AMBIDEXTERITY AND TRANSCHROMATIC PHENOMENA

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1. Introduction

Theorem 1.1. *Let G be a finite group, and let* $\mathrm{Rep}(G)$ *be the representation ring of G. Then there is a map*

$$\mathrm{Rep}(G) \to \{\textit{class functions } G \to \mathbb{C}\}$$

which becomes an equivalence after tensoring by $\mathbb{C}$ on the left hand side.

Crucial to the theory of characters are the following compatibilities:

Theorem 1.2.

(1) The character map is natural in restriction,
(2) The character map is natural in induction,
(3) The character map is natural in deflations (natural for the covariant functoriality in surjective homomorphisms), for example:

$$V^G = \frac{1}{|G|} \sum_{g \in G} \chi(V)(g).$$

We may summarize this as follows:

Theorem 1.3. *The character map is a natural transformation of commutative ring objects in* $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Glo}), \mathrm{Ab})$.

Remark 1.4. It is even compatible with enhanced ring structures (i.e. it is a map of Tambara functors). This is not something worked out in the generality we will later discuss.

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This theorem actually expresses a relation between two objects of homotopy theory:

(1) $\mathrm{Rep}(G)$ is $\mathrm{KU}^0_G(*)$,
(2) The ring of class functions is $H^0(G_{hG};\mathbb{C})$.

With this interpretation we may generalize the isomorphism by:

Theorem 1.5 (The Chern character)**.** *Let G be a finite group and let X be a finite G-space. Then there is a natural transformations*

$$\mathrm{ch}_G\colon \mathrm{KU}^0_G(X) \to H^{\mathrm{even}}(\Big(\coprod_{g\in G} X^g\Big)_{hG};\mathbb{C})$$

which becomes an isomorphism after tensoring by $\mathbb{C}$ on the left hand side.

Note that KU is (roughly) height one, while $H(-,\mathbb{C}))$ is height zero. This makes the character extremely interesting from the perspective of chromatic homotopy theory, where heigher height (i.e. more complicated) phenomena can (at least partially) be controlled by lower height computations.

Therefore the hope would be for generalizations of the isomorphism above to heigher heights. This is originally due to Hopkins–Kuhn–Ravenel. It was then generalized by Stapleton. Finally Lurie clarified the origin of these phenomena using spectral algebraic geometry.

To understand better the shape of the isomorphism we can think of the height one case, obtained (roughly) by p-completing the iso above.

Theorem 1.6 (The p-complete Chern character)**.** *Let G be a finite group and let X be a finite G-space. Fix a prime p and choose an embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$. Then there is an isomorphism*

$$\mathbb{C}\otimes_{\mathbb{Z}_p} (\mathrm{KU}^\wedge_p)^0(X_{hG}) \to H^{\mathrm{even}}(\Big(\coprod_{\alpha\colon \mathbb{Z}_p\to G} X^\alpha\Big)_{hG};\mathbb{C})$$

The work of Hopkins–Kuhn–Ravenel generalizes this to higher heights. To discuss this we need to define splitting algebras, for which we use some spectral algebraic geometry due to Lurie:

(1) We let $E \coloneqq E_n$ be height n Morava E-theory. The formal group over E induces a p-divisible group over E, which is of height n.
(2) We may base-change this to $E_{\mathbb{Q}}$, the rationalization of $E_{\mathbb{Q}}$ for $t \leq n$ to obtain a p-divisible group $\mathbb{G}_{\mathbb{Q}}$ which is still of height n.
(3) Recall that every p-divisible group $\mathbb{G}$ over a ring R sits in a short exact sequence

$$0 \to \mathbb{G}^{\circ} \to \mathbb{G} \to \mathbb{G}_{\text{ét}} \to 0,$$

with its étale part and the connected component of the identity, which is a formal group.
(4) Because $\mathbb{G}_{\mathbb{Q}}^{\circ}$ is a formal group over a rational ring, it is height zero. Therefore $\mathbb{G}_{\text{ét}}$ is height n. In other words, $\mathbb{G}_{\mathbb{Q}}$ is étale.
(5) We write C_0 for the universal faithfully flat algebra over $E_{\mathbb{Q}}$ such that $\mathbb{G}_{\text{ét}} \simeq \underline{\mathbb{Q}_p^n/\mathbb{Z}_p^n}$. That this exists is a theorem. This is the *splitting algebra* of $\mathbb{G}_{\mathbb{Q}}$.

Remark 1.7. There is a (maybe clear) analog of this for heights $0 < t < n$, due to Stapleton. Instead of basechanging to $E_{\mathbb{Q}}$ one basechanges to $L_{K(t)}E$.

Theorem 1.8 (Hopkins–Kuhn–Ravenel)**.** *Consider $t < n$ and let X be a π-finite space. We write $L_p^n X$ for the n-fold p-adic free loop space of X. Then there is a natural map*

$$E^X \to C_0^{L_p^n X},$$

called the transchromatic character map*. It is an equivalence after tensoring with C_0 on the left hand side.*

Remark 1.9. Suppose $X = BG$, then $\pi_0(L_p^n BG)$ is the set $G_{n,p}$ of n-many mutually commuting p-power torsion elements of G up to conjugation. Because C_0 is a rational algebra, we find that $C_0^{L_p^n BG} = \mathrm{Hom}(G_{n,p}/G, C_0)$. We interpret this as some find of p-typical higher character.

Hopkins–Kuhn–Ravenel also consider the interaction of this map with transfers.

Proposition 1.10. *The transchromatic character map is incoherently natural with respect to transfer along finite index covers.*

Remark 1.11. This basic kind of compatibility is already crucial for applying the transchromatic character map to understanding the power operations on E-theory.

But as we have seen Mod_E and Mod_{C_0} are both ∞-semiadditive, and so the cohomology $E^{(-)}$ and $C_0^{(-)}$ both admit transfers along arbitrary maps with π-finite fibers. In other words, both improve to ∞-commutative monoids

$$E^{(-)}, C_0^{(-)} \colon \mathrm{Span}(\mathcal{S}_{\pi\text{-fin}}) \to \mathrm{Sp}$$

in spectra. Now the functor $L_p^n \colon \mathcal{S}_{\pi\text{-fin}} \to \mathcal{S}_{\pi\text{-fin}}$ preserves pullbacks and so $C_0^{L_p^n(-)}$ is again an ∞-commutative monoid (without the descent condition).

Question 1.12. *Is the transchromatic character a map of ∞-commutative monoids?*

There is an obvious approach to this, via categorification:

Proposition 1.13. *Both*

$$\widehat{\mathrm{Mod}}_E(-) \coloneqq \mathrm{Fun}(-, \mathrm{Mod}_E(\mathrm{Sp}_{K(n)})) \quad \text{and} \quad \widehat{\mathrm{Mod}}_{C_0}(L_p^n(-)) \coloneqq \mathrm{Fun}(L_p^n(-), \mathrm{Mod}_{C_0}(\mathrm{Sp}_{K(0)}))$$

are $\mathcal{S}_{\pi\text{-fin}}$-parametrized $\mathcal{S}_{\pi\text{-fin}}$-semiadditive monoids. The ∞-commutative monoids $E^{(-)}$ and $C_0^{L_p^n(-)}$ are the canonical enhancement of the respective units to ∞-commutative monoids.

Proof. For $\mathrm{Fun}(-, \mathrm{Mod}_E(\mathrm{Sp}_{K(n)}))$ and $\mathrm{Fun}(-, \mathrm{Mod}_{C_0}(\mathrm{Sp}_{K(0)}))$, this is the ∞-semiadditivity of $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{K(0)} = \mathrm{Sp}_{\mathbb{Q}}$ due to Hopkins–Lurie. Now $L_p^n(-) \colon \mathcal{S}_{\pi\text{-fin}} \to \mathcal{S}_{\pi\text{-fin}}$ preserves pullbacks, and so $\widehat{\mathrm{Mod}}_{C_0}(L_p^n(-)) = (L_p^n)^* \widehat{\mathrm{Mod}}_{C_0}$ is again $\mathcal{S}_{\pi\text{-fin}}$-semadditive.

That the unique enhancement of C_0 to an ∞-commutative monoid agrees with $C_0^{L_p^n(-)}$ is a small but necessary check. $\square$

Remark 1.14. The $\mathcal{S}_{\pi\text{-fin}}$-parametrized category $\mathrm{Mod}_{C_0}(L_p^n(-))$ is not a parametrized category of local systems (i.e. does not satisfy descent). Therefore the generalizations of

Harpaz' universal property we have discussed previously are crucial for our strategy to work.

Theorem 1.15. *There is a $\mathcal{S}_{\pi\text{-fin}}$-semiadditive $\mathcal{S}_{\pi\text{-fin}}$-parametrized functor*

$$\widehat{\chi}\colon \widehat{\mathrm{Mod}}_E(-) \to \widehat{\mathrm{Mod}}_{C_0}(L_p^n(-))$$

whose value on the point is the functor

$$L_{K(0)}(C_0 \otimes_E -)\colon \widehat{\mathrm{Mod}}_E \to \widehat{\mathrm{Mod}}_{C_0}.$$

Remark 1.16. By the above we mean that the induced map $\widehat{\chi}\colon \mathrm{End}_E(\mathbb{1}) \to \mathrm{End}_{C_0}(\mathbb{1})$ of ∞-commutative monoids enhances the transchromatic character map.

The functor above categorifies the transchromatic character map, meaning we obtain an equivalence.

Corollary 1.17. *There is an equivalence*

$$C_0 \otimes_E E^X \simeq C_0^{L_p^n X}$$

natural in $\mathrm{Span}(\mathcal{S}_{\pi\text{-fin}})$*. Therefore we also obtain the transchromatic character map as a map of ∞-commutative monoids.*

Proof. Consider the functor

$$\widehat{\chi}\colon \widehat{\mathrm{Mod}}_E(-) \to \widehat{\mathrm{Mod}}_{C_0}(L_p^n(-)).$$

Since

(1) $\mathcal{S}_{\pi\text{-fin}}$-semiadditive functors induce functors on ∞-commutative monoids,
(2) and any object has a unique enhancement to an ∞-commutative monoid

we obtain an equivalence

$$L_{K(0)}(C_0 \otimes_E E^X) \simeq C_0^{L_p^n X}$$

of ∞-commutative monoids. A simple check shows that the application of $L_{K(0)}$ is unnecessary. ∎

Let us remark that incoherently this fact was already shown in Elliptic 3 by Lurie. So why should we care about the coherent statement? Well the coherence buys us corollaries like this:

Corollary 1.18. *Let X be a π-finite space, and recall that associated to it is an element $|X|_E \in \pi_0(E)$, the cardinality of X in Mod_E. Then there is an equivalence $\chi(|X|_E) = |L_n^p X|_{C_0}$ in $\pi_0(C_0)$.*

Proof. This follows immediately from the fact that Ξ is a map of ∞-commutative monoids. □

Definition 1.19. Let X be a connected π-finite space. We define the homotopy cardinality $|X|_{\mathbb{Q}} \in \mathbb{Q}$ to be the product

$$\prod_{i \geq 1} \pi_i(X)^{(-1)^i}$$

and the sum $|X| = \Sigma |X_i|$ if $X = \amalg_i X_i$ is the coproduct of connected spaces.

Remark 1.20. This is the cardinality of X in $\mathrm{Sp}_{\mathbb{Q}}$.

Corollary 1.21. *Let X be a π-finite space. Then $|X|_E \in \pi_0(E)$ is an element of $\mathbb{Z}_{(p)}$ and there is an equality*

$$|X|_E = |L_p^n X|_{\mathbb{Q}}.$$

Proof. The induced map $\pi_0(\mathbb{S}_{K(n)}) \to \pi_0 E$ factors through $\mathbb{Z}_p$, since these are the only elements fixed by the action of the Morava stabilizer group. Since $|X|_E$ is the image of $|X|_{\mathrm{Sp}_{K(n)}}$ under this map we conclude it is an element of $\mathbb{Z}_{(p)}$.

Now we know from before that $\chi(|X|_E) = |L_n^p X|_{C_0}$. Now we observe the map $\chi \colon \pi_0(E) \to \pi_0(C_0)$ is injective when restricted to $\mathbb{Z}_p$, since C_0 is faithfully flat over $E_{\mathbb{Q}}$. So $|X|_E = |L_n^p X|_{C_0}$.

Conversely we can consider the unit map $\mathbb{Q} \to C_0$, which is injective on π_0. As above, we find that

$$|L_n^p X|_{C_0} = |L_n^p X|_{\mathbb{Q}}.$$

Combining these two facts gives the corollary. □

2. How to categorify?

We have seen that the compatibility of the transchromatic character map with transfers follows immediately from the existence of a categorification therefore. This takes the form of a $\mathcal{S}_{\pi\text{-fin}}$-semiadditive $\mathcal{S}_{\pi\text{-fin}}$-parametrized functor

$$\widehat{\chi}\colon \widehat{\mathrm{Mod}}_E(-) \to \widehat{\mathrm{Mod}}_{C_0}(L_p^n(-))$$

enhancing the functor $L_{K(0)}(C_0 \otimes_E -)$.

The construction of such a functor follows from ideas of Lurie, who provides a common language to discuss both of the cohomology theories[1] $E^{(-)}$ and $C_0^{L_p^n(-)}$. This is the language of tempered cohomology theories. Lurie also provides a beautiful categorification of this theory, called the theory of tempered local systems.

Definition 2.1. We define $\mathrm{Glo}_{\mathrm{ab}}$ to be subcategory of Glo on the finite abelian groups and $\mathcal{S}_{\text{ab-gl}} := \mathrm{PSh}(\mathrm{Glo}_{\mathrm{ab}})$.

There is a functor $|-|\colon \mathcal{S}_{\text{ab-gl}} \to \mathcal{S}$ given by evaluation at $* \in \mathrm{Glo}_{\mathrm{ab}}$. This admits a right adjoint which we denote by

$$\mathbb{R}(-)\colon \mathcal{S} \to \mathcal{S}_{\text{ab-gl}}$$

We will always view spaces as global spaces via the functor $\mathbb{R}(-)$.

Definition 2.2. We let $\mathrm{BT}_{\mathrm{or}}$ denote the ∞-category of pairs $(R, \mathbb{G})$ of a complex periodic[2] $\mathbb{E}_\infty$-ring A together with an oriented p-divisible group $\mathbb{G}$ over A.

Remark 2.3. One should imagine that the definition of a p-divisible group is very analogous to the classical definition. An orientation of a p-divisible group $\mathbb{G}$ is an isomorphism between the connected component $\mathbb{G}^\circ$ of $\mathbb{G}$, which is a formal group over A, and the Quillen formal group $A^{\mathbb{C}P^\infty}$.

[1] One should interpret both as cohomology theories of global space

[2] weakly 2-periodic and complex oriented

Example 2.4. Suppose A is a complex periodic $K(n)$-local $\mathbb{E}_\infty$-ring. Then the Quillen formal group on A is a p-divisible group with height equal to the height of the formal group on A. We denote it $\mathbb{G}_A^{\mathcal{Q}}$

Example 2.5. Taking $\mathbb{G}_m$ over KU we obtain an oriented $\mathbf{P}$-divisible group $\mu_\infty \subset \mathbb{G}_m$ by restricting to torsion points.

Theorem 2.6 (Tempered cohomology)**.** *There is a functor*

$$\mathrm{BT}_{\mathrm{or}} \to \mathrm{Fun}^{\lim}(\mathcal{S}_{\text{ab-gl}}^{\mathrm{op}}, \mathrm{Sp}), \quad (A, \mathbb{G}) \mapsto A_{\mathbb{G}}(-).$$

which associates to every oriented p-divisible group its tempered cohomology theory*.*

Example 2.7. Suppose $\mathbb{G}$ is connected, i.e. a formal group. Then there is a natural equivalence $A_{\mathbb{G}}(X) \simeq A^{|X|}$. In other words, $A_{\mathbb{G}}$ is a Borel-equivariant theory.

Definition 2.8. We say an ab-global category is ∞-semiadditive if it is semiadditive for the class of maps $\mathcal{X} \to \mathcal{Y}$ in $\mathcal{S}_{\text{ab-gl}}$ whose fibers[3] are equivalent to $\mathbb{R}(Z)$ for π-finite spaces Z.

The following construction categorifies tempered cohomology.

Theorem 2.9 (Tempered Local Systems)**.** *There is a functor*

$$\mathrm{LocSys}\colon \mathrm{BT}_{\mathrm{or}} \to \mathrm{Fun}^{\lim}(\mathcal{S}_{\text{ab-gl}}^{\mathrm{op}}, \mathsf{Pr}_{\mathrm{L}}^{\otimes}), \quad (A, \mathbb{G}) \mapsto \mathrm{LocSys}_{\mathbb{G}}(-)$$

such that

(1) LocSys *factors through the category of presentable ab-global categories,*
(2) Each $\mathrm{LocSys}_{\mathbb{G}}$ *is ∞-semiadditive.*
(3) Suppose A is a p-local ring and $\mathbb{G}$ has height n. The there is a global left adjoint

$$\mathrm{LocSys}_{\mathbb{G}} \Rightarrow \widehat{\mathrm{Mod}}_A(|-|),$$

with fully faithful right adjoint.

[3] over all $\mathbb{B}A \to Y$

(4) it categorifies tempered cohomology, in the sense that there is a natural equivalence

$$A_{\mathbb{G}}(\mathcal{X}) \simeq p_* p^* \mathbb{1} \in \mathrm{LocSys}_{\mathbb{G}}(*) \simeq \mathrm{Mod}_A,$$

where $p\colon \mathcal{X} \to *$ is the unique map to the point.

The final property of tempered local systems which is crucial for us is the expression of character theory. We need the following definition:

Definition 2.10. Suppose $\mathcal{X}$ is a ab-global space. Write Λ for the group $\mathbb{Q}_p/\mathbb{Z}_p$. Then we define the ab-global space $\mathcal{L}_p^n(\mathcal{X})$ via the assignment

$$\mathbb{B}A \to \operatorname*{colim}_{\substack{\Lambda_0 \subset \Lambda, \\ \text{finite}}} \mathcal{X}(\mathbb{B}A \times \Lambda_0^\vee)$$

Lemma 2.11. *Let X be a π-finite space. There is an equivalence*

$$\mathcal{L}_p^n(\mathbb{R}(X)) \simeq \mathbb{R}(L_p^n(X)).$$

Theorem 2.12 (Categorified character theory)**.** *Write $\Lambda = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ for the constant p-divisible group over A. Suppose $\mathbb{G}$ is an oriented p-divisible group over A which splits as a sum $\mathbb{G}_0 \oplus \Lambda$. Then there is a symmetric monoidal fully faithful global left adjoint*

$$\Phi\colon \mathrm{LocSys}_{\mathbb{G}} \to \mathrm{LocSys}_{\mathbb{G}_0}(\mathcal{L}_p^n(-)).$$

Remark 2.13. From this equivalence, we obtain

$$A_{\mathbb{G}}(\mathcal{X}) \simeq A_{\mathbb{G}_0}(\mathcal{L}_p^n(\mathcal{X})),$$

which is a form of the character equivalence.

We now build a functor

$$\widehat{\mathrm{Mod}}_E(|-|) \to \widehat{\mathrm{Mod}}_{C_0}(|\mathcal{L}_p^n(-)|)$$

of global categories as the composite

$$\widehat{\mathrm{Mod}}_E(|-|) \hookrightarrow \mathrm{LocSys}_{\mathbb{G}_E^{\mathcal{Q}}}(-) \to \mathrm{LocSys}_{C_0 \otimes_E \mathbb{G}_E^{\mathcal{Q}}}(-) \simeq \mathrm{LocSys}_{\Lambda \oplus \mathbb{G}_{C_0}^{\mathcal{Q}}}(-) \to \dots$$
$$\dots \to \mathrm{LocSys}_{\mathbb{G}_{C_0}^{\mathcal{Q}}}(\mathcal{L}_p^n(-)) \to \mathrm{Mod}_{C_0}(|\mathcal{L}_p^n(-)|).$$

Each of the functors is either a global left or a right adjoint, and so the composite is globally semiadditive. Restricting to $\mathcal{S}_{\pi\text{-fin}} \subset \mathcal{S}_{\text{ab-gl}}$ we obtain the functor of Theorem 1.15. This concludes the talk.

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