Universal property of span 2-categories

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1 Motivation

Let C be some indexing category[1], and consider classes of morphisms

$$E, I, P \subseteq C$$

which are closed under base change and such that $I, P \subseteq E$. The following is a question that comes up in a variety of contexts:

Question 1.1. Given a functor

$$D\colon C^{\mathrm{op}} \to \mathrm{Cat}, \qquad X \mapsto D(X), \qquad f \mapsto f^*$$

which satisfies the property that the i^* have left adjoints satisfying base change and the p^* have right adjoints satisfying base change. Can we extend D to a functor

$$D\colon \mathrm{Span}(C, E) \to \mathrm{Cat}$$

where the forward functoriality $e_!\colon D(X) \to D(Y)$ is a left adjoint to e^* when $e \in E$ and a right adjoint to e^* when $e \in P$?

I have used the notations to match those used for six-functor formalisms, but this question is also relevant for various problems surrounding ambidexterity.

Even better would be if we could extend D to a suitable 2-category that actually *remembers* the data of the adjunctions $i_! \dashv i^*$ and $p^* \dashv p_*$:

Construction 1.2. There exists an $(\infty, 2)$-category $\mathrm{Span}_2(C, E)_I^P$ described as follows:

- The objects are those of C;
- The morphisms are spans

  $$X \xleftarrow{f} Z \xrightarrow{e} Y$$

  with $e \in E$;

[1] All categories are ∞-categories and all 2-categories are $(\infty, 2)$-categories

- The 2-morphisms are spans of spans

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "e"] & \\
X & Z'' \arrow[u, "p"'] \arrow[d, "i"] & Y, \\
& Z' \arrow[ul, "f'"] \arrow[ur, "e'"'] &
\end{tikzcd}$$

  where $i \in I$ and $p \in P$.

- Composition is given by pullback.

Formally, one may construct $\mathrm{Span}_2(C, E)_I^P$ for example by first constructing the functor $\mathrm{Span}(C) \to \mathrm{Cat}$ which sends X to $\mathrm{Span}(C_{/X})$, and then uses the Lu-Zheng category construction to obtain a 2-category $\mathrm{Span}_2(C)$ whose hom-categories between X and Y are $\mathrm{Span}(C_{/X\times Y})$. The 2-category $\mathrm{Span}_2(C, E)_I^P$ is then a subcategory of $\mathrm{Span}_2(C)$.

Question 1.3. Can a functor $D \colon C^{\mathrm{op}} \to \mathrm{Cat}$ be (uniquely) extended to a 2-functor $\mathrm{Span}_2(C, E)_I^P \to \mathrm{Cat}$?

There is the following conjectural statement:

Conjecture 1.4 (Gaitsgory-Rozenbluym, Hoyois)**.** *Yes, if all morphisms in $I \cap P$ are truncated and every morphism in E can be written as a composite $X \xrightarrow{i} Y \xrightarrow{p} Z$.*

Today I wish to explain partial progress towards this question:

Theorem 1.5 (Work in progress with Ben-Moshe, Lenz, Linskens, Yanovski)**.** *Yes, when $P \subseteq I = E \subseteq C$ and P is inductible.*

Remark 1.6. In the case $P = C^{\simeq}$, this recovers the known universal property of the 2-category "$\mathrm{Span}_{1.5}(C)$" where the 2-morphisms are morphisms of spans

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "e"] \arrow[dd, "i"] & \\
X & & Y. \\
& Z' \arrow[ul, "f'"] \arrow[ur, "e'"'] &
\end{tikzcd}$$

This case is conceptually simpler, as it is about freely adding left adjoints to the given category, and there is no interaction with any right adjoints.

2 Adjointable functors

As mentioned, the result is in essence a universal property of the 2-category $\mathrm{Span}_2(C, I)_I^P$, and it is convenient to formulate everything in the language of 2-categories.

Definition 2.1. Let $\mathcal{E}$ be a 2-category. A commutative square

$$\begin{tikzcd} X' \arrow[r, "h^*"] \arrow[d, "k^*"'] & X \arrow[d, "f^*"] \\ Y' \arrow[r, "g^*"] & Y \end{tikzcd}$$

is called *vertically left adjointable* if the morphisms f^* and k^* admit left adjoints $f_!$ and $k_!$ in $\mathcal{E}$ and the Beck-Chevalley map

$$\mathrm{BC}_!\colon k_!h^* \xrightarrow{\eta_f} k_!h^*f^*f_! = k_!k^*g^*f_! \xrightarrow{\varepsilon_k} g^*f_!$$

is an isomorphism in the Hom-category $\mathrm{Hom}_{\mathcal{E}}(X, Y')$.

Dually we say that the square is *vertically right adjointable* if it is left adjointable in $\mathcal{E}^{2\text{-op}}$, i.e. f^* and k^* admit *right* adjoints f_* and k_* and the Beck-Chevalley map $\mathrm{BC}_*\colon g^*f_* \to k_*h^*$ is an equivalence in $\mathrm{Hom}_{\mathcal{E}}(X, Y')$.

Definition 2.2 (cf. [EH23, Definition 2.2.5])**.** Let $\mathcal{C}$ be a small category and let $I, P \subseteq \mathcal{C}$ be two wide subcategories closed under base change. Let $\mathcal{E}$ be a 2-category, and consider a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$. For a morphism $f\colon X \to Y$ in $\mathcal{C}$ we write $f^* := F(f)\colon F(Y) \to F(X)$.

(1) We say that F is *left I-adjointable* if for every pullback square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "j"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "i"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}$ with $i \in I$, the associated square

$$\begin{tikzcd} F(Y) \arrow[r, "f^*"] \arrow[d, "i^*"'] & F(Y') \arrow[d, "j^*"] \\ F(X) \arrow[r, "g^*"] & F(X') \end{tikzcd}$$

in $\mathcal{E}$ is vertically left adjointable. In particular, i^* and j^* admit left adjoints $i_!$ and $j_!$.

(2) We say that F is *right P-adjointable* if $F^{2\text{-op}}\colon \mathcal{C}^{\mathrm{op}} = (\mathcal{C}^{\mathrm{op}})^{2\text{-op}} \to \mathcal{E}^{2\text{-op}}$ is left P-adjointable, i.e. if for every pullback square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{C}$ with $p \in P$, the associated square

$$\begin{tikzcd} F(Y) \arrow[r, "f^*"] \arrow[d, "p^*"'] & F(Y') \arrow[d, "q^*"] \\ F(X) \arrow[r, "g^*"] & F(X') \end{tikzcd}$$

in $\mathcal{E}$ is vertically right adjointable. In particular, p^* and q^* admit right adjoints p_* and q_*.

(3) We say that F is (I, P)-*bi-adjointable* if it is both left I-adjointable and right P-adjointable, and for every pullback square

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "i"] & Y \end{tikzcd}$$

in $\mathcal{C}$ with $i \in I$ and $p \in P$, the double Beck-Chevalley map $j_! p_* \to q_* i_!$ is an equivalence in $\mathrm{Hom}_{\mathcal{E}}(F(X'), F(Y))$.

If $F, G\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ are left I-adjointable, we say that a transformation $\alpha\colon F \Rightarrow G$ is *left I-adjointable* if for every morphism $i\colon X \to Y$ in I the naturality square

$$\begin{tikzcd} F(X) \arrow[r, "\alpha(X)"] \arrow[d, "f^*"'] & G(Y) \arrow[d, "f^*"] \\ F(Y) \arrow[r, "\alpha(Y)"] & G(Y) \end{tikzcd}$$

is vertically left adjointable. We may similarly define what it means for α to be *right P-adjointable*, and we say it is (P, I)-*bi-adjointable* if it is both left I-adjointable and right P-adjointable. We denote by

$$\mathrm{Fun}_{I\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}), \qquad \mathrm{Fun}_{P\text{-radj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \qquad \text{and} \qquad \mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$$

the locally full subcategories of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ spanned by the left/right/bi-adjointable functors and morphisms, respectively. If $I = P$, we will also speak of *I-bi-adjointability.*

Remark 2.3. Note that for $\mathcal{E} = \mathrm{Cat}$ we may think of F as a parametrized category. In this case:

- F is left I-adjointable if and only if it has I-colimits;
- F is right P-adjointable if and only if it has P-limits;
- F is (I, P)-bi-adjointable if and only if it has both I-colimits and P-limits and they commute with each other.

In particular, when $I = P = Q$ is an inductible subcategory, F is (Q, Q)-bi-adjointable if and only if it is Q-semiadditive.

Remark 2.4. If $P = \mathcal{C}^{\simeq}$ consists only of the equivalences in $\mathcal{C}$, then the notion of (I, P)-bi-adjointability agrees with that of left I-adjointability. Similarly, if $I = \mathcal{C}^{\simeq}$ then (I, P)-bi-adjointability agrees with right P-adjointability. For this reason, we will frequently only state and prove statements in the bi-adjointable case.

Remark 2.5. Any 2-functor $G\colon \mathcal{E} \to \mathcal{E}$ preserves adjointable squares. In particular, if $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ is (I, P)-bi-adjointable then so is the composite $G \circ F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$.

Remark 2.6. Consider a functor $H\colon \mathcal{C}' \to \mathcal{C}$ that sends pullbacks along I' and P' to pullbacks along I and P, respectively. Then for every (I, P)-bi-adjointable functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{E}$ the composite $F \circ H\colon (\mathcal{C}')^{\mathrm{op}} \to \mathcal{E}$ is (I', P')-bi-adjointable.

3 The universal bi-adjointable functor

We shall now move to a discussion of the *universal* bi-adjointable functor out of $\mathcal{C}^{\mathrm{op}}$. We continue to fix a small category $\mathcal{C}$ equipped with two wide subcategories $I, P \subseteq \mathcal{C}$ closed under base change. In light of Remark 2.5, the assignment $\mathcal{E} \mapsto \mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ determines a functor

$$\mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, -)\colon \mathrm{Cat}_{(\infty,2)} \to \mathrm{Cat}_{(\infty,2)}.$$

Proposition 3.1 (cf. [EH23, Proposition 2.3.1, Corollary 2.3.5])**.** *The functor* $\mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, -)$ *is corepresentable: there exist a (I, P)-bi-adjointable functor*

$$h = h_{(I,P)\text{-badj}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$$

such that precomposition with h induces a natural equivalence

$$h^*\colon\ \mathrm{Fun}_2(\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}), \mathcal{E}) \xrightarrow{\ \sim\ } \mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}),$$

for any $D \in \mathrm{Cat}_{(\infty,2)}$.

Notation 3.2. In the special cases where $P = \mathcal{C}^{\simeq}$, $I = \mathcal{C}^{\simeq}$ we will use the notation

$$h_{I\text{-ladj}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{I\text{-ladj}}(\mathcal{C}) \qquad \text{and} \qquad h_{P\text{-radj}}\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{P\text{-radj}}(\mathcal{C})$$

for the universal left I-adjointable or right P-adjointable functors out of $\mathcal{C}^{\mathrm{op}}$, respectively.

Proof sketch. The question is a representability question: we need to show that the functor

$$\mathrm{Map}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, -)\colon \mathrm{Cat}_{(\infty,2)} \to \mathrm{Spc}$$

is accessible and preserves small limits. We will do this by writing it as an iterated pullback of accessible functors that preserve small limits. For the condition of having adjoints, we want to consider the universal 2-category ADJ with an adjoint, i.e. it comes with a functor $[1] \to \mathrm{ADJ}$ that sends the morphism to a left adjoint morphism in ADJ, and it is universal with this property. To enforce the left adjoints for $i \in I$, we may then form the following pullback:

$$\begin{tikzcd}
\mathrm{Map}_{I-\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, -) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{C}^{\mathrm{op}}, -) \arrow[d, "(i^*)_{i \in I}"] \\
\prod_{i \in I} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ}, -) \arrow[r] & \prod_{i \in I} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}([1], -).
\end{tikzcd}$$

Similarly, to enforce the Beck-Chevalley condition we form a pullback of the form

$$\begin{tikzcd}
\mathrm{Map}_{I\text{--ladj}}(\mathcal{C}^{\mathrm{op}}, -) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}_{I-\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, -) \arrow[d] \\
\prod \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ} \times [1], -) \arrow[r] & \prod \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}([1] \times [1], -),
\end{tikzcd}$$

where this time the product is taken over all (isomorphism classes of) the relevant pullback squares. Continuing this way we may enforce right P-adjointability and (I, P)-bi-adjointability as well. $\square$

Remark 3.3 (cf. [EH23, Remark 2.3.3])**.** It follows directly from their universal properties that the assignments $(\mathcal{C}, I) \mapsto \mathbb{S}_{I\text{-ladj}}(\mathcal{C})$, $(\mathcal{C}, P) \mapsto \mathbb{S}_{P\text{-radj}}(\mathcal{C})$ and $(\mathcal{C}, I, P) \mapsto \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$ can be upgraded to 2-functors

$$\mathbb{S}_{\text{-ladj}} \colon \mathrm{SpanPair} \to \mathrm{Cat}_{(\infty,2)}, \qquad \mathbb{S}_{\text{-radj}} \colon \mathrm{SpanPair} \to \mathrm{Cat}_{(\infty,2)},$$
$$\mathbb{S}_{\text{-badj}} \colon \mathrm{SpanPair} \times_{\mathrm{Cat}} \mathrm{SpanPair} \to \mathrm{Cat}_{(\infty,2)}.$$

Similarly, the functors $h_{I\text{-ladj}}$, $h_{R\text{-radj}}$ and $h_{(I,P)\text{-badj}}$ assemble into natural transformations.

Lemma 3.4 (cf. [EH23, Lemma 2.3.7])**.** *Given $\mathcal{C}$, I and P as above, the three functors*

$$h_{I\text{-ladj}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{I\text{-ladj}}(\mathcal{C}), \qquad h_{P\text{-radj}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{P\text{-radj}}(\mathcal{C}) \qquad \textit{and} \qquad h_{(I,P)\text{-badj}} \colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$$

are essentially surjective.

Proof. The case of left I-adjointability was treated in [EH23, Lemma 2.3.7]: one observes that the functor from $\mathcal{C}^{\mathrm{op}}$ to the image of $h_{I\text{-ladj}}$ is still left L-adjointable and hence factors through $\mathbb{S}_{I\text{-ladj}}(\mathcal{C})$, when can then be used to show $h_{I\text{-ladj}}$ is essentially surjective. The proof in the other two cases is identical. $\square$

4 A recognition principle for universal bi-adjointable functors

Our eventual goal in the next section is to show that the three 2-categories obtained in the previous proposition are given by 2-categories of (iterated) spans. We will now provide a useful criterion for recognizing that a given bi-adjointable functor is universal.

Definition 4.1. A (I, P)-bi-adjointable functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}$ is called *universal* if for any other 2-category $\mathcal{E}$ restriction along F induces an equivalence of 2-categories

$$F^* \colon \mathrm{Fun}_2(\mathbb{S}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{E}).$$

Our criterion relies on the notion of *free* adjointable functors:

Definition 4.2. A (I,P)-bi-adjointable functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ is called *free on an object $A \in \mathcal{C}$* if it comes equipped with an object $1_A \in F(A)$ satisfying the property that for every other (I,P)-bi-adjointable functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ evaluation at 1_A induces an equivalence of categories

$$\mathrm{Hom}_{\mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat})}(F,G) \xrightarrow{\simeq} G(A).$$

Proposition 4.3. *For every object $A \in \mathcal{C}$, the identity morphism $\mathrm{id}_{h(A)}$ exhibits the functor*

$$\mathrm{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A), h(-))\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$$

as a free (I,P)-bi-adjointable functor.

Proof. The fact that it is a (I,P)-bi-adjointable functor with respect to $\mathcal{C}_R$ is immediate from Remark 2.5 since it may be written as a composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{h} \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}) \xrightarrow{\mathrm{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A),-)} \mathrm{Cat}_\infty,$$

where the first functor is a (I,P)-bi-adjointable functor and the second is a 2-functor. To show it is also free on A, consider another (I,P)-bi-adjointable functor $G\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. By the universal property of $h\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$, we may uniquely extend G to a 2-functor $\overline{G}\colon \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}) \to \mathrm{Cat}$. Consider now the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}_2(\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}),\mathrm{Cat})}(\mathrm{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A),-),\overline{G}) \arrow[r, "\mathrm{ev}_{\mathrm{id}_{h(A)}}"] \arrow[d, "\simeq"'] & \overline{G}(h(A)) \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat})}(\mathrm{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A),h(-)),G) \arrow[r] & G(A).
\end{tikzcd}$$

The left vertical map is induced by the equivalence $h^*\colon\ \mathrm{Fun}_2(\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}),\mathrm{Cat}) \xrightarrow{\simeq} \mathrm{Fun}_{(I,P)\text{-badj}}(\mathcal{C}^{\mathrm{op}},\mathrm{Cat})$ by passing to Hom-categories, hence is an equivalence itself. The top map is an equivalence by the 2-categorical Yoneda lemma. It follows that also the bottom map is an equivalence, as desired. ☐

Combining all the above statements, we now obtain the following recognition principle for the universal (I,P)-bi-adjointable functor out of $\mathcal{C}^{\mathrm{op}}$:

Theorem 4.4. *A (I,P)-bi-adjointable functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}$ is universal if and only if it is essentially surjective and for every object A of $\mathcal{C}$ the identity map $\mathrm{id}_{F(A)}$ exhibits the composite*

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathbb{S} \xrightarrow{\mathrm{Hom}_{\mathbb{S}}(F(A),-)} \mathrm{Cat}$$

as a free (I,P)-bi-adjointable functor on A.

Proof. The 'only if'-direction is immediate from Lemma 3.4 and Proposition 4.3. Conversely, assume that F satisfies the two condition in the theorem. By the universal property of $h\colon \mathcal{C}^{\mathrm{op}} \to$

$\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$, the functor F uniquely extends to a 2-functor $\overline{F}\colon \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C}) \to \mathbb{S}$, and it will suffice to show that $\overline{F}$ is an equivalence. Since $F = \overline{F} \circ h$ is essentially surjective by assumption, also $\overline{F}$ is essentially surjective. For fully faithfulness, we must show that for all objects $X, Y \in \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$ the induced functor

$$\overline{F}\colon \operatorname{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(X,Y) \to \operatorname{Hom}_{\mathbb{S}}(\overline{F}(X), \overline{F}(Y))$$

is an equivalence of categories. By essential surjectivity of $h\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})$, it suffices to show that for every object $A \in \mathcal{C}$ the induced transformation

$$\overline{F}\colon \operatorname{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A), h(-)) \to \operatorname{Hom}_{\mathbb{S}}(F(A), F(-))$$

is an equivalence of (I,P)-bi-adjointable functors $\mathcal{C} \to \mathrm{Cat}$. Since both sides are free on A, the left-hand side by Proposition 4.3 and the right-hand side by assumption on F, it remains to show that it sends the universal element $\mathrm{id}_{h(A)} \in \operatorname{Hom}_{\mathbb{S}_{(I,P)\text{-badj}}(\mathcal{C})}(h(A), h(A))$ to the universal element $\mathrm{id}_{F(A)} \in \operatorname{Hom}_{\mathbb{S}}(F(A), F(A))$. But this is immediate from 2-functoriality of $\overline{F}$. □

5 Universal property of span 2-categories

We will now sketch a proof of the universal property of the 2-category $\mathrm{Span}_2(\mathcal{C}, E)_I^P$, at least in a special case.

Observation 5.1. The inclusion $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(\mathcal{C}, E)_I^P$ is (I,P)-bi-adjointable:

- The left adjoint to the span $(Y \xleftarrow{i} X \xrightarrow{\mathrm{id}} X)$ is the span $(X \xleftarrow{\mathrm{id}} X \xrightarrow{i} Y)$. The unit and counit are given by

$$\begin{tikzcd} & X \arrow[ddl, bend right, "i"'] \arrow[d, dashed, "i"] \arrow[ddr, bend left, "i"] & \\ & Y \arrow[dl, equal] \arrow[dr, equal] & \\ Y & & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & X \arrow[ddl, bend right, equal] \arrow[d, dashed, "\Delta_i"] \arrow[ddr, bend left, equal] & \\ & X \times_Y X \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"'] & \\ X & & X. \end{tikzcd}$$

- The right adjoint for $p \in P$ is similar, with the 2-morphisms pointing in the other direction.

- Bi-adjointability is essentially because both the left and right adjoints are just the forward maps in the span 2-category. (One would need to unwind the Beck-Chevalley map to check that it is indeed the identity map.)

Conjecture 5.2. *If every morphism in E factors as $X \xrightarrow{i} Z \xrightarrow{p} Y$ and every morphism $e \in I \cap P$ is truncated, then $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Span}_2(\mathcal{C}, E)_I^P$ is a universal (I,P)-bi-adjointable functor.*

By the recognition principle, the statement is equivalent to the statement that for every A the functor

$$\mathrm{Hom}_{\mathrm{Span}_2(\mathcal{C},E)_I^P}(A,-)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$$

is a free (I,P)-adjointable functor. By construction of $\mathrm{Span}_2(\mathcal{C},E)_I^P$, we have an explicit description of these categories. So the conjecture is equivalent to the following:

Conjecture 5.3. *The parametrized category*

$$\underline{\mathrm{Span}}^A(I,P,I)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}, \qquad B \mapsto \mathrm{Span}(\mathcal{C}_{/A} \times_{\mathcal{C}} E_{/B}, P, I)$$

is the free pararametrized category on A which has I-colimits and P-limits which commute with each other.

Now, in this generality we do not know how to attack this question... But if we assume that $P \subseteq I = E$ is inductible, this becomes a question about semiadditivity!

Proposition 5.4. *Assume that $P \subseteq I = E$ is inductible. Then the free I-cocomplete P-semiadditive parametrized category on A is*

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}, \qquad B \mapsto \mathrm{Span}(\mathcal{C}_{/A} \times_{\mathcal{C}} I_{/B}, P, I).$$

Proof sketch. For $A = *$ the terminal object, this is essentially the universal property of the parametrized span category $\underline{\mathrm{Span}}(I,P,I)$ discussed in Talk 2. For the general case, we consider the parametrized functor

$$\begin{gathered}\underline{A} \times \underline{\mathrm{Span}}(I,P,I) \to \underline{\mathrm{Span}}^A(I,P,I)\\ \mathrm{Hom}_C(B,A) \times \mathrm{Span}(I_{/B},P,I) \to \mathrm{Span}(\mathcal{C}_{/A} \times_{\mathcal{C}} I_{/B}, P, I)\\ (f\colon B \to A, i\colon C \to B) \mapsto (A \xleftarrow{fi} C \xrightarrow{i} B).\end{gathered}$$

Then one can prove that for any parametrized category C with I-colimits, the restriction functor

$$\mathrm{Fun}(\underline{\mathrm{Span}}^A(I,P,I),\mathcal{C}) \to \mathrm{Fun}(\underline{A} \times \underline{\mathrm{Span}}(I,P,I),\mathcal{C}) \simeq \mathrm{Fun}(\underline{A}, \underline{\mathrm{Fun}}(\underline{\mathrm{Span}}(I,P,I),\mathcal{C}))$$

restricts to an equivalence between:

- Parametrized functors $\underline{\mathrm{Span}}^A(I,P,I) \to \mathcal{C}$ that preserve I-colimits;
- Functors $\underline{A} \times \underline{\mathrm{Span}}(I,P,I) \to \mathcal{C}$ that preserve I-colimits in the second variable.

The inverse is given by parametrized right Kan extension. Now, if $\mathcal{C}$ happens to be P-semiadditive, then $\underline{\mathrm{Fun}}_{I\text{-ladj}}(\underline{\mathrm{Span}}(I,P,I),\mathcal{C}) \simeq \mathcal{C}$, and $\mathrm{Fun}(\underline{A},\mathcal{C}) \simeq \mathcal{C}(A)$, so we get the desired universal property. □

References

[EH23] Elden Elmanto and Rune Haugseng. "On distributivity in higher algebra. I: The universal property of bispans". *Compos. Math.* 159.11 (2023), pp. 2326–2415.