Parametrized higher semiadditivity

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Joint work with Tobias Lenz and Sil Linskens.

Thanks for the organizers for giving me the opportunity to speak at this conference.

The topic of my talk is perhaps not directly connected to the research of Emanuel Farjoun, but it has received a lot of interest from the Israeli school in general and in particular the University of Jerusalem: higher semiadditivity.

1 Motivation

All my 'categories' will be ∞-categories unless otherwise specified.

- $\mathcal{C}$ is *pointed* if $0 = \operatorname{colim} \xrightarrow{\sim} \lim = 1$;
- $\mathcal{C}$ is *semiadditive* if $\coprod_{i \in I} X_i \xrightarrow{\sim} \prod_{i \in I} X_i$ for I finite.

Hopkins-Lurie: vast generalization called *ambidexterity*. Examples:

1) Representation theory

Let $H \leqslant G$ be finite groups, V an H-representation. Get G-repr. $\operatorname{ind}_H^G(V) := \bigoplus_{[g] \in G/H} V$. Have adjunctions

$$\begin{tikzcd} \mathrm{Rep}_G \arrow[r, "\mathrm{res}^G_H"] & \mathrm{Rep}_H \arrow[l, bend right, "\mathrm{ind}^G_H"'] \arrow[l, bend left, "\mathrm{ind}^G_H"] \end{tikzcd} .$$

2) Equivariant homotopy theory

For $H \leq G$ finite, have

$$\begin{tikzcd} \mathrm{Sp}_G \arrow[r, "\mathrm{res}^G_H"] & \mathrm{Sp}_H . \arrow[l, bend right=40, "\mathrm{ind}^G_H"'] \arrow[l, bend left=40, "\mathrm{coind}^G_H"] \end{tikzcd}$$

There is an iso $\mathrm{ind}^G_H \xrightarrow{\sim} \mathrm{coind}^G_H$ called ‘Wirthmüller isomorphism’.

3) Sheaf theory

For $f \colon X \to Y$ finite covering map in Top, get

$$\begin{tikzcd} \mathrm{Shv}(Y;\mathrm{Sp}) \arrow[r, "f^*"] & \mathrm{Shv}(X;\mathrm{Sp}) \arrow[l, bend right=40, "f_\sharp"'] \arrow[l, bend left=40, "f_*"] \end{tikzcd}$$

and an iso $f_\sharp \xrightarrow{\sim} f_*$.

4) Six-functor formalisms

More generally, replacing $\mathrm{Shv}(-;\mathrm{Sp})$ by an arbitrary six-functor formalism $D(-)$ in the sense of Scholze, we get for any ‘finite étale map’ $f \colon Y \to X$

$$\begin{tikzcd} D(Y) \arrow[r, "f^*"] & D(X) \arrow[l, bend right=40, "f_\sharp"'] \arrow[l, bend left=40, "f_*"] \end{tikzcd}$$

and $f_\sharp \xrightarrow{\sim} f_*$.

5) Representation theory

For V a complex G-representation,

$$\begin{aligned} V/G &\xrightarrow{\sim} V^G \\ [v] &\mapsto \sum_g gv. \end{aligned}$$

6) Tate vanishing

Theorem 1.1 (Hovey-Sadofsky)**.** *For $X \colon BG \to \mathrm{Sp}_{K(n)}$, we have*

$$X_{hG} \xrightarrow{\sim} X^{hG}.$$

7) Higher semiadditivity

Theorem 1.2 (Hopkins-Lurie)**.** *For a π-finite space A and $X \colon A \to \mathrm{Sp}_{K(n)}$, we have*

$$\operatorname{colim}_A X \xrightarrow{\sim} \lim_A X.$$

2 Parametrized (higher) semiadditivity

Captured by Hopkins-Lurie (different presentation):

- Let $\mathcal{A}$ be an ∞-category of 'indexing objects', e.g. Fin, Fin_G, FinGrpd, Top, etcetera
- Let $\mathcal{C} \colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a sheaf of ∞-categories;
- Let $\mathcal{Q} \subseteq \mathcal{A}$ be a wide subcategory closed under base change (the '*finite étale maps*').

Definition 2.1. We say that $\mathcal{Q}$ is *inductible* if:

- for $(q \colon A \to B) \in \mathcal{Q}$, also $(\Delta_q \colon A \to A \times_B A) \in \mathcal{Q}$;
- every morphism q in $\mathcal{Q}$ is n-truncated for some n.

Definition 2.2. We say $\mathcal{C}$ is *$\mathcal{Q}$-semiadditive* if for every $(q \colon A \to B) \in \mathcal{Q}$, $q^* \colon C(B) \to C(A)$ admits left/right adjoints $f_!$ and f_* satisfying base change, and the *norm map* $\mathrm{Nm}_q \colon q_! \to q_*$ is an equivalence.

Here Nm_q is defined inductively, assuming we already have $\mathrm{Nm}_\Delta \colon \Delta_! \xrightarrow{\sim} \Delta_*$:

$$q^* q_! \simeq \mathrm{pr}_{2!} \, \mathrm{pr}_1^* \xrightarrow{\mu_\Delta} \mathrm{pr}_{2!} \Delta_! \Delta^* \, \mathrm{pr}_1^* = \mathrm{id}\,.$$

Examples of $\mathcal{Q}$-semiadditivity:

- ordinary
- G-
- global
- higher / m-
- p-typical
- tempered ambidexterity
- six-functor formalisms.

3 Universality of commutative monoids

Semiadditivity is closely linked to the structure of commutative monoids:

- CMon = universal semiadditive category;
- $\mathcal{C}$ semiadditive $\implies$ canonical enrichment in commutative monoids.

Our goal: get similar results for $\mathcal{Q}$-semiadditivity.

Construction 3.1. Recall the *span category* $\mathrm{Span}(\mathcal{A}, \mathcal{Q})$ has

- Objects: $X \in \mathcal{A}$;
- Morphisms: spans

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "q"] & \\ X & & Y. \end{tikzcd}$$

- Composition via pullback.

Similarly, we obtain $\mathrm{Span}(\mathcal{Q}_{/X})$ for every $X \in \mathcal{Q}$.
We vastly generalize this.

Theorem A (CLL). *The free $\mathcal{Q}$-semiadditive sheaf $\mathcal{A}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is*

$$\underline{\mathrm{Span}}(\mathcal{Q}): \qquad A \mapsto \mathrm{Span}(Q_{/A}).$$

Remark 3.2. This result was known in two special cases:

- (Harpaz): When $\mathcal{A} = \mathcal{Q} = \mathrm{Spc}_m$, m-finite spaces, where one restricts to functors $\mathcal{C} \colon \mathrm{Spc}_m^{\mathrm{op}} \to \mathrm{Cat}_\infty$ Kan extended from a point;
- (Nardin) When $\mathcal{A} = \mathcal{Q} = \mathrm{Fin}_G$, finite G-sets, for finite-product preserving functors $\mathcal{C} \colon \mathrm{Fin}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

As a consequence, we may deduce universal way of turning any parametrized category into a $\mathcal{Q}$-semiadditive one:

Theorem B (CLL). *The inclusion $\mathrm{Cat}_{\mathcal{A}}^{\mathcal{Q}-\oplus} \subseteq \mathrm{Cat}_{\mathcal{A}}^{\mathcal{Q}-\times}$ admits a right adjoint given by sending $\mathcal{C}$ to*

$$\mathrm{CMon}^{\mathcal{Q}}(\mathcal{C}) := \underline{\mathrm{Fun}}^{\mathcal{Q}-\times}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{C}),$$

the parametrized category of $\mathcal{Q}$-commutative monoids in $\mathcal{C}$.

Of special interest will be the universal *presentable* case, obtained by letting $\mathcal{C} = \underline{\mathrm{Spc}}_{\mathcal{A}}$ be a parametrized version of the category of spaces.

Definition 3.3. An $\mathcal{A}$-*sheaf with* $\mathcal{Q}$-*transfers* is a functor

$$F\colon \operatorname{Span}(\mathcal{A}, \mathcal{Q}) \to \mathrm{Spc}$$

whose restriction $F|_{\mathcal{A}^{\mathrm{op}}}\colon \mathcal{A}^{\mathrm{op}} \to \mathrm{Spc}$ is a sheaf.

Note that in Harpaz's context this is precisely an *m-commutative monoid*, while in Nardin's context this is a *Mackey functor*.

Theorem C (CLL)**.** *There is an equivalence*

$$\mathrm{CMon}^{\mathcal{Q}}(\underline{\mathrm{Spc}}_{\mathcal{A}}) \simeq \mathrm{Shv}^{\mathcal{Q}}(\mathcal{A}).$$

Corollary 3.4. *If $\mathcal{C}$ is $\mathcal{Q}$-semiadditive, then $\mathcal{C}(X)$ is canonically enriched in sheaves with $\mathcal{Q}$-transfers.*

4 Applications

1) Equivariant homotopy theory

- Recover spectral Mackey functor description of Sp_G;
- Extend to the global case (Lenz) and G-global case (Pützstück);
- Obtain a universal property for:
  - Kaledin's *Mackey profunctors*;
  - Krause-McCandless-Nikolaus's *quasifinitely genuine G-spectra*.

2) Six-functor formalisms

Every six-functor formalism is uniquely enriched in sheaves with transfers for "finite étale maps".

3) Character theory

In certain situations, one may construct *character maps* where both the source and the target admit higher commutative monoid structures coming from higher semiadditivity. Our result can be used to show additional coherences for such character maps.

Theorem 4.1 (Carmeli, C., Ramzi, Yanovski)**.** *If $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is m-semiadditive, then the character maps*

$$\chi\colon (\mathcal{C}^{\mathrm{dbl}})^{A} \to \mathrm{End}(\mathbb{1})^{LA}$$
$$(V\colon A \to \mathcal{C}^{\mathrm{dbl}}(\mapsto (\lambda \mapsto \mathrm{tr}(\gamma_*\colon V \to V))$$

are natural in $A \in \operatorname{Span}(\mathrm{Spc}_m)$.

Theorem 4.2 (Ben-Moshe)**.** *The transchromatic character map*

$$\chi\colon E_n^X \to C_t^{L_p^{n-t}X}$$

with $L_p^{n-t}X := \mathrm{Map}(B\,\mathbb{Z}_p^{n-t}, X)$ *is natural in* $X \in \mathrm{Span}(\mathrm{Spc}_\pi)$.

Here

- E_n = Lubin-Tate spectrum of height n;
- C_t is a certain $L_{K(t)}E_n$-algebra.

5 Proof sketch Theorem A

NTS: for any $\mathcal{Q}$-semiadditive $\mathcal{C}$

$$\underline{\mathrm{Fun}}^{Q-\oplus}(\underline{\mathrm{Span}}(\mathcal{Q}), \mathcal{C}) \xrightarrow{\sim} \mathcal{C}.$$

Following Harpaz, prove this by induction on 'truncation level' of $\mathcal{Q}$:

- First reduce to $\mathcal{Q} = \mathcal{Q}_{\leq n}$;
- Show that we may reduce to

  $$\underline{\mathrm{Span}}(\mathcal{Q}_{\leq n}, \mathcal{Q}_{\leq n-1}) \subseteq \underline{\mathrm{Span}}(\mathcal{Q}_{\leq n})$$

  via restriction/left Kan extension.
- Show that we may reduce to

  $$\underline{\mathrm{Span}}(\mathcal{Q}_{\leq n-1}) \subseteq \underline{\mathrm{Span}}(\mathcal{Q}_{\leq n}, \mathcal{Q}_{\leq n-1})$$

  via restriction/right Kan extension.
- Conclude by induction.