m-semiadditive (∞, n)-categories

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1 Iterated spans

We denote by $m\mathrm{Fin}$ the ∞-category of m-finite spaces.

Recall that in previous talks we exhibited the span category $\mathrm{Span}_1(m\mathrm{Fin})$ as the free m-semiadditive ∞-category generated by the singleton object $1 = \{*\}$ in the sense that for every other m-semiadditve ∞-category $\mathcal{D}$, the evaluation functor

$$\mathrm{ev}_1 \colon \mathrm{Fun}^{\oplus-m}(\mathrm{Span}_1(m\mathrm{Fin}), \mathcal{D}) \to \mathcal{D}$$

is an equivalence of ∞-categories.

The goal of this project is to extend this universal property to higher spans. For this we consider the (∞, n)-category $\mathrm{Span}_n(m\mathrm{Fin})$ which can be informally described as follows:

- objects are m-finite spaces
- for each A, B, the $(\infty, n-1)$-category of morphisms between them is
$$\mathrm{Span}_{n-1}(m\mathrm{Fin}_{/A\times B}) = \mathrm{Span}_{n-1}(m\mathrm{Fin}^{B\times A}) = \mathrm{Span}_{n-1}(m\mathrm{Fin})^{A\times B}$$
- composition is induced by the matrix multiplication functors
$$\begin{aligned} m\mathrm{Fin}^{C\times B} \times m\mathrm{Fin}^{B\times A} &\longrightarrow m\mathrm{Fin}^{C\times A} \\ (M, N) &\longmapsto L; \qquad\qquad \text{with } L_{c,a} = \sum_{b:B} {}_cM_b \times {}_bN_a \end{aligned}$$
or, equivalently, pullback of spans:
$$\begin{tikzcd} & & L \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner", rotate=-45] & & \\ & N \arrow[dl] \arrow[dr] & & M \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

(We will see a more precise formulation of this later.)

2 m-semiadditive (∞, n)-categories

We aim to define the notion of m-semiadditivity for (∞, n)-categories, such that $\mathrm{Span}_n(m\mathrm{Fin})$ becomes the free such (∞, n)-category generated by a point. The first guess would be that an m-semiadditive (∞, n)-category $\mathbb{D}$ is defined by the property

- For every m-finite space A and every diagram $X\colon A \to \mathbb{D}$, the limit and colimit
$$\operatorname{colim} X \simeq \lim X$$
exists and are canonically identified.

The problem with this definition is that any semiadditive $(\infty, 1)$-category $\mathcal{D}$, viewed as an (∞, n)-category, would satisfy it. This is an issue, because for $n \geq 2$ every additive functor $\mathrm{Span}_n(m\mathrm{Fin}) \to \mathcal{D}$ is necessarily 0 because every zero map $0 \to A$ has an adjoint in $\mathrm{Span}_2(m\mathrm{Fin})$, hence becomes an equivalence in the $(\infty, 1)$-category $\mathcal{D}$.

Analyzing the recursive structure of the iterated span category, we instead arrive at the following definition/characterization:

Lemma & Definition 2.1. Let $n \geq 2$. An (∞, n)-category $\mathbb{D}$ is called m-semiadditive if

(E) it is enriched in m-semiadditive $(n-1)$-categories

and satisfies any of the following equivalent conditions:

(S1) the underlying $(\infty, 1)$-category $\mathbb{D}^{(1)}$ of $\mathbb{D}$ is m-semiadditive;

(S2) for every diagram $X\colon S \to \mathbb{D}^{(1)} \subset \mathbb{D}$ indexed by an m-finite space one of the following equivalent conditions holds:

(a) X admits a limit in $\mathbb{D}^{(1)}$;

(b) X admits a limit in $\mathbb{D}$;

(c) X admits a colimit in $\mathbb{D}^{(1)}$;

(d) X admits a colimit in $\mathbb{D}$.

(e) X admits both a limit and a colimit in $\mathbb{D}$ and they are canonically identified.

A functor $\mathbb{F}\colon \mathbb{C} \to \mathbb{D}$ between m-semiadditive (∞, n)-categories is called m-semiadditive if it satisfies any of the following equivalent conditions

(F1) The underlying functor $\mathbb{F}^{(1)}\colon \mathbb{C}^{(1)} \to \mathbb{D}^{(1)}$ is m-semiadditive.

(F2) $\mathbb{F}$ preserves limits indexed by m-finite spaces;

(F3) $\mathbb{F}$ preserves colimits indexed by m-finite spaces;

(F4) for each pair X, Y of objects in $\mathbb{C}$, the induced functor

$$\mathbb{F}_{X,Y} \colon \mathbb{C}(X,Y) \to \mathbb{D}(\mathbb{F}X, \mathbb{F}Y)$$

on hom-$(\infty, n-1)$-categories is m-semiadditive.

Remark 2.2. Let us clarify what we mean by condition (E). In general, "being enriched" is of course (a large amount of) extra structure. However, from the Lemma/Definition it is clear by induction (starting from the base case $n = 1$, which we understand well from previous talks) that "being m-semiadditive" is a property both for (∞, n)-categories and for functors between them. Therefore we can say that an (∞, n)-category $\mathbb{D}$ *is enriched* in $(\infty, n-1)$-categories precisely if

- for every pair of objects X, Y in $\mathbb{D}$, the hom-(∞, n)-category $\mathbb{D}(X, Y)$ is m-semiadditive, and
- for each triple of objects X, Y, Z, the composition map

$$\mathbb{D}(Y,Z) \times \mathbb{D}(X,Y) \to \mathbb{D}(X,Y)$$

  is a m-semiadditive functor of $(\infty, n-1)$-categories in each variable.

Remark 2.3. Note that this easy inductive definition does not work for $n = 1$. While it makes perfect sense to define a m-semiadditive 0-category to just be an $(\infty, 0)$-category (i.e., an ∞-groupoid) equipped with the structure of an m-commutative monoid, this is a priori not just a property.

It turns out that an m-semiadditive $(\infty, 1)$-category always carries a unique enrichment in m-commutative monoids, but this is not clear a priori.

The main construction of Lemma & Definition 2.1 is the following (and its dual); everything else then follows rather easily.

Construction 2.4. Let $\mathbb{D}$ be an (∞, n)-category enriched in m-semiadditive $(n-1)$-categories (in fact, it suffices that $\mathbb{D}$ is enriched in $(n-1)$-categories with m-finite colimits). Let $X \colon S \to \mathbb{D}^{(1)} \subset \mathbb{D}$ be a diagram indexed by an m-finite space and

$$P = (p_s \colon L \to X_s)_{s:S}$$

a limit cone in $\mathbb{D}^{(1)}$ over X.

By defintion this means that for each $Y : \mathbb{D}$ the induced map

$$(p_s \circ -)_{s:S} \colon \mathbb{D}(Y,L)^{\simeq} \xrightarrow{\simeq} \lim_{s:S} \mathbb{D}(Y, X_s)^{\simeq}$$

is an equivalence of spaces. In particular, we can set $Y \coloneqq X_t$ functorially in $t : S$ and find a unique cocone I via

$$\begin{aligned} \lim\nolimits_{t:S} \mathbb{D}(X_t, L)^{\simeq} &\xrightarrow{\quad\simeq\quad} \lim\nolimits_{t:S} \lim\nolimits_{s:S} \mathbb{D}(X_t, X_s)^{\simeq} \\ I = (i_t)_{t:S} &\longmapsto \Big(\textstyle\bigoplus_{q:S(t,s)} X_q\Big)_{t,s}, \end{aligned}$$

where the direct sum is taken in $\mathbb{D}(X_t, X_s)$. (Note that we implicitly use the identification $S = S^{\mathrm{op}}$ coming from the fact that S is a groupoid.)

Lemma 2.5. *In the setting of Construction 2.4, the cone $P\colon L \Rightarrow X$ and the cocone $I\colon X \Rightarrow L$ satisfy the relations*

$$p_s \circ i_t = \bigoplus_{q:S(t,s)} X_q \tag{1}$$

$$\bigoplus_{t:S} i_t \circ p_t = \mathrm{id}_L \tag{2}$$

taking place in $\mathbb{D}(X_t, X_s)^{\simeq}$ (functorially in $s, t : S$) and in $\mathbb{D}(L, L)^{\simeq}$, respectively.

Proof. The first identification holds by construction of I, so it remains to verify the second one. For this we perform the computation

$$\begin{aligned}
p_s \circ \bigoplus_{t:S} i_t \circ p_t &= \bigoplus_{t:S} p_s \circ i_t \circ p_t && p_s \circ - \text{ preserves } \oplus \\
&= \bigoplus_{t:S} \left(\bigoplus_{q:S(t,s)} X_q \right) \circ p_t && \text{definition of } I \\
&= \bigoplus_{t:S} \bigoplus_{q:S(t,s)} X_q \circ p_t && - \circ p_t \text{ preserves } \oplus \\
&= \bigoplus_{(t,q):\sum_{t:S} S(t,s)} X_q \circ p_t && \text{formula for iterated colimits} \\
&= X_{\mathrm{id}_s} \circ p_s = p_s \circ \mathrm{id}_L && \{(s, \mathrm{id}_s)\} \xrightarrow{\simeq} \sum_{t:S} S(t,s)
\end{aligned}$$

in $\lim_{s:S} \mathbb{D}(L, X_s)^{\simeq}$, which implies the desired relation because $(p_s \circ -)$ is an equivalence of spaces by assumption. $\square$

The crucial observation that makes Lemma & Definition 2.1 work is then the following:

Lemma 2.6. *Let $X\colon S \to \mathbb{D}^{(1)} \subset \mathbb{D}$ be a diagram indexed by an m-finite space. If $P\colon L \Rightarrow X$ and $I\colon X \Rightarrow L$ are a cone and a cocone satisfying the relations of Lemma 2.5, then automatically P is a limit cone in $\mathbb{D}$ and I is a colimit cone in $\mathbb{D}$ (not just in $\mathbb{D}^{(1)}$).*

In particular, the limit and colimt of X are both identified with L, hence with each other.

Proof. We have to show that for every object $Y : D$, the map

$$\Phi \coloneqq (p_s \circ -)_{s:S}\colon \mathbb{D}(Y, L) \longrightarrow \lim_{s:S} \mathbb{D}(Y, X_s)$$

is an equivalence of $(\infty, n-1)$-categories (not just on the underlying groupoids); and dually for I. We claim that the following functor is an explicit inverse:

$$\Gamma := \bigoplus_{t:S}(i_t \circ (-)_t) \colon \lim_{t:S} \mathbb{D}(Y, X_t) \xrightarrow{\lim_{t:S}(i_t \circ -)} \lim_{t:S} \mathbb{D}(Y, L) \xrightarrow{\bigoplus_{t:S}} \mathbb{D}(Y, L)$$

This can be computed explicitly using the given formulas:

$$\begin{aligned}
\Gamma \circ \Phi &= \bigoplus_{t:S} i_t \circ p_t \circ - \\
&= \left(\bigoplus_{t:S} i_t \circ p_t\right) \circ - && - \circ - \text{ preserves } \oplus \text{ in the first variable} \\
&= \mathrm{id}_L \circ - = \mathrm{id}_{\mathbb{D}(Y,L)} && \text{formula (2)}
\end{aligned}$$

and (functorially in $s : S$):

$$\begin{aligned}
(\Phi \circ \Gamma)_s &= p_s \circ \bigoplus_{t:S} i_t \circ (-)_t \\
&= \bigoplus_{t:S} p_s \circ i_t \circ (-)_t && - \circ - \text{ preserves } \oplus \text{ in the second variable} \\
&= \bigoplus_{t:S} \left(\bigoplus_{q:S(t,s)} X_q \right) \circ (-)_t && \text{formula (1)} \\
&= \bigoplus_{t:S} \bigoplus_{q:S(t,s)} X_q \circ (-)_t && - \circ - \text{ preserves } \oplus \text{ in the first variable} \\
&= X_{\mathrm{id}_s} \circ (-)_s && \{(s, \mathrm{id}_s)\} \xrightarrow{\simeq} \sum_{t:S} S(t,s) \\
&= \left(\mathrm{id}_{\lim_{s:S} \mathbb{D}(Y, X_s)}\right)_s
\end{aligned}$$

□

Proof of Lemma & Definition 2.1. From Construction 2.4, Lemma 2.5 and Lemma 2.6 we get that every limit in $\mathbb{D}^{(1)}$ indexed by an m-finite space is already both a limit and a colimit in $\mathbb{D}$. From this we get all the equivalent characterizations of m-semiadditive (∞, n)-categories.

Next we consider a functor $\mathbb{F}$ between m-semiadditive (∞, n)-categories. If this functor preserves the enrichment, then it preserves the equations of Lemma 2.5 hence also all limits and colimits indexed by m-finite spaces.

It remains to prove the converse. So assume that the functor $\mathbb{F}$ preserves limits and colimits indexed by m-finite spaces. (It suffices to have one or the other, because on the underlying semiadditive $(\infty, 1)$-categories, we know that preservation of one implies the other.) For every pair of objects X, Y in $\mathbb{C}$ and m-finite space S, set $L := \lim_{s:S} X$ exhibited by the cone P and the cocone I

and consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}(Y,L) \arrow[r, "(p_s\circ -)_s", "\simeq"'] \arrow[d, "\mathbb{F}"] & \lim_{s:S}\mathbb{C}(Y,X) \arrow[r, "\bigoplus_{s:S}"] \arrow[d, "\mathbb{F}"] & \mathbb{C}(Y,X) \arrow[d, "\mathbb{F}"] \\
\mathbb{D}(\mathbb{F}Y,\mathbb{F}L) \arrow[r, "(\mathbb{F}p_s\circ -)_s", "\simeq"'] & \lim_{s:S}\mathbb{D}(\mathbb{F}Y,\mathbb{F}X) \arrow[r, "\bigoplus_{s:S}"] & \mathbb{D}(\mathbb{F}Y,\mathbb{F}X)
\end{tikzcd} \tag{3}$$

where the downward arrows are induced by $\mathbb{F}$; note that the lower left horizontal map is an equivalence because $\mathbb{F}$ preserves limits. Note that the upper vertical composite is

$$\bigoplus_{s:S} p_s \circ - = \left(\bigoplus_{s:S} p_s\right) \circ - = \nabla_X \circ -$$

because we have

$$\left(\bigoplus_{s:S} p_s\right) \circ i_t = \bigoplus_{s:S} \bigoplus_{q:S(t,s)} \mathrm{id}_X = \mathrm{id}_X,$$

(for $t : S$), which is the defining equation of the codiagonal $\nabla_X \colon L \to X$. Similarly, the lower composite is identified with $\nabla_{\mathbb{F}X}$. It follows that the outer rectangle commutes because $\mathbb{F}$ preserves the codiagonal. Thus it follows that the right square also commutes which means that $\mathbb{F}_{Y,X} \colon \mathbb{C}(Y,X) \to \mathbb{D}(\mathbb{F}Y,\mathbb{F}X)$ preserves limits/colimits, hence is m-semiadditive. This means precisely that $\mathbb{F}$ is an enriched functor. (For a complete argument one has to track the actual 2-cells appearing in the diagram (3); we omit the details.) □

Remark 2.7. It is a recurring phenomenon that in the presence of certain enrichments one can characterize certain limits or colimits "by formulas" in terms of the enrichment. This then always implies that these limits/colimits are automatically preserved by *every* enriched functor. The special thing about these formulas in "semiadditive-like" settings, is that they exhibit simultaneously a limit and a colimit, thus yielding an identification of the two. We will see this same pattern again next week in the context of *lax* semiadditivity.

Vice versa, if one can characterize the enrichment in terms of limits/colimits, then every functor that preserves them is automatically enriched. In our setting this was particularly easy, since preserving the enrichment is not an additional structure but just a property. Again, contrast with the case $n = 1$, where on the homs we have a morphism of m-commutative monoids.

2.1 Goal

We want to prove that $\mathrm{Span}_n(m\mathrm{Fin})$ is the free m-semiadditive (∞, n)-category on one generator.

Conjecture 2.8. *Let $\mathbb{D}$ be an m-semiadditive (∞, n)-category. The evaluation functor at the singleton induces an equivalence*

$$\mathrm{Map}^{\oplus-m}(\mathrm{Span}_n(m\mathrm{Fin}), \mathbb{D}) \xrightarrow{\mathrm{ev}_1} \mathbb{D}^{\simeq}$$

of spaces (which one can then bootstrap to an equivalence of $(\infty, n)-$*categories).*

The goal of this talk will be to sketch how we would construct an inverse. In other words, given an object $d : \mathbb{D}^{\simeq}$, how would we construct the unique m-semiadditive functor $\mathrm{Span}_n(m\mathrm{Fin}) \to \mathbb{D}$ determined by it.

3 Flagged enriched categories

Throughtout this section, let $\mathcal{V} = (\mathcal{V}, \otimes)$ be a symmetric monoidal ∞-category. For us, this will mostly be the ∞-category of $(\infty, n-1)$-categories or the sub-operad of m-semiadditive $(\infty, n-1)$-categories.

Definition 3.1. A 1-flagged $\mathcal{V}$-category consists of a $\mathcal{V}$-enriched category $\mathbb{D}$, together with an essentially surjective functor $\mathcal{C} \to \mathbb{D}^{(1)}$ from an ∞-category $\mathcal{C}$.

Here the underlying ∞-category is defined by transporting the enrichement along the lax monoidal functor $\mathcal{V}(1, -) \colon \mathcal{V} \to \mathrm{Spaces}$, where 1 is the monoidal unit. (This recovers the usual notion of underlying ∞-category of an (∞, n)-category.)

Construction 3.2. We consider the (∞, n)-category $\mathrm{Span}_n(m\mathrm{Fin})$ with its tautological 1-flagging by $(\mathrm{Span}_n(m\mathrm{Fin}))^{(1)} = \mathrm{Span}_1(m\mathrm{Fin})$.

Then, for an arbitrary m-semiadditive (∞, n)-category $\mathbb{D}$, and let $d : \mathbb{D}^{\simeq}$. Since the undelying ∞-category of $\mathbb{D}$ is m-semiadditive, we obtain a unique m-semiadditive functor $d \colon \mathrm{Span}_1(m\mathrm{Fin}) \to \mathbb{D}^{(1)} \subset \mathbb{D}$ and we denote by $\langle d \rangle \subseteq \mathbb{D}$ its essential image.

Every putative m-semiadditive functor $\mathrm{Span}_n(m\mathrm{Fin}) \to \mathbb{D}$ with $1 \mapsto d$ must then induce a commutative square

$$\begin{tikzcd}
\mathrm{Span}_1(m\mathrm{Fin}) \arrow[r] \arrow[d, equal] & \mathrm{Span}_n(m\mathrm{Fin}) \arrow[d, dashed] \\
\mathrm{Span}_1(m\mathrm{Fin}) \arrow[r, "d"] & \langle d \rangle
\end{tikzcd}$$

i.e., a morphism of $\mathrm{Span}_1(m\mathrm{Fin})$-flagged (∞, n)-categories.

We can thus replace $(\mathbb{D}, d)$ by $(\langle d \rangle, d)$ (where we view d interchangeably as an object and as a (m-semiadditive) 1-flagging by $\mathrm{Span}_1(m\mathrm{Fin})$) and show the following:

Conjecture 3.3. $\mathrm{Span}_n(m\mathrm{Fin})$ *is the initial* m*-semiadditive* $\mathrm{Span}_1(m\mathrm{Fin})$*-flagged* (∞, n)*-category.*

3.1 A convenient model for 1-flagged enriched categories

So far we have skirted around the topic of how we want to rigorously work with higher/enriched categories. While there are many ways to do it, for us it will be convenient to adopt the following approach due to Hinich.

Construction 3.4 (Hinich)**.** Let $\mathcal{C}$ be an ∞-category. One can construct a (non-symmetric, colored) operad $H(\mathcal{C})$ whose algebras are enriched categories 1-flagged by $\mathcal{C}$.

- Colors are pairs (X, Y) of objects of $\mathcal{C}$.
- An n-ary operation $(X_0, Y_1), \dots, (X_{n-1}, Y_n) \to (Y_0, X_n)$ is a diagram of the form

$$\begin{tikzcd}
(X_0 & , & Y_1) \arrow[r, bend right, "f_1"'] & (X_1 & \cdots & Y_{n-1}) \arrow[r, bend right, "f_{n-1}"'] & (X_{n-1} & , & Y_n) \arrow[dlll, "f_n"] \\
 & & & (Y_0 \arrow[ulll, "f_0"] & , & X_n) & & &
\end{tikzcd}$$

  where each $f_i \colon Y_i \to X_i$ is a morphism in $\mathcal{C}$.

- Note that in the case $n = 0$ this means that a 0-ary operation $\emptyset \xrightarrow{f} (X, Y)$ is just a morphism $f \colon X \to Y$.
- Composition is induced by the composition in $\mathcal{C}$. [Draw a picture on the blackboard]

Let us explain how to think of an $H(\mathcal{C})$-algebra $\mathbb{D}$ in $\mathcal{V}$ as an enriched category:

- To each pair (X, Y) of objects of $\mathcal{C}$ it assigns a hom-object $\mathbb{D}(X, Y)$ in $\mathcal{V}$.
- Given a list (X, Y, Z), one obtains a composition map

$$\mathbb{D}(X, Y) \otimes \mathbb{D}(Y, Z) \to \mathbb{D}(X, Z)$$

  assigned to the tautological operation

$$\begin{tikzcd}
(X & , & Y) \arrow[r, bend right, "="'] & (Y & , & Z) \arrow[dl, "="] \\
 & (X \arrow[ul, "="] & & , & Z) &
\end{tikzcd}$$

  (So far, this would only make $\mathbb{D}$ into an enriched category 0-flagged by $\mathcal{C}^{\simeq}$.)

- The action on 0-ary operations yields for each pair (X, Y) a morphism

$$\begin{aligned}
\mathcal{C}(X, Y) &\longrightarrow \mathcal{V}(1, \mathbb{D}(X, Y)) \\
f &\longmapsto \mathbb{D}(\emptyset \xrightarrow{f} (X, Y))
\end{aligned}$$

- The action on 1-ary operations lets us define maps

$$f \circ -\colon \mathbb{D}(X,Y) \to \mathbb{D}(X,Y') \quad \text{and} \quad - \circ f\colon \mathbb{D}(Y',Z) \to \mathbb{D}(Y,Z)$$

  for each $f\colon \mathcal{C}(Y,Y')$ which explain what it means to "compose with arrows in $\mathcal{C}$".

- A general n-ary operation encodes maps of the form $f_n \circ - \cdots - \circ f_1 - \circ f_0$.

3.2 Enriched matrix categories

Let $\mathcal{A}$ be an associative (but not necessarily commutative) algebra in $\mathcal{V}$, i.e., a lax monoidal functor $\mathcal{A}\colon \{*\} \to \mathcal{V}$. If the ∞-category $\mathcal{V}$ is m-semiadditive with compatible monoidal structure (preserving m-finite direct sums in each variable), then by the 1-categorical (lax monoidal) universal property of spans we get a unique lax monoidal extension, which we also denote by $\mathcal{A}$:

$$\begin{tikzcd}
\{*\} \arrow[d, hook] \arrow[dr, "\mathcal{A}"] & \\
\mathrm{Span}_1(m\mathrm{Fin}) \arrow[r, dashed, "\mathcal{A}"'] & \mathcal{V}
\end{tikzcd}$$

One way to think about this extension is that it encodes not just the multiplication of $\mathcal{A}$ (which is already encoded in the original lax monoidal functor $* \to \mathcal{V}$), but also the unique m-commutative addition on the object $A \coloneqq \mathcal{A}(*)$ given by

$$\sum_S\colon \bigoplus_{s:S} A = \mathcal{A}(S) \xrightarrow{\mathcal{A}(S \to *)} \mathcal{A}(*) = A.$$

(for each m-finite space S).

From this data we expect to be able to construct the 0-flagged enriched matrix category $\mathrm{Mat}_{\mathcal{A}}$ with

- objects are m-finite spaces
- the hom-objects are

$$\mathrm{Mat}_{\mathcal{A}}(S,T) \coloneqq \mathcal{A}(S,T) = \bigoplus_{T \times S} A$$

- and whose composition is given by matrix multiplication

$$\begin{aligned} \mathrm{Mat}_{\mathcal{A}}(T,U) \otimes \mathrm{Mat}_{\mathcal{A}}(S,T) &\longrightarrow \mathrm{Mat}_{\mathcal{A}}(S,U) \\ m \otimes n &\longmapsto l; \qquad \text{with } l_{u,s} = \textstyle\sum_{t:T} m_{u,t} \cdot n_{t,s} \end{aligned} \tag{4}$$

Construction 3.5. Since every object in $\mathrm{Span}_1(m\mathrm{Fin})$ is self-dual, we obtain a canonical self-enrichment. Moreover, we can tautologically flag this enriched category by $\mathrm{Span}_1(m\mathrm{Fin})$. Together, this amounts to an algebra

$$\mathbb{K}\colon H(\mathrm{Span}_1(m\mathrm{Fin})) \to \mathrm{Span}_1(m\mathrm{Fin})^{\otimes}$$

On colors it sends $(X,Y) \mapsto X \times Y$ and for example a 2-ary operation of the form

$$\begin{tikzcd}
(X_0 & , & Y_1) & & (X_1 & , & Y_2) \\
F_0 \arrow[u] \arrow[d] & & & F_1 \arrow[ul] \arrow[ur] & & & F_2 \arrow[u] \arrow[d] \\
(Y_0 & & & , & & & X_2)
\end{tikzcd}$$

to the map $(X_0 \times Y_1, X_1 \times Y_2) \to (Y_0 \times X_2)$ over $[2] \leftarrow [1]$ corresponding to the span

$$\begin{tikzcd}
(X_0 \times Y_1) \times (X_1 \times Y_2) \\
F_0 \times F_1 \times F_2 \arrow[u] \arrow[d] \\
Y_0 \times X_2
\end{tikzcd} \tag{5}$$

We can then define $\mathrm{Mat}_{\mathcal{A}}$ as the composition of operad maps

$$H(\mathrm{Span}_n(m\mathrm{Fin})) \xrightarrow{\mathbb{K}} \mathrm{Span}_n(m\mathrm{Fin})^{\otimes} \to \mathcal{V}^{\otimes}$$

yielding an enriched category which is not just 0-flagged by $m\mathrm{Fin}^{\simeq}$ but 1-flagged by $\mathrm{Span}_n(m\mathrm{Fin})$. We see that indeed we have $\mathrm{Mat}_{\mathcal{A}}(S,T) = \mathcal{A}(S,T) = \bigoplus_{T\times S} A$ and that in the case where $X_i = F_i = Y_i$, the induced morphism (5) yields precisely the matrix multiplication formula (4).

We can now formulate the conjectural universal property of enriched matrix categories:

Conjecture 3.6. *Let $\mathcal{V}$ be a symmetric monoidal m-semiadditive ∞-category. Then restriction along*

$$\{*\}^{\otimes} = H(*) \to H(\mathrm{Span}_1(m\mathrm{Fin}))$$

induces an equivalence between

- *associative algebras $\mathcal{A}$ in $\mathcal{V}$ and*

- $\mathrm{Span}_1(m\mathrm{Fin})$*-flagged* $\mathcal{V}$*-categories* $\mathbb{D}$ *which are* m*-semiadditive in the following sense: for each pair of* m*-finite spaces* S *and* T*, the map*

  $$\mathbb{D}(S,T) \to \bigoplus_{S\times T} \mathbb{D}(*,*)$$

  induced by the family of 1*-ary operations* $(S,T) \xrightarrow{(i_s,p_t)} (*,*)$ *(for* $s : S$ *and* $t : T$*) is an equivalence.*

which can be described explicitly by

$$\mathrm{Mat}_- : \{\textit{ass. algebas}\} \overset{\simeq}{\leftrightarrows} \{\textit{m-s.a.}\ \mathrm{Span}_1(m\mathrm{Fin})\textit{-flagged categories}\} : \mathrm{End}_-(*).$$

Remark 3.7. Conjecture 3.6 is an $(\infty,1)$-categorical statement. Indeed, $H(*) \to H(\mathrm{Span}_n(m\mathrm{Fin}))$ is "just" a morphism of ∞-operads which we claim induces an equivalence between (certain of) their algebras.

While we do not have a proof of this conjecture yet, we believe that an operadic analog of Harpaz' alternating left/right Kan extension argument will be able to interpolate between $H(*) = H(\mathrm{Span}_1((-2)\mathrm{Fin}))$ and $H(\mathrm{Span}_1(m\mathrm{Fin}))$.

The following observation ties this construction to the way spans were introduced at the beginning:

Observation 3.8. Let $\mathcal{A} \coloneqq (\mathrm{Span}_{n-1}(m\mathrm{Fin}), \times)$, viewed as an associative algebra in the symmetric monoidal $(\infty,1)$-category of m-semiadditive $(\infty,n-1)$-categories (which is itself m-semiadditive). Then we can identify

$$\mathrm{Mat}_{\mathcal{A}} = \mathrm{Span}_n(m\mathrm{Fin})$$

as a $\mathrm{Span}_1(m\mathrm{Fin})$-flagged (∞,n)-category.

Remark 3.9. Depending on which model one initially takes for the iterated span categories, this observation can be easier or harder to make precise. For the purpose of this talk we will take it as the construction. This construction is of course not quite complete, because one still needs to construct the monoidal structure to inductively be able to continue with the next application of Mat_-.

Note that assuming Conjecture 3.6 one easily gets a comparison with other models: indeed it is typically rather easy to show that (any version of) the iterated span category $\mathrm{Span}_n(m\mathrm{Fin})$ is m-semiadditive and it is immediate that its endomorphism algebra at the singleton is $\mathrm{Span}_{n-1}(m\mathrm{Fin})$.

4 The universal property of iterated spans

Finally, we sketch how to deduce the desired universal property of the iterated span category.

Given an m-semiadditive (∞,n)-category $\mathbb{D}$ and an object $d : \mathbb{D}^{\simeq}$, we have identifications between

- m-semiadditive functors $\mathrm{Span}_n(m\mathrm{Fin}) \to \mathbb{D}$ with $* \mapsto d$;
- m-semiadditive maps $\mathrm{Span}_n(m\mathrm{Fin}) \to (\langle d \rangle_{\mathbb{D}}, d)$ of $\mathrm{Span}_1(m\mathrm{Fin})$-flagged (∞, n)-categories;
- m-semiadditive monoidal functors $(\mathrm{Span}_{n-1}(m\mathrm{Fin}), \times) \to (\mathrm{End}_{\mathbb{D}}(d), \circ)$

By using the universal property of $\mathrm{Span}_{n-1}(m\mathrm{Fin})$ by induction, we then see that such a functor is uniquely determined by the unitality constraint $* \mapsto \mathrm{id}_d$. (Here one needs argue that this fully determines the monoidal functor and not just the underlying functor of $(\infty, n-1)$-categories).