\alpha)$$

Definition 4.2. A natural endomorphism $\beta\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is called a *semi-inverse* of α if for each α-divisible X, we have $\mathrm{id}_X = \alpha\beta$ in $\pi_0\mathrm{Map}(X, X)$.

Lemma 4.3. *If an object Y is α-complete then it is $(\mathrm{id} - \alpha\beta)$-divisible.*

Proof. Note that for each $r \in \mathbb{N}$, α is nilpotent on Y/α^r, hence $\mathrm{id} - \alpha\beta$ is invertible. Passing to the limit, it follows that $(\mathrm{id} - \alpha\beta)$ is invertible on the completion $\widehat{Y_\alpha}$. □

Proposition 4.4. *The recollement* (4) *associated to $\alpha\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is trivial if and only if α admits a semi-inverse.*

Proof. Assume that α has a semi-inverse β. Then for any α-divisible object X and α-complete object Y the endomorphism $\mathrm{id} - \alpha\beta$ acts by 0 on X and invertibly on Y; it follows that on $\mathcal{C}(Y, X)$ it acts both by 0 and invertibly, which means that $\mathcal{C}(Y, X) = 0$.

Conversely, if we have a trivial recollement $\mathcal{C} = \mathcal{C}[\alpha^{-1}] \times \widehat{\mathcal{C}_\alpha}$, then $\beta := \alpha^{-1}|_{\mathcal{C}[\alpha^{-1}]} \times 0$ is a semi-inverse. □

5 Semiadditive Height

For a while, we fix a prime p. We abbreviate $p_{(n)} := |\mathrm{B}^n C_p|$ (this cardinality of course depends on $\mathcal{C}$, but we suppress that from the notation).

Definition 5.1. Let $0 \le n \le m < \infty$. Let $\mathcal{C} \in \mathcal{C}\mathrm{at}^{\oplus_p - m}$,i.e., $\mathcal{C}$ is p-typically m-semiadditive). For each $X \in \mathcal{C}$

- $\mathrm{ht}(X) \le n$ if and only if X is $p_{(n)}$-divisible;
- $\mathrm{ht}(X) > n$ if and only if X is $p_{(n)}$-complete;
- $\mathrm{ht}(X) = n$ if and only if X is $p_{(n)}$-divisible and $p_{(n-1)}$-complete;
- $\mathrm{ht}(X) = \infty$ if and only if X is $p_{(k)}$-complete for all $k \ge 0$;
- $-1 < \mathrm{ht}(X) \le \infty$ always;
- $\mathrm{ht}(X) \le -1$ or $\mathrm{ht}(X) > \infty$ if and only if $X = 0$.

We write $\mathcal{C}_{\le n}$, $\mathcal{C}_{>n}$ and $\mathcal{C}_n$ for the full subcategories of objects of height $\le n$, $> n$ and $= n$, respectively. We say $\mathrm{Ht}(\mathcal{C}) \le n$ if and only if $\mathcal{C} = \mathcal{C}_{\le n}$, etc.

Example 5.2. If the prime p is invertible in $\mathcal{C}$ (e.g. if $\mathcal{C}$-local for some $\ell \ne p$), then $\mathrm{Ht}(\mathcal{C}) \le 0$ (with respect to p) because $p = |\mathrm{B}^0 C_p|$.

Remark 5.3. All these subcategories are closed under limits, hence they inherit (p-typical) m-semiadditivity from $\mathcal{C}$.

Lemma 5.4 (Height is monotone)**.** *For $0 \le n \le n' \le m < \infty$:*

1. $\mathrm{ht}(X) \le n \implies \mathrm{ht}(X) \le n'$
2. $\mathrm{ht}(X) > n' \implies \mathrm{ht}(X) > n$.

Proof. Part 1 from Corollary 1.6, because $\Omega \mathrm{B}^{n+1} C_p = \mathrm{B}^n C_p$. So that $p_{(n)} = |\mathrm{B}^n C_p|$ invertible implies $p_{(n+1)} = |\mathrm{B}^{n+1} C_p|$ invertible (as long as $n+1 \le m$ so that everything is $\mathcal{C}$-ambidextrous). Part 2 follows by passing to right orthogonal complements. $\square$

5.1 Bounded Height

Proposition 5.5. *Let $\mathcal{C} \in \mathcal{C}\mathrm{at}^{\oplus_p - n}$ which admits $\mathrm{B}^{n+1} C_p$-limits and colimits. The following are equivalent:*

- *$\mathrm{B}^n C_p$ is $\mathcal{C}$-amenable, i.e., $\mathrm{Ht}(\mathcal{C}) \le n$*
- *$\mathrm{B}^{n+1} C_p$ is $\mathcal{C}$-acyclic,*
- *$\mathrm{B}^{n+2} C_p$ is $\mathcal{C}$-trivial.*

Proof. Immediate from Section 2. $\square$

Theorem 5.6. *Let $\mathcal{C}$ be 0-semiadditive and p-local. Assume that $\mathcal{C}$ is p-typically n-semiadditive with all $(n+1)$-finite limits and colimits. If $\mathcal{C}$ is of height $\mathrm{Ht}(\mathcal{C}) \leq n$, then $\mathcal{C}$ is already ∞-semiadditive.*

Proof. We only prove that $\mathcal{C}$ is p-typically ∞-semiadditive. Dealing with the other primes $\ell \neq p$ is not hard, because ℓ is invertible on $\mathcal{C}$.

By induction on $k \geq n$ we prove that

- $\mathrm{B}^k C_p$ is $\mathcal{C}$-amenable (hence ambidextrous) and $\mathcal{C}$ admits $\mathrm{B}^{k+1}C_p$-colimits.

Case $k = n$ is the assumption. Assuming the induction hypothesis for k, Corollary 1.4 yields that $\mathrm{B}^{k+1}C_p$ is $\mathcal{C}$-ambidextrous (delooping amenable gives ambidextrous) and then $\mathrm{B}^{k+1}C_p$ is also amenable ($p_{(k)}$ is inverse to $p_{(k+1)}$). The colimits of shape $\mathrm{B}^{k+2}C_p$ exist for trivial reasons, because the diagonal functor $\mathcal{C} \to \mathcal{C}^{\mathrm{B}^{k+1}C_p}$ is even an equivalence. □

By bootstrapping Proposition 5.5 along the Postnikov tower and dealing with the invertible primes one obtains:

Theorem 5.7. *Let $\mathcal{C}$ be ∞-semiadditive and p-local with $\mathrm{Ht}(\mathcal{C}) \leq n$. For every π-finite space A we have:*

- *A is $n-1$-connected and nilpotent $\implies$ A is $\mathcal{C}$-amenable;*
- *A is n-connected $\implies$ A is $\mathcal{C}$-acyclic;*
- *A is $n+1$-connected $\implies$ A is $\mathcal{C}$-trivial;*

(By convention, a space A is n-connected if all $\pi_i(A)$ are trivial for $i \leq n$. For example, $\mathrm{B}^n C_p$ is $n-1$-connected)

6 Semiadditive Redshift

Recall that the categories $\mathcal{C}\mathrm{at}^{\oplus-m} \subseteq \mathcal{C}\mathrm{at}_{m-\mathrm{fin}}$ are themselves m-semiadditive. Hence for a given $\mathcal{C}$ we can consider $\mathrm{ht}(\mathcal{C})$ (with respect to some prime p) as an object of $\mathcal{C}\mathrm{at}^{\oplus-m}$ or of $\mathcal{C}\mathrm{at}_{m-\mathrm{fin}}$ and compare it to $\mathrm{Ht}(\mathcal{C})$.

Theorem 6.1. *Let $0 \leq n \leq m \leq \infty$ and $\mathcal{C} \in \mathcal{C}\mathrm{at}_{(m+1)-\mathrm{fin}}$. Assume that $\mathcal{C}$ is p-typically m-semiadditive. Then (with respect to p):*

1. $\mathrm{Ht}(\mathcal{C}) \leq n \iff \mathrm{ht}(\mathcal{C}) \leq n+1$
2. $\mathrm{Ht}(\mathcal{C}) > n \iff \mathrm{ht}(\mathcal{C}) > n+1$

Proof. 1. We have $\mathrm{Ht}(\mathcal{C}) \leq n$

if and only if $\mathrm{B}^{n+2}C_p$ is $\mathcal{C}$-trivial (by Proposition 5.5)

if and only if $C \xrightarrow{\Delta} C^{\mathrm{B}^{n+2}C_p}$ is an equivalence

if and only if $|\mathrm{B}^{n+1}C_p|_{\mathcal{C}}$ is invertible (by Proposition 2.4) if and only if $\mathcal{C}$ is $p_{(n+1)}$-divisible

if and only if $\mathrm{ht}(\mathcal{C}) \leq n+1$.

2. $\Longrightarrow$ Assume that $\mathrm{Ht}(\mathcal{C}) > n$. Let $\mathcal{D} \in \mathcal{C}\mathrm{at}_{(m+1)-\mathrm{fin}}$ with $\mathrm{ht}(\mathcal{D}) \leq n+1$. We need to show that every functor $F\colon \mathcal{D} \to \mathcal{C}$ that preserves $(m+1)$-finite colimits is zero. Write $B \coloneqq \mathrm{B}^{n+1}C_p$. The fact that $\mathrm{ht}(\mathcal{D}) \leq n+1$, means that the composite $p_{(n+1)}\colon \mathcal{D} \xrightarrow{B^*} \mathcal{D}^B \xrightarrow{B_!} \mathcal{D}$ is an equivalence which implies that the counit $X[B] = B_! B^* X \to X$ is an equivalence for each X.[1] Applying the functor F, which preserves the tensoring with B, we get an equivalence $FX[B] \xrightarrow{\simeq} FX$, which implies that $|\Omega B| = p_{(n)}$ is invertible on FX, i.e. $\mathrm{ht}(FX) \leq n$. But $\mathcal{C} = \mathcal{C}_{>n}$ by assumption, hence $FX = 0$.

$\Longleftarrow$ Assume that $\mathrm{ht}(\mathcal{C}) > n+1$. The full subcategory $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ is of height $\mathrm{Ht}(()\mathcal{C}_{\leq n}) \leq n$ and is again p-typically m-semiadditive. Thus by part 1 that we already proved, we have $\mathrm{ht}(\mathcal{C}_{\leq n}) \leq n+1$. Since by definition there cannot be any non-zero maps from height $\leq n+1$ to height $> n+1$, the inclusion $\mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ must be 0. This implies that $\mathcal{C}_{>n} \coloneqq \mathcal{C}_{\leq n}^{\perp} = 0^{\perp} = \mathcal{C}$, so that we have $\mathrm{Ht}(\mathcal{C}) > n$, as desired. □

7 Height decomposition

Theorem 7.1. *Let $\mathcal{C}$ be stable with sequential limits and colimits. Let $m \leq \infty$ and assume that $\mathcal{C}$ is m-semiadditive.*

1. *If $m < \infty$, then $\mathcal{C}$ decomposes as*

$$\mathcal{C} = \mathcal{C}_0 \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_{m-1} \times \mathcal{C}_{>m}.$$

2. *If $m = \infty$, then $\mathcal{C}$ admits a recollement*

$$\begin{tikzcd} \mathcal{C}_\infty \arrow[r, hook] & \mathcal{C} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \arrow[r] & \prod_{n\in\mathbb{N}} \mathcal{C}_n \arrow[l, bend left, "\perp"] \end{tikzcd}$$

(Recall that $\mathcal{C}_n \subseteq \mathcal{C}$ denotes the full subcategory of height-n objects.)

Proof. 1. By induction, the main step is showing that for each $n < m$, the divisible-complete-recollement

$$\begin{tikzcd} \mathcal{C}_{\leq n} \arrow[r, hook] & \mathcal{C} \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"] \arrow[r] & \mathcal{C}_{>n} \arrow[l, bend left, "\perp"] \end{tikzcd} \tag{5}$$

[1]Here we use a little known fact about adjunctions: the counit $FG \to \mathrm{id}$ of an adjunction $F \vdash G$ is an equivalence if and only if the composite FG is an equivalence.

induced by the endomorphism $\alpha \coloneqq p_{(n)} \colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is trivial. This follows from Proposition 4.4, because $p_{(n+1)}$ is a semi-inverse of $p_{(n)}$ by Corollary 1.6. Note that this step breaks down for $n = m$ because $\mathrm{B}^{m+1}C_p$ need not be $\mathcal{C}$-ambidextrous (and Corollary 1.4 doesn't help unless $\mathrm{B}^m C_p$ is amenable, in which case $\mathcal{C}_{>m} = 0$)

2. Follows from Proposition 3.3 applied to the descending chain

$$\mathcal{C}_\infty \subseteq \cdots \subseteq \mathcal{C}_{>2} \subseteq \mathcal{C}_{>1} \subseteq \mathcal{C}_{>0} \subseteq \mathcal{C}$$

Note that this uses the "wrong" inclusion $\mathcal{C}_{>n} \subseteq \mathcal{C}$ of the recollements (5). This is not an issue because all these recollements are trivial so that we may exchange the role of the two components $\mathcal{C}_{\leq n}$ and $\mathcal{C}_{>n}$. $\square$

References

[CSY20] Shachar Carmeli, Tomer M. Schlank, and Lior Yanovski. Ambidexterity in chromatic homotopy theory, 2020.