Introduction to stable homotopy theory

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Introduction

Stable homotopy theory started off as a branch of algebraic topology that emerged in the mid-20th century as mathematicians sought to understand the behavior of homotopy groups in high dimensions. The key insight was the *Freudenthal suspension theorem* [Fre37]: if X is a CW-complex with one 0-cell and no k-cells for $1 \leq k \leq n-1$, then the canonical map $X \to \Omega\Sigma X$ from X to the loop space of its suspension induces an isomorphism

$$\pi_k(X) \xrightarrow{\sim} \pi_k(\Omega\Sigma X) = \pi_{k+1}(\Sigma X)$$

on k-th homotopy groups for all $0 \leq k \leq 2n-2$ (and a surjection for $k = 2n-1$). Since ΣX is another CW-complex which now has no k-cells for $1 \leq k \leq n$, we may apply this theorem another time to ΣX, and proceeding in this fashion we see that the sequence

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \to \pi_k(\Omega^2\Sigma^2 X) \to \dots$$

eventually stabilizes for every natural number k. The limiting group in this sequence is known as the *k-th stable homotopy group* of X and is denoted by $\pi_k^{\mathrm{st}}(X)$. These stable homotopy groups, while difficult to compute, contain deep geometric and topological information.

Given pointed CW-complexes X and Y, we may more generally consider the set $[X,Y]_*^{\mathrm{st}}$ of (homotopy classes of) *stable maps* from X to Y, defined as the colimit of the sequence

$$[X,Y]_* \to [X,\Omega\Sigma Y]_* \to [X,\Omega^2\Sigma^2 Y]_* \to \dots;$$

note that we have $\pi_k^{\mathrm{st}}(X) = [S^k, X]_*^{\mathrm{st}}$ as a special case. Finite pointed CW-complexes and stable maps assemble into a category known as the *Spanier-Whitehead category* [SW53].

The introduction of generalized (co)homology theories by Eilenberg and Steenrod [ES52] reinforced the centrality of the notion of stability in homotopy theory: the stability isomorphism $E_k(X) \cong E_{k+1}(\Sigma X)$ was taken as one of the core axioms. At the same time, it also made clear that the Spanier-Whitehead category was not quite the right category to consider: not every generalized cohomology theory can be represented by an object from this category. Several solutions were proposed, for example by Boardman, Lima, Kan and Whitehead. It eventually lead to Adams' definition of the 'stable homotopy category'

[Ada74], which contains the Spanier-Whitehead category as a full subcategory. Its objects are known as *spectra*:[1]

Definition. A *spectrum* is a sequence of pointed spaces $X = \{X_n\}_{n\geq 0}$ together with homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for all $n \geq 0$, where Ω denotes the loop space functor. A spectrum X is called *connective* if the homotopy groups $\pi_k(X_n)$ vanish for $k < n$.

Spectra are absolutely fundamental in stable homotopy theory, and their role is best compared with the role of abelian groups in commutative algebra. Indeed, most of the familiar operations on abelian groups admit 'homotopical' reflections at the level of spectra:

- Given two spectra X and Y, we may form their *direct sum* $X \oplus Y$, which is both the product and the coproduct in the category of spectra. In particular, the category of spectra is *additive*;
- Given two spectra X and Y, we may form their *tensor product* $X \otimes Y$. This gives rise to the notion of a *ring spectrum*: a spectrum R equipped with a multiplication operation $m\colon R \otimes R \to R$ that is 'coherently' associative and unital;
- We may talk about *modules* over a ring spectrum. Familiar notions for modules over (commutative) rings, like the properties of being projective, perfect, or flat, admit generalizations to the setting of ring spectra;
- Given a spectrum X and a prime p, we may form its *p-localization* $X_{(p)}$ and its *p-completion* $X_p^\wedge$, which have similar behavior as their analogues for abelian groups;
- And so on and so forth...

The techniques and concepts of stable homotopy theory, while rooted in algebraic topology, have found significant applications in numerous neighboring fields of mathematics, like (derived) algebraic geometry, algebraic K-theory, geometric topology and representation theory.

Higher category theory

To fully appreciate the pervasive role of (stable) homotopy theory in modern mathematics, we must first internalize what I call the 'fundamental principle of homotopy theory':

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing a homotopy/isomorphism between them.

[1] In the older literature these are often called 'Ω-spectra'.

This principle is already familiar to most mathematicians when applied to mathematical *objects*: for instance, when we want to express that two abelian groups A and B are 'the same', we naturally construct an isomorphism $A \cong B$ between them. In homotopy theory, we extend this principle to *morphisms* between objects; for example, two continuous maps $f, g\colon X \to Y$ are considered 'equal' when equipped with a homotopy $f \sim g$ between them. More broadly, in homotopy theory, the notion of 'sameness' is not a *property* (a binary yes/no question) but rather *structure* that must be explicitly provided (a homotopy, isomorphism, etc.). This shift in perspective originated in algebraic topology, but has by now become fundamental in many neighboring fields of mathematics.

While this principle appears straightforward, it rapidly gives rise to intricate 'coherence problems'. Consider, for instance, the task of defining a homotopical version of an abelian group A. Following the Fundamental Principle, expressing the associativity of addition $+\colon A \times A \to A$ requires providing a homotopy between the two maps $A \times A \times A \to A$ given by $(a, b, c) \mapsto a + (b + c)$ and $(a, b, c) \mapsto (a + b) + c$. Once such a homotopy is given, one can construct two distinct homotopies between the maps $(a, b, c, d) \mapsto a + (b + (c + d))$ and $(a, b, c, d) \mapsto ((a + b) + c) + d$. Requiring these to coincide necessitates the introduction of *homotopies between homotopies*. This process continues indefinitely, leading to an infinite hierarchy of higher homotopies, as laid out for instance by Stasheff [Sta63].

The modern solution to such coherence problems lies in the theory of *∞-categories*. Conceptually, an ∞-category behaves like an ordinary category, but one in which we take the Fundamental Principle to its logical conclusion. Like an ordinary category, an ∞-category has objects X and morphisms $f\colon X \to Y$, but it also allows for homotopies $f \sim g$ between morphisms f and g, as well as homotopies between these homotopies, and so on ad infinitum. Homotopical versions of familiar categorical constructions are conveniently formulated within this framework: for example, we may define for any ∞-category C with finite products a new ∞-category $\mathrm{CGrp}(C)$ of *commutative groups* in C, where all the required higher coherences are automatically built into the objects.

As we will see in the course, there are ∞-categories $\mathcal{S}$ and Sp whose objects are spaces[2] and spectra, respectively, and whose 'higher homotopies' are the expected ones. We will discuss the following theorem, which provides crucial insight into the nature of stable homotopy theory:

Recognition Principle for Infinite Loop Spaces (Boardman–Vogt, May and Segal)**.** There is a fully faithful functor of ∞-categories

$$\mathrm{CGrp}(\mathcal{S}) \hookrightarrow \mathrm{Sp}$$

[2]For reasons that will be explained, we will call these objects 'animae' in the course, and write An rather than $\mathcal{S}$ for the resulting ∞-category.

whose image consists of the connective spectra.[3]

In other words, we may think of a connective spectrum as a "homotopy coherent" analogue of an abelian group. This result is part of a much broader phenomenon: many algebraic structures have natural "homotopy coherent" analogues in the world of spectra (think for example of the ring spectra mentioned before).

While coherence problems in homotopy theory can be (and historically have been) handled without ∞-categories, the ∞-categorical framework provides a natural setting where the Fundamental Principle is built into the foundations. For this reason, a significant portion of this course will be devoted to developing the essential aspects of ∞-category theory, with a focus on its applications to stable homotopy theory.

Content of the course

This course will cover the following topics:

(1) Spectra, and their fundamental role through Brown representability.

(2) The language of ∞-categories, building on the Fundamental Principle of Homotopy Theory discussed above.

(3) Stable ∞-categories, with the ∞-category Sp of spectra as our main example.

(4) Algebraic structures in homotopy theory, including monoids, commutative monoids and symmetric monoidal ∞-categories. We will prove May's recognition principle for infinite loop spaces mentioned above.

(5) Vector bundles and K-theory.

(6) Localization and completion techniques in stable homotopy theory, including Bousfield localizations, localization at a prime, and arithmetic completion.

There are two further chapters containing additional material not covered during the course:

(7) Thom spectra, and its relation to classical differential topology.

(8) Duality in stable homotopy theory, culminating in Atiyah duality.

Finally, there are two appendices:

[3]The difference between spectra and connective spectra is similar to the difference between arbitrary chain complexes and non-negatively graded chain complexes in homological algebra.

(A) Additional background material on ∞-category theory.

(B) A discussion of the relation between homotopy pushouts/pullbacks in a model category C and pushouts/and pullbacks in the underlying ∞-category $C[W^{-1}]$.

Acknowledgments

In writing these lecture notes, I am taking inspiration from various sources on (stable) homotopy theory. Especially influential are the following three sets of lecture notes:

- The lecture notes *Introduction to stable homotopy theory* by Denis Nardin [Nar21];
- The lecture notes by Ferdinand Wagner from a lecture course by Fabian Hebestreit on *algebraic and Hermitian K-theory* [HW21];
- The lecture notes by Jack Davies and Liz Tatum on *stable and chromatic homotopy theory* [DT24].

Classical developments of the theory of spectra may be found for example in the textbooks by Switzer [Swi75] and Adams [Ada74]; see also Barnes and Roitzheim [BR20] for a recent account of spectra via model categories.

The book *Higher algebra* by Jacob Lurie [Lur17] has been highly influential for the modern approach to stable homotopy theory via ∞-categories, and is still one of the major references in the field.

The approach to ∞-categories taken in this course is based on joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [Cis+24]; see here for details.

Various words of thanks are in order. I thank Marc Hoyois for pointing out various mathematical mistakes in the notes. I thank the participants of the course for their feedback, which has led to various useful clarifications and additions to the notes. I thank Tim Henke for various suggestions improving the exposition of these notes. Finally I wish to thank Denis Nardin, Fabian Hebestreit, Jack Davies and Liz Tatum for the excellent sets of lecture notes that have formed the basis for these notes; the lectures of Fabian were particularly influential to me since they were where I learned most of the basics on ∞-category theory.

In the process of writing these notes, I have now and then made use of Claude 1.5 Sonnet to find reformulations that best convey what I want to express.

1 Spectra and Brown representability

In this chapter, we will introduce the most fundamental objects in stable homotopy theory: spectra.

1.1 Homotopy pullbacks and homotopy pushouts

We start with some recollections regarding homotopy pullbacks and homotopy pushouts.

Definition 1.1.1. Consider maps $f\colon X \to Z$ and $g\colon Y \to Z$ of topological spaces. We define their *homotopy pullback* $X \times_Z^h Y$ as

$$\begin{aligned} X \times_Z^h Y &:= X \times_Z Z^{[0,1]} \times_Z Y \\ &= \{(x, p, y) \in X \times Z^{[0,1]} \times Y \mid p(0) = f(x), p(1) = g(y)\}. \end{aligned}$$

We denote by $\mathrm{pr}_X\colon X \times_Z^h Y \to X$ and $\mathrm{pr}_Y\colon X \times_Z^h Y \to Y$ the two projection maps, and we denote by $H\colon (X \times_Z^h Y) \times [0,1] \to Z$ the homotopy given by $((x, p, y), t) \mapsto p(t)$. Note that H satisfies $H_0 = f \circ \mathrm{pr}_X$ and $H_1 = g \circ \mathrm{pr}_Y$, so that it defines a homotopy between the two outer composites in the following square:

$$\begin{tikzcd} X \times_Z^h Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] \arrow[ur, phantom, "\overset{H}{\Nearrow}\,\sim"] & Z. \end{tikzcd}$$

In fact, observe that this square is in a sense *universal* with this data: for an arbitrary space T, a map $T \to X \times_Z^h Y$ is the same as a pair of maps $t_X\colon T \to X$ and $t_Y\colon T \to Y$ together with a homotopy $f \circ t_X \sim g \circ t_Y$. We will discuss the resulting universal property of homotopy pullbacks in more detail in Chapter 2, see Section 2.3.2.

Example 1.1.2. Given a map $f\colon X \to Y$, the *homotopy fiber* of f at a point $y \in Y$ is the homotopy pullback of f along the map $y\colon * \to Y$.

Example 1.1.3. The *loop space* ΩX of a pointed space (X,x) is defined as the homotopy pullback of the map $x\colon * \to X$ along itself:

$$\Omega X = \Omega(X,x) := \{p\colon [0,1] \to X \mid p(0) = p(1) = x\}.$$

We regard ΩX again as a pointed space with basepoint given by the constant loop at x.

Given a pointed map $f\colon (X,x) \to (Y,y)$, we obtain an induced map $\Omega f\colon \Omega X \to \Omega Y$ given by $\Omega f(p) := f \circ p$. This defines a functor

$$\Omega\colon \mathrm{Top}_* \to \mathrm{Top}_*,$$

where Top_* denotes the category of pointed topological spaces and pointed maps.

Definition 1.1.4. Consider maps $f\colon Z \to X$ and $g\colon Z \to Y$ of topological spaces. We define their *homotopy pushout* $X \sqcup_Z^h Y$ as

$$\begin{aligned} X \sqcup_Z^h Y &:= X \sqcup_Z (Z \times [0,1]) \sqcup_Z Y \\ &= (X \sqcup Z \times [0,1] \sqcup Y)/\simeq, \end{aligned}$$

where the equivalence relation $\simeq$ is generated by identifying $(z,0) \in Z \times [0,1]$ with $f(z) \in X$ and $(z,1) \in Z \times [0,1]$ with $g(z) \in Y$ for all $z \in Z$. We denote by $\iota_X\colon X \hookrightarrow X \sqcup_Z^h Y$, $x \mapsto x$ and $\iota_Y\colon Y \hookrightarrow X \sqcup_Z^h Y$, $y \mapsto y$ the two canonical inclusions, and we denote by $H\colon Z \times [0,1] \to X \sqcup_Z^h Y$ the homotopy given by $(z,t) \mapsto (z,t)$. Note that H satisfies $H_0 = \iota_X \circ f$ and $H_1 = \iota_Y \circ g$, so that it defines a homotopy between the two outer composites in the following square:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"'] \arrow[ur, phantom, "\overset{H}{\underset{\sim}{\Nearrow}}"] & X \sqcup_Z^h Y. \end{tikzcd}$$

Again, note that this square is *universal* with this data: for another topological space T, a map $X \sqcup_Z^h Y \to T$ is precisely the data of maps $t_X\colon X \to T$ and $t_Y\colon Y \to T$ together with a homotopy $t_X \circ f \sim t_Y \circ g$.

Definition 1.1.5. Given a map $f\colon X \to Y$ of topological spaces, we define the *homotopy cofiber* $C(f)$ as

$$C(f) = Y \sqcup_X^h * = Y \sqcup_X (X \times [0,1])/X \times \{1\}.$$

Given a map $g\colon Y \to Z$ equipped with a *nullhomotopy* $g \circ f \sim \mathrm{const}_z$, i.e. a map $H\colon X \times [0,1] \to Z$ with $H_0 = gf$ and $H_1 = \mathrm{const}_z$, we obtain a canonical map

$$C(f) \to Z, \qquad y \mapsto g(y), \qquad (x,t) \mapsto H(x,t).$$

We say that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a *cofiber sequence* if this map $C(f) \to Z$ is a homotopy equivalence.

Example 1.1.6. When $f = \mathrm{id}_X\colon X \to X$ is the identity, the cofiber $C(\mathrm{id}_X)$ is known as the *cone of* X and is denoted $C(X)$:

$$C(X) = (X \times [0,1])/X \times \{1\}.$$

Note that $C(f) = Y \sqcup_X C(X)$.

Remark 1.1.7. If (X,x) is a pointed topological space, we often want to consider the *reduced cone*

$$\tilde{C}(X) := C(X)/\{x\} \times [0,1].$$

Similarly, if $f\colon X \to Y$ is a pointed map, we may consider the *reduced cofiber* $\tilde{C}(f) = C(f) = Y \sqcup_X \tilde{C}(X)$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of *pointed* spaces is called a *cofiber sequence in* CW_* if the composite gf comes with a null-homotopy $H\colon gf \sim \mathrm{const}_*$ such that the induced map $\tilde{C}(f) \to Z$ is a homotopy equivalence.

Example 1.1.8. The *suspension* SX of a topological space X is the homotopy cofiber of $X \to *$:

$$SX := * \sqcup_X (X \times [0,1]) \sqcup_X *.$$

The *reduced suspension* ΣX of a pointed topological space (X,x) is the quotient

$$\Sigma X := SX/(\{x\} \times [0,1]).$$

Note that the latter defines a functor $\Sigma\colon \mathrm{Top}_* \to \mathrm{Top}_*$ given on a pointed map $f\colon (X,x) \to (Y,y)$ by $(\Sigma f)[(x,t)] := [(f(x),t)]$.

Lemma 1.1.9. *There is an adjunction* $\Sigma\colon \mathrm{Top}_* \rightleftarrows \mathrm{Top}_* : \Omega$.

Proof. Consider pointed spaces (X,x) and (Y,y). The data of a pointed map $\Sigma X \to Y$ is that of a map $H\colon X \times [0,1] \to Y$ such that $H_0 = H_1 = \mathrm{const}_y$ and $H_t(x) = y$ for all $t \in [0,1]$. This in turn corresponds to a pointed map $\tilde{H}\colon X \to \Omega(Y,y) \subseteq \mathrm{Map}([0,1],Y)$, as desired. □

Recall that the *homotopy category* hTop_* is the category whose objects are the pointed topological spaces (X,x), and whose morphisms are the (pointed) homotopy classes of pointed maps $(X,x) \to (Y,y)$:

$$\mathrm{Hom}_{\mathrm{hTop}_*}((X,x),(Y,y)) = [X,Y]_*.$$

Corollary 1.1.10. *There is an adjunction* $\Sigma\colon \mathrm{hTop}_* \rightleftarrows \mathrm{hTop}_* : \Omega$.

Proof. It remains to show that two pointed maps $f,g\colon \Sigma X \to Y$ are pointed homotopic if and only if their adjoint maps $\tilde{f},\tilde{g}\colon X \to \Omega Y$ are, which is clear from chasing through the previous proof. □

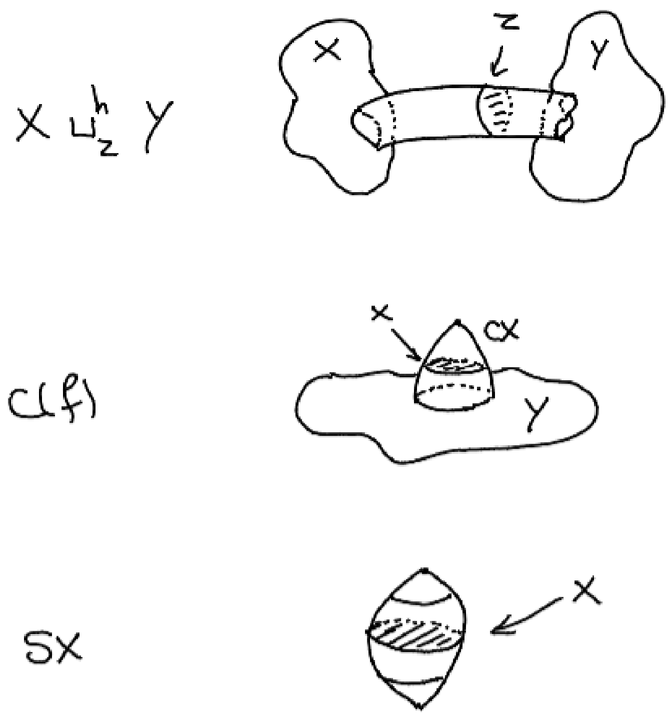


Figure 1.1: Illustration of the homotopy pushout, cofiber and suspension.

Remark 1.1.11. Recall that the k-th homotopy group of a pointed topological space (X,x) is defined as

$$\pi_k(X,x) := [S^k, X]_*.$$

The previous corollary thus in particular provides canonical isomorphisms

$$\pi_k(\Omega X) = [S^k, \Omega X]_* \cong [\Sigma S^k, X]_* \cong [S^{k+1}, X]_* = \pi_{k+1}(X).$$

1.2 Spaces versus topological spaces

In homotopy theory, we are interested in topological spaces up to weak homotopy equivalence, i.e. we regard two topolological spaces as 'the same' if there is a zig-zag of weak homotopy equivalences between them. Recall from your previous topology courses the following two theorems:

Theorem 1.2.1 (CW-approximation)**.** *For every topological space X there exists a CW-complex and a weak homotopy equivalence $Z \xrightarrow{\sim} X$.*

Theorem 1.2.2 (Whitehead theorem)**.** *Let X and Y be CW-complexes. Then every weak homotopy equivalence $X \xrightarrow{\sim} Y$ is already an actual homotopy equivalence.*

Because of these two theorems, we may safely work with CW-complexes up to homotopy equivalence. We will now introduce some terminology which emphasizes this perspective.

Convention 1.2.3. We define a *space* to be a topological space which is homotopy equivalent to a CW-complex. Similarly we define a *pointed space* to be a topological space which is pointed homotopy equivalent to a pointed CW-complex. We denote by

$$\mathrm{h}\mathcal{S} \subseteq \mathrm{hTop} \qquad \text{and} \qquad \mathrm{h}\mathcal{S}_* \subseteq \mathrm{hTop}_*$$

the full subcategories of (pointed) spaces. Since any homotopy equivalence $X \simeq Y$ becomes an isomorphism in hTop, the category $\mathrm{h}\mathcal{S}$ is equivalent to the homotopy category hCW of CW-complexes, and similarly $\mathrm{h}\mathcal{S}_* \simeq \mathrm{hCW}_*$.

Although this terminology may be quite confusing at first, it is very useful to distinguish the two different usages of topological spaces in homotopy theory:

- When we say 'topological space', we are talking about a specific set X equipped with a topology. In particular, it makes sense to speak of the set of points of X or the open subsets of X.
- Whenever we use the word 'space' without the adjective 'topological', we do not allow ourselves to say anything that is not invariant under homotopy equivalences. For example, we may speak of the homology or homotopy groups of a space, but we do not allow ourselves to speak about the 'set of points' of a space, or its 'open subsets', or its 'dimension': all of these will change if we replace the space X by a homotopy equivalent one.

When using the second perspective, we are in a sense forgetting the actual topological structure of the topological space, and we only remember its purely homotopical 'spirit', also known as its underlying *anima*.[1] See also Remarks 2.6.2 and 2.6.39 below.

Let us now check that all the relevant constructions on topological spaces preserve homotopy equivalences.

Exercise 1.2.4. Consider a diagram of topological spaces

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Z \arrow[d, "\gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta"] \\
X' \arrow[r, "f'"] & Z' & Y' \arrow[l, "g'"']
\end{tikzcd}$$

which commutes up to homotopy, in the sense that we are given homotopies $H\colon \gamma \circ f \sim f' \circ \alpha$ and $K\colon \gamma \circ g \sim g' \circ \beta$.

[1]The word 'anima' means 'soul' in Latin.

- Construct a map $\alpha \times_\gamma \beta\colon X \times^h_Z Y \to X' \times^h_{Z'} Y'$. (This map will depend on the choices of homotopies H and K, though this is not reflected in our notation.)
- Show that $\alpha \times_\gamma \beta$ is a homotopy equivalence whenever α, β and γ are homotopy equivalences.

Exercise 1.2.5. Formulate and prove the analogous statement for homotopy pushouts.

Exercise 1.2.6. Show that if $f\colon Z \to X$ and $g\colon Z \to Y$ are maps of spaces, then the homotopy pushout $X \sqcup^h_Z Y$ is again a space.

Hint: use Exercise 1.2.5 and the cellular approximation theorem.

The dual statement for homotopy *pullbacks* is also true but highly non-trivial:

Theorem 1.2.7 (Mather [Mat76, Lemma 36], Milnor [Mil59, Theorem 3])**.** *If $f\colon X \to Z$ and $g\colon Y \to Z$ are maps of spaces, then the homotopy pullback $X \times^h_Z Y$ is again a space.*

As a special case of the above exercise and theorem, we see that for a pointed space X the reduced suspension ΣX and the loop space ΩX are again pointed spaces. In particular, the adjunction from Corollary 1.1.10 restricts to an adjunction

$$\Sigma\colon \mathrm{h}\mathcal{S}_* \rightleftarrows \mathrm{h}\mathcal{S}_* :\Omega.$$

1.3 Cohomology theories and Brown representability

Recall from the introduction the following definition:

Definition 1.3.1. A *spectrum* X is a sequence of pointed spaces $X = (X_n)_{n\geq 0}$ together with pointed homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for all $n \geq 0$.

Given another spectrum $Y = (Y_n)_{n\geq 0}$, a *morphism of spectra* $f\colon X \to Y$ is a sequence of pointed maps $(f_n\colon X_n \to Y_n)_{n\geq 0}$ together with homotopies H_n making all the following diagrams commute up to homotopy:

$$\begin{tikzcd} X_n \arrow[r, "\simeq"] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \arrow[dl, phantom, "\overset{\sim}{\swarrow} H_n"] \\ Y_n \arrow[r, "\simeq"'] & \Omega Y_{n+1}. \end{tikzcd}$$

As we will now discuss, the notion of a spectrum is intimately tied to that of a (generalized) cohomology theory. Recall that a *graded abelian group* is a collection $(A_n)_{n\in\mathbb{Z}}$ of abelian groups, indexed by the integers. A *morphism of graded abelian groups* from $(A_n)_{n\in\mathbb{Z}}$ to $(B_n)_{n\in\mathbb{Z}}$ is a collection of maps $f_n\colon A_n \to B_n$. We let $\mathrm{Ab}^{\mathbb{Z}}$ denote the resulting category of graded abelian groups.

Definition 1.3.2. A *(generalized) cohomology theory* is a pair (E^*, ∂) consisting of:

- A functor $E^* \colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$;
- A natural isomorphism
$$\partial \colon E^*(-) \xrightarrow{\cong} E^{*+1}(\Sigma(-))$$
of functors $\mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$, called the *suspension isomorphism*.

This pair is required to satisfy the following two conditions:

- (Wedge axiom) For every small collection of pointed spaces $\{X_i\}_{i\in I}$, the inclusions $\iota_i \colon X_i \hookrightarrow \bigvee_{i\in I} X_i$ induce an isomorphism
$$(\iota_i^*)_{i\in I} \colon E^*\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} E^*(X_i).$$
In particular, for $I = \emptyset$ we have $E^*(*) \cong 0$.
- (Exactness) For any cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of pointed spaces, the sequence
$$E^*(Z) \xrightarrow{g^*} E^*(Y) \xrightarrow{f^*} E^*(X)$$
is exact.

Example 1.3.3. For an abelian group A, we have a functor $\widetilde{H}^*(-;A) \colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{N}}$ given by the *singular cohomology with coefficients in A*. By defining $\widetilde{H}^n(X;A) = 0$ for $n < 0$, this defines a generalized cohomology theory.

Later in the course we will see other cohomology theories, like complex K-theory and cobordism.

Remark 1.3.4. Given a cofiber sequence $X \to Y \to Z$, the cofiber of $Y \to Z$ is equivalent to $\Sigma(X)$, and the cofiber of $Z \to \Sigma(X)$ is $\Sigma(Y)$. Proceeding this way, we get a chain of cofiber sequences known as the *Puppe sequence*:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X) \xrightarrow{\Sigma(f)} \Sigma(Y) \xrightarrow{\Sigma(g)} \Sigma(Z) \to \Sigma^2(X) \to \dots.$$

Using that $E^n(\Sigma X) \cong E^{n-1}(X)$, we thus obtain a long exact sequence

$$\dots \xrightarrow{g^*} E^{n-1}(Y) \xrightarrow{f^*} E^{n-1}(X) \to E^n(Z) \xrightarrow{g^*} E^n(Y) \xrightarrow{f^*} E^n(X) \to E^{n+1}(Z) \xrightarrow{g^*} E^{n+1}(Y) \xrightarrow{f^*} \dots.$$

See Exercise 2.3 on page 248 for more details.

Remark 1.3.5. Every cohomology theory E^* in the above sense defines a cohomology theory on CW-pairs (X, A) by setting

$$E^*(X, A) := E^*(X/A),$$

where we regard the quotient space X/A as a pointed space with basepoint A/A. We then recover the long exact sequence of a pair by applying the previous remark to the cofiber sequence

$$A_+ \to X_+ \to X/A.$$

Every spectrum gives rise to a cohomology theory on pointed spaces:

Construction 1.3.6. Let $E = (E_n)_{n\geq 0}$ be a spectrum. We define a functor $E^*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ as follows:

$$E^k(X) := \begin{cases} [X, E_k]_* & k \geq 0 \\ [\Sigma^{-k}X, E_0]_* & k < 0. \end{cases}$$

Given the homotopy equivalences $E_k \simeq \Omega E_{k+1} \simeq \Omega^2 E_{k+2}$, these sets of homotopy classes admit canonical stuctures of abelian groups via concatenation of loops. We further define the suspension isomorphism ∂ for $k \geq 0$ as

$$\partial\colon E^k(X) = [X, E_k]_* \cong [X, \Omega E_{k+1}]_* \cong [\Sigma X, E_{k+1}]_* = E^{k+1}(\Sigma X),$$

where all these identifications are natural in X. For $k < 0$ we take $\delta\colon E^k(X) \xrightarrow{\cong} E^{k+1}(\Sigma X)$ to be the identity map of $[\Sigma^{-k}X, E_0]_* = [\Sigma^{-(k+1)}\Sigma X, E_0]_*$.

Proposition 1.3.7. *Let E be a spectrum. Then the pair (E^*, ∂) constructed above defines a cohomology theory.*

Proof. First note that for two homotopic maps $f \sim g\colon X \to Y$, the induced maps $f^*, g^*\colon E^k(Y) \to E^k(X)$ are equal, so that the functor $E^k\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}$ is well-defined. Given pointed spaces $\{X_i\}_{i\in I}$, pointed maps from $\bigvee_{i\in I} X_i$ to E_k correspond to collections of pointed maps $X_i \to E_k$, and since this process preserves pointed homotopy classes we obtain a bijection

$$E^k(\bigvee_{i\in I} X_i) = [\bigvee_{i\in I} X_i, E_k]_* \cong \prod_{i\in I}[X_i, E_k]_* = \prod_{i\in I} E^k(X_i);$$

the argument for $k < 0$ is analogous, using that Σ^{-k} preserves wedge sums. Finally, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a cofiber sequence of pointed spaces, we claim that for any space E the sequence

$$[Z, E_k]_* \xrightarrow{g^*} [Y, E_k]_* \xrightarrow{f^*} [X, E_k]_*$$

is exact: the image of g^* agrees with the preimage $(f^*)^{-1}(\mathrm{const}_*)$. Since $g \circ f$ is null-homotopic, it is clear that $\mathrm{im}(g^*) \subseteq (f^*)^{-1}(\mathrm{const}_*)$. Conversely, let $h\colon Y \to E$ be a pointed map such that the composite $h \circ f\colon X \to E$ is nullhomotopic. Choosing a nullhomotopy then defines an extension of h to the mapping cone $C(f) \to E$, showing that h is in the image of the restriction map $[C(f), E]_* \to [Y, E]_*$. But by assumption we have $Z \simeq C(f)$ and the claim follows. $\square$

The main goal of this section is to prove a converse of Proposition 1.3.7: using the Brown representability theorem, stated below, we will show that every cohomology comes from some spectrum E. Let us start by introducing some terminology.

Definition 1.3.8. Consider a functor $F\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$.

(1) We say that F is *representable* if it is of the form[2] $[-, Z]_*\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$ for some pointed space Z. (By the Yoneda lemma the object Z is then uniquely determined up to isomorphism as an object of $\mathrm{h}\mathcal{S}_*$.)

(2) We say that F satisfies the *Wedge Axiom* if for every small collection of pointed spaces $\{X_i\}_{i\in I}$ the map

$$F\left(\bigvee_{i\in I} X_i\right) \xrightarrow{\cong} \prod_{i\in I} F(X_i)$$

induced by the inclusions $X_i \hookrightarrow \bigvee_{i\in I} X_i$ is a bijection (equivalently: the functor F preserves arbitrary products);

(3) We say that F satisfies *Mayer-Vietoris* if for every homotopy pushout square

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X \end{tikzcd}$$

of connected pointed spaces, the induced map

$$(i^*, j^*)\colon F(X) \to F(A) \times_{F(C)} F(B)$$

is surjective.

Remark 1.3.9. If F satisfies the Wedge-Axiom, then $F(\Sigma X)$ is automatically a group for every X, and $F(\Sigma^2 X)$ is even an abelian group. To see this, first observe that ΣX is a *cogroup* in $\mathrm{h}\mathcal{S}_*$: there are pointed maps

$$\Sigma X \to \mathrm{pt}, \qquad \mathrm{pinch}\colon \Sigma X \to \Sigma X \vee \Sigma X$$

where the first map is the unique map to the point and where the second map is the *pinch map* defined by identifying $\Sigma X \vee \Sigma X$ with the quotient of $X \times [0,1]$ at the subspace

[2] By 'of the form' we mean 'naturally isomorphic to a functor of the form'.

$X \times \{0, \frac{1}{2}, 1\} \cup \{x_0\} \times [0,1]$. We may more abstractly describe this, using the definition of ΣX as a homotopy pushout, as the map

$$\Sigma X = \mathrm{pt} \sqcup^h_X \mathrm{pt} \simeq \mathrm{pt} \sqcup^h_X X \sqcup^h_X \mathrm{pt} \to \mathrm{pt} \sqcup^h_X \mathrm{pt} \sqcup^h_X \mathrm{pt} \simeq \Sigma X \vee \Sigma X$$

induced by the unique map $X \to \mathrm{pt}$. This map satisfies counitality and coassociativity, and it has a coinverse map $i \colon \Sigma X \to \Sigma X$ given by 'inverting the homotopy'. We leave the details as an exercise to the reader. (Eventually we will see a much more general statement in the context of ∞-categories: see Exercise 7.7 and Remark 7.8 on page 259.)

Applying the contravariant functor F then turns the cogroup object ΣX into a group $F(\Sigma X)$: the unit, multiplication and inversion are given by the maps

$$\mathrm{pt} = F(\mathrm{pt}) \to F(\Sigma X),$$
$$F(\Sigma X) \times F(\Sigma X) \simeq F(\Sigma X \vee \Sigma X) \xrightarrow{F(\mathrm{pinch})} F(\Sigma X),$$
$$F(\Sigma X) \xrightarrow{F(i)} F(\Sigma X).$$

The space $\Sigma^2 X$ is a cogroup in two a priori distinct ways, but it follows from the Eckmann-Hilton argument (see below) that they agree in $\mathrm{h}\mathcal{S}_*$ and are in fact (co)commutative. It follows that $F(\Sigma^2 X)$ is even an abelian group for every space X.

With similar reasoning one observes that for any natural transformation $F \to F'$ the induced map $F(\Sigma X) \to F'(\Sigma X)$ is a group morphism for every pointed space X.

Lemma 1.3.10 (Eckmann–Hilton argument)**.** *Let X be a set equipped with two binary unital operations $\circ$ and $\otimes$: there are elements $1_\circ$ and $1_\otimes$ satisfying*

$$1_\circ \circ a = a = a \circ 1_\circ \qquad \text{and} \qquad 1_\otimes \otimes a = a = a \otimes 1_\otimes$$

for all $a \in X$. Assume that the two operations commute with each other, in the sense that

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d)$$

for all $a, b, c, d \in X$. Then the operations $\circ$ and $\otimes$ are the same and in fact commutative and associative. □

Remark 1.3.11. If F is given as a functor $F \colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}$ into the category of *abelian groups*, then the Mayer-Vietoris property is equivalent to saying that the sequence

$$F(X) \xrightarrow{(i^*, j^*)} F(A) \times F(B) \xrightarrow{k^* - l^*} F(C)$$

is exact. If F also satisfies the Wedge Axiom, then one may use the cofiber sequence $A \vee B \to X \to \Sigma C$ to extend this to a long exact sequence

$$\cdots \to F(\Sigma A) \times F(\Sigma B) \xrightarrow{(\Sigma k)^* - (\Sigma l)^*} F(\Sigma C) \xrightarrow{\partial} F(X) \xrightarrow{(i^*, j^*)} F(A) \times F(B) \xrightarrow{k^* - l^*} F(C).$$

Some hints for obtaining this sequence are provided in Exercise 2.5 on page 248.

The following is the main theorem of this section:

Theorem 1.3.12 (Brown representability theorem)**.** *Let* $\mathrm{h}\mathcal{S}_*^{\geq 1}$ *denote the homotopy category of connected pointed spaces. Then a functor* $F\colon (\mathrm{h}\mathcal{S}_*^{\geq 1})^{\mathrm{op}} \to \mathrm{Set}$ *is representable if and only if it satisfies the Wedge Axiom and Mayer-Vietoris.*

Remark 1.3.13. Counterexamples to this theorem exist if one either drops the pointedness or the connectedness assumption.

We will prove this below, after some preparations. First note that for every connected pointed space Z and every element $\xi \in F(Z)$ we obtain a natural transformation

$$T_\xi\colon [X, Z]_* \to F(X), \qquad (f\colon X \to Z) \mapsto f^*\xi,$$

where we write $f^* := F(f)\colon F(Z) \to F(X)$. (In fact, by the Yoneda lemma every natural transformation $[-, Z]_* \to F(-)$ is of this form.) We say that an element $\xi \in F(Z)$ is *n-universal* for $n \in \mathbb{N}$ if the map

$$T_\xi\colon \pi_k(Z) = [S^k, Z]_* \to F(S^k)$$

is a bijection for $1 \leq k < n$ and a surjection for $k = n$. We will say that ξ is *universal* if ξ is n-universal for all $n \geq 0$. The core ingredient of the proof of Brown representability is the following statement:

Proposition 1.3.14. *Let* $F\colon (\mathrm{h}\mathcal{S}_*^{\geq 1})^{\mathrm{op}} \to \mathrm{Set}$ *be a functor satisfying the Wedge Axiom and Mayer-Vietoris. For every pointed space* X *and every element* $\eta \in F(X)$*, there exists a pointed space* Z_X*, a pointed map* $f_X\colon X \to Z_X$ *and a universal element* $\xi_X \in F(Z_X)$ *satisfying* $f_X^*(\xi_X) = \eta$*.*

Proof. Step 1: We start by producing a sequence

$$X =: Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} \dots$$

of pointed spaces and pointed maps, together with n-universal elements $\xi_n \in F(Z_n)$ satisfying $f_n^*(\xi_{n+1}) = \xi_n \in F(Z_n)$, where $\xi_0 = \eta \in F(X)$. We proceed by induction on n. For $n = 1$, we define

$$Z_1 := X \vee \bigvee_{m \geq 1} \bigvee_{\gamma \in F(S^m)} S^m.$$

By the Wedge Axiom there is a bijection

$$F(Z_1) \cong F(X) \times \prod_{m \geq 1} \prod_{\gamma \in F(S^m)} F(S^m),$$

and we let $\xi_1 \in F(Z_1)$ be the element whose first component is $\eta \in F(X)$ and whose component at $\gamma \in F(S^m)$ is $\gamma \in F(S^m)$. Note that for every $k \geq 1$ the map

$$T_{\xi_1} \colon [S^k, Z_1]_* \to F(S^k)$$

is surjective, and in particular ξ_1 is 1-universal.

Now for $n \geq 1$ assume that we have constructed Z_n together with an n-universal element $\xi_n \in F(Z_n)$ that satisfies the additional condition that $T_{\xi_n} \colon [S^k, Z_n]_* \to F(S^k)$ is a surjection for all $k \in \mathbb{N}$; our task is to define Z_{n+1} and ξ_{n+1}. Recall from Remark 1.3.9 that $F(S^n)$ is a group, which is abelian for $n \geq 2$. Consider the set

$$K_n := \{[f] \in [S^n, Z_n] \mid f^*\xi_n = 0 \in F(S^n)\}.$$

We may then define the pointed space Z_{n+1} by attaching an $(n+1)$-cell to Z_n along the map $f \colon S^n \to Z_n$ for every element of $[f] \in K_n$, i.e. we consider the following homotopy pushout:

$$\begin{tikzcd}
\bigvee_{[f]\in K_n} S^n \arrow[r] \arrow[d] & Z_n \arrow[d, "f_n"] \\
* \simeq \bigvee_{[f]\in K_n} D^{n+1} \arrow[r] & Z_{n+1}.
\end{tikzcd}$$

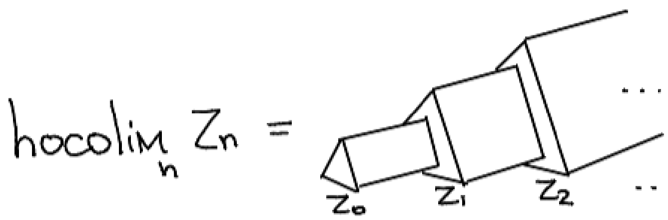
By assumption we have $f^*\xi_n = 0 \in F(S^n)$ for all $[f] \in K_n$, and so by Mayer-Vietoris there exists an element $\xi_{n+1} \in F(Z_{n+1})$ satisfying $f_n^*\xi_{n+1} = \xi_n \in F(Z_n)$. We claim that ξ_{n+1} is $(n+1)$-universal, i.e. the map

$$[S^k, Z_{n+1}]_* \to F(S^k), \qquad f \mapsto f^*(\xi_{n+1})$$

is a bijection for all $1 \leq k \leq n$ and a surjection for $k = n+1$. We already know it is a surjection for all k. By the cellular approximation theorem we further see that the map $f_n \colon Z_n \to Z_{n+1}$ induces bijections $[S^k, Z_n]_* \xrightarrow{\sim} [S^k, Z_{n+1}]_*$ for $0 \leq k < n$, and since ξ_n is n-universal we deduce that also ξ_{n+1} is n-universal. It remains to see that the map

$$[S^n, Z_{n+1}]_* \to F(S^n), \qquad g \mapsto g^*\xi_{n+1}$$

is injective. By Remark 1.3.9 this is a group homomorphism, and so it suffices to show it has trivial kernel. Let $[g] \in [S^n, Z_{n+1}]_*$ be an element satisfying $g^*\xi_{n+1} = 0$. Again by the cellular approximation theorem we may assume that g factors as a map $S^n \xrightarrow{h} Z_n \xrightarrow{f_n} Z_{n+1}$. But then we have $h^*\xi_n = g^*\xi_{n+1} = 0$ hence $[h] \in K_n$. But then by construction the composite $g = f_n \circ h \colon S^n \to Z_n \to Z_{n+1}$ is null-homotopic, i.e. we have $[g] = 0 \in [S^n, Z_{n+1}]_*$. We conclude that ξ_{n+1} is $(n+1)$-universal. This finishes the induction, and completes the proof of Step 1.



Step 2: We will now define Z_X as the homotopy colimit of the sequence $\{Z_n\}$. Recall that the homotopy colimit is given by the *mapping telescope*

$$Z_X := \mathrm{hocolim}_n Z_n := \left(\bigvee_{n\in\mathbb{N}} Z_n \wedge [n,n+1]_+ \right) /\sim,$$

where the equivalence relation $\sim$ is given by $(x_n, n+1) \sim (f(x_n), n+1)$. Note that a map out of the mapping telescope into some other space T corresponds to a family of pointed maps $t_n \colon Z_n \to T$ together with pointed homotopies between $t_n \colon Z_n \to T$ and $t_{n+1} \circ f_n \colon Z_n \to T$.

One can show (see Exercise 2.1 on page 246) that the homotopy colimit may also be given by the following homotopy pushout:

$$\begin{tikzcd}
\bigvee_{n\geq 0} Z_n \arrow[r] \arrow[d] & \bigvee_{k\geq 0} Z_{2k} \arrow[d] \\
\bigvee_{k\geq 0} Z_{2k+1} \arrow[r] & Z = \mathrm{hocolim}_n Z_n.
\end{tikzcd}$$

Here the horizontal map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k}$ for $n = 2k$ even, and the map $f_{2k+1}\colon Z_{2k+1} \to Z_{2k+2}$ for $n = 2k+1$ odd. The vertical map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k+1}$ for $n = 2k+1$ odd, and the map $f_{2k}\colon Z_{2k} \to Z_{2k+1}$ for $n = 2k$ even. By the Mayer-Vietoris property, we then obtain an element $\xi \in F(Z)$ that restricts to $\xi_n \in F(Z_n)$ for all $n \geq 0$, hence in particular to $\eta \in F(X)$. We claim that ξ is universal. To see this, consider for $k \geq 0$ the following diagram:

$$\begin{tikzcd}
{[S^k, Z_0]_*} \arrow[r] \arrow[drrr, bend right, "T_{\xi_0}"'] & \dots \arrow[r] & {[S^k, Z_k]} \arrow[r] \arrow[dr, dashed, "T_{\xi_k}"'] & {[S^k, Z_{k+1}]} \arrow[r, "\cong"] \arrow[d, dashed, "T_{\xi_{k+1}}"] & {[S^k, Z_{k+2}]} \arrow[r, "\cong"] \arrow[dl, dashed, "T_{\xi_{k+2}}"] & \dots \arrow[r, "\cong"] & {[S^k, Z]} \arrow[dlll, bend left, "T_\xi"] \\
 & & & F(S^k) & & &
\end{tikzcd}$$

By assumption, all the maps T_{ξ_n} are bijections for $n > k$. Also, since Z_{n+1} is obtained from Z_n by attaching $(n+1)$-cells, the top horizontal maps $[S^k, Z_n] \to [S^k, Z_{n+1}]$ are bijections for $n > k$. It follows that the map T_ξ is a bijection for every k, as desired. $\square$

We are now ready to prove the Brown representability theorem:

Proof of Theorem 1.3.12. If $F \cong [-,Z]_*$ is representable, then the Wedge Axiom and Mayer-Vietoris are satisfied by the universal properties of the wedge and the homotopy pushout, just like in the proof of Proposition 1.3.7.

Conversely, assume that F satisfies the Wedge Axiom and Mayer-Vietoris. To find the required pointed space Z, we take $X = \mathrm{pt}$, and we let $\eta := 0 \in F(\mathrm{pt})$ be the unique element. Applying Proposition 1.3.14 we obtain a pointed space $Z := Z_{\mathrm{pt}}$ and a universal element $\xi := \xi_{\mathrm{pt}}$. We have to show that for every pointed space X the induced map

$$T_\xi \colon [X,Z]_* \to F(X), \qquad (f\colon X \to Z) \mapsto f^*\xi$$

is a bijection.

Surjectivity: Pick an element $\eta \in F(X)$, and consider the space $\widetilde{X} := X \vee Z$. We let $\widetilde{\eta} \in F(\widetilde{X})$ be the element corresponding to the pair (η,ξ) under the bijection $F(\widetilde{X}) \cong F(X) \times F(Z)$ from the Wedge Axiom. Applying Proposition 1.3.14 to $(\widetilde{X},\widetilde{\eta})$, we thus find a pointed space $\widetilde{Z}$, a pointed map $\widetilde{f}\colon \widetilde{X} \to \widetilde{Z}$ and a universal element $\widetilde{\xi} \in F(\widetilde{Z})$ such that $\widetilde{f}^*(\widetilde{\xi}) = \widetilde{\eta}$. In particular, if we let $g\colon Z \to \widetilde{Z}$ denote the composite of $\widetilde{f}$ with the inclusion $Z \hookrightarrow X \vee Z = \widetilde{X}$ we see that $g^*(\widetilde{\xi}) = \xi$, and in particular the following diagram commutes for all $k \geq 0$:

$$\begin{tikzcd} \pi_k(Z) = [S^k,Z]_* \arrow[rr, "g_*"] \arrow[dr, "\cong", "T_\xi"'] & & {[S^k,\widetilde{Z}]_* = \pi_k(\widetilde{Z})} \arrow[dl, "\cong"', "T_{\widetilde{\xi}}"] \\ & F(S^k). & \end{tikzcd}$$

Since ξ and $\widetilde{\xi}$ are both universal, the bottom two maps are bijections. The top map is then also a bijection and it follows that the map $g\colon Z \to \widetilde{Z}$ is a weak homotopy equivalence. Since Z and $\widetilde{Z}$ are homotopy equivalent to CW-complexes this means that g is even a homotopy equivalence (Theorem 1.2.2). It then follows from the commutative diagram

$$\begin{tikzcd} {[X,Z]_*} \arrow[rr, "g_*", "\cong"'] \arrow[dr, "T_\xi"'] & & {[X,\widetilde{Z}]_*} \arrow[dl, "T_{\widetilde{\xi}}"] \\ & F(X) & \end{tikzcd}$$

that the composite $X \to \widetilde{Z} \xrightarrow{\sim} Z$ is an element in $[X,Z]_*$ which is mapped to η by T_ξ.

Injectivity: Suppose that $f,g\colon X \to Z$ are two pointed maps satisfying $\eta := f^*(\xi) = g^*(\xi)$. Define the pointed space $\widetilde{X}$ as the following homotopy pushout:

$$\begin{tikzcd} X \vee X \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{id}_X,\mathrm{id}_X)}"'] & Z \arrow[d] \\ X \arrow[r] & \widetilde{X} := X \sqcup^h_{X\vee X} Z. \end{tikzcd}$$

By the Mayer-Vietoris property of F, there exists an element $\widetilde{\eta} \in F(\widetilde{X})$ which restricts to $\eta \in F(X)$ along the bottom map and to $\xi \in F(Z)$ along the right vertical map. Applying the claim to $(\widetilde{X}, \widetilde{\eta})$ we find a pointed space $\widetilde{Z}$ with a pointed map $\widetilde{X} \to \widetilde{Z}$ and a universal element $\widetilde{\xi} \in F(\widetilde{Z})$ which restricts to $\widetilde{\eta}$. Arguing as before, we see that the composite $Z \to \widetilde{X} \to \widetilde{Z}$ is a homotopy equivalence. But this implies that there is a map $\widetilde{f}$ fitting in a homotopy commutative diagram as follows:

$$\begin{tikzcd} X \vee X \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{id}_X, \mathrm{id}_X)}"'] & Z \\ X. \arrow[ur, dashed, "\widetilde{f}"'] & \end{tikzcd}$$

We conclude that f and g are both homotopic to $\widetilde{f}$ and in particular that $[f] = [g] \in [X, Z]_*$, as desired. This finishes the proof of the theorem. $\square$

Having established Brown representability, we may now deduce that every cohomology theory is represented by a spectrum:

Proposition 1.3.15. *Let $h^* \colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Ab}^{\mathbb{Z}}$ be a cohomology theory. Then there exists a spectrum E such that h^* is isomorphic to the cohomology theory E^* associated to E as in Construction 1.3.6.*

Proof. Step 1: We start by showing that for $n \geq 0$ the functor $h^n \colon (\mathrm{h}\mathcal{S}_*^{\geq 1})^{\mathrm{op}} \to \mathrm{Ab}$ satisfies the hypotheses of the Brown representability theorem. The Wedge Axiom holds by assumption. For the Mayer-Vietoris property, consider a homotopy pushout square

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X \end{tikzcd}$$

By the pasting law of homotopy pushout squares (Exercise 2.2), the induced map on cofibers $C(k) \to C(j)$ is a homotopy equivalence. We thus obtain a diagram of long exact sequences

$$\begin{tikzcd} h^n(C(j)) \arrow[r, "q^*"] \arrow[d, "\cong"'] & h^n(X) \arrow[r, "j^*"] \arrow[d, "i^*"] & h^n(B) \arrow[r, "\partial"] \arrow[d, "l^*"] & h^{n+1}(C(j)) \arrow[d, "\cong"] \\ h^n(C(k)) \arrow[r, "p^*"] & h^n(A) \arrow[r, "k^*"] & h^n(C) \arrow[r, "\partial"] & h^{n+1}(C(k)). \end{tikzcd}$$

Consider elements $\alpha \in h^n(A)$ and $\beta \in h^n(B)$ such that $k^*(\alpha) = l^*(\beta) \in h^n(C)$. We must show that there exists an element $\gamma \in h^n(X)$ satisfying $\alpha = i^*(\gamma)$ and $\beta = j^*(\gamma)$. Because the bottom sequence is exact we get $\partial l^*\beta = \partial k^*\alpha = 0$, and as the first vertical map is a bijection it follows that $\partial\beta = 0$. We may thus find an element $\gamma_0 \in h^n(X)$ such that $\beta = j^*\gamma_0$:

$$\begin{tikzcd}
\gamma_0 \arrow[r, mapsto, dashed, "j^*"] & \beta \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "l^*"] & \partial\beta = 0 \arrow[d, mapsto, "\cong"] \\
\alpha \arrow[r, mapsto, "k^*"] & k^*\alpha = l^*\beta \arrow[r, mapsto, "\partial"] & \partial l^*\beta = 0.
\end{tikzcd}$$

While γ_0 might not be mapped to α under i^*, we see that

$$k^*(\alpha - i^*\gamma_0) = k^*\alpha - k^*i^*\gamma_0 = l^*\beta - l^*j^*\gamma_0 = 0,$$

and thus we see that $\alpha - i^*\gamma_0$ lies in the image of $p^*\colon h^n(C(k)) \to h^n(A)$. Since the left vertical map is a bijection, we may thus pick some $\gamma_1 \in h^n(C(j))$ such that $\alpha - i^*\gamma_0 = i^*q^*\gamma_1$. But then the element $\gamma := \gamma_0 + q^*\gamma_1$ satisfies

$$i^*(\gamma) = i^*\gamma_0 + i^*q^*\gamma_1 = \alpha \qquad \text{and} \qquad j^*(\gamma) = j^*\gamma_0 + j^*q^*\gamma_1 = \beta + 0 = \beta.$$

We may summarize this diagram chase as follows:

$$\begin{tikzcd}
\gamma_1 \arrow[r, mapsto, "q^*"] \arrow[d, mapsto, dashed, "\cong"] & q^*\gamma_1 \arrow[r, mapsto, dashed, "j^*"] \arrow[d, mapsto, dashed, "i^*"] & 0 \\
\gamma_1' \arrow[r, mapsto, dashed, "p^*"] & \alpha - i^*\gamma_0 \arrow[r, mapsto, "k^*"] & 0,
\end{tikzcd}
\qquad
\begin{tikzcd}
\gamma := \gamma_0 + q^*\gamma_1 \arrow[r, mapsto, "j^*"] \arrow[d, mapsto, "i^*"] & \beta = \beta + 0 \arrow[d, mapsto, "l^*"] \\
\alpha = i^*\gamma_0 + (\alpha - i^*\gamma_0) \arrow[r, mapsto, "k^*"] & k^*\alpha.
\end{tikzcd}$$

This finishes the proof that h^n satisfies Mayer-Vietoris.

Step 2: By Brown representability, we obtain pointed connected spaces Z_n together with natural isomorphisms

$$h^n(-) \cong [-, Z_n]_*\colon (\mathrm{h}\mathcal{S}_*^{\geq 1})^{\mathrm{op}} \to \mathrm{Set},$$

for all $n \geq 0$. Since ΣX is connected for all X, we then also obtain natural isomorphisms

$$h^n(X) \cong h^{n+1}(\Sigma X) \cong [\Sigma X, Z_{n+1}]_* \cong [X, \Omega Z_{n+1}] = [X, E_n]_*,$$

where we set $E_n := \Omega Z_{n+1}$. Since representing objects are unique up to isomorphism in $\mathrm{h}\mathcal{S}_*$, we then obtain pointed homotopy equivalences $E_n \simeq \Omega E_{n+1}$ which make the following diagram commute:

$$\begin{tikzcd}
h^n(X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h^{n+1}(\Sigma X) \arrow[d, "\cong"] \\
{[X, E_n]_*} \arrow[r, dashed, "\cong"'] & {[X, \Omega E_{n+1}]_*}.
\end{tikzcd}$$

We conclude that the sequence $(E_n)_{n\in\mathbb{N}}$ defines a spectrum and that the cohomology theory h^* is naturally isomorphic to E^*. $\square$

In a similar way we can show that every morphism of cohomology theories is represented by a morphism of spectra:

Lemma 1.3.16. *Let E and F be spectra and let $f^*\colon E^* \to F^*$ be a morphism of cohomology theories. Then there exists a morphism of spectra $f\colon E \to F$ that induces f^*.*

Proof. The map f^n has the form of a natural transformation $[-, E_n]_* \to [-, F_n]_*$, for $n \geq 0$. By the Yoneda lemma, it is given by postcomposition with a certain pointed map $f_n\colon E_n \to F_n$. In light of the commutative diagram

$$\begin{tikzcd} {[-, E_n]_*} \arrow[r, "\cong"] \arrow[d, "f_n \circ -"'] & {[-, \Omega E_{n+1}]_*} \arrow[d, "\Omega f_{n+1} \circ -"] \\ {[-, F_n]_*} \arrow[r, "\cong"] & {[-, \Omega F_{n+1}]_*} \end{tikzcd}$$

it follows that the diagram

$$\begin{tikzcd} E_n \arrow[r, "\sim"] \arrow[d, "f_n"'] & \Omega E_{n+1} \arrow[d, "\Omega f_{n+1}"] \\ F_n \arrow[r, "\sim"] & \Omega F_{n+1} \end{tikzcd}$$

commutes in $\mathrm{h}\mathcal{S}_*$. In particular we may pick a homotopy filling this diagram, turning the maps (f_n) into a morphism of spectra. □

Warning 1.3.17. There may be different maps of spectra which induce the same map at the level of cohomology theories! These are closely related to the famous *phantom maps*.

2 Introduction to ∞-categories

The goal of this chapter is to get the reader up to speed with the theory of ∞-categories. The approach we will take is quite different from most other foundational sources on ∞-categories: instead of *defining* ∞-categories (e.g. in terms of simplicial sets), we will take ∞-categories as primitive objects whose behavior we have to *axiomatize* (just like how in ordinary mathematics one cannot *define* what a 'set' is).

2.1 The fundamental principle of homotopy theory

Before introducing ∞-categories, let me say a word about why I will not define what an ∞-category is in terms of more basic structures, like we would do for most other definitions in mathematics. The reason is that an ∞-category only becomes a natural concept to consider when one already has the mindset of a homotopy theorist, which requires a fundamental shift in perspective on the notion of *equality*:

> **Fundamental Principle of Homotopy Theory:** When expressing that two entities are equal, we must always specify *how* they are equal by providing a homotopy/isomorphism between them.

In other words, equality in homotopy theory is not a *property* (a binary yes/no question) but is additional *structure* that needs to be provided. This sits in stark contrast with the perspective of ordinary mathematics, in which equality is merely a property; e.g. two elements of a set are either equal or they are not equal.

Although it is possible to develop ∞-category theory entirely within the classical framework of set theory (for example in terms of *quasicategories*, see Section 2.5 below) I think doing this is actually more confusing than helpful: it inherently goes against the Fundamental Principle. To understand what ∞-category theory 'really' is, it is not very relevant how precisely ∞-categories can be implemented within ordinary mathematics; it is much more important to become fluent in the 'language of ∞-category theory'. For this reason, I have chosen to take a more axiomatic approach to ∞-category theory which aims to directly introduce such a language, building on in-progress joint work with Denis-Charles Cisinski, Kim Nguyen and Tashi Walde [Cis+24].

We will start introducing this language in the next section. Before that, let us illustrate and motivate the Fundamental Principle by discussing various instances of it with which you may already be familiar:

Example 2.1.1 (Algebraic topology)**.** Algebraic topology provides the most natural example of the Fundamental Principle:

- Two continuous maps $f, g\colon X \to Y$ are considered 'equal up to homotopy' if we provide a homotopy $H\colon X \times [0,1] \to Y$ between f and g;
- Two spaces X and Y are then considered 'homotopically the same' when equipped with a *homotopy equivalence*: continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ together with chosen homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$.
- When studying diagrams of spaces, we similarly replace strict commutativity by homotopy commutativity. For instance, given a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "h"] \\ Z \arrow[r, "k"] & W \end{tikzcd}$$

  we do not ask for a strict equality $h \circ f = k \circ g$, but instead require a specified homotopy $H\colon h \circ f \sim k \circ g$ witnessing how the two compositions are 'the same'. This perspective naturally leads to *homotopy coherent diagrams*, where we must also specify compatibilities between different choices of homotopies.
- The homotopical viewpoint also changes how we think about uniqueness of a structure: it means that the *space of all such structures* is contractible. A key example appears when studying loop spaces: there are many ways to compose n loops in a space, corresponding to different ways of subdividing the unit interval, but the space of all possible n-fold compositions is contractible.[1]
- As a final example, also the notion of quotient gets enhanced. When a group G acts on a space X, the ordinary quotient X/G simply identifies points in the same orbit.

[1] More precisely, for each of the $n!$ possible orderings there is a contractible space of parametrizations for composing n loops in that given order.

The *homotopy quotient* $X /\!\!/ G$ instead adds paths between points in the same orbit, connecting x to gx for each group element g. In particular, points can become 'equal to themselves' in non-trivial ways.

Example 2.1.2 (Homological algebra)**.** The phenomenon of 'equality as structure' also appears naturally in homological algebra, where it leads us to think about chain complexes not as rigid algebraic objects but as representatives of more flexible homotopical entities:

- Instead of asking whether two chain maps $f, g\colon C_\bullet \to D_\bullet$ between chain complexes are *equal*, we ask whether they are *chain homotopic*. Recall that a chain homotopy between f and g is a family of maps $H_n\colon C_n \to D_{n+1}$ satisfying $f_n - g_n = \partial^D \circ H_n + H_{n-1} \circ \partial^C$. Just like in topology, we should keep track of these homotopies as additional structure.

- The notion of chain homotopy leads to the notion of a *chain homotopy equivalence* $C_\bullet \to D_\bullet$: a chain map which admits an inverse up to chain homotopy.

- The analogue in homological algebra of the weak homotopy equivalences is played by the *quasi-isomorphisms*: chain maps that induce isomorphisms on all homology groups. There is also an analogue of Whitehead's theorem (Theorem 1.2.2) in this setting: chain map between two *projective* chain complexes is a quasi-isomorphism if and only if it is a chain homotopy equivalence. We may think of the projective chain complexes as the homological analogues of CW-complexes.

- In nice abelian categories, every chain complex $C_\bullet$ admits a *projective resolution*, i.e., a projective chain complex $P_\bullet$ equipped with a quasi-isomorphism to $C_\bullet$. We may think of this as a homological analogue of the CW-approximation theorem (Theorem 1.2.1).

- Instead of working with strict pullbacks and pushouts of chain complexes, one often wants to work with *homotopy* pullbacks/pushouts. For example, the exact sequences in the abelian category are replaced by so-called *distinguished triangles*

$$C_\bullet \xrightarrow{f} D_\bullet \to C(f),$$

  where $C(f)$ is known as the *mapping cone* and is the homological analogue of the homotopy cofiber from Definition 1.1.5.

Example 2.1.3 (Category theory)**.** Two objects X and Y in a category C are considered 'the same' when equipped with an isomorphism $f\colon X \xrightarrow{\sim} Y$. We need to keep track of these isomorphisms in our definitions:

- Two functors $F, G\colon C \to D$ are regarded 'the same' if we specify a natural isomorphism $\eta\colon F \overset{\cong}{\Longrightarrow} G$ between them: the chosen isomorphisms $\eta_X\colon F(X) \xrightarrow{\sim} G(X)$ are required to be compatible with the functors F and G.

- In the theory of monoidal categories, associativity and unitality are not strict equalities but rather natural isomorphisms: for objects X, Y, and Z, we have an associator isomorphism $\alpha_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and unitor isomorphisms $\lambda_X\colon 1 \otimes X \xrightarrow{\sim} X$ and $\rho_X\colon X \otimes 1 \xrightarrow{\sim} X$. These must satisfy coherence conditions like the Mac Lane pentagon.

- Universal properties are typically defined up to unique isomorphism. For instance, a product of objects X and Y is unique only up to unique isomorphism.

We will get more used to the Fundamental Principle throughout the course, and from my point of view it is the most important aspect of ∞-category theory that one needs to learn.

2.2 The language of ∞-category theory

As explained in the previous section, the theory of ∞-categories requires a language that embraces the fundamental principle of homotopy theory. In this section we will introduce such a language.

Remark 2.2.1. The language we introduce should be regarded as some kind of 'meta theory' which mimics the behavior of an *∞-category of ∞-categories*. While we use the words 'axioms', we do not provide a fully formal axiomatization of ∞-category theory; this is not a feasible task for a one-semester course. The main purpose of the axioms is to provide concrete enough rules for the reader to start working with the theory.

2.2.1 Categories, functors and natural isomorphisms

We start with the basic objects of this language:

Axiom A.1. (0) There exists mathematical structures called *∞-categories*, denoted C, D, E, etcetera.

(1) Given ∞-categories C and D, we may speak of *functors* $f\colon C \to D$ between them. Every ∞-category C has an *identity functor* $\mathrm{id}_C\colon C \to C$. Given two functors $f\colon C \to D$ and $g\colon D \to E$, there is a *composite functor* $g \circ f\colon C \to E$.

(2) Given two functors $f, f' \colon C \to D$, we may speak of *natural isomorphisms* $\alpha \colon f \cong f'$. We may sometimes display natural isomorphisms diagrammatically as follows:

$$\begin{tikzcd} C \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=L}] & D. \arrow[from=U, to=L, Rightarrow, "\cong", "\alpha"'] \end{tikzcd}$$

Every functor f has an *identity isomorphism* $\mathrm{id}_f \colon f \cong f$. Every natural isomorphism α has an *inverse isomorphism* $\alpha^{-1} \colon f' \cong f$. Given two natural isomorphisms $\alpha \colon f \cong f'$ and $\beta \colon f' \cong f''$, there is a *(vertical) composite isomorphism* $\beta \circ \alpha \colon f \cong f''$.

To avoid cluttering of notation, we will frequently work with natural isomorphisms $f \cong f'$ that have not been given an explicit name.

(3) Given two natural isomorphisms $\alpha, \alpha' \colon f \cong f'$, we may speak of *isomorphisms of natural isomorphisms* $\alpha \cong \alpha'$, also called 3-*isomorphisms* for short. We may again speak of identity 3-isomorphisms $\alpha \cong \alpha$, of composite 3-isomorphisms $\alpha \cong \alpha''$ and of inverse 3-isomorphisms $\alpha' \cong \alpha$.

(4+) Proceeding in this way, we demand for every natural n a notion of an *n-isomorphism* between two $(n-1)$-isomorphisms, and we may speak of identity n-isomorphisms, of inverse n-isomorphisms, and of composite n-isomorphisms.

We should think of a natural isomorphism $f \cong f'$ between two functors as expressing that f and f' are 'equal'. Similarly two n-isomorphisms are regarded as 'equal' if we are given an $(n+1)$-isomorphism between them.

Axiom A.2. The composition of functors is unital and associative up to isomorphism: for functors $f \colon C \to D$, $g \colon D \to E$ and $h \colon E \to F$, we have natural isomorphisms[2]

$$\lambda_f \colon \mathrm{id}_D \circ f \cong f, \qquad \rho_f \colon f \circ \mathrm{id}_C \cong f \qquad \text{and} \qquad \alpha_{f,g,h} \colon (h \circ g) \circ f \cong h \circ (g \circ f).$$

The composition of (higher) isomorphisms is similarly unital and associative up to isomorphism, and for any natural isomorphism $\alpha \colon f \cong f'$ there are preferred isomorphisms $\alpha^{-1} \circ \alpha \cong \mathrm{id}_f$ and $\alpha \circ \alpha^{-1} \cong \mathrm{id}_{f'}$.

Since we think of isomorpisms as 'equalities', we better make sure that all constructions we do preserve isomorphisms.

Axiom A.3. Composition of functors respects natural isomorphisms: given a natural isomorphism $\alpha \colon f \cong f'$ of functors $C \to D$ and a natural isomorphism $\beta \colon g \cong g'$ of functors $D \to E$, we obtain a new natural isomorphism $\beta * \alpha \colon g \circ f \cong g' \circ f'$ of functors $C \to E$, called the *horizontal composite*.

[2]We have given them names to make clear that the axiom provides preferred choices for such isomorphisms, but in practice we often don't write out the names for these isomorphisms each time.

To be fully precise one needs to write down analogous rules for horizontal composition of n-isomorphisms for all n, and one needs to express their compatibility with vertical composition. We will not go into the details here and instead refer to [Cis+24, Section 1.1.2].

Definition 2.2.2. A *commutative square of functors*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ C' \arrow[r, "f'"'] & D' \end{tikzcd}$$

consists of four functors f, g, h and f' together with a specified natural isomorphism $\alpha\colon h \circ f \cong f' \circ g$. A *commutative triangle of functors*

$$\begin{tikzcd} & D \arrow[dr, "g"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"'] & & E \end{tikzcd}$$

consists of three functors f, g and h together with a specified natural isomorphism $\alpha\colon h \cong g \circ f$. We may similarly define a *commutative square/triangle of n-isomorphisms*.

Definition 2.2.3. A functor $f\colon C \to D$ is said to be an *equivalence* if there exists another functor $g\colon D \to C$ and natural isomorphisms $\mathrm{id}_C \cong gf$ and $\mathrm{id}_D \cong fg$. In this case we call g an *inverse* to f.

Exercise 2.2.4. Show that a functor f is an equivalence if and only if it admits both a section (i.e., a g such that $\mathrm{id}_D \cong fg$) and a retraction (i.e. an h such that $\mathrm{id}_C \cong hf$), and that in this case we have $g \cong h$.

Notation 2.2.5. It follows immediately from the previous exercise that if g and h are both inverses to f, then there is an isomorphism $g \cong h$. So up to isomorphism there is a unique inverse to f, which we will denote by $f^{-1}\colon D \to C$.

Exercise 2.2.6. Show that if $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.

Lemma 2.2.7. *Equivalences satisfy the 2-out-of-3 property: given functors $f\colon C \to D$ and $g\colon D \to E$, if two of the three functors f, g and gf are equivalences, then so is the third.*

Proof. If g and f are equivalences, then so is gf by Exercise 2.2.6. If f and gf are equivalences, then we may rewrite g as the composite $g \cong g \circ \mathrm{id}_D \cong g \circ f \circ f^{-1} = gf \circ f^{-1}$, so that g is the composite of two equivalences and hence itself an equivalence. A similar argument shows that f is an equivalence whenever g and gf are equivalences. $\square$

Exercise 2.2.8. Show that equivalences of ∞-categories satisfy the 2-out-of-6 property: given functors $f: C \to D$, $g: D \to E$ and $h: E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.

2.2.2 Objects, morphisms and commutative triangles

Constructions from ordinary category may be imported to the setting of ∞-categories:[3]

Axiom C. Every ordinary category is an ∞-category. Every ordinary functor $f: C \to D$ is a functor of ∞-categories, and every functor arises in this way. Every ordinary natural isomorphism $\alpha: f \xrightarrow{\cong} f'$ is a natural isomorphism of functors of ∞-categories, and every natural isomorphism arises in this way. This process preserves composition of functors and natural isomorphisms.

The assumption that ordinary categories are ∞-categories provides a convenient way to define objects, morphisms and commutative triangles in an ∞-category C.

Notation 2.2.9. For $n \geq 0$, we write $[n]$ for the following partially ordered set:

$$[n] := \{0 \leq 1 \leq \cdots \leq n\}.$$

For $n > 1$ and $0 \leq i \leq n$, we define the *face map* $d_i: [n-1] \to [n]$ as the unique injective map of posets whose image leaves out i. Similarly we define the *degeneracy map* $s_i: [n+1] \to [n]$ as the unique surjective map of posets that hits i twice.

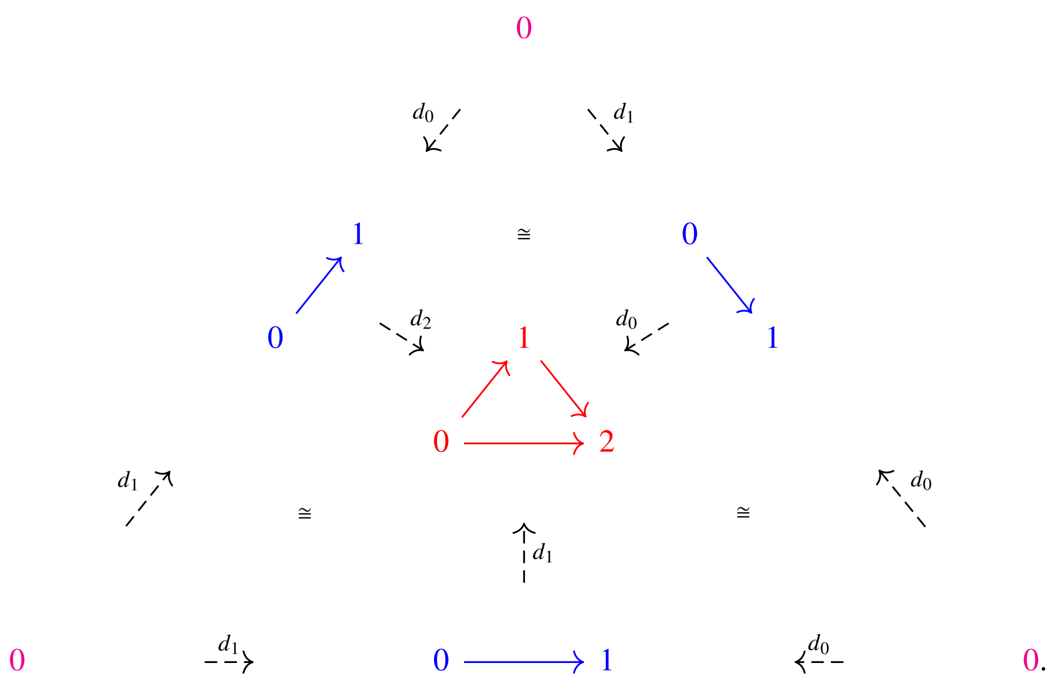
Recall that every partially ordered set $(P, \leq)$ can be turned into a category: given $p, q \in P$ we add a unique morphism $p \to q$ between them whenever $p \leq q$. The resulting categories $[0]$, $[1]$ and $[2]$ may be displayed as follows:

$$0, \qquad 0 \longrightarrow 1 \qquad \text{and} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd}$$

The five face maps $d_0, d_1: [0] \to [1]$ and $d_0, d_1, d_2: [1] \to [2]$ and the relations between

[3] In a fully axiomatic approach, as we do in [Cis+24], one would not introduce such an axiom. In this course I wish to take a hybrid approach, since developing the full axiomatic theory would take too long.

them may be displayed as follows:



The two degeneracy maps $s_0, s_1\colon [2] \to [1]$ may be interpreted the following 'projections':

$$
\begin{tikzcd}
0 \arrow[r] \arrow[dr] & 1 \arrow[d] & & 0 \arrow[d] & & & 0 \arrow[d] \arrow[dl] \\
& 2 \arrow[rr, phantom, "\overset{s_0}{\dashrightarrow}" description] & & 1 & & 2 \arrow[ll, phantom, "\overset{s_1}{\dashleftarrow}" description] & 1. \arrow[l]
\end{tikzcd}
$$

Definition 2.2.10. Let C be an ∞-category.

(1) An *object of C* is a functor $x\colon [0] \to C$;

(2) A *morphism of C* is a functor $f\colon [1] \to C$. We write $f(0)$ for its *source/domain* and $f(1)$ for its *target/codomain*, which are defined as the following composites:

$$f(0)\colon [0] \xrightarrow{d_1} [1] \xrightarrow{f} C \qquad \text{and} \qquad f(1)\colon [0] \xrightarrow{d_0} [1] \xrightarrow{f} C.$$

We will also write $f\colon x \to y$ to indicate that f comes with isomorphisms $x \cong f(0)$ and $y \cong f(1)$.

(3) A *commutative triangle in C* is a functor $\sigma\colon [2] \to C$. We will frequently denote commutative triangles by

$$
\sigma \quad = \quad
\begin{tikzcd}
& y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & & z,
\end{tikzcd}
$$

where $f = \sigma \circ d_2$, $g = \sigma \circ d_0$ and $h = \sigma \circ d_1$. We will often leave commutative triangles unnamed, and only label their three edges.

Warning 2.2.11. While we may speak of *objects of* C, we may not speak of the *set of objects of* C! This is similar to how we may speak of points of a space X, but we may not speak of the *set of points*, cf. Convention 1.2.3.

Remark 2.2.12. We want to think of a commutative triangle σ as saying that h is a composite of f and g. Note that at this stage we do not yet say anything about the existence nor the uniqueness of composites; we will get back to this in Section 2.4.

Example 2.2.13. Given an object x of C, we define the *identity morphism* $\mathrm{id}_x\colon x \to x$ as the composite $[1] \to [0] \xrightarrow{x} C$. Since the composites

$$[0] \xrightarrow{d_1} [1] \xrightarrow{s_0} [0] \qquad \text{and} \qquad [0] \xrightarrow{d_0} [1] \xrightarrow{s_0} [0]$$

are naturally isomorphic to the identity functor $\mathrm{id}_{[0]}$, we see that the source and target of id_x are indeed isomorphic to x.

Exercise 2.2.14. Construct for every morphism $f\colon x \to y$ commutative triangles in C of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

After having composition of morphisms available, the exercise will imply the expected relations $f \simeq f \circ \mathrm{id}_x$ and $f \simeq \mathrm{id}_y \circ f$, see Lemma 2.4.7.

2.3 Basic constructions of ∞-categories

There are various ways to build new ∞-categories out of old ones: we may form products, coproducts, pullbacks and functor categories. We will introduce each of these constructions via the universal properties they are supposed to satisfy.

Remark 2.3.1. In our meta theory we only impose non-coherent versions of these universal properties. We will see in Proposition 2.3.34 below that this is good enough: internally the theory 'believes' that coherent forms of the universal properties are in fact satisfied.

2.3.1 Products and coproducts

We start by introducing the terminal and initial ∞-categories and products/coproducts of ∞-categories.

Axiom B.1 (Terminal ∞-category)**.** The category $[0]$ is the *terminal ∞-category*: For every ∞-category C there is a functor $p_C \colon C \to [0]$, and for any two functors $f, f' \colon C \to [0]$ there is a natural isomorphism $f \cong f'$.

Notation 2.3.2. We will frequently also write $*$ for $[0]$.

Definition 2.3.3. Let C and D be ∞-categories and let x be an object of D. We define the *constant functor* $\mathrm{const}_x \colon C \to D$ as the composite $x \circ p_C \colon C \to D$:

$$\mathrm{const}_x : C \xrightarrow{p_C} [0] \xrightarrow{x} D.$$

Definition 2.3.4 (Contractible category)**.** An ∞-category C is called *contractible* if the functor $p_C \colon C \to *$ is an equivalence.

Remark 2.3.5. Observe that an ∞-category C is contractible if and only if there exists an absolute object $x \colon * \to C$ such that the composite $\mathrm{const}_x \colon C \to C$ is naturally isomorphic to the identity $\mathrm{id}_C \colon C \to C$. A natural isomorphism $H \colon \mathrm{const}_x \cong \mathrm{id}_C$ is called a *contraction of C onto x*.

Axiom B.2 (Initial ∞-category)**.** The empty category $\emptyset$ is the *initial ∞-category*: For every ∞-category C there is a functor $\emptyset \to C$, and for any two functors $f, f' \colon \emptyset \to C$ there is a natural isomorphism $f \cong f'$.

Axiom B.3 (Product of ∞-categories)**.** The *product* of two ∞-categories C and D is an ∞-category $C \times D$ equipped with two functors $\mathrm{pr}_C \colon C \times D \to C$ and $\mathrm{pr}_D \colon C \times D \to D$ called the *projection functors*. Given functors $f \colon T \to C$ and $g \colon T \to D$, we obtain a functor $(f,g) \colon T \to C \times D$ equipped with natural isomorphisms $\mathrm{pr}_C \circ (f,g) \cong f$ and $\mathrm{pr}_D \circ (f,g) \cong g$. Given two functors $h, k \colon T \to C \times D$ and two natural isomorphisms $\alpha \colon \mathrm{pr}_C \circ h \cong \mathrm{pr}_C \circ k$ and $\beta \colon \mathrm{pr}_D \circ h \cong \mathrm{pr}_D \circ k$, we obtain a natural isomorphism $(\alpha,\beta) \colon h \cong k$ satisfying $\mathrm{pr}_C \circ (\alpha,\beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha,\beta) \cong \beta$.

Remark 2.3.6. Taking $T = *$ we see that objects of $C \times D$ have the form (x,y), where x is an object of C and y is an object of D.

Axiom B.4 (Coproduct of categories)**.** The *coproduct* (or *disjoint union*) of two ∞-categories C and D is an ∞-category $C \sqcup D$ equipped with two functors $i_C \colon C \to C \sqcup D$ and $i_D \colon D \to C \sqcup D$ called the *inclusion functors*. Given functors $f \colon C \to E$ and $g \colon D \to E$, we obtain a functor $\langle f,g \rangle \colon C \sqcup D \to E$ equipped with natural isomorphisms $\langle f,g \rangle \circ i_C \cong f$ and $\langle f,g \rangle \circ i_D \cong g$. Given two functors $h, k \colon C \sqcup D \to E$ and two natural isomorphisms $\alpha \colon h \circ i_C \cong k \circ i_C$ and $\beta \colon h \circ i_D \cong k \circ i_D$, we obtain a natural isomorphism $\langle \alpha,\beta \rangle \colon h \cong k$ satisfying $\langle \alpha,\beta \rangle \circ i_C \cong \alpha$ and $\langle \alpha,\beta \rangle \circ i_D \cong \beta$.

From these simple rules, we may deduce that products and coproducts are commutative, unital and associative. For commutativity of products, one argues as follows:

Lemma 2.3.7. *For ∞-categories C and D, the functor* $(\mathrm{pr}_D, \mathrm{pr}_C)\colon C\times D \to D\times C$ *is an equivalence, with inverse given by* $(\mathrm{pr}_C, \mathrm{pr}_D)\colon D\times C \to C\times D$.

Proof. We need to provide natural isomorphisms $(\mathrm{pr}_C, \mathrm{pr}_D)\circ(\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{id}_{C\times D}$ and $(\mathrm{pr}_D, \mathrm{pr}_C)\circ(\mathrm{pr}_C, \mathrm{pr}_D) \cong \mathrm{id}_{D\times C}$. By symmetry, it will suffice to produce the first natural isomorphism. For this, it will in turn suffice to produce natural isomorphisms after composing with the functors pr_C and pr_D. For the composition with pr_C we compute that

$$\mathrm{pr}_C\circ((\mathrm{pr}_C, \mathrm{pr}_D)\circ(\mathrm{pr}_D, \mathrm{pr}_C)) \cong (\mathrm{pr}_C\circ(\mathrm{pr}_C, \mathrm{pr}_D))\circ(\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C\circ(\mathrm{pr}_D, \mathrm{pr}_C) \cong \mathrm{pr}_C,$$

which is indeed isomorphic to $\mathrm{pr}_C\circ\mathrm{id}_{C\times D}$. The case for composition with pr_D is analogous. This finishes the proof. □

Unitality may be argued similarly:

Lemma 2.3.8. *For every ∞-category C, the projection functor* $\mathrm{pr}_C\colon C\times * \to C$ *is an equivalence, with inverse given by the functor* $(\mathrm{id}_C, p_C)\colon C \to C\times *$.

Proof. The composite $\mathrm{pr}_C\circ(\mathrm{id}_C, p_C)\colon C\to C$ is isomorphic to id_C by the defining property of (id_C, p_C). To show that the composite $(\mathrm{id}_C, p_C)\circ\mathrm{pr}_C\colon C\times *\to C\times *$ is isomorphic to $\mathrm{id}_{C\times *}$, it will suffice to do so after composing with pr_C and with pr_*. The case for pr_* is clear, since any two functors $C\times * \to *$ are isomorphic. In the case of pr_C we compute

$$\mathrm{pr}_C\circ((\mathrm{id}_C, p_C)\circ\mathrm{pr}_C) \cong (\mathrm{pr}_C\circ(\mathrm{id}_C, p_C))\circ\mathrm{pr}_C \cong \mathrm{id}_C\circ\mathrm{pr}_C \cong \mathrm{pr}_C \cong \mathrm{pr}_C\circ\mathrm{id}_{C\times *},$$

where the first isomorphism is associativity, the second is the relation proved in the first paragraph, and the third and fourth are unitality. This finishes the proof. □

Exercise 2.3.9. Formulate and prove the associativity of the product of ∞-categories.

Exercise 2.3.10. Show that the coproduct of ∞-categories is associative, commutative and unital:

$$(C\sqcup D)\sqcup E \xrightarrow{\sim} C\sqcup(D\sqcup E), \qquad C\sqcup D \xrightarrow{\sim} D\sqcup C, \qquad \emptyset\sqcup C \xrightarrow{\sim} C \xrightarrow{\sim} C\sqcup\emptyset.$$

Construction 2.3.11. The formation of products and coproducts of ∞-categories is functorial: given two functors $f\colon C\to C'$ and $g\colon D\to D'$, we obtain new functors

$$\begin{aligned} f\times g &:= (f\circ\mathrm{pr}_C, g\circ\mathrm{pr}_D)\colon C\times D\to C'\times D' \\ f\sqcup g &:= \langle i_{C'}\circ f, i_{D'}\circ g\rangle\colon C\sqcup D\to C'\sqcup D'. \end{aligned}$$

We leave it to the reader to verify that there are natural isomorphisms

$$\begin{aligned} &\mathrm{id}_C\times\mathrm{id}_D \cong \mathrm{id}_{C\times D} \quad \text{and} \quad (f'\times g')\circ(f\times g) \cong (f'\circ f)\times(g'\circ g), \\ &\mathrm{id}_C\sqcup\mathrm{id}_D \cong \mathrm{id}_{C\sqcup D} \quad \text{and} \quad (f'\sqcup g')\circ(f\sqcup g) \cong (f'\circ f)\sqcup(g'\circ g). \end{aligned}$$

for functors $f\colon C\to C'$, $f'\colon C'\to C''$, $g\colon D\to D'$ and $g'\colon D'\to D''$.

2.3.2 Pullbacks of ∞-categories

We will now axiomatize the *pullback* of two functors $f\colon C \to E$ and $g\colon D \to E$ of ∞-categories. This will be similar to the axiom for the product $C \times D$, but it is more involved due to the fact that we need to record compatibilities with the structure maps to E.

Axiom B.5 (Pullbacks of ∞-categories)**.** Consider two functors $f\colon C \to E$ and $g\colon D \to E$. The *pullback* of f and g, also called the *fiber product of C and D over E*, is an ∞-category $C \times_E D$. It comes equipped with two functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$, or in other words, a commutative square

$$\begin{tikzcd} C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"] & E. \end{tikzcd}$$

Given functors $t_C\colon T \to C$ and $t_D\colon T \to D$ equipped with a natural isomorphism $\alpha\colon f \circ t_C \cong g \circ t_D$, we obtain a functor $t = (t_C, t_D)\colon T \to C \times_E D$. This functor comes equipped with natural isomorphisms $\mathrm{pr}_C \circ t \cong t_C$ and $\mathrm{pr}_D \circ t \cong t_D$. Furthermore, the composite isomorphism

$$f \circ t_C \cong f \circ \mathrm{pr}_C \circ t \cong g \circ \mathrm{pr}_D \circ t \cong g \circ t_D$$

is isomorphic to α. We summarize this with the following diagram:

$$\begin{tikzcd} T \arrow[drr, bend left, "t_C"] \arrow[dr, dashed, "{(t_C,t_D)}"] \arrow[ddr, bend right, "t_D"'] & & \\ & C \times_E D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_D"'] & C \arrow[d, "f"] \\ & D \arrow[r, "g"] & E. \end{tikzcd}$$

Consider now two functors $t, t'\colon T \to C \times_E D$, and assume we are given two natural isomorphisms $\alpha\colon \mathrm{pr}_C \circ t \cong \mathrm{pr}_C \circ t'$ and $\beta\colon \mathrm{pr}_D \circ t \cong \mathrm{pr}_D \circ t'$ such that the following square of natural isomorphisms commutes:

$$\begin{tikzcd} f \circ \mathrm{pr}_C \circ t \arrow[r, "f \circ \alpha", "\cong"'] \arrow[d, "\cong"'] & f \circ \mathrm{pr}_C \circ t' \arrow[d, "\cong"] \\ g \circ \mathrm{pr}_D \circ t \arrow[r, "g \circ \beta", "\cong"'] & g \circ \mathrm{pr}_D \circ t'. \end{tikzcd}$$

Then we obtain a natural isomorphism $(\alpha, \beta)\colon t \cong t'$ satisfying $\mathrm{pr}_C \circ (\alpha, \beta) \cong \alpha$ and $\mathrm{pr}_D \circ (\alpha, \beta) \cong \beta$, and inducing an isomorphic isomorphism of natural isomorphisms in the previous square.

Remark 2.3.12. We refer to [Cis+24, Axiom B.5] for a precise formulation of the last condition.

Remark 2.3.13. While Axiom B.5 may look somewhat complicated, let us point out that it has the same form as that of Axioms B.1 and B.3:

- An ∞-category X is introduced, together with certain additional structure;
- For any other ∞-category T equipped with the same structure, we are given a functor $T \to X$ that is compatible with this structure;
- If we have *two* functors $T \to X$ that are compatible with this structure, then we are given a natural isomorphism between them that is compatible with this structure.

The Axioms B.2 and B.4 have a similar form, except that they talk about functors *out of* X.

Exercise 2.3.14. Show that the fiber product is commutative, associative and unital: for functors $C \to E$, $D \to E$ and $B \to E$, there are preferred equivalences

$$C \times_E D \xrightarrow{\sim} D \times_E C, \qquad B \times_E (C \times_E D) \xrightarrow{\sim} (B \times_E C) \times_E D, \qquad C \times_E E \xrightarrow{\sim} C.$$

Construction 2.3.15. The construction of pullbacks of ∞-categories is functorial: given a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi"'] & E \arrow[d, "\chi"] & D \arrow[l, "g"'] \arrow[d, "\psi"] \\ C' \arrow[r, "f'"'] & E' & D' \arrow[l, "g'"] \end{tikzcd}$$

there is an induced functor

$$\varphi \times_\chi \psi := (\varphi \circ \mathrm{pr}_C, \psi \circ \mathrm{pr}_D) \colon C \times_E D \to C' \times_{E'} D'.$$

Exercise 2.3.16. In the above situation, show that if each of the functors φ, ψ and χ is an equivalence, then so is $\varphi \times_\chi \psi$. (This is essentially identical to Exercise 1.2 on Sheet 1.)

Corollary 2.3.17. *Equivalences are closed under pullback: if $g \colon D \to E$ is an equivalence and $f \colon C \to E$ is an arbitrary functor, then also $C \times_E D \to C$ is an equivalence.*

Proof. This is a special case of Exercise 2.3.16 by taking $C' = C$, $E' = D' = E$, $\varphi = \mathrm{id}_C$, $\chi = \mathrm{id}_E$ and $\psi = g$. $\square$

Pullbacks of categories lead to the notion of a *fiber* of a functor:

Definition 2.3.18. Let $f\colon C \to D$ be a functor. For every object x of D, we define the *fiber* of f over x as the pullback

$$\begin{tikzcd} C_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ * \arrow[r, "x"] & D. \end{tikzcd}$$

Pullback squares

The definition of pullbacks of ∞-categories naturally leads to the notion of a pullback square.

Definition 2.3.19 (Pullback square)**.** A commutative square

$$\begin{tikzcd} T \arrow[r, "t"] \arrow[d, "s"'] & C \arrow[d, "f"] \\ D \arrow[r, "g"'] & E \end{tikzcd}$$

is called a *pullback square* if the induced functor $(s,t)\colon T \to C \times_E D$ is an equivalence.

Lemma 2.3.20 (Pasting lemma for pullback squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

(1) There is a preferred[4] *equivalence $D_1 \times_{D_3} C_3 \xrightarrow{\sim} D_1 \times_{D_2} (D_2 \times_{D_3} C_3)$.*

(2) If the right-hand square is a pullback square, then the left-hand square is a pullback square if and only if the outer rectangle is a pullback square.

Proof. For part (1), we define the functor as $(\mathrm{pr}_{D_1}, (h_1 \circ \mathrm{pr}_{D_1}, \mathrm{pr}_{C_3}))$. We claim that an inverse is given by the functor

$$(\mathrm{pr}_{D_1}, \mathrm{pr}_{C_3} \circ \mathrm{pr}_{D_2 \times_{D_3} C_3})\colon D_1 \times_{D_2} (D_2 \times_{D_3} C_3) \to D_1 \times_{D_3} C_3.$$

To show that the two composites are isomorphic to the respective identities, it suffices to construct these isomorphisms after projecting to the two components and verifying that these isomorphisms agree with composed with the functors to D_3. But after unwinding

[4]Whenever we use the word 'preferred', this is meant to indicate that the real content of the statement is not just that some equivalence/isomorphism exists, but that there is a specific choice of such an equivalence we are interested in, to be constructed in the proof.

the definitions, this holds tautologically true by the very construction of the two functors in both directions. We leave the details to the reader.

Part (2) is an immediate consequence of the 2-out-of-3 property from Lemma 2.2.7 applied to the following commutative square:

$$\begin{tikzcd} C_1 \arrow[r, "(f_1,g_1)"] \arrow[d, "(f_1,g_2g_1)"'] & D_1 \times_{D_2} C_2 \arrow[d, "\simeq"] \\ D_1 \times_{D_3} C_3 \arrow[r, "\simeq"] & D_1 \times_{D_2} (D_2 \times_{D_3} C_3), \end{tikzcd}$$

where the right vertical map is an equivalence by applying Exercise 2.3.16 to the equivalence $C_2 \xrightarrow{\sim} D_2 \times_{D_3} C_3$. □

Lemma 2.3.21. *Consider a commutative square*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ E \arrow[r, "k"] & F \end{tikzcd}$$

and assume that h is an equivalence. Then g is an equivalence if and only if the square is a pullback square.

Proof. Since h is an equivalence, also the projection map $\mathrm{pr}_E\colon D \times_F E \to E$ is an equivalence by Corollary 2.3.17. Since the composite of the map $(f,g)\colon C \to D \times_F E$ with pr_E is isomorphic to g, the claim follows from the 2-out-of-3 property of equivalences, Lemma 2.2.7. □

2.3.3 Universality of coproducts

So far the axioms for products and coproducts of ∞-categories are symmetric: all statements for products have dual analogues for coproducts. We will now introduce the *universality of coproducts* which breaks this symmetry:

Axiom B.2' (Universality of initial category)**.** The initial category is *strictly* initial: every functor $C \to \emptyset$ is an equivalence.

Axiom B.4' (Universality of coproducts)**.** (1) For functors $f\colon C' \to C$ and $g\colon D' \to D$, the commutative squares

$$\begin{tikzcd} C' \arrow[r, "i_{C'}"] \arrow[d, "f"'] & C' \sqcup D' \arrow[d, "f \sqcup g"] \\ C \arrow[r, "i_C"] & C \sqcup D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D' \arrow[r, "i_{D'}"] \arrow[d, "g"'] & C' \sqcup D' \arrow[d, "f \sqcup g"] \\ D \arrow[r, "i_D"] & C \sqcup D \end{tikzcd}$$

are pullback squares.

(2) For a functor $h\colon E \to C \sqcup D$, the functor

$$\langle \mathrm{pr}_E, \mathrm{pr}_E\rangle\colon (E \times_{C\sqcup D} C) \sqcup (E \times_{C\sqcup D} D) \to E$$

is an equivalence.

Remark 2.3.22. To make sense of these two axioms, we may think of them as the categorical versions of the following two facts about sets:

(1) If a set S admits a map $S \to \emptyset$ to the empty set, then it must itself be empty;

(2) If a set S admits a map $S \to T_0 \sqcup T_1$ to a disjoint union of two sets, then we may write S as the disjoint union of the preimages of T_0 and T_1.

Lemma 2.3.23. *For functors $f\colon C \to \Gamma$, $g\colon D \to \Gamma$ and $\varphi\colon \Gamma' \to \Gamma$, the commutative square*

$$\begin{tikzcd}
(C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \arrow[r] \arrow[d] & C \sqcup D \arrow[d, "\langle f, g\rangle"] \\
\Gamma' \arrow[r, "\varphi"] & \Gamma
\end{tikzcd}$$

is a pullback square.

Proof. We need to show that the induced functor

$$(C \times_\Gamma \Gamma') \sqcup (D \times_\Gamma \Gamma') \to (C \sqcup D) \times_\Gamma \Gamma'$$

is an equivalence. But this is a special case of Axiom B.4' applied to the functor $f = \mathrm{pr}_1\colon (C \sqcup D) \times_\Gamma \Gamma' \to C \sqcup D$: by the pasting law of pullback squares there are equivalences

$$C \times_{C\sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq C \times_\Gamma \Gamma' \qquad \text{and} \qquad D \times_{C\sqcup D} (C \sqcup D) \times_\Gamma \Gamma' \simeq D \times_\Gamma \Gamma'.$$

This finishes the proof. □

Corollary 2.3.24 (Distributivity)**.** *For ∞-categories C, D and E, the functor*

$$\langle E \times i_C, E \times i_D\rangle\colon E \times C \sqcup E \times D \to E \times (C \sqcup D)$$

is an equivalence.

Proof. This is an instance of the previous lemma by taking $\Gamma = *$, $\Gamma' = E$, $f = p_C\colon C \to *$, $g = p_D\colon D \to *$, and $\varphi = p_E\colon E \to *$. □

2.3.4 Functor categories

Given two ∞-categories C and D, we are interested in forming the *functor category* $\mathrm{Fun}(C,D)$ whose objects are precisely the functors from C to D.

Axiom B.6 (Functor category)**.** The *functor category* between two ∞-categories C and D is an ∞-category denoted by $\mathrm{Fun}(C,D)$. Given another ∞-category E, we obtain for every functor $f\colon E\times C\to D$ a functor $f_c\colon E\to\mathrm{Fun}(C,D)$ obtained from f by *currying*. Conversely, there is for every functor $g\colon E\to\mathrm{Fun}(C,D)$ a functor $g^u\colon E\times C\to D$ obtained by *uncurrying*. These operations come equipped with natural isomorphisms $f\cong(f_c)^u$ and $g\cong(g^u)_c$. In other words, currying and uncurrying determine a 'one-to-one correspondence' between functors $E\to\mathrm{Fun}(C,D)$ and functors $E\times C\to D$.

The operations of currying and uncurrying respect natural isomorphisms: given a natural isomorphism $\alpha\colon f\cong f'$ of functors $f\colon E\times C\to D$, we obtain a natural isomorphism $\alpha^c\colon f_c\cong f'_c$ between their curried functors. Conversely, an isomorphism $\beta\colon g\cong g'$ gives $\beta^u\colon g^u\cong(g')^u$. Again, we demand that[5] $(\alpha_c)^u\cong\alpha$ and $(\beta^u)_c\cong\beta$.

By considering the case of the terminal category $E=*$, we observe that objects $*\to\mathrm{Fun}(C,D)$ of the functor category correspond to functors $C\simeq C\times *\to D$ from C to D, justifying the notation.

Axiom B.7 (Functoriality uncurrying in E)**.** The operation of uncurrying is assumed to be functorial in E. More precisely, given functors $g\colon E\to\mathrm{Fun}(C,D)$ and $h\colon E'\to E$, then the uncurrying of the composite $g\circ h\colon E'\to\mathrm{Fun}(C,D)$ comes with a preferred isomorphism to the composite $E'\times C\xrightarrow{h\times\mathrm{id}_C}E\times C\xrightarrow{g^u}D$. Similarly, given an isomorphism $\beta\colon g\cong g'$, the uncurrying of the isomorphism $\beta\circ h\colon g\circ h\cong g'\circ h$ is isomorphic to $\beta^u\circ(h\times\mathrm{id}_C)\colon g^u\circ(h\times\mathrm{id}_C)\cong(g')^u\circ(h\times\mathrm{id}_C)$.

Exercise 2.3.25. Show that also the operation of currying is functorial in E: given functors $f\colon E\times C\to D$ and $h\colon E'\to E$, there is a preferred isomorphism between the currying of the functor $E'\times C\xrightarrow{h\times\mathrm{id}_C}E\times C\xrightarrow{f}D$ and the composite $E'\xrightarrow{h}E\xrightarrow{f_c}\mathrm{Fun}(C,D)$.

Construction 2.3.26 (Evaluation functor)**.** Given ∞-categories C and D, the *evaluation functor* $\mathrm{ev}\colon\mathrm{Fun}(C,D)\times C\to D$ is defined as $\mathrm{ev}:=(\mathrm{id}_{\mathrm{Fun}(C,D)})^u$, i.e. as the functor obtained by uncurrying the identity functor $\mathrm{id}_{\mathrm{Fun}(C,D)}\colon\mathrm{Fun}(C,D)\to\mathrm{Fun}(C,D)$. Given an object $x\in C$, we write $\mathrm{ev}_x\colon\mathrm{Fun}(C,D)\to D$ for the composite

$$\mathrm{ev}_x\colon\mathrm{Fun}(C,D)\simeq\mathrm{Fun}(C,D)\times *\xrightarrow{\mathrm{id}\times x}\mathrm{Fun}(C,D)\times C\xrightarrow{\mathrm{ev}}D,$$

and refer to it as the *evaluation functor at x*.

[5]These isomorphisms should really be interpreted as commutative squares of natural isomorphisms.

Definition 2.3.27 (Arrow category)**.** Let C be an ∞-category. We refer to the functor category $\mathrm{Fun}([1],C)$ as the *arrow category* of C. Observe that objects of $\mathrm{Fun}([1],C)$ are precisely morphisms ('arrows') in C.

Definition 2.3.28 (Hom anima)**.** Let C be any ∞-category. Given objects x and y of C, we define the *hom anima* $\mathrm{Hom}_C(x,y)$ via the following pullback square:

$$\begin{tikzcd} \mathrm{Hom}_C(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ * \arrow[r, "(x,y)"] & C \times C. \end{tikzcd}$$

Observe that the objects of $\mathrm{Hom}_C(x,y)$ are triples (f,α,β), where $f\colon x' \to y'$ is a morphism in C and $\alpha\colon x \cong x'$ and $\beta\colon y \cong y'$ are isomorphisms in C. We will often abuse notation and pretend that $x = x'$ and $y = y'$.

The use of the word 'anima' will be explained below, see Exercise 2.6.3.

Exercise 2.3.29. Let C and D be ∞-categories and let $x,x' \in C$ and $y,y' \in D$ be objects. Produce an equivalence

$$\mathrm{Hom}_{C\times D}((x,y),(x',y')) \simeq \mathrm{Hom}_C(x,x') \times \mathrm{Hom}_D(y,y').$$

Exercise 2.3.30. Let $f\colon C \to E$ and $g\colon D \to E$ be functors, and let $x,x' \in C \times_E D$ be objects. Show that there is a pullback square

$$\begin{tikzcd} \mathrm{Hom}_{C\times_E D}(x,x') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_C(\mathrm{pr}_C(x),\mathrm{pr}_C(x')) \arrow[d] \\ \mathrm{Hom}_D(\mathrm{pr}_D(x),\mathrm{pr}_D(x')) \arrow[r] & \mathrm{Hom}_E(f\,\mathrm{pr}_C(x), f\,\mathrm{pr}_C(x')). \end{tikzcd}$$

Definition 2.3.31 (Natural transformations)**.** If C and D are ∞-categories, we define a *natural transformation* of functors $C \to D$ to be a morphism in $\mathrm{Fun}(C,D)$. By (un)currying, this may equivalently be encoded as a functor $[1] \times C \to D$. We denote the hom anima in $\mathrm{Fun}(C,D)$ by $\mathrm{Nat}(f,g)$:

$$\mathrm{Nat}(f,g) := \mathrm{Hom}_{\mathrm{Fun}(C,D)}(f,g).$$

The functoriality of the product naturally provides functoriality for the functor categories:

Construction 2.3.32 (Functoriality of functor categories in C and D)**.** Given a functor $g\colon D \to D'$, we define the functor $g \circ -\colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D')$ as the currying of the composite

$$\mathrm{Fun}(C,D) \times C \xrightarrow{\mathrm{ev}} D \xrightarrow{g} D'.$$

Similarly, given a functor $f\colon C' \to C$, we define the functor $-\circ f\colon \operatorname{Fun}(C',D) \to \operatorname{Fun}(C,D)$ as the currying of the composite

$$\operatorname{Fun}(C',D)\times C \xrightarrow{\mathrm{id}\times f} \operatorname{Fun}(C',D)\times C' \xrightarrow{\mathrm{ev}} D.$$

In a completely analogous way, every natural isomorphism $\beta\colon g \cong g'$ of functors $D \to D'$ induces a natural isomorphism $(\alpha\circ -)\colon (g\circ -) \cong (g'\circ -)$ of functors $\operatorname{Fun}(C,D) \to \operatorname{Fun}(C,D')$, and similarly for the construction $-\circ f$.

Alternative notations for these functors that we will frequently use are g_* and f^*:

$$g_* := g\circ -\colon \operatorname{Fun}(C,D) \to \operatorname{Fun}(C,D'), \qquad f^* := -\circ f\colon \operatorname{Fun}(C',D) \to \operatorname{Fun}(C,D).$$

Exercise 2.3.33. Formulate and prove that the assignments $g \mapsto (g\circ -)$ and $f \mapsto (-\circ f)$ are functorial, in the sense that they respect identity functors and composition of functors.

Proposition 2.3.34. *Let T, C, D and E be ∞-categories, and let $f\colon C \to E$ and $g\colon D \to E$ be functors.*

(1) The functor $p_{\operatorname{Fun}(T,*)}\colon \operatorname{Fun}(T,*) \to *$ is an equivalence.

(2) The functor

$$(\mathrm{pr}_C\circ -, \mathrm{pr}_D\circ -)\colon \operatorname{Fun}(T,C\times D) \to \operatorname{Fun}(T,C)\times\operatorname{Fun}(T,D)$$

is an equivalence.

(3) The functor

$$(\mathrm{pr}_C\circ -, \mathrm{pr}_D\circ -)\colon \operatorname{Fun}(T,C\times_E D) \to \operatorname{Fun}(T,C)\times_{\operatorname{Fun}(T,E)}\operatorname{Fun}(T,D)$$

induced by the isomorphism $(f\circ -)\circ(\mathrm{pr}_C\circ -) \cong (g\circ -)\circ(\mathrm{pr}_D\circ -)$ is an equivalence.

(4) The functor $\operatorname{Fun}(\emptyset,T) \to *$ is an equivalence.

(5) The functor

$$(-\circ i_C,\ -\circ i_D)\colon \operatorname{Fun}(C\sqcup D,T) \to \operatorname{Fun}(C,T)\times\operatorname{Fun}(D,T)$$

is an equivalence.

(6) The functor $\mathrm{ev}_*\colon \operatorname{Fun}(*,C) \to C$ is an equivalence.

(7) The functor

$$\operatorname{Fun}(T,\operatorname{Fun}(C,D)) \to \operatorname{Fun}(T\times C,D)$$

obtained by currying the composite

$$\operatorname{Fun}(T,\operatorname{Fun}(C,D))\times T\times C \xrightarrow{\mathrm{ev}\times\mathrm{id}_C} \operatorname{Fun}(C,D)\times C \xrightarrow{\mathrm{ev}} D.$$

is an equivalence.

Proof. Each of these claims is a direct (though somewhat tedious) consequence of the universal properties of all the constructions involved. Because of time reasons, we will not prove most of these claims and instead refer the reader to [Cis+24, Section 1.4]. But to give just a flavor of the proof, let us show how to prove (1).

We will show that an inverse to the functor $\mathrm{Fun}(T,*) \to *$ is given by the functor $(p_{*\times T})_c \colon * \to \mathrm{Fun}(T,*)$, obtained by currying the functor $p_{*\times T} \colon * \times T \to *$. The composite $p_{\mathrm{Fun}(T,*)} \circ (p_{*\times T})_c \colon * \to *$ is automatically isomorphic to $\mathrm{id}_* \colon * \to *$, hence it remains to show that the composite $(p_{*\times T})_c \circ p_{\mathrm{Fun}(T,*)} \colon \mathrm{Fun}(T,*) \to \mathrm{Fun}(T,*)$ is isomorphic to $\mathrm{id}_{\mathrm{Fun}(T,*)}$. It will suffice to prove that the two uncurried functors $\mathrm{Fun}(T,*) \times T \to *$ are isomorphic. But this is again automatic by the universal property of $*$. □

2.3.5 Pushouts of ∞-categories

Using functor categories, we may dualize the definition of a pullback square of ∞-categories:

Definition 2.3.35 (Pushout square)**.** A commutative square of ∞-categories

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ D' \arrow[r, "v"] & D \end{tikzcd}$$

is called a *pushout square* (or *cocartesian*) if, for every ∞-category E, the induced square

$$\begin{tikzcd} \mathrm{Fun}(D,E) \arrow[r, "v^*"] \arrow[d, "f^*"'] & \mathrm{Fun}(D',C) \arrow[d, "(f')^*"] \\ \mathrm{Fun}(C,E) \arrow[r, "u^*"] & \mathrm{Fun}(C',E) \end{tikzcd}$$

is a pullback square.

Lemma 2.3.36 (Pasting lemma for pushout squares)**.** *Consider a commutative diagram*

$$\begin{tikzcd} C_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & C_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & C_3 \arrow[d, "f_3"] \\ D_1 \arrow[r, "h_1"'] & D_2 \arrow[r, "h_2"'] & D_3. \end{tikzcd}$$

If the left-hand square is a pushout square, then the right-hand square is a pushout square if and only if the outer rectangle is a pushout square.

Proof. This follows immediately from Lemma 2.3.20. □

2.3.6 Summary

Here is a brief summary of the constructions of ∞-categories we have introduced so far:

Axiom B. We demand the existence of the following ∞-categories:

(B.1) A *terminal ∞-category* $*$: for every C there is a functor $p_C\colon C \to *$, and for any two functors $f,g\colon C \to *$ there is a natural isomorphism $f \cong g$.

(B.2) An *initial ∞-category* $\emptyset$: for every C there is a functor $\emptyset \to C$, and for any two functors $f,g\colon \emptyset \to C$ there is a natural isomorphism $f \cong g$.

(B.2') The ∞-category $\emptyset$ is strictly initial: every functor $C \to \emptyset$ is an equivalence.

(B.3) For every two ∞-categories C and D a *product* $C \times D$ with functors $\mathrm{pr}_C\colon C \times D \to C$ and $\mathrm{pr}_D\colon C \times D \to D$: functors $E \to C \times D$ are specified by giving the two components $E \to C$ and $E \to D$, and natural isomorphisms between functors $E \to C \times D$ are specified by giving natural isomorphisms between the components.

(B.4) Similarly there is a *coproduct* $C \sqcup D$ with functors $i_C\colon C \to C \sqcup D$ and $i_D\colon D \to C \sqcup D$: functors $C \sqcup D \to E$ are specified by giving the two components $C \to E$ and $D \to E$, and natural isomorphisms between functors $C \sqcup D \to E$ are specified by giving natural isomorphisms between the components.

(B.4') Coproducts are universal: a functor $E \to C \sqcup D$ into a coproduct can essentially uniquely be written in the form $E_C \sqcup E_D \to C \sqcup D$ for two functors $E_C \to C$ and $E_D \to D$.

(B.5) For functors $f\colon C \to E$ and $g\colon D \to E$ there exists a *pullback* $C \times_E D$ with functors $\mathrm{pr}_C\colon C \times_E D \to C$ and $\mathrm{pr}_D\colon C \times_E D \to D$ and a natural isomorphism $f \circ \mathrm{pr}_C \cong g \circ \mathrm{pr}_D$. Functors $T \to C \times_E D$ are specified by giving two components $t\colon T \to C$ and $s\colon T \to D$ together with a natural isomorphism $f \circ t \cong g \circ s$. One may similarly specify natural isomorphisms between two functors $T \to C \times_E D$.

(B.6) For ∞-categories C and D there is a *functor category* $\mathrm{Fun}(C,D)$. Every functor $f\colon E \times C \to D$ gives rise to a *curried functor* $f_c\colon E \to \mathrm{Fun}(C,D)$ and conversely every functor $g\colon E \to \mathrm{Fun}(C,D)$ gives rise to an *uncurried functor* $g^u\colon E \times C \to D$, satisfying $f \cong (f_c)^u$ and $g \cong (g^u)_c$. We may similarly curry/uncurry natural isomorphisms between functors. Currying is demanded to be functorial in E.

2.4 The commutative square, Segal and Rezk axioms

In Section 2.2.2 we introduced the notions of morphisms and commutative triangles in an ∞-category C. We will now impose three additional axioms that guarantee that these notions behave as expected, which may informally be stated as follows:

(D) *Commutative square axiom*: the data of a commutative square $[1]\times[1] \to C$ is the same as that of two commutative triangles $[2] \to C$ that agree along their diagonal;

(E) *Segal axiom*: for every pair of composable morphisms $f\colon x \to y$ and $g\colon y \to z$, there is an essentially unique commutative triangle $\sigma\colon [2] \to C$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"'] & & z. \end{tikzcd}$$

This leads to a *composition map* $-\circ-\colon \mathrm{Hom}_C(y,z)\times \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(x,z)$.

(F) *Rezk axiom*: the data of an isomorphism $x \cong y$ between two objects x and y is the same as that of a morphism $f\colon x \to y$ that admits an inverse $g\colon y \to x$ satisfying $gf \cong \mathrm{id}_x$ and $fg \cong \mathrm{id}_y$.

In the next three subsections we will provide precise formulations of these three axioms.

2.4.1 The commutative square axiom

Throughout, C denotes an ∞-category.

Definition 2.4.1 (Commutative square)**.** We define a *commutative square in C* to be a functor $[1]\times[1] \to C$. We will often display commutative squares as follows:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

Remark 2.4.2. Once we have the ∞-category of ∞-categories available, see Section 2.6.8 below, the meta-theoretical notion of a commutative square of functors from Definition 2.2.2 will be an instance of Definition 2.4.1 applied to $C = \mathrm{Cat}_\infty$.

Definition 2.4.3. We define two functors $j_0, j_1\colon [2] \to [1]\times[1]$ as

$$j_0 := (s_0, s_1) \qquad \text{and} \qquad j_1 := (s_1, s_0).$$

These two commutative triangles in $[1]\times[1]$ may be visualized as follows:

$$j_0 \quad = \quad \begin{tikzcd} (0,0) \arrow[r] \arrow[dr] & (0,1) \arrow[d] \\ & (1,1), \end{tikzcd} \qquad j_1 \quad = \quad \begin{tikzcd} (0,0) \arrow[d] \arrow[dr] & \\ (1,0) \arrow[r] & (1,1). \end{tikzcd}$$

The simplicial relations $s_0 \circ d_1 \cong \mathrm{id}_{[1]} \cong s_1 \circ d_1$ provide an isomorphism making the following square commute:

$$\begin{tikzcd} {[1]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[2]} \arrow[d, "j_0"] \\ {[2]} \arrow[r, "j_1"'] & {[1]\times[1]}. \end{tikzcd}$$

Axiom D (Commutative square axiom)**.** For every ∞-category C, the commutative square

$$\begin{tikzcd} \mathrm{Fun}([1]\times[1],C) \arrow[r, "j_1^*"] \arrow[d, "j_0^*"'] & \mathrm{Fun}([2],C) \arrow[d, "d_1^*"] \\ \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] & \mathrm{Fun}([1],C) \end{tikzcd}$$

is a pullback square.

In other words, providing a commutative square in C is the same as providing two commutative triangles that agree on their diagonals:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "l"] & y \arrow[d, "h"] \\ z \arrow[r, "k"'] & w. \end{tikzcd}$$

Corollary 2.4.4. *For every ∞-category C the restriction functors*

$$j_0^* \colon \mathrm{Fun}([1]\times[1],C) \to \mathrm{Fun}([2],C) \qquad \textit{and} \qquad j_1^* \colon \mathrm{Fun}([1]\times[1],C) \to \mathrm{Fun}([2],C)$$

admit sections

$$p_0^* \colon \mathrm{Fun}([2],C) \to \mathrm{Fun}([1]\times[1],C) \qquad \textit{and} \qquad p_2^* \colon \mathrm{Fun}([2],C) \to \mathrm{Fun}([1]\times[1],C)$$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "h"'] & y \arrow[d, "g"] \\ & z \end{tikzcd} \mapsto \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\ x \arrow[r, "h"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, "f"'] \arrow[dr, "h"] & \\ y \arrow[r, "g"'] & z \end{tikzcd} \mapsto \begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "h"] & z \arrow[d, "\mathrm{id}_z"] \\ y \arrow[r, "g"'] & z. \end{tikzcd}$$

Proof. Under the equivalence $\mathrm{Fun}([1]\times[1],C) \xrightarrow{\sim} \mathrm{Fun}([2],C)\times_{\mathrm{Fun}([1],C)}\mathrm{Fun}([2],C)$, we take one of the components to be the identity of $\mathrm{Fun}([2],C)$ and we take the other one to be given by the degenerate triangle, i.e. given by precomposition with

$$[2] \xrightarrow{s_0} [1] \xrightarrow{d_1} [2] \qquad \text{or} \qquad [2] \xrightarrow{s_1} [1] \xrightarrow{d_1} [2]. \qquad \square$$

Remark 2.4.5. Applying the corollary to $C = [2]$, we in particular obtain functors

$$p_0 := p_0^*(\mathrm{id}_{[2]})\colon [1]\times[1] \to [2] \qquad \text{and} \qquad p_2 := p_2^*(\mathrm{id}_{[2]})\colon [1]\times[1] \to [2],$$

and in fact p_0^* and p_2^* are given by precomposition with p_0 and p_2, respectively. We may pictorially represent these two commutative squares in $[2]$ by the following diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] \arrow[dr] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad\qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

2.4.2 The Segal axiom

Given a commutative triangle $\sigma\colon [2] \to C$ we may extract two morphisms $f := \sigma\circ d_2$ and $g := \sigma\circ d_0$:

$$\sigma \qquad = \qquad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr] & & z. \end{tikzcd}$$

By the simplicial identities we have $d_2\circ d_0 = d_0\circ d_1\colon [0] \to [2]$, so that the target of f does indeed agree with the source of g.

Axiom E (Segal axiom)**.** For every ∞-category C, the restriction functor

$$(d_0^*, d_2^*)\colon \mathrm{Fun}([2],C) \to \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C), \qquad \sigma \mapsto (g,f)$$

is an equivalence of ∞-categories. In fact, we require more generally that for $n \geq 2$ the restriction functor

$$(d_0^*, e_1^*)\colon \mathrm{Fun}([n],C) \to \mathrm{Fun}([n-1],C)\times_{\mathrm{Fun}([0],C)}\mathrm{Fun}([1],C)$$

is an equivalence of ∞-categories, where $e_1\colon [1] \cong \{0 \leq 1\} \hookrightarrow [n]$ is the inclusion.

Given morphisms $f\colon x \to y$ and $g\colon y \to z$, the axiom provides an essentially unique commutative triangle $\sigma\colon [2] \to C$ satisfying $d_2^*(\sigma) \cong f$ and $d_0^*(\sigma) \cong g$. The resulting morphism $g\circ f := d_1^*(\sigma)\colon [1] \to C$ is called the *composite of f and g*. The assignment $(g,f) \mapsto g\circ f$ is functorial:

Construction 2.4.6 (Composition functor)**.** Consider the zig-zag

$$\mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xleftarrow[\sim]{(d_0^*,d_2^*)} \mathrm{Fun}([2],C) \xrightarrow{d_1^*} \mathrm{Fun}([1],C).$$

The first map is an equivalence by the Segal axiom, and hence it admits an inverse, providing a composite functor

$$-\circ-\colon\ \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xrightarrow{\sim} \mathrm{Fun}([2],C) \xrightarrow{d_1^*} \mathrm{Fun}([1],C).$$

Passing to fibers over $(x,y,z)\in C^3$ and $(x,z)\in C^2$, respectively, then produces a functor

$$-\circ-\colon\ \mathrm{Hom}_C(y,z)\times \mathrm{Hom}_C(x,y)\to \mathrm{Hom}_C(x,z), \qquad (g,f)\mapsto g\circ f,$$

which we call the *composition functor*.

We will now show that composition in an ∞-category is unital and associative.

Lemma 2.4.7 (Unitality)**.** *For every morphism $f\colon x\to y$ in an ∞-category C, there are natural isomorphisms*

$$\mathrm{id}_y\circ f \cong f \cong f\circ \mathrm{id}_x$$

in $\mathrm{Fun}([1],C)$*. Moreover, these isomorphisms are natural in f, in the sense that they form natural isomorphisms of functors* $\mathrm{Fun}([1],C)\to\mathrm{Fun}([1],C)$.

Proof. For a fixed morphism f, the isomorphisms are an immediate consequence of Exercise 2.2.14, using the degenerate commutative triangles $s_1^*(f) := f\circ s_1$ and $s_0^*(f) := f\circ s_0$. For the naturality of the relation $f\circ\mathrm{id}_x\cong f$ we have to show that the composite

$$\mathrm{Fun}([1],C) \xrightarrow{\sim} \mathrm{Fun}([1],C)\times_C C \xrightarrow{1\times p_{[1]}^*} \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xrightarrow{-\circ-} \mathrm{Fun}([1],C)$$

is isomorphic to the identity functor. This follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}([1],C) \arrow[r, "s_0^*"] \arrow[d, "\simeq"'] \arrow[rr, bend left, equal] & \mathrm{Fun}([2],C) \arrow[r, "d_1^*"] \arrow[d, "\simeq"] & \mathrm{Fun}([1],C) \arrow[d, equal] \\
\mathrm{Fun}([1],C)\times_C C \arrow[r, "1\times p_{[1]}^*"'] & \mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \arrow[r, "-\circ-"'] & \mathrm{Fun}([1],C).
\end{tikzcd}$$

A similar discussion applies to the relation $f\cong \mathrm{id}_y\circ f$. $\square$

Proposition 2.4.8 (Associativity)**.** *For composable morphisms $f\colon x\to y$, $g\colon y\to z$ and $h\colon z\to w$ in an ∞-category C, there is a natural isomorphism*

$$h\circ(g\circ f)\cong (h\circ g)\circ f$$

in $\mathrm{Fun}([1],C)$.

Proof. Consider the following two morphisms in $\mathrm{Fun}([1],C)$, regarded as objects of $\mathrm{Fun}([1],\mathrm{Fun}([1],C)) \simeq \mathrm{Fun}([1]\times[1],C)$:

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f" description] & y \arrow[d, "g"] \\ x \arrow[r, "g\circ f"'] & z \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} y \arrow[r, "h\circ g"] \arrow[d, "g"'] \arrow[dr, "h\circ g" description] & w \arrow[d, "\mathrm{id}_w"] \\ z \arrow[r, "h"'] & w. \end{tikzcd}$$

These morphisms are natural in f, g and h: we have functors

$$\mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C) \xrightarrow{\simeq} \mathrm{Fun}([2],C) \xrightarrow{p_0^*} \mathrm{Fun}([1]\times[1],C),$$

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[d, "g"] \\ & z \end{tikzcd} \mapsto \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g\circ f"'] & y \arrow[d, "g"] \\ & z \end{tikzcd} \mapsto \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr, "g\circ f" description] & y \arrow[d, "g"] \\ x \arrow[r, "g\circ f"'] & z, \end{tikzcd}$$

and similarly for the other diagram. Forming the composite of these two morphisms in $\mathrm{Fun}([1],C)$, we thus obtain a commutative square in C of the form

$$\begin{tikzcd} x \arrow[r, "(h\circ g)\circ f"] \arrow[d, "\mathrm{id}_x"'] \arrow[dr] & w \arrow[d, "\mathrm{id}_w"] \\ x \arrow[r, "h\circ(g\circ f)"'] & w, \end{tikzcd}$$

natural in f, g and h. Using Lemma 2.4.7, the diagonal of this square is then naturally isomorphic to both $(h\circ g)\circ f$ and $h\circ(g\circ f)$, finishing the proof. $\square$

2.4.3 The Rezk axiom

The notions of identity morphisms and composition of morphisms in ∞-categories, provided by the Segal axiom, lead to the notion of an *isomorphism* in an ∞-category:

Definition 2.4.9 (Isomorphisms)**.** Consider a morphism $f: x \to y$ in an ∞-category C. We say that f is *invertible*, or that it is an *isomorphism*,[6] if there are commutative triangles in C of the form

$$\begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "\mathrm{id}_y"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "\mathrm{id}_x"'] & y \arrow[d, "h"] \\ & x. \end{tikzcd}$$

[6] In the literature these are often called ‘equivalences’. I might sometimes accidentally call them ‘equivalences’ as well.

A priori, this use of the word 'isomorphism' clashes with the other notion of 'isomorphism' that we have, namely that of natural isomorphisms between functors $x, y\colon * \to C$. For this reason, we will now introduce the *Rezk axiom*, which enforces that these two notions of isomorphisms agree with each other. We start by defining the ∞-category $\mathrm{Iso}(C)$ of isomorphisms in C.

Definition 2.4.10. Given a ∞-category C, we define the ∞-category $\mathrm{Iso}(C)$ via the following pullback diagram:

$$\begin{tikzcd}
\mathrm{Iso}(C) \arrow[r, dashed] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, dashed, bend left, "\pi_{\mathrm{Iso}}"] \arrow[dd] & \mathrm{Fun}([2],C)\times_{d_0^*,\mathrm{Fun}([1],C),d_2^*}\mathrm{Fun}([2],C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\Delta"] \\
& \mathrm{Fun}([2],C)\times\mathrm{Fun}([2],C) \arrow[r, "d_0^*\times d_2^*"] \arrow[d, "d_1^*\times d_1^*"] & \mathrm{Fun}([1],C)\times\mathrm{Fun}([1],C) \\
C\times C \arrow[r, "{(x,y)\mapsto(\mathrm{id}_y,\mathrm{id}_x)}"] & \mathrm{Fun}([1],C)\times\mathrm{Fun}([1],C). &
\end{tikzcd}$$

We may think of the objects of $\mathrm{Iso}(C)$ as commutative diagrams in C of the form

$$\begin{tikzcd}
& x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"] & \\
y \arrow[ur, "g"] \arrow[dr, "\mathrm{id}_y"'] & & x. \\
& y \arrow[ur, "h"'] &
\end{tikzcd}$$

(Note that this is not a commutative square!) More formally, an object of $\mathrm{Iso}(C)$ consists of objects x and y of C, a morphism f in C and two commutative triangles σ_1 and σ_2 in C together with isomorphisms $d_0^*(\sigma_1) \cong f \cong d_2^*(\sigma_2)$, $d_1^*(\sigma_1) \cong \mathrm{id}_x$ and $d_1^*(\sigma_2) \cong \mathrm{id}_y$.

We denote the composite functor at the top of the diagram by

$$\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \to \mathrm{Fun}([1],C).$$

As one expects, the identity map $\mathrm{id}_x\colon x \to x$ is invertible in C for every term x of C:

Construction 2.4.11. We construct a functor $i\colon C \to \mathrm{Iso}(C)$ lifting the functor $C \to \mathrm{Fun}([1],C)\colon x \mapsto \mathrm{id}_x$. Informally, i is given by sending an object x to the following commutative diagram in C:

$$\begin{tikzcd}
& x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "\mathrm{id}_x"] & \\
x \arrow[ur, "\mathrm{id}_x"] \arrow[dr, "\mathrm{id}_x"'] & & x. \\
& x \arrow[ur, "\mathrm{id}_x"'] &
\end{tikzcd}$$

More formally, restriction along the map $p_{[2]}\colon [2] \to [0]$ induces a functor $p_{[2]}^*\colon C \to \mathrm{Fun}([2],C)$, giving a functor $(p_{[2]}^*, p_{[2]}^*)\colon C \to \mathrm{Fun}([2],C) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([2],C)$. Since the two relevant edges of this diagram are both the identity on x, this functor factors through $\mathrm{Iso}(C)$, providing the desired functor $i\colon C \to \mathrm{Iso}(C)$.

Axiom F (Rezk axiom)**.** For every ∞-category C the functor $i\colon C \to \mathrm{Iso}(C)$ is an equivalence.

The Rezk axiom expresses that two objects x and y in C may be identified with each other as soon as we find an invertible morphism $f\colon x \to y$ between them:

Corollary 2.4.12. *Let x and y be objects of C and assume there exists an invertible morphism $f\colon x \to y$. Then there exists an isomorphism $x \cong y$ in C.*

Proof. Since the functor $i\colon C \to \mathrm{Iso}(C)$ is an equivalence, there exists an object z of C together with an isomorphism $i(z) \cong f$ in $\mathrm{Iso}(C)$. Using the source and target functors $s,t\colon \mathrm{Iso}(C) \to C$, this isomorphism in $\mathrm{Iso}(C)$ produces isomorphisms $z \cong x$ and $z \cong y$ in C. By inversion we obtain $x \cong z$ and by composition we obtain $x \cong y$, as desired. □

Remark 2.4.13. To be completely honest with you: the Rezk axiom doesn't *really* identify the two notions of isomorphisms that we have, it really just says that every isomorphism in C is isomorphic to an identity map. In the way I have set things up in these lectures, it is difficult to do much better than that: the notion of "natural isomorphism" is simply a primitive notion and we cannot really say very much about it. A more satisfying approach would have been to not just introduce the notion of a natural isomorphism $\alpha\colon f \cong g$, but to instead introduce a whole ∞-category $f \cong g$ of natural isomorphisms, and then define a single natural isomorphism as a functor $\alpha\colon * \to f \cong g$. If one proceeds this way, one can just ask this ∞-category to fit into a pullback square of the form

$$\begin{tikzcd} f \cong g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Iso}(\mathrm{Fun}(C,D)) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ * \arrow[r, "(f,g)"] & \mathrm{Fun}(C,D) \times \mathrm{Fun}(C,D), \end{tikzcd}$$

and then one has *truly* identified the primitive notion of natural isomorphism with that of an invertible natural transformation (which is what I had always intended).

I don't want to change Axiom A.1 anymore so we'll just leave it like it is, but from now on it should be understood that the notion of a natural isomorphism is supposed to be just the same as that of an invertible natural transformation!

Axioms E and F are inspired by Rezk's notion of *complete Segal spaces* [Rez01]; see Appendix A.5 for a definition of these objects.

2.5 Digression: Quasicategories

So far, we have been treating ∞-category theory very axiomatically, focusing on the essential features of the theory itself and ignoring how precisely one might model these objects within ordinary mathematics. For completeness, we will now discuss the most common choice of model for ∞-categories: *quasicategories*. I will give a definition of these objects, give the main examples, and sketch why these objects do indeed satisfy the axioms we have introduced so far. This material will not be used in the remainder of the course and will not appear in the exam (with the exception of Definition 2.5.4, which is important independently of quasicategories).

Remark 2.5.1. Some words on history are in order. Quasicategories have played a pivotal role in the acceptance of ∞-category theory as a legitimate part of mathematics. The conceptual idea of 'doing category theory up to homotopy' was around for a while, but a major difficulty was how to make this idea rigorous within ordinary mathematics. While the definition of a quasicategory already appeared in the work of Boardman and Vogt [BV73] under the name 'weak Kan complexes', it was through deep insights and important foundational work by André Joyal [Joy08] and subsequently Jacob Lurie [Lur09b; Lur17] that the theory obtained a rigorous, well-developed and practical form that could immediately be used by researchers.

Definition 2.5.2. The *simplex category* $\mathbf{\Delta}$ is the category of non-empty finite linearly ordered sets and order-preserving maps between them.

Observe that any object in $\mathbf{\Delta}$ is isomorphic to the linear order $[n] = \{0 \leq 1 \leq \cdots \leq n\}$ for some n, and that the isomorphism is unique if it exists. We will therefore frequently denote general objects of $\mathbf{\Delta}$ simply by $[n]$.

Observation 2.5.3. Every morphism in $\mathbf{\Delta}$ can be written as a finite composite of face maps $d_i\colon [n-1] \to [n]$ and degeneracy maps $s_i\colon [n+1] \to [n]$ (see Notation 2.2.9). The identities these maps satisfy are known as the *simplicial identities*.

Definition 2.5.4. Let C be an ∞-category. A *simplicial object in C* is defined to be a functor $\mathbf{\Delta}^{\mathrm{op}} \to C$. We will denote the ∞-category of simplicial objects by

$$\mathrm{s}C := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

In light of Observation 2.5.3, we may informally think of a simplicial object in C as a collection of objects X_n of C for every natural number n together with face maps $d_i\colon X_n \to X_{n-1}$ and degeneracy maps $s_i\colon X_n \to X_{n_1}$ satisfying the simplicial identities. We frequently draw simplicial objects as follows:

$$\ldots \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} X_2 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} X_1 \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} X_0.$$

Since this looks somewhat chaotic, it is common to leave the degeneracy maps implicit and only draw the face maps:

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow}\; X_0.$$

We will now specialize to $C = \mathrm{Set}$. For a simplicial set $X \in \mathrm{sSet}$, we refer to the set X_n as its *set of n-simplices*.

Definition 2.5.5 (Simplices)**.** For every natural number n, we define the *n-simplex* Δ^n as the simplicial set represented by $[n]$:

$$\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}.$$

By the Yoneda lemma this defines a fully faithful functor

$$\Delta^\bullet \colon \mathbf{\Delta} \hookrightarrow \mathrm{sSet}.$$

Again by the Yoneda lemma there is a natural bijection $X_n \cong \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X)$, so that we may think of an n-simplex in X as a morphism $\sigma \colon \Delta^n \to X$ of simplicial sets.

Definition 2.5.6 (Horns)**.** For $n \geq 1$ and $0 \leq k \leq n$, we define the *k-horn* Λ^n_k as the subsimplicial set

$$\Lambda^n_k \subseteq \Delta^n$$

whose m-simplices are those maps $\varphi \colon [m] \to [n]$ in $\mathbf{\Delta}$ such that there is some element $i \in [n] \setminus \{k\}$ which is not in the image of φ. Equivalently, it is the union inside Δ^n of all of its faces except for the k-th face.

Example 2.5.7. For $n = 1$, the two horns Λ^1_0 and Λ^1_1 may be displayed as follows:

$$\begin{array}{ccccc} 0 & & 0 \longrightarrow 1 & & 1 \\ & \hookrightarrow & & \hookleftarrow & \\ \Lambda^1_0 & & \Delta^1 & & \Lambda^1_1 \end{array}$$

Example 2.5.8. For $n = 2$ the three horns Λ^2_0, Λ^2_1 and Λ^2_2 may be displayed as follows:

$$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad \Lambda^2_1 \qquad\qquad \Lambda^2_2$$

Definition 2.5.9. Let X be a simplicial set.

(1) We say that X is a *Kan complex* if for every $n \geq 1$ and $0 \leq k \leq n$, every morphism of simplicial sets $\Lambda^n_k \to X$ extends to a morphism $\Delta^n \to X$:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n . \arrow[ur, dashed] & \end{tikzcd}$$

(2) We say that X is a *quasicategory* if the above condition holds for all $n \geq 1$ but only for $0 < k < n$.

We denote by

$$\mathrm{Kan} \subseteq \mathrm{qCat} \subseteq \mathrm{sSet}$$

the full subcategories on the Kan complexes and the quasicategories, respectively.

We will now give the two main examples of quasicategories:

Example 2.5.10 (Singular complex)**.** Let X be a topological space. We will construct a simplicial set $\mathrm{Sing}(X) \in \mathrm{sSet}$ called the *singular complex of* X.

- For every $n \geq 0$, let $|\Delta^n|$ denote the topological n-simplex:

$$|\Delta^n| := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \cdots + x_n = 1,\ x_i \geq 0 \text{ for } i = 0, \ldots, n\}.$$

- For every order-preserving map $\varphi \colon [n] \to [m]$ we get a continuous map

$$|\Delta^\varphi| \colon |\Delta^n| \to |\Delta^m|, \quad (x_0, \ldots, x_n) \mapsto (y_0, \ldots, y_m), \qquad y_i := \sum_{j \in \varphi^{-1}(i)} x_j.$$

  This defines a functor

$$|\Delta^\bullet| \colon \mathbf{\Delta} \to \mathrm{Top}, \qquad [n] \mapsto |\Delta^n|, \qquad \varphi \mapsto |\Delta^\varphi|.$$

- We now define the simplicial set $\mathrm{Sing}(X)$ as

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X),$$

  i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{|\Delta^\bullet|} \mathrm{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Top}}(-,X)} \mathrm{Set}$.

You may know this simplicial set from your algebraic topology courses: it is used in the definition of the *singular (co)homology* of X. It is not difficult to see that $\mathrm{Sing}(X)$ is functorial in X, so that it defines a functor

$$\mathrm{Sing} \colon \mathrm{Top} \to \mathrm{sSet}.$$

Exercise 2.5.11. Show that the singular complex $\mathrm{Sing}(X)$ of a topological space X is a Kan complex.

Example 2.5.12 (Nerve of a category)**.** Let C be a (small) ordinary category. We define the *nerve of* C as the simplicial set $N(C) \in \mathrm{sSet}$ given by

$$N(C)_n := \mathrm{Hom}_{\mathrm{Cat}}([n], C),$$

i.e. it is the composite $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}}(-,C)} \mathrm{Set}$. Here Cat denotes the ordinary category of small categories. Again this construction is functorial in C, resulting in a functor

$$N\colon \mathrm{Cat} \to \mathrm{sSet}.$$

Exercise 2.5.13. Show that the nerve $N(C)$ of a category C is a quasicategory. Provide an example for which it is not a Kan complex.

Exercise 2.5.14. Show that the functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful, and provide a characterization of its essential image in terms of lifting conditions similar to Definition 2.5.9.

We will now discuss the *homotopy-coherent nerve* of a Kan-enriched category.

Definition 2.5.15. A *Kan-enriched category* (resp. a simplicially enriched category) is a category $\mathbf{C}$ such that for all objects x, y of $\mathbf{C}$ the hom set $\mathrm{Hom}_{\mathbf{C}}(x, y)$ is the set of vertices of some Kan complex $\mathrm{Hom}^{\Delta}(x, y)$ (resp. simplicial set) and the composition in $\mathbf{C}$ is induced by a morphism of Kan complexes (resp. simplicial sets)

$$-\circ-\colon\ \mathrm{Hom}^{\Delta}(y,z) \times \mathrm{Hom}^{\Delta}(x,y) \to \mathrm{Hom}^{\Delta}(x,z)$$

which satisfy unitality and associativity.

Example 2.5.16. The category sSet has an internal Hom: for a simplicial set X, the functor $X \times -\colon \mathrm{sSet} \to \mathrm{sSet}$ admits a right adjoint that we denote by

$$F(X,-)\colon \mathrm{sSet} \to \mathrm{sSet}.$$

Using this, the category Kan can be turned into a Kan-enriched category **Kan** by taking

$$\mathrm{Hom}^{\Delta}_{\mathbf{Kan}}(X,Y) := F(X,Y),$$

the internal Hom in Kan.

Example 2.5.17. Similarly, the category qCat can be turned into a Kan-enriched category **qCat** by taking

$$\mathrm{Hom}^{\Delta}_{\mathbf{qCat}}(C,D) := F(C,D)^{\simeq},$$

where $C^{\simeq} \subseteq C$ denotes the largest Kan complex contained in C (consisting of those n-simplices all of whose edges are invertible in C).

The vertices of the Kan complex $\mathrm{Hom}^{\Delta}_{\mathbf{C}}(x,y)$ are simply the morphisms $f\colon x \longrightarrow y$ in $\mathbf{C}$. We refer to 1-simplices in $\mathrm{Hom}^{\Delta}_{\mathbf{C}}(x,y)$ as *homotopies* between morphisms, denoted $f \sim g$. We would now like to construct a variant of the nerve for a Kan-enriched category in which the diagrams 'commute up to homotopy'. For example, the 2-simplices of this homotopy-coherent nerve should consist of three morphisms

$$\begin{tikzcd}
& x_1 \arrow[dr, "f_{12}"] & \\
x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & x_2
\end{tikzcd}$$

together with a homotopy $f_{012}\colon f_{12} \circ f_{01} \simeq f_{02}$. Similarly, the 3-simplices should consist of the data of six morphisms

$$\begin{tikzcd}
& x_1 \arrow[dr, "f_{12}"] \arrow[rr, "f_{13}"] & & x_3 \\
x_0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] \arrow[urrr, "f_{03}"] & & x_2 \arrow[ur, "f_{23}"'] &
\end{tikzcd}$$

together with four homotopies

$$f_{012}\colon f_{12} \circ f_{01} \simeq f_{02}, \quad f_{013}\colon f_{13} \circ f_{01} \simeq f_{03}, \quad f_{023}\colon f_{23} \circ f_{02} \simeq f_{03}, \quad f_{123}\colon f_{23} \circ f_{12} \simeq f_{13}$$

and a homotopy of homotopies $f_{0123}\colon \Delta^1 \times \Delta^1 \longrightarrow \mathrm{Hom}^{\Delta}_{\mathbf{C}}(x_0,x_3)$ representing the following diagram

$$\begin{tikzcd}
f_{23} \circ f_{12} \circ f_{01} \arrow[r, "f_{123} \circ f_{01}"] \arrow[d, "f_{23} \circ f_{012}"'] & f_{13} \circ f_{01} \arrow[d, "f_{013}"] \\
f_{23} \circ f_{02} \arrow[r, "f_{023}"'] & f_{03}.
\end{tikzcd}$$

Definition 2.5.18. For every $n \geq 0$, we denote by $\mathfrak{C}[\Delta^n]$ the simplicially enriched category defined as follows:

- The objects of $\mathfrak{C}[\Delta^n]$ are the elements of $[n]$;

- Given $i, j \in [n]$, we define the simplicial set $\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j)$ to be empty if $j < i$, and otherwise as the nerve of the poset $P_{i,j}$ defined by

$$P_{i,j} := \{ I \subseteq \{i, i+1, \ldots, j-1, j\} \mid i, j \in I \} \subseteq \mathcal{P}(\{i, \ldots, j\}),$$

  ordered by *reverse* inclusion.

- For $i,j,k \in [k]$, the composition map

$$\mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(j,k) \times \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,j) \to \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,k)$$

  is induced from taking the nerve of the map of partially ordered sets $P_{j,k} \times P_{i,j} \to P_{i,k}$ which sends (I_1, I_2) to the union $I_1 \cup I_2$.

This defines a functor

$$\mathfrak{C}[\Delta^\bullet] \colon \mathbf{\Delta} \to \mathrm{Cat}_\Delta$$

from $\mathbf{\Delta}$ to the category of (small) simplicially enriched categories.

Definition 2.5.19 (Homotopy coherent nerve)**.** Let $\mathbf{C}$ be a simplicially enriched category. We define its *homotopy coherent nerve* $N^\Delta(\mathbf{C}) \in \mathrm{sSet}$ by

$$N^\Delta(\mathbf{C})_n := \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathbf{C}),$$

i.e. as the composite $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathfrak{C}[\Delta^\bullet]} (\mathrm{Cat}_\Delta)^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\Delta}(-,\mathbf{C})} \mathrm{Set}$.

Proposition 2.5.20 (Cordier and Porter [CP86])**.** *Assume that* $\mathbf{C}$ *is Kan-enriched. Then the simplicial set* $N^\Delta(\mathbf{C})$ *is a quasicategory.*

Exercise 2.5.21. Give a proof of Proposition 2.5.20.

We now relate the theory of quasicategories to our axioms of ∞-category theory:

Proposition 2.5.22. *Quasicategories satisfy axioms A-F.*

Proof sketch.

- A *functor* of quasicategories is a morphism $f \colon C \to D$ of simplicial sets. Composition is strictly unital and associative;
- The ∞-categories $*$ and $\emptyset$ are the initial and terminal objects of sSet;
- Products and coproducts are given by categorical products and coproducts of simplicial sets;
- Pullbacks are given by *homotopy pullbacks* of quasicategories;
- The functor category $\mathrm{Fun}(C,D)$ is the internal hom in qCat: the functor $\mathrm{Fun}(C,-) \colon \mathrm{qCat} \to \mathrm{qCat}$ is right adjoint to $C \times - \colon \mathrm{qCat} \to \mathrm{qCat}$;
- For every quasicategory C we may define an quasicategory $\mathrm{Iso}(C)$ just like in Definition 2.4.10, but where now all the pullbacks are *strict* pullbacks taken in sSet. It comes with a forgetful functor $(s,t) \colon \mathrm{Iso}(C) \to C \times C$. We define a *natural isomorphism* $\alpha \colon f \cong g$ to be a functor $\alpha \colon C \to \mathrm{Iso}(D)$ such that $s(\alpha) = f$ and $t(\alpha) = g$. Similarly, we define an *n-isomorphism* for $n \geq 3$ to be a functor $C \to \mathrm{Iso}^{n-1}(D) = \mathrm{Iso}(\ldots\mathrm{Iso}(D))$.

- One can check that commutative square axiom, Segal axiom and Rezk axiom are satisfied. Since this requires a lot of simplicial combinatorics not relevant to the actual practice of ∞-category theory, we will not go into the details here. $\square$

Remark 2.5.23. One can prove that a simplicial set X is a quasicategory if and only if the morphism of simplicial sets

$$F(\Delta^2, X) \to F(\Lambda^2_1, X) \cong F(\Delta^1, X) \times_X F(\Delta^1, X)$$

induced by the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ is a so-called 'trivial fibration' of simplicial sets. As a consequence this morphism is an equivalence of quasicategories for every quasicategory X. So even though a priori a quasicategory only guarantees the *existence* of a 2-simplex given any pair of composable 1-simplices $f\colon x \to y$ and $g\colon y \to z$, the filling conditions for all the higher-dimensional horns guarantee that this 2-simplex is in some sense 'unique up to homotopy', which is precisely what the Segal axiom is expressing.

From this point forward, the reader who chooses to do so may think of ∞-categories as quasicategories. Nevertheless, I will continue writing everything in a way that does not refer to simplicial sets and respects the fundamental principle of homotopy theory.

2.6 More aspects of ∞-category theory

The theory of ∞-category theory is vast, and I could keep telling you about more things for the rest of the course if I wanted. But since this course is ultimately about stable homotopy theory, we do not have the time to go into all possible aspects of the theory. In this section I will quickly go through the remaining constructions of ∞-category that we need, and state various important results as black boxes.

Remark. Later in the course, we will need more advanced concepts in ∞-category theory, like adjunctions, Kan extensions and (co)final functors. Readers interested in learning about abstract ∞-category theory may find detailed discussions of these notions in Appendix A.

2.6.1 Animae

The notion of a 'groupoid' from ordinary category theory has a natural analogue in the setting of ∞-categories:

Definition 2.6.1. An ∞-category C is called an *anima*, or an *∞-groupoid*, if every morphism in C is invertible.

Remark 2.6.2. The plural of 'anima' is either 'anima' or 'animae', depending on taste. (I personally prefer 'animae' but since many people use 'anima' I will probably often mix them up.)

The terminology 'anima' was introduced relatively recently by Clausen and Scholze. For historical reasons, the word 'space' is often used for 'anima' in the literature: it turns out that the ∞-category An of animae can be obtained from the ordinary category of topological spaces by inverting the weak homotopy equivalences, see Theorem 2.6.38 below. While it is certainly useful at times to think of animae as spaces (in the sense of Convention 1.2.3) it is such a fundamental concept in homotopy theory that it definitely deserves its own special name as well.

Exercise 2.6.3 (See Exercise 6.6 on page 256 for some hints)**.** Let C be an ∞-category and let x and y be objects in C. Show that the Hom anima $\mathrm{Hom}_C(x, y)$ is indeed an anima.

Recall that every category C has a largest subgroupoid $C^{\simeq}$ contained in it, called the *core* of C, which has the same objects as C but whose morphisms are only the *isomorphisms* in C. It is uniquely characterized by the property that the inclusion $C^{\simeq} \hookrightarrow C$ is universal among functors from a groupoid into C. We will use this universal property to axiomatize the ∞-categorical version of the core:

Axiom G (Groupoid core axiom)**.** For every ∞-category C, there is an anima $C^{\simeq}$ called the *(groupoid) core of* C, which comes equipped with a functor $\gamma_C\colon C^{\simeq} \to C$. Every functor $F\colon X \to C$ from an anima X factors through γ_C, and for functors $G, H\colon X \to C^{\simeq}$ every natural isomorphism $\gamma_C \circ G \cong \gamma_C \circ H$ may be lifted to a natural isomorphism $G \cong H$.

Furthermore, the groupoid core of $[1]$ is equivalent to $* \sqcup *$.

Definition 2.6.4. For ∞-categories C and D, we write $\mathrm{Map}(C, D)$ for the anima $\mathrm{Fun}(C, D)^{\simeq}$.

Exercise 2.6.5. Show that for an anima X and an ∞-category C the inclusion $C^{\simeq} \hookrightarrow C$ induces an equivalence

$$\mathrm{Map}(X, C^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, C).$$

Exercise 2.6.6. Let $F\colon C \to D$ be a functor. Show that the following conditions are equivalent:

(1) The functor F is an equivalence;

(2) For every ∞-category E the functor $F \circ -\colon \mathrm{Map}(E, C) \to \mathrm{Map}(E, D)$ is an equivalence;

(3) For every ∞-category E the functor $- \circ F\colon \mathrm{Map}(D, E) \to \mathrm{Map}(C, E)$ is an equivalence.

2.6.2 Fully faithful and essentially surjective functors are equivalences

Recall that a functor between ordinary categories is an equivalence if and only if it is both fully faithful and essentially surjective. The same statement is true for ∞-categories as well, but we will treat it as a black box since its proof requires some techniques that we haven't introduced.

Definition 2.6.7. A functor $F\colon C \to D$ is called *fully faithful* if for every pair of objects x and y of C, the induced map of animae

$$F\colon \operatorname{Hom}_C(x,y) \to \operatorname{Hom}_D(Fx,Fy)$$

is an equivalence.

Definition 2.6.8. A functor $F\colon C \to D$ is called *essentially surjective* if for every object y of D there exists an object x of C together with an isomorphism $Fx \cong y$.

Theorem 2.6.9 (Joyal, [Lan21, Theorem 2.3.20], [Cis+24, Theorem 6.3.1])**.** *A functor $F\colon C \to D$ is an equivalence if and only if it is fully faithful and essentially surjective.*

2.6.3 Pointwise natural isomorphisms are invertible

Another statement we will treat as a black box is the following:

Theorem 2.6.10 (Joyal, [Lan21, Theorem 2.2.1], [Cis+24, Theorem 6.2.10])**.** *Let $F,G\colon C \to D$ be two functors and let $\alpha\colon F \Rightarrow G$ be a natural transformation. Then α is a natural isomorphism if and only if for every object x of C the morphism $\alpha(x)\colon F(x) \to G(x)$ is an isomorphism in D.*

2.6.4 Opposite categories

Axiom H. For every ∞-category C there is an *opposite category* C^{op}. Similarly every functor $F\colon C \to D$ induces an opposite functor $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$, and this process respects composition of functors. We have equivalences $(C^{\mathrm{op}})^{\mathrm{op}} \simeq C$ and $(F^{\mathrm{op}})^{\mathrm{op}} \simeq F$.

If C and D are ordinary categories then C^{op} and F^{op} agree with the usual constructions of opposite categories.

If X is an anima then we have an equivalence $X^{\mathrm{op}} \simeq X$.

Note that functors $[1] \to C^{\mathrm{op}}$ correspond to functors $[1] \simeq [1]^{\mathrm{op}} \to (C^{\mathrm{op}})^{\mathrm{op}} \simeq C$, so that morphisms in C^{op} are just morphisms in C but with source and target flipped. Also since $[2] \simeq [2]^{\mathrm{op}}$, this correspondence between morphisms in C and morphisms in C^{op} preserves composition. In particular C is an anima if and only if C^{op} is an anima.

Proposition 2.6.11. *The construction $C \mapsto C^{\mathrm{op}}$ is compatible with all constructions of ∞-categories:*

(1) There are equivalences $*^{\mathrm{op}} \simeq *$ and $\emptyset^{\mathrm{op}} \simeq \emptyset$.

(2) For ∞-categories C and D there are equivalences

$$\begin{aligned} (C\times D)^{\mathrm{op}} &\xrightarrow{\sim} C^{\mathrm{op}}\times D^{\mathrm{op}}, \\ C^{\mathrm{op}}\sqcup D^{\mathrm{op}} &\xrightarrow{\sim} (C\sqcup D)^{\mathrm{op}}, \\ \mathrm{Fun}(C^{\mathrm{op}},D^{\mathrm{op}}) &\simeq \mathrm{Fun}(C,D)^{\mathrm{op}}. \end{aligned}$$

(3) For functors $F\colon C\to E$ and $G\colon D\to E$ there is an equivalence

$$(C\times_E D)^{\mathrm{op}} \simeq C^{\mathrm{op}}\times_{E^{\mathrm{op}}} D^{\mathrm{op}}.$$

(4) For an ∞-category C there is an equivalence $(C^{\mathrm{op}})^{\simeq} \simeq (C^{\simeq})^{\mathrm{op}}$.

Proof. All of these properties follow readily from the universal properties of these constructions. For illustration, let us merely sketch the proof that $(-)^{\mathrm{op}}$ preserves products: functors $T\to (C\times D)^{\mathrm{op}}$ correspond to functors $T^{\mathrm{op}}\to C\times D$, which in turn are pairs of functors $(T^{\mathrm{op}}\to C, T^{\mathrm{op}}\to D)$, which in turn translate back into pairs of functors $(T\to C^{\mathrm{op}}, T\to D^{\mathrm{op}})$, i.e. functors $T\to C^{\mathrm{op}}\times D^{\mathrm{op}}$. □

Corollary 2.6.12. *For objects x and y in an ∞-category C we have an equivalence*

$$\mathrm{Hom}_{C^{\mathrm{op}}}(x,y) \simeq \mathrm{Hom}_C(y,x).$$

Proof. Applying $(-)^{\mathrm{op}}$ to the pullback square defining $\mathrm{Hom}_C(y,x)$ and using the equivalence $\mathrm{Fun}([1],C)^{\mathrm{op}} \simeq \mathrm{Fun}([1]^{\mathrm{op}},C^{\mathrm{op}}) \simeq \mathrm{Fun}([1],C^{\mathrm{op}})$, we obtain a pullback square of the form

$$\begin{tikzcd} \mathrm{Hom}_C(y,x)^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}([1],C^{\mathrm{op}}) \arrow[d, "{(\mathrm{ev}_1,\mathrm{ev}_0)}"] \\ * \arrow[r, "{(y,x)}"] & C^{\mathrm{op}}\times C^{\mathrm{op}}, \end{tikzcd}$$

where the evaluation at 0 and 1 have swapped roles because of the equivalence $[1]^{\mathrm{op}}\simeq[1]$ that was used. This shows that $\mathrm{Hom}_{C^{\mathrm{op}}}(x,y)\simeq \mathrm{Hom}_C(y,x)^{\mathrm{op}}$. But since $\mathrm{Hom}_C(y,x)$ is an anima by Exercise 2.6.3, we have $\mathrm{Hom}_C(y,x)\simeq \mathrm{Hom}_C(y,x)^{\mathrm{op}}$, finishing the proof. □

2.6.5 Subcategories

Next up, we would like to define subcategories.

Definition 2.6.13. A functor $F: C \to D$ is called a *monomorphism*, or an *embedding*, if the commutative square

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "F"] \\ C \arrow[r, "F"'] & D \end{tikzcd}$$

is a pullback square, i.e. the functor $\Delta_F := (\mathrm{id}_C, \mathrm{id}_C): C \to C \times_D C$ is an equivalence of ∞-categories.

Remark 2.6.14. Note that this means that two functors $G, H: T \to C$ are naturally isomorphic whenever the functors $F \circ G$ and $F \circ H$ are naturally isomorphic, hence this is a homotopical version of the usual notion of monomorphisms in a category.

We denote embeddings as $F: C \hookrightarrow D$. If y is an object of D, we say that *y is in C*, sometimes written as $y \in C$, if there exists some object x of C satisfying $F(x) \cong y$. By the following lemma this object is then unique up to contractible choice and we will often abusively again denote it by y.

Lemma 2.6.15. *Let $F: C \hookrightarrow D$ be a monomorphism, and let y be an object of D which is contained in C. Then the fiber C_y of F at y is contractible.*

Proof. By definition C_y sits in a pullback square as follows:

$$\begin{tikzcd} C_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "F"] \\ * \arrow[r, "y"] & D. \end{tikzcd}$$

Since y is contained in C, there exists some object $x: * \to C_y$ of C_y. We claim that this functor is an equivalence between C_y and $*$. To this end, consider the following commutative diagram:

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[d, "x"'] & C_y \arrow[r] \arrow[d, "(x, \mathrm{id}_{C_y})"] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d, "x"] \\ C_y \arrow[r, "\Delta_{C_y}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_y \times C_y \arrow[r, "\mathrm{pr}_1"] \arrow[d] & C_y \\ C \arrow[r, "\Delta_F", "\simeq"'] & C \times_D C. & \end{tikzcd}$$

Since the bottom square is a pullback square and Δ_F is an equivalence, also Δ_{C_y} is an equivalence. To show that $x: * \to C_y$ is an equivalence, it thus remains to show that the top left square is a pullback square. But this follows from the pasting law of pullback squares, since both the top right square and the top outer rectangle are pullback squares. □

Definition 2.6.16. Let C be an ∞-category. A *collection of morphisms in* C is a monomorphism $M \hookrightarrow \mathrm{Map}([1],C)$. We say it is *closed under composition* if the following two conditions are satisfied:

- For a morphism $f\colon x \to y$ in C, if $f \in M$ then also $\mathrm{id}_x \in M$ and $\mathrm{id}_y \in M$;
- For morphisms $f\colon x \to y$ and $g\colon y \to z$ in C, if $g, f \in M$ then also $g \circ f \in M$.

Axiom I (Subcategory axiom)**.** Consider an ∞-category C equipped with a collection of morphisms $m\colon M \hookrightarrow \mathrm{Map}([1],C)$ of C which is closed under composition. Then there exists a ∞-category $\langle M\rangle_C$ equipped with a functor $i_M\colon \langle M\rangle_C \to C$ satisfying:

- The induced functor $(i_M)_*\colon\ \mathrm{Map}([1],\langle M\rangle_C) \to \mathrm{Map}([1],C)$ factors through M;
- The functor i_M is *universal* with respect to the previous property: every other functor $f\colon D \to C$ whose induced functor $f_*\colon\ \mathrm{Map}([1],D) \to \mathrm{Map}([1],C)$ factors through M admits an essentially unique factorization through $\langle M\rangle_C$.

The last condition means more precisely that there is a pullback square of the form

$$\begin{tikzcd}
\mathrm{Map}(D,\langle M\rangle_C) \arrow[rr] \arrow[d] & & \mathrm{Map}(D,C) \arrow[d] \\
\mathrm{Map}(\mathrm{Map}([1],D),M) \arrow[r, hook] & \mathrm{Map}(\mathrm{Map}([1],D),\mathrm{Map}([1],C)). &
\end{tikzcd}$$

We refer to $\langle M\rangle_C$ as the *(non-full) subcategory spanned by the morphisms in* M.

Exercise 2.6.17. Show that the induced functor $(i_M)_*\colon\ \mathrm{Map}([1],\langle M\rangle_C) \to M$ is an equivalence.

We may also form *full* subcategories in which we only select a collection of objects and then take *all* morphisms between those objects:

Construction 2.6.18. Consider an embedding $\Gamma \hookrightarrow C^{\simeq}$. We define M_Γ as the following pullback:

$$\begin{tikzcd}
M_\Gamma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}([1],C) \arrow[d, "{(s,t)}"] \\
\Gamma \times \Gamma \arrow[r, hook] & C^{\simeq} \times C^{\simeq}.
\end{tikzcd}$$

We refer to $\langle M_\Gamma\rangle_C$ as the *full subcategory spanned by the objects in* Γ.

2.6.6 Localizations

Another important ∞-categorical construction is that of *localizations*, where we formally invert a collection of morphisms in an ∞-category:

Axiom J.1 (Localization axiom)**.** For every ∞-category C and every collection of morphisms $W \hookrightarrow \mathrm{Map}([1], C)$, there exists a functor $l\colon C \to C[W^{-1}]$ exhibiting $C[W^{-1}]$ as a localization of C at the morphisms in W:

- The functor l sends the morphisms in W to isomorpisms $C[W^{-1}]$;
- The functor l is *universal* with respect to the previous property: for every other ∞-category D, precomposition with l induces a fully faithful functor

  $$l^*\colon \mathrm{Fun}(C[W^{-1}], D) \hookrightarrow \mathrm{Fun}(C, D)$$

  whose essential image is the full subcategory of $\mathrm{Fun}(C, D)$ spanned by those functors $f\colon C \to D$ which sends morphisms of C contained in W to isomorphisms in D.

The last condition in particular means that there is a pullback square of the form

$$\begin{tikzcd} \mathrm{Map}(C[W^{-1}], D) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(C, D) \arrow[d] \\ \mathrm{Map}(W, \mathrm{Iso}(D)^{\simeq}) \arrow[r, hook] & \mathrm{Map}(W, \mathrm{Map}([1], D)). \end{tikzcd}$$

Definition 2.6.19. Given a collection of morphisms W in an ∞-category C, we say that a functor $l\colon C \to D$ *exhibits D as a localization of C at the morphisms in W* if it satisfies properties (1) and (2) of the axiom. In this case, l induces a functor $C[W^{-1}] \to D$ which is an equivalence.

Example 2.6.20 (Exercise 6.4)**.** The functor $p_{[1]}\colon [1] \to *$ exhibits $*$ as a localization of $[1]$ at all its morphisms.

Example 2.6.21 (Exercise 5.2)**.** Recall the functors

$$p_0\colon [1]\times[1] \to [2] \qquad \text{and} \qquad p_2\colon [1]\times[1] \to [2]$$

from Remark 2.4.5, pictorially represented by the following two diagrams:

$$p_0 \;=\; \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad\qquad p_2 \;=\; \begin{tikzcd} 0 \arrow[r] \arrow[d] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

Then p_0 exhibits $[2]$ as a localization of $[1]\times[1]$ at the morphism $(0,0) \to (0,1)$, while p_2 exhibits $[2]$ as a localization of $[1]\times[1]$ at the morphism $(0,1) \to (1,1)$.

In the case where W consists of *all* morphisms in C, we denote the localization by $|C|$ and call it the *geometric realization of* C.

Axiom J.2 (Geometric realization axiom)**.** For every ∞-category C, the geometric realization $|C|$ is an anima.

Remark 2.6.22. The terminology 'geometric realization' will be justified in Lemma A.5.6, where we will show that $|C|$ is equivalent to the geometric realization (i.e. colimit) of the nerve $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ of C.

Observation 2.6.23. Let C and D be ∞-categories and let W and W' collections of morphisms in C and D, respectively. Then the maps $(C\times D)[(W\times W')^{-1}] \to C[W^{-1}]$ and $(C\times D)[(W\times W')^{-1}] \to D[W'^{-1}]$ induced by the projection maps define an equivalence

$$(C\times D)[(W\times W')^{-1}] \xrightarrow{\sim} C[W^{-1}]\times D[W'^{-1}].$$

Indeed, this follows immediately from the universal properties of both sides, together with the fact that a functor $F\colon C\times D \to E$ inverts all morphisms in $W\times W'$ if and only if it inverts all morphisms of the form $w\times \mathrm{id}_d$ for $d\in D$ and $w\in W$ and all morphisms of the form $\mathrm{id}_c \times w'$ for $c\in C$ and $w'\in W'$.

As aspecial case, we see that the projections induce an equivalence

$$|C\times D| \xrightarrow{\sim} |C|\times|D|$$

on geometric realizations.

2.6.7 Limits and colimits

Very important in category theory are the notions of limits and colimits. When formulated correctly, their definition immediately provides a definition for limits and colimits in the ∞-categorical setting as well:

Definition 2.6.24 (Limit/colimit)**.** Let I and C be ∞-categories and let $F\colon I\to C$ be a functor.

(1) A *cone on* F is an object X in C together with a natural transformation $\varepsilon\colon \mathrm{const}_X \to F$ of functors $I\to C$.

(2) A cone (X,ε) is called a *limit cone* if for every other object Y of C the composite

$$\mathrm{Hom}_C(Y,X) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_Y,\mathrm{const}_X) \xrightarrow{\varepsilon\circ -} \mathrm{Nat}(\mathrm{const}_Y, F)$$

is an equivalence.

In this case, the object X is called the *limit* of F. It is unique up to contractible choice, and will be denoted either by $\lim_I F$ or $\lim_{i\in I} F(i)$.

Dually, one defines a *cocone* to be a natural transformation $\eta\colon F \to \mathrm{const}_X$. It is a *colimit cone* if for every Y the composite

$$\mathrm{Hom}_C(X,Y) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_X, \mathrm{const}_Y) \xrightarrow{-\circ\eta} \mathrm{Nat}(F, \mathrm{const}_Y)$$

is an equivalence.

Observation 2.6.25. Note that a functor $F\colon I \to C$ admits a limit in C if and only if the opposite functor $F^{\mathrm{op}}\colon I^{\mathrm{op}} \to C^{\mathrm{op}}$ admits a colimit in C^{op}.

Definition 2.6.26. We say an ∞-category C *admits all I-indexed (co)limits* if for every functor $F\colon I \to C$ there exists a (co)limit (co)cone for F. If $G\colon C \to D$ is a functor between ∞-categories that admit I-indexed (co)limits, then we say that G *preserves I-indexed (co)limits* if for every functor $F\colon I \to C$, the functor G sends (co)limit (co)cones of F in C to (co)limit (co)cones of GF in D.

It is sometimes useful to rephrase (co)cones in terms of functors rather than natural transformations:

Definition 2.6.27 (Cocone ∞-categories)**.** For an ∞-category I, we define the ∞-category $I^{\triangleright}$, called the *cocone of I*, via the following pushout square of ∞-categories:

$$\begin{tikzcd} I\times\{1\} \arrow[r, hook] \arrow[d] & I\times[1] \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & I^{\triangleright}. \end{tikzcd}$$

We will often denote the object classified by the bottom morphism as ∞. Note that a functor $\overline{F}\colon I^{\triangleright} \to C$ consists of a functor $I\times[1] \to C$ whose restriction to $I\times\{1\}$ is constant. Equivalently, it consists of a functor $F := \overline{F}|_{I\times\{0\}}\colon I \to C$ and an object $W := \overline{F}(\infty)$ in C together with a cocone $F \to \mathrm{const}_W$.

We say that a functor $\overline{F}\colon I^{\triangleright} \to C$ is a *colimit diagram* if the associated cocone $F \to \mathrm{const}_W$ is a colimit cocone.

We dually define $I^{\triangleleft}$, called the *cone of I*, as the following pushout of ∞-categories:

$$\begin{tikzcd} I\times\{0\} \arrow[r, hook] \arrow[d] & I\times[1] \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & I^{\triangleleft}. \end{tikzcd}$$

A functor $\overline{F}\colon I^{\triangleleft} \to C$ consists of a functor $F\colon I \to C$ together with a cone $\mathrm{const}_W \to F$. We say that $\overline{F}$ is a *limit diagram* if this cone is a limit cone.

The notion of limits and colimits are closely related to that of *adjunctions*, introduced in Appendix A.2. We discuss the relations between (co)limits and adjunctions in Appendix A.2.3.

Let us discuss some special cases of limits/colimits that will be relevant.

Initial and terminal objects

Definition 2.6.28 (Initial/terminal objects)**.** An object X of an ∞-category C is called *terminal* if for every other object Y the Hom anima $\mathrm{Hom}_C(Y,X)$ is contractible. We say that X is *initial* if for every other Y the Hom anima $\mathrm{Hom}_C(X,Y)$ is contractible.

Lemma 2.6.29. *The terminal objects of C are precisely the limits of the unique functor $F\colon \emptyset \to C$.*

Proof. First assume that X is a limit of $F\colon \emptyset \to C$. Since $\mathrm{Fun}(\emptyset, C)$ is contractible, it follows that $\mathrm{Nat}(\mathrm{const}_Y, F)$ is contractible for all Y, hence also $\mathrm{Hom}_C(Y,X)$ is contractible. This shows that X is terminal.

Conversely, assume that X is terminal. Since $\mathrm{Fun}(\emptyset, C) \simeq *$, there is a unique natural transformation $\varepsilon\colon \mathrm{const}_X \to F$. We need to show that for every Y the map $\mathrm{Hom}_C(Y,X) \to \mathrm{Nat}(\mathrm{const}_Y, F)$ is an equivalence. But this is clear since both sides are contractible. $\square$

Pullbacks and pushouts

Definition 2.6.30. Consider the category $\lrcorner$ that has three objects 0, 1 and 2 and two non-identity morphisms $0 \to 2$ and $1 \to 2$:

$$\lrcorner \quad := \quad \begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

The two non-identity morphisms induce two functors $[1] \to \lrcorner$ whose targets agree, so that we get a commutative square of the following form:

$$\begin{tikzcd} {[0]} \arrow[r, "d_0"] \arrow[d, "d_0"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & \lrcorner . \end{tikzcd} \tag{2.1}$$

Axiom K (Pullback diagram axiom)**.** The commutative square (2.1) is a pushout square of ∞-categories.

Unwinding definitions, this axiom says that a functor $F\colon \lrcorner \to C$ corresponds to a pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in C.

We will now provide an explicit description of limits over $\lrcorner$-indexed diagrams. Given a fourth object W of C, a cone on $F\colon \lrcorner \to C$ a priori consists of a commutative diagram

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d] & W \arrow[r, equal] \arrow[d] & W \arrow[d] \\ X \arrow[r, "f"] & Z & Y. \arrow[l, "g"'] \end{tikzcd}$$

The following lemma lets us equivalently encode this data as a commutative square:

Lemma 2.6.31. *The functor $\lrcorner \times [1] \to [1]\times[1]$ described informally by the diagram*

$$\begin{tikzcd} (0,0) \arrow[r, equal] \arrow[d] & (0,0) \arrow[r, equal] \arrow[d] & (0,0) \arrow[d] \\ (1,0) \arrow[r] & (1,1) & (0,1) \arrow[l] \end{tikzcd}$$

exhibits $[1]\times[1]$ as a localization of $\lrcorner \times [1]$ at the morphisms $(0,0) \to (2,0)$ and $(1,0) \to (2,0)$.

Proof. Since $\lrcorner \times [1]$ is a pushout of two copies of $[1]\times[1]$ and $[1]\times[1]$ is a pushout of two copies of $[2]$, we may construct such a functor by gluing two copies of the functor $p_0\colon [1]\times[1] \to [2]$ from Remark 2.4.5. We need to show that for any ∞-category C, precomposition with this functor induces a fully faithful functor

$$\mathrm{Fun}([1]\times[1], C) \hookrightarrow \mathrm{Fun}(\lrcorner \times [1], C)$$

whose essential image consists of those functors $\lrcorner \times [1] \to C$ inverting the two morphisms $(0,0) \to (2,0)$ and $(1,0) \to (2,0)$. Since the ∞-categories $[1]\times[1]$ and $\lrcorner \times [1]$ are pushouts, we may rewrite this map (up to equivalence) as follows:

$$p_0^* \times_{\mathrm{id}} p_0^*\colon \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([2], C) \to \mathrm{Fun}([1]\times[1]) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([1]\times[1], C).$$

The claim now follows from the fact that $p_0\colon [1]\times[1] \to [2]$ is a localization at the morphism $(0,0) \to (0,1)$, see Example 2.6.21. □

Corollary 2.6.32. *Consider a functor $F\colon \lrcorner \to C$, corresponding to morphisms $f\colon X \to Z$ and $g\colon Y \to Z$, and let W be an object of C. Then the functor from the lemma induces an equivalence of ∞-categories*

$$\left\{ \text{Commutative squares } \begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd} \right\} \xrightarrow{\sim} \mathrm{Nat}\left(\begin{tikzcd} & W \arrow[d, equal] & & X \arrow[d, "f"] \\ W \arrow[r, equal] & W, & Y \arrow[r, "g"] & Z \end{tikzcd} \right).$$

Proof. By definition, the right-hand side is defined as the Hom anima in $\mathrm{Fun}(\lrcorner, C)$ between the diagrams const_W and F. Because we have an equivalence

$$\mathrm{Fun}([1], \mathrm{Fun}(\lrcorner, C)) \xrightarrow{\sim} \mathrm{Fun}(\lrcorner \times [1], C),$$

this means that the right-hand side is equivalent to the fiber over (const_W, F) of the functor

$$\mathrm{Fun}(\lrcorner \times [1], C) \to \mathrm{Fun}(\lrcorner \times \{0\}, C) \times \mathrm{Fun}(\lrcorner \times \{1\}, C).$$

By the previous lemma, this fiber is equivalent to the fiber over (W, F) of the functor

$$\mathrm{Fun}([1] \times [1], C) \to \mathrm{Fun}(\{(0,0)\}, C) \times \mathrm{Fun}(\lrcorner, C),$$

which is by definition what the left-hand side of the corollary means. $\square$

Definition 2.6.33. We say a commutative square in C of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a *pullback square* if the corresponding cone $\mathrm{const}_W \to F$ is a limit cone in C. In this case, we also write $W = X \times_Z Y$.

The story for pushouts is entirely dual:

Definition 2.6.34 (Pushout squares)**.** We may consider the category $\ulcorner := \lrcorner^{\mathrm{op}}$, which we may display as follows:

$$\ulcorner \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2. & \end{tikzcd}$$

Using Proposition 2.6.11, we see that the commutative square

$$\begin{tikzcd} {[0]} \arrow[r, "d_1"] \arrow[d, "d_1"'] & {[1]} \arrow[d] \\ {[1]} \arrow[r] & \ulcorner \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout square of ∞-categories, and hence a functor $F \colon \ulcorner \to C$ corresponds to a pair of morphisms $f \colon Z \to X$ and $g \colon Z \to Y$. By dualizing the argument for cones on $\lrcorner$ from before (replacing the functor p_0 by $p_2 \colon [1] \times [1] \to [2]$) one sees that a cocone $F \to \mathrm{const}_W$ may equivalently be encoded as a commutative square in C of the form

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] \\ Y \arrow[r] & W. \end{tikzcd}$$

We say that it is a *pushout square* if it corresponds to a colimit cocone. In this case we write $W = X \sqcup_Z Y$.

2.6.8 The ∞-categories Cat_∞ and An

We will introduce the ∞-categories Cat_∞ and An of (small) ∞-categories and (small) animae.

Axiom L. There is an ∞-category Cat_∞ whose objects are called *small ∞-categories*. Every object $c \in \mathrm{Cat}_\infty$ defines an ∞-category C. For objects $c, d \in \mathrm{Cat}_\infty$ corresponding to ∞-categories C and D, there is an equivalence of animae

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(c, d) \xrightarrow{\sim} \mathrm{Map}(C, D),$$

and in particular morphisms in Cat_∞ correspond to functors of ∞-categories. Composition in Cat_∞ corresponds to composition of functors.

The ∞-category Cat_∞ admits all I-limits and I-colimits for every small ∞-category I.

The ∞-category Cat_∞ is closed under all constructions of ∞-categories introduced so far: initial/terminal ∞-categories, (co)products, pullbacks, functor categories, groupoid cores, subcategories, and localizations. The ∞-categories $[n]$ are small for all $n \geq 0$.

Remark 2.6.35. Axiom L is a somewhat ad hoc and informal approximation of a ‘true’ axiom that would introduce Cat_∞ via a concept called ‘directed univalence’. Setting up the required theory here would take too much time, so we will have to do with Axiom L. The interested reader may consult [Cis+24, Chapter 7] for more details.

Remark 2.6.36. There is a functor $\mathrm{qCat} \to \mathrm{Cat}_\infty$ which exhibits Cat_∞ as the localization of qCat at the equivalences of quasicategories.

In terms of quasicategories, the ∞-category Cat_∞ may be modeled as the homotopy-coherent nerve $N^\Delta(\mathbf{qCat})$ of the Kan-enriched category **qCat** from Example 2.5.17. We will not use this perspective.

Definition 2.6.37. We define the *∞-category of animae* as the full subcategory

$$\mathrm{An} \subseteq \mathrm{Cat}_\infty$$

spanned by the animae.

Theorem 2.6.38. *The subcategory* $\mathrm{An} \subseteq \mathrm{Cat}_\infty$ *is closed under all limits and colimits. Each of the following four vertical functors*

$$\begin{tikzcd}
\mathrm{CW} \arrow[r, hook] \arrow[drr, "\Pi_\infty"'] & \mathrm{Top} \arrow[r, "\mathrm{Sing}"] \arrow[dr, "\Pi_\infty"] & \mathrm{Kan} \arrow[r, hook] \arrow[d, "\Pi_\infty"] & \mathrm{sSet} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) \arrow[dl, "|-|:=\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}"] \\
& & \mathrm{An} &
\end{tikzcd}$$

is a localization, inducing equivalences of ∞-categories

$$\begin{aligned}\mathrm{An} &\simeq \mathrm{CW}[\text{homotopy equivalences}^{-1}]\\ &\simeq \mathrm{Top}[\text{weak homotopy equivalences}^{-1}]\\ &\simeq \mathrm{Kan}[\text{homotopy equivalences}^{-1}]\\ &\simeq \mathrm{sSet}[\text{weak equivalences}^{-1}].\end{aligned}$$

When X is a topological space/CW-complex/Kan complex, we refer to the anima $\Pi_\infty(X)$ as its *fundamental ∞-groupoid* or *underlying anima*.

Remark 2.6.39. Since the ∞-category An is a localization of the category of topological spaces, it is often called the *∞-category of spaces* in the literature and is denoted either by $\mathcal{S}$ or by Spc. This explains Convention 1.2.3: we use the word 'topological space' if we want to think of X as an object of Top, but if we use the word 'space' then it means we are thinking of it as an object of An. Throughout the rest of the course, we will mostly use the word 'anima' instead of 'space', though at times it is certainly useful for intuition to think of an anima as a space.

We will see in Propositions B.2.8 and B.3.8 below that the localization functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ sends homotopy pushouts/pullbacks of spaces to pushouts/pullbacks in An.

Remark 2.6.40. In terms of quasicategories, the ∞-category An can be modeled by the quasicategory $N^\Delta(\mathbf{Kan})$, the homotopy-coherent nerve of the Kan-enriched category **Kan** from Example 2.5.16. We will not use this perspective.

Remark 2.6.41. The Hom animae $\mathrm{Hom}_C(X,Y)$ assemble into a *Hom functor*[7]

$$\mathrm{Hom}_C\colon C^{\mathrm{op}} \times C \to \mathrm{An}.$$

In particular every object $X \in C$ defines a functor $Y_X := \mathrm{Hom}_C(-,X)\colon C^{\mathrm{op}} \to \mathrm{An}$.

The following theorem is again taken as a black box:

Theorem 2.6.42 (Yoneda lemma)**.** *For a functor $F\colon C^{\mathrm{op}} \to \mathrm{An}$ and an object $X \in C$ there is an equivalence*

$$\mathrm{Nat}(Y_X, F) \xrightarrow{\sim} F(X), \qquad (\varphi\colon Y_X \to F) \mapsto \varphi(\mathrm{id}_X).$$

In particular the Yoneda embedding

$$Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}), \qquad X \mapsto \mathrm{Hom}_C(-,X)$$

is fully faithful.

[7] Formally defining this functor takes some work.

Theorem 2.6.43. *The functor* $\mathrm{Hom}_C(-,-)\colon C^{\mathrm{op}}\times C \to \mathrm{An}$ *preserves limits in both variables separately.*

Proof sketch. The claim means that for an object X in C the functors

$$\mathrm{Hom}_C(X,-)\colon C \to \mathrm{An} \qquad \text{and} \qquad \mathrm{Hom}_C(-,X)\colon C^{\mathrm{op}} \to \mathrm{An}$$

preserve limits. Note that the claim for $\mathrm{Hom}_C(-,X)$ may be reduced to that for $\mathrm{Hom}_C(X,-)$ by replacing C by C^{op}. For $\mathrm{Hom}_C(X,-)$, we must show that for any functor $F\colon I \to C$ whose limit exists in C the canonical map

$$\mathrm{Hom}_C(X,\lim_{i\in I} F(i)) \to \lim_{i\in I} \mathrm{Hom}_C(X,F(i))$$

is an equivalence. By definition of a limit, we know that the map of animae

$$\mathrm{Hom}_C(X,\lim_{i\in I} F(i)) \to \mathrm{Nat}(\mathrm{const}_X,F)$$

is an equivalence, where the right-hand side denotes the Hom anima in $\mathrm{Fun}(I,C)$. The proof thus essentially boils down to computing Hom animae in functor categories. This is non-trivial to do, but let us just say that in this case there is an equivalence

$$\mathrm{Nat}(\mathrm{const}_X,F) \xrightarrow{\sim} \lim_{i\in I} \mathrm{Hom}_C(X,F(i))$$

compatible with the two maps from $\mathrm{Hom}_C(X,\lim_{i\in I}(F(i)))$. □

2.6.9 Adjunctions

The final aspect of ∞-category theory that I wish to introduce is the notion of an *adjunction*:

Definition 2.6.44. Let $F\colon C \to D$ and $G\colon D \to C$ be two functors. An *adjunction* between C and D consists of a natural isomorphism

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}}\times D \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism has been given.

As in ordinary category theory, adjunctions may equivalently be characterized in terms of a unit $\eta\colon \mathrm{id}_C \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}_D$ satisfying the triangle identities $(\varepsilon F)(F\eta) \cong \mathrm{id}_F$ and $(G\varepsilon)(\eta G) \cong \mathrm{id}_G$. An important feature of adjunctions is that they can be characterized *pointwise*: given a functor $F\colon C \to D$, if one can provide for every object Y of D a suitable object $G(Y)$ of C equipped with a map $\varepsilon_Y\colon FG(Y) \to Y$ satisfying a suitable adjointness condition, then these objects automatically assemble into a right adjoint $G\colon D \to C$, and the maps ε_Y assemble into a counit ε for the adjunction.

The theory of adjunctions is very interesting and there is a lot to say about it. Details on the pointwise characterization of adjunctions, along with related concepts such as Kan extensions, (co)final functors, and Bousfield localizations, can be found in Appendix A.2.

3 Stable ∞-categories

In this chapter we introduce the notion of a *stable* ∞-category. A good reference for this material is Chapter 1 of Lurie's book *Higher Algebra* [Lur17].

3.1 Pointed ∞-categories

Recall from Section 2.6.7 that an object X of an ∞-category C is called *terminal* if $\mathrm{Hom}_C(Y,X)$ is contractible for all $Y \in C$. Dually, X is *initial* if instead $\mathrm{Hom}_C(X,Y)$ is contractible for all $Y \in C$.

Definition 3.1.1 (Zero object)**.** We call an ∞-category C *pointed* if it has a *zero object*, i.e. an object 0 that is both initial and terminal.

Example 3.1.2. The category of abelian groups is pointed: the zero object is the trivial group 0. More generally, any additive category is pointed.

There is a universal way to turn an ∞-category C with a terminal object $*$ into a pointed ∞-category:

Definition 3.1.3 (Pointed objects)**.** If C is an ∞-category with a terminal object $*$, we define its *∞-category of pointed objects* via the following pullback square:

$$\begin{tikzcd} C_* \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_0"] \\ * \arrow[r, "*"] & C. \end{tikzcd}$$

In other words, C_* is the slice category $C_{*/}$ from Definition A.2.26. Objects of C_* are objects X of C equipped with a map $x\colon * \to X$ from the terminal object. We will frequently denote such objects by (X,x).

Lemma 3.1.4. *Let C be an ∞-category with a terminal object $*$.*

(1) The ∞-category C_* is pointed;

(2) The ∞-category C is pointed if and only if the forgetful functor $C_* \to C$ is an equivalence.

Proof. (1) Recall from Lemma A.2.27 that the Hom anima in C_* between two pointed objects (X,x) and (Y,y) may be computed as the fiber

$$\begin{tikzcd}
\mathrm{Hom}_{C_*}((X,x),(Y,y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_C(X,Y) \arrow[d, "-\circ x"] \\
* \arrow[r, "y"] & \mathrm{Hom}_C(*,Y).
\end{tikzcd}$$

It follows immediately from this formula that the pointed object $(*, \mathrm{id}_*\colon * \to *)$ is both initial and terminal in C_*, and hence a zero object.

(2) The 'only if' part is clear from (1). Conversely, assume that C is pointed. Then the object $*$ is also an initial object in C, and hence the target functor $C_* = C_{*/} \to C$ is an equivalence by Lemma A.2.29. □

For future purposes, we will record the precise sense in which C_* is the 'universal' pointed ∞-category mapping to C.

Notation 3.1.5. Let Cat_∞^* denote the subcategory of Cat_∞ spanned by ∞-categories with a terminal object and functors which preserve terminal objects. Let $\mathrm{Cat}_\infty^{\mathrm{pt}} \subseteq \mathrm{Cat}_\infty^*$ denote the full subcategory spanned by the pointed ∞-categories.

Corollary 3.1.6. *The inclusion $\mathrm{Cat}_\infty^{\mathrm{pt}} \hookrightarrow \mathrm{Cat}_\infty^*$ admits a right adjoint*

$$\mathrm{Cat}_\infty^* \to \mathrm{Cat}_\infty^{\mathrm{pt}}, \qquad C \mapsto C_*,$$

with counit given by the forgetful functor $t\colon C_* \to C$.

Proof. We will use the (dual of the) criterion from Proposition A.2.63. The construction $C \mapsto C_*$ determines a functor $R := (-)_*\colon \mathrm{Cat}_\infty^* \to \mathrm{Cat}_\infty^*$, and the forgetful functors $C_* \to C$ determine a natural transformation $\varepsilon\colon R \to \mathrm{id}_{\mathrm{Cat}_\infty^*}$. We need to check that for every C the two functors

$$\varepsilon_{C_*}\colon (C_*)_* \to C_* \qquad \text{and} \qquad (\varepsilon_C)_*\colon (C_*)_* \to C_*$$

are equivalences. For the first this is an instance of Lemma 3.1.4. The second follows immediately from the first, using the fact that the 'swap of components'-map $\mathrm{Fun}([1]\times[1],C) \simeq \mathrm{Fun}([1]\times[1],C)$ induces an equivalence $(C_*)_* \simeq (C_*)_*$ which makes the following triangle commute:

$$\begin{tikzcd}
(C_*)_* \arrow[rr, "\simeq"] \arrow[dr, "\varepsilon_{C_*}"'] & & (C_*)_* \arrow[dl, "(\varepsilon_C)_*"] \\
& C_*. &
\end{tikzcd}$$

□

Corollary 3.1.7. *Let $C, D \in \mathrm{Cat}_\infty^*$ and assume that D is pointed. Then the forgetful functor $t\colon C_* \to C$ induces an equivalence*

$$t_*\colon \mathrm{Fun}^*(D, C_*) \xrightarrow{\sim} \mathrm{Fun}^*(D, C),$$

where $\mathrm{Fun}^*(-,-)$ *denotes the full subcategory of the functor category spanned by those functors that preserve the terminal object.*

Proof. By the Yoneda lemma it suffices to show that

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(E, \mathrm{Fun}^*(D, C_*)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cat}_\infty}(E, \mathrm{Fun}^*(D, C))$$

for all $E \in \mathrm{Cat}_\infty$. By currying and uncurrying, this is equivalent to the condition that the map

$$\mathrm{Hom}_{\mathrm{Cat}_\infty^*}(D, \mathrm{Fun}(E, C_*)) \to \mathrm{Hom}_{\mathrm{Cat}_\infty^*}(D, \mathrm{Fun}(E, C))$$

is an equivalence. Since $\mathrm{Fun}(E,-)$ preserves pullbacks we see that $\mathrm{Fun}(E, C_*) \simeq \mathrm{Fun}(E, C)_*$, and so this is an instance of the adjunction from Corollary 3.1.6. $\square$

3.2 Stable ∞-categories

Recall from Section 2.6.7 the definitions of pushouts and pullbacks in an ∞-category, defined as colimits and limits along the indexing categories $\ulcorner$ and $\lrcorner$, respectively.

Definition 3.2.1 (Stable ∞-category)**.** An ∞-category C is called *stable* if:

(1) There exists a zero object $0 \in C$;

(2) Pushouts and pullbacks exist in C;

(3) A commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\ W \arrow[r, "k"] & Z \end{tikzcd}$$

in C is a pullback square if and only if it is a pushout square.

The squares in (3) are called *exact squares*.

Important examples of exact squares are the *exact sequences*:

Definition 3.2.2 (Exact sequence)**.** A *null-sequence* in a pointed ∞-category C is a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z, \end{tikzcd}$$

where 0 is a zero object. We call it a *fiber sequence* if it is a pullback square in C and a *cofiber sequence* if it is a pushout square. In these cases, we write $X = \mathrm{fib}(g)$ and $Z = \mathrm{cofib}(f)$, respectively. We will call it an *exact sequence* if it is both a fiber sequence as well as a cofiber sequence. We will frequently use the simplified notation

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

for null-sequences, leaving the null-homotopy of $gf\colon X \to Z$ implicit in the notation.

Definition 3.2.3 (Loop functor)**.** Let C be a pointed ∞-category that admits fibers. For an object $X \in C$ we define ΩX as the fiber of $0 \to X$:

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd} \tag{3.1}$$

Remark 3.2.4. The assignment $X \mapsto \Omega X$ can be upgraded to a functor $\Omega\colon C \to C$. To see this, we will show that sending X to the diagram (3.1) determines a functor $C \to \mathrm{Fun}(\lrcorner, C)$; we may then compose with the limit functor to obtain Ω. Using the equivalence $C \simeq C_* \subseteq \mathrm{Fun}([1], C)$ from Lemma 3.1.4, the functoriality of (3.1) is then given by the following composite:

$$C \simeq C \times_C C \simeq C_* \times_C C_* \hookrightarrow \mathrm{Fun}([1], C) \times_{\mathrm{ev}_1, C, \mathrm{ev}_1} \mathrm{Fun}([1], C) \simeq \mathrm{Fun}(\lrcorner, C).$$

Definition 3.2.5 (Suspension functor)**.** Let C be a pointed ∞-category that admits cofibers. For $X \in C$ we define the *suspension* ΣX of X as the cofiber of $X \to 0$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \Sigma X. \end{tikzcd}$$

The assignment $X \mapsto \Sigma X$ assembles into a functor $\Sigma\colon C \to C$ just as in Remark 3.2.4.

Exercise 3.2.6. Let C be a pointed ∞-category that admits fibers and cofibers. Show that the functor $\Sigma\colon C \to C$ is left adjoint to $\Omega\colon C \to C$: for all $X, Y \in C$ there is a natural equivalence

$$\mathrm{Hom}_C(\Sigma X, Y) \simeq \mathrm{Hom}_C(X, \Omega Y).$$

It turns out that stability may be completely characterized in terms of fibers and cofibers:

Theorem 3.2.7. *For a pointed ∞-category C, the following three conditions are equivalent:*

(1) The ∞-category C is stable;

(2) The ∞-category C admits both fibers and cofibers, and a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a fiber sequence if and only if it is a cofiber sequence;

(3) The ∞-category C admits fibers and the functor $\Omega\colon C \to C$ is an equivalence;

(4) The ∞-category C admits cofibers and the functor $\Sigma\colon C \to C$ is an equivalence.

Proof. It is clear that (1) implies (2) by taking $W = 0$ in the defining property of stability. If we assume (2), then the fiber sequence

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X
\end{tikzcd}$$

is by assumption also a pushout square and thus the counit map $\Sigma\Omega X \to X$ is an equivalence. Similarly the unit map $X \to \Omega\Sigma X$ is an equivalence. It follows that Σ and Ω are inverse equivalences, so that (3) and (4) hold.

We now show that (3) implies (1); applying the argument to C^{op} will then show that also (4) implies (1). We will first show that C admits pushouts and that pushout squares agree with pullback squares. To this end, consider the full subcategory

$$P \subseteq \mathrm{Fun}([1] \times [1], C)$$

spanned by the pullback squares. We then have the following claim:

Claim: The restriction functor $P \to \mathrm{Fun}(\ulcorner, C)$ is an equivalence.

Proof of claim: To construct an inverse, consider a diagram $z \leftarrow x \to y$ in C, and consider the following pullback diagram (ignoring the dashed morphisms):

$$\begin{tikzcd}
\Omega x \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Omega z \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[d] & \\
\Omega y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & a \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & b \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[d] \\
0 \arrow[r] & c \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & x \arrow[r] \arrow[d] & y \arrow[d, dashed] \\
& 0 \arrow[r] & z \arrow[r, dashed] & \Omega^{-1} a.
\end{tikzcd}$$

The desired inverse functor is now given by

$$\left(\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}\right) \quad \mapsto \quad \Omega^{-1}\left(\begin{tikzcd} \Omega x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Omega y \arrow[d] \\ \Omega z \arrow[r] & a \end{tikzcd}\right).$$

Note that this functor does indeed land in P since the equivalence $\Omega^{-1}\colon C \xrightarrow{\sim} C$ preserves pullback squares. Furthermore, it provides an extension of the original diagram, hence defines a right inverse. Finally, if we start with any pullback square, we may rewrite it in the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & y \arrow[d] \\ z \arrow[r] & \Omega^{-1}a, \end{tikzcd}$$

and we see that the above procedure recovers the original pullback square, showing that it is also a left inverse. This finishes the proof of the claim. □

We will now use the claim to produce pushouts in C. Consider again a diagram $z \leftarrow x \to y$ in C, and extend it uniquely to a pullback diagram:

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & y \arrow[d] \\ z \arrow[r] & w. \end{tikzcd}$$

We claim that this square is also a pushout square in C. Indeed, for every other object a in C we have

$$\mathrm{Nat}\left(\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}, \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] & a \\ a & \end{tikzcd}\right) \simeq \mathrm{Nat}\left(\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & y \arrow[d] \\ z \arrow[r] & w \end{tikzcd}, \begin{tikzcd} a \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner" very near start] & a \arrow[d, equal] \\ a \arrow[r, equal] & a \end{tikzcd}\right) \simeq \mathrm{Hom}_C(w,a),$$

where the last equivalence holds because $[1]\times[1]$ has a terminal object $(1,1)$. This computation shows that C admits pushouts and that pushout squares agree with pullback squares.

Note that in particular C admits cofibers and that $\Sigma\colon C \to C$ is an equivalence, so applying the above logic to C^{op} we see that C also admits all pullbacks, and hence is stable. This finishes the proof. □

Remark 3.2.8 (Abelian categories)**.** To get some intuition for stable ∞-categories, observe that condition (2) in the previous proposition is very similar to the definition of an abelian category. Recall that an abelian category $\mathcal{A}$ is an ordinary category satisfying:

(1) $\mathcal{A}$ is semiadditive: it admits a zero object 0 and binary biproducts $X \oplus Y$;

(2) $\mathcal{A}$ admits all kernels and cokernels;

(3) All monomorphisms and epimorphisms in $\mathcal{A}$ are normal:

   a) Every monomorphism $i\colon A \hookrightarrow B$ is the kernel of the quotient map $B \twoheadrightarrow \mathrm{coker}(i)$;

   b) Every epimorphism $p\colon B \twoheadrightarrow C$ is the cokernel of the inclusion $\ker(p) \hookrightarrow B$.

Note that condition (3) is equivalent to the condition that for every pair of morphisms

$$A \overset{i}{\hookrightarrow} B \overset{p}{\twoheadrightarrow} C$$

satisfying $pi = 0$, if i is a monomorphism and p is an epimorphism then this sequence is a fiber sequence (i.e. $A \cong \ker(p)$) if and only if it is a cofiber sequence (i.e. $C = \mathrm{coker}(i)$). In a stable ∞-category, the notions of 'monomorphism' and 'epimorphism' are not well-behaved, so we simply demand condition (3) for *all* morphisms i and p. It turns out that in this case we may weaken the assumption in (1): it turns out that stable ∞-categories are automatically additive (see Lemma 3.3.3 below).

Warning 3.2.9. Based on questions during/after the lecture, let me emphasize: when we regard an abelian category $\mathcal{A}$ as an ∞-category, it is *never* stable unless it is the trivial category (since the loop functor $\Omega\colon \mathcal{A} \to \mathcal{A}$ is the zero-functor but by Theorem 3.2.7 it needs to be an equivalence in order for $\mathcal{A}$ to be stable). More generally, an ordinary category can never be a stable ∞-category. The previous remark merely illustrates an analogy which may help to evoke some intuitions.

Something we *can* do is that we may associate to every abelian category $\mathcal{A}$ a stable ∞-category $\mathcal{D}(\mathcal{A})$ called the *derived ∞-category* of $\mathcal{A}$, obtained from the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ by inverting the quasi-isomorphisms:

$$\mathcal{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

It comes with a fully faithful inclusion $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$. Usual constructions from homological algebra (like derived tensor products or derived Hom (a.k.a. Ext) functors) are naturally formulated within $\mathcal{D}(\mathcal{A})$.

Exact functors

When working with stable ∞-categories, we usually want to restrict attention to *exact* functors between them:

Definition 3.2.10 (Exact and pointed functors)**.** A functor $F: C \to D$ between pointed ∞-categories is *pointed* if it preserves zero objects. If C and D are stable, then a functor $F: C \to D$ is called *exact* if it is pointed and sends exact sequences to exact sequences.

Proposition 3.2.11. *Let $F: C \to D$ be a functor between stable ∞-categories. Then the following are equivalent:*

(1) The functor F is exact;

(2) The functor F preserves initial objects and pushouts;

(3) The functor F preserves terminal objects and pullbacks.

Proof. Since C and D are pointed, it is clear that F is pointed in each of the three cases. The argument from Theorem 3.2.7 shows that a pointed functor preserves exact sequences if and only if it preserves pushout squares if and only if it preserves pullback squares. $\square$

Since exact functors are clearly closed under composition, the following definition makes sense:

Definition 3.2.12. We define the *∞-category of small stable ∞-categories* as the (non-full) subcategory

$$\mathrm{Cat}_\infty^{\mathrm{st}} \subseteq \mathrm{Cat}_\infty$$

spanned by the stable ∞-categories and the exact functors between them.

Theorem 3.2.13 ([Lur17, Theorem 1.1.4.4])**.** *The ∞-category* $\mathrm{Cat}_\infty^{\mathrm{st}}$ *admits small limits, and the inclusion* $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty$ *preserves small limits.*

3.3 The structure of stable ∞-categories

Let us now establish some general properties of stable ∞-categories.

3.3.1 Additivity

We start with additivity:

Definition 3.3.1 ((Semi)additive ∞-category)**.** An ∞-category C is *semiadditive* if

(1) It admits finite products and coproducts;

(2) It has a zero-object 0;

(3) It admits *biproducts*: for any $X, Y \in C$, the map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$

is an equivalence.

In this case, we will write $X \oplus Y$ for the biproduct of X and Y. We call C *additive* if it is semiadditive and the following further condition is satisfied:

(4) For any $X \in C$, the "shear map"

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \sqcup X \to X \times X$$

is an equivalence.

Remark 3.3.2. As we will see in Section 4.3, the Hom anima $\mathrm{Hom}_C(X,Y)$ of a semiadditive ∞-category automatically becomes a *commutative monoid*. The sum $f+g$ of two morphisms $f, g \in \mathrm{Hom}_C(X,Y)$ is given by the composite

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y,$$

where Δ and ∇ denote the diagonal and codiagonal. If C is additive, then $\mathrm{Hom}_C(X,Y)$ is even a commutative *group*. See Exercise 7.6 for details.

Lemma 3.3.3. *Every stable ∞-category C is additive.*

Proof. Products and coproducts are special cases of pullbacks and pushouts, and the zero-object exists by pointedness of C. To show C has biproducts, let X and Y be objects in C, and consider the following commutative square:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "(\mathrm{id},0)"] \\ Y \arrow[r, "(0,\mathrm{id})"] & X \times Y. \end{tikzcd}$$

Since it is a product of the two pullback squares

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r, "\mathrm{id}"] & Y, \end{tikzcd}$$

it is again a pullback square. Since C is stable, it is also a pushout square. This means that the map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$

is an equivalence, i.e. that C is semiadditive. For additivity, we consider the commutative square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "{(\mathrm{id},\mathrm{id})}"] \\ X \arrow[r, "{(0,\mathrm{id})}"] & X \times X. \end{tikzcd}$$

We claim that this is a pullback square as well. To see this, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "{(\mathrm{id},\mathrm{id})}"] & \\ X \arrow[r, "{(0,\mathrm{id})}"] \arrow[d] & X \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "pr_1"] & X \arrow[d] \\ 0 \arrow[r] & X \arrow[r] & 0. \end{tikzcd}$$

The right bottom square, the bottom rectangle and the left rectangle are pullback squares. By the pasting-property of pullback diagrams, the upper right square is a pullback square and thus the upper left square is a pullback square. Since C is stable, it means that this is also a pushout square, and so the map

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \sqcup X \to X \times X$$

is an equivalence. This finishes the proof that C is additive. □

3.3.2 Functor categories

Proposition 3.3.4. *Let C be a stable ∞-category and I an arbitrary ∞-category. Then* $\mathrm{Fun}(I,C)$ *is a stable ∞-category.*

Idea of proof. For this proof, recall from Lemma A.2.23 and Theorem 2.6.10 the following facts:

(1) If C admits K-indexed (co)limits, then so does $\mathrm{Fun}(I,C)$;

(2) For every $i \in I$ the functor $\mathrm{ev}_i\colon \mathrm{Fun}(I,C) \to C$ preserves all K-limits and K-colimits;

(3) The functors $(\mathrm{ev}_i)_{i\in I}$ are jointly conservative: a natural transformation $\alpha\colon F \to G$ is a natural isomorphism (i.e. an isomorphism in $\mathrm{Fun}(I,C)$) if and only if each morphism $\alpha(i)\colon F(i) \to G(i)$ is an isomorphism in C.

From these facts it follows that the constant functor $\mathrm{const}_0\colon I \to C$ is both terminal and initial in $\mathrm{Fun}(I,C)$, hence a zero-object. Furthermore, the ∞-category $\mathrm{Fun}(I,C)$ admits pushouts and pullbacks. Given a commutative square

$$\begin{tikzcd} E \arrow[r] \arrow[d] & F \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

in $\mathrm{Fun}(I,C)$, it is a pushout/pullback square if and only if it is so after evaluating at each $i \in I$, and hence by stability of C these two conditions are equivalent. □

3.3.3 Finite limits and colimits

Stable ∞-categories admit all finite limits and colimits:

Definition 3.3.5 (Finite ∞-categories)**.** We denote by

$$\mathrm{Cat}_\infty^{\mathrm{fin}} \subseteq \mathrm{Cat}_\infty$$

the smallest full subcategory which contains the ∞-categories $\emptyset$, $*$ and $[1]$, and which is closed under pushouts of ∞-categories. A small ∞-category I is called *finite* if it is contained in $\mathrm{Cat}_\infty^{\mathrm{fin}}$.

Example 3.3.6. The following ∞-categories are finite:

- The ∞-categories $\emptyset$, $*$ and $[1]$;
- The ∞-category $* \sqcup *$;
- The ∞-categories $\lrcorner$ and $\ulcorner$;
- The ∞-categories $[n]$.

Remark 3.3.7. There is a functor $(-)^{\mathrm{op}}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ implementing the assignment $C \mapsto C^{\mathrm{op}}$. This functor is an equivalence, hence in particular preserves pushouts and full subcategories. Since each of the ∞-categories $\emptyset$, $*$ and $[1]$ are equivalent to their opposites, it follows that an ∞-category C is finite if and only if its opposite C^{op} is finite.

Definition 3.3.8. An ∞-category C is called *left exact*, or said to *admit finite limits*, if it admits all I-indexed limits for all finite ∞-categories I. Similarly, a functor $F: C \to D$ between left exact ∞-categories is called *left exact* if it preserves I-indexed limits for all $I \in \mathrm{Cat}_\infty^{\mathrm{fin}}$. We denote by

$$\mathrm{Cat}_\infty^{\mathrm{lex}} \subseteq \mathrm{Cat}_\infty$$

the subcategory consisting of the left exact ∞-categories and left exact functors.

Theorem 3.3.9 (Lurie [Lur09b, Corollary 4.4.2.4])**.** *(1) An ∞-category C admits finite limits and only if it has a terminal object and admits pullbacks.*

(2) A functor $F: C \to D$ is left exact if and only if it preserves terminal objects and pullbacks.

Dually C has finite colimits if and only if it has an initial object and admits pushouts, and F preserves finite colimits if and only if it preserves initial objects and pushouts.

Idea of proof. One direction is clear. The other direction needs the non-trivial fact that when the ∞-category I can be written as a colimit $I = \mathrm{colim}_j I_j$, we get an alternative description of $\lim_{i\in I} F(i)$ as $\lim_{j\in J^{\mathrm{op}}} \lim_{i\in I_j} F(i)$. (See Theorem A.2.25 for a more precise formulation). ◻

Corollary 3.3.10. *Every stable ∞-category admits finite limits and colimits. Every exact functor $F: C \to D$ between stable ∞-categories preserves finite limits and colimits.* ◻

We record an alternative characterization of stable ∞-categories, due to Moritz Groth [Gro16]:

Proposition 3.3.11. *An ∞-category C is stable if and only if it has all finite limits and colimits and finite limits commute with finite colimits.*

Proof. Let C be a stable ∞-category. If I is a finite ∞-category, then C has I-indexed colimits by the previous corollary, hence there is a functor $\mathrm{colim}: \mathrm{Fun}(S, C) \to C$. Its domain is also stable and colim preserves finite colimits, because colimits commute with colimits (exercise). So the functor is exact and hence preserves finite limits as well.

Conversely, assume that C is finitely complete and cocomplete and that finite limits commute with colimits in C. Considering the empty category, the fact that empty limits commute with empty colimits expresses that C is pointed. To show that C is stable, consider for every

$X \in C$ the following diagram:

$$\begin{tikzcd}
X \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 & X \arrow[l] \arrow[r] & 0 \\
0 \arrow[u] & X \arrow[l] \arrow[r] \arrow[u] & X. \arrow[u]
\end{tikzcd}$$

Taking pushouts of the rows we get the diagram $0 \to \Sigma X \leftarrow 0$ whose pullback is $\Omega\Sigma X$. If instead we take pullbacks of the columns, we get $X \leftarrow X \to X$, whose pushout is X. Pullbacks commuting with pushouts tell us then the canonical map $X \to \Omega\Sigma X$ is an equivalence. A similar argument shows that $\Sigma\Omega X \to X$ is an equivalence, so that C is stable by Theorem 3.2.7. □

3.3.4 Exercises on stable ∞-categories

Recall that there are exercise sheets at the end of the notes. Since some of the exercises from sheet 7 are especially important, I will also state them here as part of the lecture notes. Solutions to these exercises may be found here.

Exercise 3.3.12. Let $f : X \to Y$ be a morphism in a stable ∞-category. Show that there is an isomorphism

$$\operatorname{cofib}(f) \cong \operatorname{fib}(f)[1].$$

Exercise 3.3.13. Let $f : X \to Y$ be a morphism in a stable ∞-category C. Show that the following are equivalent:

(1) The morphism f is an isomorphism in C;

(2) The cofiber $\operatorname{cofib}(f)$ of f is a zero object in C;

(3) The fiber $\operatorname{fib}(f)$ of f is a zero object in C.

Recall that the *splitting lemma* provides equivalent conditions for a short exact sequence in an abelian category to be a *split short exact sequence*. In the following exercise we will prove a splitting lemma for stable ∞-categories:

Exercise 3.3.14. Let C be a stable ∞-category and let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be an exact sequence in C. Show that the following are equivalent:

(1) The associated morphism $\operatorname{fib}(i) \to X$ is null-homotopic;

(2) The associated morphism $Z \to \mathrm{cofib}(p)$ is null-homotopic;

(3) The map i admits a retract: there is a morphism $r\colon Y \to X$ satisfying $ri \simeq \mathrm{id}_X$;

(4) The map p admits a section: there is a morphism $s\colon Z \to Y$ satisfying $ps \simeq \mathrm{id}_Z$;

(5) There is an isomorphism $Y \cong X \oplus Z$ in C making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "{(\mathrm{id}_X,0)}"'] & Y \arrow[r, "p"] \arrow[d, "\cong"] & Z \\ & X \oplus Z. \arrow[ur, "\mathrm{pr}_Z"'] & \end{tikzcd}$$

For the following exercise, you may use the fact (to be proved in two different ways in Exercise 7.6 and Exercise 7.7 below) that in a stable ∞-category C the set $[X,Y] := \pi_0 \mathrm{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y in C admits a canonical abelian group structure, and that the composition maps

$$- \circ f\colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ -\colon [X,Y] \to [X,Y']$$

are group homomorphisms for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C.

Exercise 3.3.15. Let C be a stable ∞-category and consider a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "g"'] & W. \end{tikzcd}$$

Show that the following conditions are equivalent:

(1) The square is exact;

(2) The sequence

$$X \xrightarrow{(f,u)} Y \oplus Z \xrightarrow{(v,-g)} W$$

is an exact sequence.

3.4 Stabilization of ∞-categories

Given any ∞-category with finite limits, there is a universal way to turn it into a stable ∞-category:

Definition 3.4.1. Let C be an ∞-category admitting finite limits. We define its *stabilization* $\mathrm{Sp}(C)$ as the $\mathbb{N}^{\mathrm{op}}$-indexed limit

$$\mathrm{Sp}(C) := \lim(\ldots \xrightarrow{\Omega} C_* \xrightarrow{\Omega} C_* \xrightarrow{\Omega} C_*)$$

in the (very large[1]) ∞-category Cat_∞ of ∞-categories. An object of $\mathrm{Sp}(C)$ is called a *spectrum object in C*.

The forgetful functor to the copy of C_* indexed by $n \in \mathbb{N}$ is denoted by

$$\Omega^{\infty-n} \colon \mathrm{Sp}(C) \to C_*.$$

We write $\Omega^\infty = \Omega^{\infty-0} \colon \mathrm{Sp}(C) \to C_*$ for the forgetful functor to the final copy.

Definition 3.4.2. We define the ∞-category Sp of *spectra* as the stabilization of the ∞-category An:

$$\mathrm{Sp} := \mathrm{Sp}(\mathrm{An}) = \lim(\cdots \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_* \xrightarrow{\Omega} \mathrm{An}_*).$$

Remark 3.4.3. By the universal property of the limit, an object X of $\mathrm{Sp}(C)$ consists of a sequence $(X_n)_{n\in\mathbb{N}}$ of objects in C_* equipped with isomorphisms $\sigma_n \colon X_n \xrightarrow{\cong} \Omega X_{n+1}$ in C_*. For $C = \mathrm{An}$ this is precisely the definition of a spectrum we saw in Definition 1.3.1!

Similarly, given spectrum objects $X = (X_n)_{n\in\mathbb{N}}$ and $Y = (Y_n)_{n\in\mathbb{N}}$ in C, the Hom anima in $\mathrm{Sp}(C)$ may be computed as

$$\mathrm{Hom}_{\mathrm{Sp}(C)}(X, Y) \simeq \lim_n \mathrm{Hom}_{C_*}(X_n, Y_n),$$

so that a morphism $X \to Y$ consists of a collection of morphisms $f_n \colon X_n \to Y_n$ in C_* together with a collection of homotopies filling the following squares:

$$\begin{tikzcd}
X_n \arrow[r, "\cong"] \arrow[d, "f_n"'] & \Omega X_{n+1} \arrow[d, "\Omega f_{n+1}"] \\
Y_n \arrow[r, "\cong"] & \Omega Y_{n+1}.
\end{tikzcd}$$

Again for $C = \mathrm{An}$ these are precisely the morphisms of spectra from Definition 1.3.1!

Remark 3.4.4. A morphism $f \colon X \to Y$ of spectrum objects in C is an isomorpihsm if and only if each induced map $f_n \colon X_n \to Y_n$ is an isomorphism. Indeed, this is an instance of the more general claim that for an ∞-category D of the form $D = \lim_{i\in I} D_i$ for some diagram $D_\bullet \colon I \to \mathrm{Cat}_\infty$, a morphism $f \colon X \to Y$ in D is an isomorphism if and only if each $f_i \colon X_i \to Y_i$ is an isomorphism in D_i. To see this, it suffices by the Yoneda lemma

[1] I'm sweeping set-theoretical issues under the rug here: since I want to allow C to be a large ∞-category, I should work in a 'very large' universe $\widehat{\mathrm{Cat}}_\infty$ whose objects are the large ∞-categories. For those interested, the key word to look up here is 'Grothendieck universes'.

to see that for each Z the induced map $\mathrm{Hom}_D(Z,X) \to \mathrm{Hom}_D(Z,Y)$ is an isomorphism. The Hom animae in D are computed as limits $\lim_i \mathrm{Hom}_{D_i}(X_i,Z_i)$, and since each map $\mathrm{Hom}_{D_i}(Z_i,X_i) \to \mathrm{Hom}_{D_i}(Z_i,Y_i)$ is an isomorphism it remains to show that isomorphisms are closed under limits. But this is a consequence of Theorem 2.6.10.

Let us prove that $\mathrm{Sp}(C)$ is automatically stable. We start by discussing limits and colimits in $\mathrm{Sp}(C)$:

Lemma 3.4.5. *Let I and C be ∞-categories and assume C has finite limits.*

(1) If C has I-limits, then also $\mathrm{Sp}(C)$ has I-limits and the functors $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C$ preserve them for all $n \geq 0$.

(2) If C_* has I-colimits that are preserved by $\Omega\colon C_* \to C_*$, then $\mathrm{Sp}(C)$ has I-colimits and the functors $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C_*$ preserve them for all $n \geq 0$.

Proof. This is a special case of a the more general fact that if $C_\bullet\colon K \to \mathrm{Cat}_\infty$ is a diagram of small ∞-categories C_k for $k \in K$ such that C_k admits I-(co)limits for all k and such that $C_k \to C_{k'}$ preserves I-(co)limits for all $k \to k'$ in K, then also the limit $\lim_{k\in K} C_k$ in Cat_∞ has I-(co)limits and each functor $\lim_{k\in K} C_k \to C_k$ preserves them. By the description of limits in terms of an adjunction on functor categories from Corollary A.2.17, the claim then follows by combining the equivalence

$$\mathrm{Fun}(I, \lim_{k\in K} C_k) \simeq \lim_{k\in K} \mathrm{Fun}(I, C_k)$$

with the fact that a limit of adjunctions is an adjunction. We omit the details.

In the case at hand, if C has I-limits then so does C_*, and since $\Omega\colon C_* \to C_*$ preserves all I-limits (as it is a right adjoint) also $\mathrm{Sp}(C)$ has all limits. For the colimits, we need to specifically demand that Ω preserves them. □

We are now ready to prove that $\mathrm{Sp}(C)$ is stable. The proof is a bit more tricky than one might naively expect, and hence we will provide a more detailed proof than usual. I would like to thank Marc Hoyois for pointing out to me that my original proof sketch was too naive.

Theorem 3.4.6. *For every ∞-category C with finite limits, the stabilization $\mathrm{Sp}(C)$ is a stable ∞-category.*

Proof. By the previous corollary, the ∞-category $\mathrm{Sp}(C)$ has finite limits. The terminal spectrum object is at each leven n by the terminal object of C_*, but since this is also an initial object and $\Omega\colon C_* \to C_*$ preserves initial objects we get from Lemma 3.4.5 that it is also an initial object, so that $\mathrm{Sp}(C)$ is pointed.

By part (3) of Theorem 3.2.7, it thus remains to show that the functor

$$\Omega\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C)$$

is an equivalence. Let us start by giving a more explicit description of the spectrum object $\Omega(X)$ for $X \in \mathrm{Sp}(C)$. Since each of the functors $\Omega^{\infty-n}\colon \mathrm{Sp}(C) \to C_*$ preserve limits, they will commute with taking loops, and hence we get

$$\Omega(X)_n = \Omega(X_n)$$

for all n. We need to be careful with the structure maps though: since $\Omega\colon C_* \to C_*$ preserves limits, we have for all finite categories I a commutative square

$$\begin{tikzcd} \mathrm{Fun}(I, C_*) \arrow[r, "\lim_I"] \arrow[d, "\Omega"'] & C_* \arrow[d, "\Omega"] \\ \mathrm{Fun}(I, C_*) \arrow[r, "\lim_I"] & C_*, \end{tikzcd}$$

and in particular there is a commutative square

$$\begin{tikzcd} C_* \arrow[r, "\Omega"] \arrow[d, "\Omega"'] & C_* \arrow[d, "\Omega"] \\ C_* \arrow[r, "\Omega"] & C_*. \end{tikzcd}$$

We refer to the resulting natural isomorphism $\mathrm{swap}\colon \Omega^2 \cong \Omega^2$ as the *swap of coordinates*, since it swaps the roles of the two instances of Ω. Most importantly: it is *not* the identity isomorphism! All in all, we see that the structure morphisms of the spectrum object $\Omega(X)$ are given by

$$\sigma_n^{\Omega(X)}\colon \Omega(X)_n = \Omega(X_n) \xrightarrow[\cong]{\Omega(\sigma_n^X)} \Omega(\Omega(X_{n+1})) \xrightarrow[\cong]{\mathrm{swap}} \Omega(\Omega(X_{n+1})) = \Omega(\Omega(X)_{n+1}).$$

Now, to show that $\Omega\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C)$ is an equivalence, we claim that an inverse is given by the 'shift functor'

$$[1]\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C), \qquad X \mapsto X[1] = \{X_{n+1}\}_{n\in\mathbb{N}},$$

where the structure maps for $X[1]$ are the isomorphisms

$$\sigma_n^{X[1]}\colon X[1]_n = X_{n+1} \xrightarrow[\cong]{\sigma_{n+1}^X} \Omega(X_{n+2}) = \Omega(X[1]_{n+1})$$

inherited from X. For this, first note that the shift functor is an equivalence of ∞-categories. Indeed, consider the other shift functor

$$[-1]\colon \mathrm{Sp}(C) \to \mathrm{Sp}(C), \qquad X \mapsto X[-1] := \{Y_n\}_{n\in\mathbb{N}}, \qquad Y_n := \begin{cases} X_{n-1} & n \geq 1 \\ \Omega X_0 & n = 0, \end{cases}$$

where the structure maps are the isomorphisms

$$\sigma_n^{X[-1]}\colon X[-1]_n = X_{n-1} \xrightarrow[\cong]{\sigma_{n-1}^X} \Omega(X_n) = \Omega(X[-1]_{n+1})$$

for $n \geq 1$, and the identity map $\sigma_0^{X[-1]} := \mathrm{id}_{\Omega X_0}$ when $n = 0$. Since we are simply shifting up and down it is easy to produce natural isomorphisms

$$X \cong X[1][-1] \qquad \text{and} \qquad X \cong X[-1][1],$$

showing that the two shift functors are inverse to one another.

To show that Ω is also inverse to the shift functor, it hence remains to construct a natural isomorphism

$$\alpha\colon X \xrightarrow{\cong} \Omega(X[1])$$

of functors $\mathrm{Sp}(C) \to \mathrm{Sp}(C)$; by uniqueness of left-inverses of the shift functor this then results in a natural isomorphism $\Omega \cong [-1]$ and hence Ω is also a *right*-inverse of $[1]$. To construct α, note that for any n we have an isomorphism $\sigma_n\colon X_n \xrightarrow{\cong} \Omega(X_{n+1}) = \Omega(X[1])_n$. However, due to the swap maps that we have in the structure maps of $\Omega(X[1])$ these maps will not immediately define an isomorphism of spectrum objects. Instead we are going to use the following trick to get rid of the swap maps:

Claim: The forgetful functors $\Omega^{\infty-2k}\colon \mathrm{Sp}(C) \to C_*$ induce an equivalence

$$\mathrm{Sp}(C) \simeq \lim(\dots \xrightarrow{\Omega^2} C_* \xrightarrow{\Omega^2} C_* \xrightarrow{\Omega^2} C_*).$$

Indeed, this is an immediate consequence of the fact that the even natural numbers are cofinal in $\mathbb{N}$, so any $\mathbb{N}^{\mathrm{op}}$-indexed limit may be computed as a limit over just the even natural numbers. In particular, a spectrum object in C may equivalently be described as a collection $(X_{2k})_{k\in\mathbb{N}}$ of pointed objects of C together with structure isomorphisms $\sigma'_{2k}\colon X_{2k} \xrightarrow{\cong} \Omega^2 X_{2k+2}$. The key fact is now the following lemma:

Lemma 3.4.7. *For any pointed category D with finite limits, the natural isomorphism*

$$\Omega(\Omega^2 X) \xrightarrow{\cong} \Omega^2(\Omega X)$$

coming from the fact that Ω commutes with Ω^2 is homotopic to the identity map on $\Omega^3(X)$.

Proof sketch. There is a *universal* pointed ∞-category with finite limits: the opposite $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}$ of the ∞-category of finite pointed animae. Universality means that evaluation at the object $S^0 \in \mathrm{An}_*^{\mathrm{fin}}$ determines an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}((\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}, D) \xrightarrow{\sim} D$$

between D and the ∞-category of left-exact functors $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \to D$. Universality is satisfies essentially by definition of $\mathrm{An}_*^{\mathrm{fin}}$: it is the smallest subcategory of An_* generated under finite colimits by S^0. (See also Proposition A.4.6 for the non-pointed version of this claim, with $C = *$.) Given an object X of D, we denote the resulting functor by

$$X^{(-)}\colon (\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}} \to D,$$

and for a pointed anima A we refer to the object X^A as the *tensoring of X by A*. It will then suffice to show that the relation is satisfied in $(\mathrm{An}_*^{\mathrm{fin}})^{\mathrm{op}}$. Passing to the opposite category, this means that we must show that the canonical comparison map

$$\tau\colon \Sigma^2(\Sigma(S^0)) \xrightarrow{\cong} \Sigma(\Sigma^2(S^0))$$

is homotopic to the identity map.

Note that both sides of the map τ are the 3-dimensional sphere S^3, hence this pointed map determines an element of $\pi_3(S^3)$. Recall that assigning to a map $f\colon S^3 \to S^3$ its degree determines an isomorphism $\deg\colon \pi_3(S^3) \xrightarrow{\cong} \mathbb{Z}$, and hence it will suffice to show that the degree of τ is 1. In order to do this, consider for each pointed anima X the equivalence $\sigma_X\colon \Sigma(\Sigma(X)) \xrightarrow{\sim} \Sigma(\Sigma(X))$ obtained from the fact that Σ canonically commutes with itself (since both are colimits). We may then rewrite the above comparison map as the following composite:

$$\Sigma(\Sigma(\Sigma(S^0))) \xrightarrow{\Sigma(\sigma_{S^0})} \Sigma(\Sigma(\Sigma(S^0))) \xrightarrow{\sigma_{\Sigma(S^0)}} \Sigma(\Sigma(\Sigma(S^0))).$$

We then conclude that

$$\deg(\tau) = \deg(\Sigma(\sigma_{S^0}))\cdot\deg(\sigma_{\Sigma(S^0)}) = \deg(\sigma_{S^0})^2 = 1,$$

where the last equality holds since by invertibility of the map σ_{S^0} we have $\deg(\sigma_{S^0}) = \pm 1$. $\square$

Now to finish the proof of the theorem, i.e. to produce the natural isomorphism $\alpha\colon X \xrightarrow{\cong} \Omega(X[1])$, we consider the maps

$$\alpha_{2k} := \sigma_{2k}\colon X_{2k} \xrightarrow{\cong} \Omega(X_{2k+1}) = \Omega(X[1])_{2k}.$$

We must then produce commutative diagrams as follows:

$$\begin{tikzcd}
X_{2k} \arrow[r, "\sigma^X_{2k}"] \arrow[d, "\sigma^X_{2k}"', "\cong"] & \Omega(X_{2k+1}) \arrow[d, "\sigma^{\Omega(X[1])}_{2k}"] \\
\Omega(X_{2k+1}) \arrow[d, "\Omega(\sigma^X_{2k+1})"', "\cong"] & \Omega^2(X_{2k+2}) \arrow[d, "\Omega(\sigma^{\Omega(X[1])}_{2k+1})"] \\
\Omega^2(X_{2k+2}) \arrow[r, "\Omega^2(\sigma^X_{2k+2})"] & \Omega^3(X_{2k+3}).
\end{tikzcd}$$

Unwinding definitions, we see that the two maps differ only by the swap map $\Omega(\Omega^2(X_{2k+3})) \cong \Omega^2(\Omega(X_{2k+3}))$, hence the claim follows from Lemma 3.4.7. $\square$

Notation 3.4.8. The 'shift notation' is actually convenient in an arbitrary stable ∞-category C: for $n \geq 0$ we write

$$[n] := \Sigma^n : C \xrightarrow{\sim} C.$$

We also write

$$[-n] := \Omega^n : C \xrightarrow{\sim} C.$$

Exercise 3.4.9 (See Exercise 8.1)**.** Show that the assignment $C \mapsto \mathrm{Sp}(C)$ can be turned into a functor

$$\mathrm{Sp}(-) : \mathrm{Cat}_\infty^{\mathrm{lex}} \to \mathrm{Cat}_\infty^{\mathrm{st}}$$

from left-exact ∞-categories to stable ∞-categories which is right adjoint to the inclusion $\mathrm{Cat}_\infty^{\mathrm{st}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{lex}}$: for every stable ∞-category C and every left-exact ∞-category D, postcomposition with $\Omega^\infty : \mathrm{Sp}(D) \to D$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{lex}}(C, \mathrm{Sp}(D)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(C, D).$$

3.5 The ∞-category of spectra

We will now have a closer look at the main example of a stabilization: the ∞-category of spectra from Definition 3.4.2.

Proposition 3.5.1. *The ∞-category* Sp *of spectra admits all small limits and filtered colimits, which are preserved by the functor* $\Omega^\infty : \mathrm{Sp} \to \mathrm{An}$. *In particular,* Sp *admits all sequential (i.e.* $\mathbb{N}$*-indexed) colimits.*

Proof. The statement about limits is an instance of Lemma 3.4.5: An admits all small limits and colimits. For the statement about filtered colimits, we recall from Theorem A.2.46 the non-trivial fact that finite limits commute with filtered colimits in An. Since $\Omega : \mathrm{An}_* \to \mathrm{An}_*$ is a finite limit, it thus commutes with filtered colimits, and so we may apply Lemma 3.4.5 to get that Sp admits filtered colimits. □

Example 3.5.2 (Eilenberg-MacLane spectra)**.** Let A be an abelian group. We define the *Eilenberg-MacLane spectrum HA* of A as the spectrum given in degree n by the Eilenberg-MacLane space

$$(HA)_n := K(A, n),$$

characterized up to homotopy equivalence by the property that the only non-zero homotopy group of $K(A,n)$ is $\pi_n(K(A,n)) \cong A$. Since the loop space operation shifts homotopy groups, cf. Remark 1.1.11, we obtain equivalences $\Omega K(A, n+1) \simeq K(A,n)$ turning this into a spectrum.

Another important class of examples is suspension spectra:

Construction 3.5.3 (Suspension spectrum)**.** Consider the functor $Q\colon \mathrm{An}_* \to \mathrm{An}_*$ defined by the formula

$$Q(X) := \mathrm{colim}_{n\geq 0}\,\Omega^n\Sigma^n X,$$

where the comparison maps in the diagram are induced by the unit $X \to \Omega\Sigma X$ of the adjunction $\Sigma \dashv \Omega$. Again using the fact that Ω preserves filtered colimits, we see that

$$\Omega Q(\Sigma(X)) = \Omega(\mathrm{colim}_{n\geq 0}\,\Omega^n\Sigma^{n+1}X) \simeq \mathrm{colim}_{n\geq 0}\,\Omega^{n+1}\Sigma^{n+1} \simeq Q(X).$$

Hence we obtain a spectrum $\Sigma^\infty(X)$, called the *suspension spectrum of* X:

$$\Sigma^\infty(X)_n := Q(\Sigma^n X).$$

This construction is functorial in X, giving rise to a functor

$$\Sigma^\infty\colon \mathrm{An}_* \to \mathrm{Sp}.$$

Proposition 3.5.4. *The suspension spectrum functor* $\Sigma^\infty\colon \mathrm{An}_* \to \mathrm{Sp}$ *is left adjoint to* $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}_*$.

Proof. We will show condition (2) from Proposition A.2.2. For a pointed anima X, we have a map

$$\eta_X\colon X = \Omega^0\Sigma^0 X \to \mathrm{colim}_n\,\Omega^n\Sigma^n X = QX = \Omega^\infty\Sigma^\infty(X).$$

We claim that this is the unit map for an adjunction. In other words, we have to show that for every spectrum Y the composite

$$\mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \xrightarrow{\Omega^\infty} \mathrm{Hom}_{\mathrm{An}_*}(\Omega^\infty\Sigma^\infty X, \Omega^\infty Y) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$$

is an equivalence. To see this, consider the following commutative diagram of animae:

$$\begin{tikzcd}
\ddots \arrow[dr, "\simeq"'] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
& \mathrm{Hom}_{\mathrm{An}_*}(\Sigma^2 X, Y_2) \arrow[r, "\Omega"] \arrow[dr, "\simeq"'] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega\Sigma^2 X, Y_1) \arrow[r, "\Omega"] \arrow[d] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega^2\Sigma^2 X, Y_0) \arrow[d] \\
& & \mathrm{Hom}_{\mathrm{An}_*}(\Sigma X, Y_1) \arrow[r, "\Omega"] \arrow[dr, "\simeq"'] & \mathrm{Hom}_{\mathrm{An}_*}(\Omega\Sigma X, Y_0) \arrow[d] \\
& & & \mathrm{Hom}_{\mathrm{An}_*}(X, Y_0).
\end{tikzcd}$$

The bottom-right corner is $\mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$, and since the diagonal maps are all equivalences we see that the limit of the diagonals is also equivalent to $\mathrm{Hom}_{\mathrm{An}_*}(X, \Omega^\infty(Y))$. But an alternative way of computing this limit is by first taking the limit of the vertical columns and then taking a limit along the resulting horizontal diagram. For the limit of the k-th column we have

$$\lim_n \mathrm{Hom}_{\mathrm{An}_*}(\Omega^n\Sigma^{n+k}X, Y_k) \simeq \mathrm{Hom}_{\mathrm{An}_*}(\mathrm{colim}_n\, \Omega^n\Sigma^n\Sigma^k X, Y_k) \simeq \mathrm{Hom}_{\mathrm{An}_*}(Q\Sigma^k(X), Y_k).$$

The resulting horizontal limit then gives

$$\lim_k \mathrm{Hom}_{\mathrm{An}_*}(Q\Sigma^k(X), Y_k) = \mathrm{Hom}_{\lim_k \mathrm{An}_*}((Q\Sigma^k(X))_{k\in\mathbb{N}}, (Y_k)_{k\in\mathbb{N}}) = \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, Y).$$

This finishes the proof. □

Definition 3.5.5 (Unreduced suspension spectrum)**.** Let X be an anima. We define its *unreduced suspension spectrum* $\mathbb{S}[X] \in \mathrm{Sp}$ as

$$\mathbb{S}[X] := \Sigma^\infty(X_+),$$

where $X_+ := X \sqcup *$. This defines a functor

$$\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}.$$

We define the *sphere spectrum* as

$$\mathbb{S} := \mathbb{S}[\mathrm{pt}] = \Sigma^\infty(S^0).$$

By the following corollary, we may think of the unreduced suspension spectrum as the 'free spectrum on X':

Corollary 3.5.6. *The functor* $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ *is left adjoint to* $\Omega^\infty\colon \mathrm{Sp} \to \mathrm{An}$*.*

Proof. This is immediate from Proposition 3.5.4 in light of the observation that the functor

$$(-)_+\colon \mathrm{An} \to \mathrm{An}_*, \qquad X \mapsto X_+$$

is a left adjoint to the forgetful functor $\mathrm{An}_* \to \mathrm{An}$. □

We have the following table with analogies between ordinary algebra and 'higher algebra':

Ordinary algebra	Higher algebra
Abelian group	Spectrum
Integers $\mathbb{Z}$	Sphere spectrum $\mathbb{S}$
Underlying set	Underlying anima $X_0 = \Omega^\infty(X)$
Free abelian group $\mathbb{Z}[S]$	Unreduced suspension spectrum $\mathbb{S}[X]$.

Recall that a *presentation* of an abelian group A is a way to write A as a cokernel of a map of free abelian groups:

$$A = \operatorname{coker}(\mathbb{Z}[T] \xrightarrow{f} \mathbb{Z}[S]).$$

There is always a *standard presentation* where we take S to be the underlying set of A, and take T the underlying set of $\mathbb{Z}[A]$. We will now discuss an analogous story for spectra.

Definition 3.5.7 (Prespectrum)**.** A *prespectrum* is a collection $(X_n)_{n\in\mathbb{N}}$ of pointed animae together with *non-invertible* maps $\sigma_n\colon X_n \to \Omega(X_{n+1})$. A morphism $f\colon X \to Y$ of prespectra is a collection of pointed maps $f_n\colon X_n \to Y_n$ together with homotopies making the following squares commute:

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d, "\sigma_n^X"'] & Y_n \arrow[d, "\sigma_n^Y"] \\ \Omega(X_{n+1}) \arrow[r, "\Omega(f_{n+1})"] & \Omega(Y_{n+1}). \end{tikzcd}$$

We want to think of a prespectrum as a *presentation* of a spectrum. Note that every spectrum is canonically a prespectrum. Conversely we can turn every prespectrum into a spectrum as follows:

Construction 3.5.8. Given a prespectrum X, we define its *associated spectrum* X^{sp} as the following colimit in Sp:

$$X^{\mathrm{sp}} := \operatorname{colim}(\Sigma^\infty X_0 \to (\Sigma^\infty X_1)[-1] \to (\Sigma^\infty X_2)[-2] \to (\Sigma^\infty X_3)[-3] \to \dots),$$

where the comparison maps in the colimit are given levelwise by

$$(\Sigma^\infty X_n)[-n]_k = (\Sigma^\infty X_n)_{k-n} = Q\Sigma^{k-n} X_n \xrightarrow{\sigma_n} Q\Sigma^{k-n}\Omega X_{n+1} \xrightarrow{\varepsilon} Q\Sigma^{k-(n+1)} X_{n+1} = (\Sigma^\infty X_{n+1})[-(n+1)]_k,$$

where ε is the counit of the suspension-loop adjunction on An_*.

Remark 3.5.9. One may define an ∞-category PSp of prespectra, and one can show that the above construction defines a left adjoint

$$(-)^{\mathrm{sp}}\colon \mathrm{PSp} \to \mathrm{Sp}$$

to the inclusion $\mathrm{Sp} \hookrightarrow \mathrm{PSp}$.

Proposition 3.5.10 (Standard presentation)**.** *Every spectrum X is naturally isomorphic to the spectrum associated to its underlying prespectrum:*

$$\operatorname{colim}_n((\Sigma^\infty X_n)[-n]) \xrightarrow{\cong} X.$$

Proof. By the Yoneda lemma, it suffices to show that the two functors $\mathrm{Sp}^{\mathrm{op}} \to \mathrm{An}$ represented by both sides are equivalent. For a spectrum Y, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}}(\mathrm{colim}_n((\Sigma^\infty X_n)[-n]), Y) &\simeq \lim_n \mathrm{Hom}_{\mathrm{Sp}}((\Sigma^\infty X_n)[-n], Y) \simeq \lim_n \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X_n, Y[n]) \\ &\simeq \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, \Omega^\infty Y[n]) = \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, Y_n) \simeq \mathrm{Hom}_{\mathrm{Sp}}(X, Y).\end{aligned}$$

All of this is natural in Y, finishing the proof. □

Corollary 3.5.11. *The ∞-category* Sp *admits all small colimits: given a diagram $X\colon I \to$* Sp*, their colimit is given by*

$$\mathrm{colim}_{i\in I} X(i) = ((\mathrm{colim}_{i\in I} X(i)_n)_{n\in\mathbb{N}})^{\mathrm{sp}},$$

where the structure maps of the prespectrum are the maps

$$\mathrm{colim}_{i\in I} X(i)_n \xrightarrow[\cong]{\mathrm{colim}_i(\sigma_n)} \mathrm{colim}_{i\in I} \Omega X(i)_{n+1} \to \Omega(\mathrm{colim}_{i\in I} X(i)_{n+1}).$$ □

3.6 Tensor products and mapping spectra

Using the standard presentation of spectra, we can now define the *tensor product* of spectra. This is supposed to be a functor

$$-\otimes-\colon \mathrm{Sp}\times\mathrm{Sp} \to \mathrm{Sp}$$

which preserves colimits in both variables. By resolving every spectrum as a colimit of suspension spectra via their standard presentations, it thus suffices to define the tensor product of suspension spectra. Since the functor $\Sigma^\infty\colon \mathrm{An}_* \to \mathrm{Sp}$ is supposed to be symmetric monoidal, we wish to define the tensor product on suspension spectra by

$$\Sigma^\infty(X)\otimes\Sigma^\infty(Y) := \Sigma^\infty(X\wedge Y).$$

Here the smash product $X\wedge Y$ of two pointed animae is defined via the following pushout square:

$$\begin{tikzcd} X\vee Y \arrow[r] \arrow[d] & X\times Y \arrow[d] \\ * \arrow[r] & X\wedge Y. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

All in all, we see that the following definition is forced upon us:

Definition 3.6.1. The *tensor product* $X\otimes Y$ of two spectra X and Y is defined as the following colimit in Sp:

$$X\otimes Y := \mathrm{colim}_{n\in\mathbb{N}}\,\mathrm{colim}_{m\in\mathbb{N}}\,(\Sigma^\infty(X_n\wedge X_m)[-(n+m)])\,.$$

This defines a functor

$$-\otimes-\colon \mathrm{Sp}\times\mathrm{Sp} \to \mathrm{Sp}.$$

We will now prove that $-\otimes Y\colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits for every spectrum Y by producing an explicit right adjoint:

Construction 3.6.2. Let Y and Z be spectra. We define the *mapping spectrum* $\mathrm{map}(Y,Z) \in \mathrm{Sp}$ as

$$\mathrm{map}(Y,Z)_n := \mathrm{Hom}_{\mathrm{Sp}}(Y, Z[n]) \simeq \lim_k \mathrm{Hom}_{\mathrm{An}_*}(Y_k, Z_{k+n}),$$

where the structure maps are given by

$$\mathrm{Hom}_{\mathrm{Sp}}(Y, Z[n]) \cong \mathrm{Hom}_{\mathrm{Sp}}(Y, \Omega(Z[n+1])) \cong \Omega\,\mathrm{Hom}_{\mathrm{Sp}}(Y, Z[n+1]).$$

This defines a functor

$$\mathrm{map}\colon \mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp} \to \mathrm{Sp}.$$

Proposition 3.6.3. *For every spectrum Y, the functor* $\mathrm{map}(Y,-)\colon \mathrm{Sp} \to \mathrm{Sp}$ *is right adjoint to* $-\otimes Y\colon \mathrm{Sp} \to \mathrm{Sp}$*: there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Sp}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathrm{Sp}}(X, \mathrm{map}(Y,Z))$$

for spectra X and Y.

Proof. Plugging in the definition of $X \otimes Y$ and unwinding everything, we get

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sp}}(X \otimes Y, Z) &= \mathrm{Hom}_{\mathrm{Sp}}(\mathrm{colim}_{n,m}\, \Sigma^\infty(X_n \wedge Y_m)[-(n+m)], Z)\\
&\cong \lim_{n,m} \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty(X_n \wedge Y_m), Z[n+m])\\
&\cong \lim_{n,m} \mathrm{Hom}_{\mathrm{An}_*}(X_n \wedge Y_m, Z_{n+m})\\
&\cong \lim_n \lim_m \mathrm{Hom}_{\mathrm{An}_*}(X_n, \mathrm{Hom}_{\mathrm{An}_*}(Y_m, Z_{n+m}))\\
&\cong \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, \lim_m \mathrm{Hom}_{\mathrm{An}_*}(Y_m, Z_{n+m}))\\
&\cong \lim_n \mathrm{Hom}_{\mathrm{An}_*}(X_n, \mathrm{map}(Y,Z)_n)\\
&\cong \mathrm{Hom}_{\mathrm{Sp}}(X, \mathrm{map}(Y,Z)),
\end{aligned}$$

finishing the proof. $\square$

Corollary 3.6.4. *For every spectrum Z, the functor* $\mathrm{map}(-,Z)\colon \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$ *is right adjoint to* $\mathrm{map}(-,Z)\colon \mathrm{Sp} \to \mathrm{Sp}^{\mathrm{op}}$.

Proof. Combining the previous proposition with the symmetry of the tensor product, we see that

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Sp}}(X, \mathrm{map}(Y,Z)) &\simeq \mathrm{Hom}_{\mathrm{Sp}}(X \otimes Y, Z)\\
&\simeq \mathrm{Hom}_{\mathrm{Sp}}(Y \otimes X, Z)\\
&\simeq \mathrm{Hom}_{\mathrm{Sp}}(Y, \mathrm{map}(X,Z))\\
&\simeq \mathrm{Hom}_{\mathrm{Sp}^{\mathrm{op}}}(X, \mathrm{map}(Y,Z)).
\end{aligned}$$

$\square$

Corollary 3.6.5. *The functors*

$$Y \otimes \mathrm{Sp} \to \mathrm{Sp}, \qquad \mathrm{map}(Y,-)\colon \mathrm{Sp} \to \mathrm{Sp} \qquad \textit{and} \qquad \mathrm{map}(-,Z)\colon \mathrm{Sp}^{\mathrm{op}} \to \mathrm{Sp}$$

are exact for all $Y, Z \in \mathrm{Sp}$. □

3.7 Homotopy, homology and cohomology groups

Important invariants of spectra are their *homotopy groups*:

Definition 3.7.1 (Homotopy groups)**.** For a spectrum X and an integer k, we define its *k-th homotopy group* $\pi_k(X) \in \mathrm{Ab}$ as

$$\pi_k(X) := [\mathbb{S}[k], X] := \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}[k], X).$$

Observe that the assignment $X \mapsto \pi_*(X)$ assembles into a functor $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}^{\mathbb{Z}}$.

Exercise 3.7.2. Show that we have $\pi_k(X[n]) \simeq \pi_{k-n}(X)$ for all $k, n \in \mathbb{Z}$.

Exercise 3.7.3. Show that for $k \geq n$ we have $\pi_k(X) \cong \pi_{n+k}(X_n)$. In particular we have $\pi_k(X) = \pi_k(\Omega^\infty X)$ for $k \geq 0$.

Corollary 3.7.4. *The functors* $\pi_k\colon \mathrm{Sp} \to \mathrm{Ab}$ *for* $k \in \mathbb{Z}$ *are jointly conservative: a morphism of spectra* $f\colon X \to Y$ *is an isomorphism if and only if the induced map* $\pi_k(f)\colon \pi_k(X) \to \pi_k(Y)$ *is an isomorphism.*

Proof. By Remark 3.4.4, f is an isomorphism if and only if each map $f_n\colon X_n \to Y_n$ of animae is an isomorphism. By the Whitehead Theorem for animae, see Proposition 4.2.11 below, this is the case if f_n induces isomorphisms on all homotopy groups. By Exercise 3.7.3 this condition is equivalent to the condition that f induces isomorphisms on all homotopy groups. □

Proposition 3.7.5 (Long exact sequence of homotopy groups)**.** *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be an exact sequence of spectra. Then there is a long exact sequence of homotopy groups of the form*

$$\ldots \to \pi_1(Y) \xrightarrow{g_*} \pi_1(Z) \to \pi_0(X) \xrightarrow{f_*} \pi_0(Y) \xrightarrow{g_*} \pi_0(Z) \to \pi_{-1}(X) \xrightarrow{f_*} \pi_{-1}(Y) \to \ldots.$$

Proof. Since every exact sequence of spectra $X \xrightarrow{f} Y \xrightarrow{g} Z$ leads to two new exact sequences

$$Z[-1] \to X \xrightarrow{f} Y \qquad \text{and} \qquad Y \xrightarrow{g} Z \to X[1],$$

it inductively suffices to prove that the sequence $\pi_0(X) \xrightarrow{f} \pi_0(Y) \xrightarrow{g} \pi_0(Z)$ is exact. It is clear that the composite is the zero map. Conversely, consider a class $[\sigma] \in \pi_0(Y)$, represented by a morphism of spectra $\sigma\colon \mathbb{S} \to Y$, and assume that the class $[g \circ \sigma] \in \pi_0(Z)$ is zero. Then $\sigma\colon \mathbb{S} \to Y$ factors through the fiber X of $g\colon Y \to Z$, and hence is of the form $f \circ \tau$ for some $\tau\colon \mathbb{S} \to X$. It follows that $[\sigma]$ is in the image of $f\colon \pi_0(X) \to \pi_0(Y)$, as desired. □

The special case of homotopy groups of tensor products of spectra or of mapping spectra have a special name:

Definition 3.7.6 ((Co)homology groups)**.** If E is a spectrum and X is an anima, we define the *E-homology of X* as the homotopy groups of $\mathbb{S}[X] \otimes E$:

$$E_k(X) := \pi_k(\mathbb{S}[X] \otimes E).$$

Dually we define the *E-cohomology groups of X* as the homotopy groups of the mapping spectrum $\mathrm{map}(\mathbb{S}[X], E)$:[2]

$$E^k(X) := \pi_{-k}(\mathrm{map}(\mathbb{S}[X], E)).$$

Exercise 3.7.7. Show that this definition of $E^k(X)$ agrees with the one from Construction 1.3.6.

Corollary 3.7.8. *Let E be a spectrum and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be cofiber sequences of animae. Then there are long exact sequences of the form*

$$\ldots \to E_1(Y) \xrightarrow{g_*} E_1(Z) \to E_0(X) \xrightarrow{f_*} E_0(Y) \xrightarrow{g_*} E_0(Z) \to E_{-1}(X) \xrightarrow{f_*} E_{-1}(Y) \to \ldots.$$

and

$$\ldots \to E^{-1}(Y) \xrightarrow{g^*} E^{-1}(Z) \to E^0(X) \xrightarrow{f^*} E^0(Y) \xrightarrow{g^*} E^0(Z) \to E^1(X) \xrightarrow{f^*} E^1(Y) \to \ldots.$$

Proof. The functor $\mathbb{S}[-]\colon \mathrm{An} \to \mathrm{Sp}$ is a left adjoint and thus sends cofiber sequences to exact sequences. The functors $- \otimes E$ and $\mathrm{map}(-, E)$ are exact by Corollary 3.6.5 and thus preserve exact sequences. The claim thus follows from Proposition 3.7.5. □

[2] The minus sign is mostly for historical reasons: this expression shows that it would be more natural to always use homological grading and simply put cohomology in negative degrees.

4 Algebraic structures in homotopy theory

As discussed in the introduction of these notes, the fundamental principle of homotopy theory implies that homotopical analogues of familiar algebraic structures often come with infinite hierarchies of coherence conditions: each time we demand a certain relation to hold, this relation will be exhibited by some homotopy, and then these homotopies themselves need to satisfy even higher coherences, ad infinitum. For this reason, we cannot generally define algebraic structures in homotopy theory by writing down a list of relations that need to be satisfied; we need to find cleverer ways to encode all this data. In this chapter we will discuss some example of such algebraic structures: monoids, commutative monoids and commutative algebras.

4.1 Monoids and groups

We start with the notion of a monoid.

Definition 4.1.1 (Monoid)**.** Let C be an ∞-category that admits finite products. We define a *monoid in* C, also known as an E_1*-monoid*, to be a simplicial object

$$M\colon \mathbf{\Delta}^{\mathrm{op}} \to C, \quad [n] \mapsto M_n := M([n]),$$

satisfying the *Segal condition*: for every $[n] \in \mathbf{\Delta}$ the map

$$(e_i^*)_{i=1}^n\colon M_n \to \prod_{i=1}^n M_1$$

induced by the n inclusion maps $e_i\colon [1] \cong \{i-1 \le i\} \hookrightarrow [n]$ is an isomorphism. In particular, taking $n = 0$ we see that M_0 is a terminal object of C.

We write

$$\mathrm{Mon}(C) \subseteq \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C)$$

for the full subcategory spanned by the monoids in C. We refer to M_1 as the *underlying object* of M.

Note that the unique map $s_0\colon [1] \to [0]$ provides a map

$$e := s_0^*\colon * = M_0 \to M_1$$

which we will call the *unit* of M. Similarly, the inclusion $d_1\colon [1] \cong \{0 \le 2\} \hookrightarrow [2]$ induces a map

$$m := d_1^*\colon M_1 \times M_1 \simeq M_2 \to M_1$$

which we call the *multiplication* of M. The unitality and associativity are encoded by the following commutative diagrams in $\mathbf{\Delta}$:

$$\begin{tikzcd} & {[1]} \arrow[dl, equal] \arrow[d, "(0,2)"] \arrow[dr, equal] & \\ {[1]} & {[2]} \arrow[l, "(0,0,1)"] \arrow[r, "(0,1,1)"'] & {[1]} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} {[1]} \arrow[r, "(0,2)"] \arrow[d, "(0,2)"'] & {[2]} \arrow[d, "(0,2,3)"] \\ {[2]} \arrow[r, "(0,1,3)"] & {[3]}. \end{tikzcd}$$

Similarly, all the expected 'higher coherences' relating the various ways of multiplying four, five, or more elements in M will all follow from suitable higher dimensional diagrams in the category $\mathbf{\Delta}$ by composing with $M\colon \mathbf{\Delta}^{\mathrm{op}} \to C$.

Instead of merely encoding the two-fold multiplication $M_1 \times M_1 \to M_1$, the monoid M encodes all n-fold multiplications $\prod_{i=1}^n x_i$ simultaneously, together with all their compatibilities: for a map $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ we should think of the resulting map $\varphi^*\colon M^n \to M^m$ as follows:

$$\varphi^*(x_1, \ldots, x_n) := (y_1, \ldots, y_m), \qquad y_i := \prod_{\varphi(i-1) < j \le \varphi(i)} x_j.$$

Exercise 4.1.2. Let C be an ordinary category. Show that the category $\mathrm{Mon}(C)$ is equivalent to the usual category of monoids in C.

Definition 4.1.3. We say that a monoid $M \in \mathrm{Mon}(C)$ is *grouplike*, or a *group in C*, if the so-called 'shear map'

$$(\mathrm{pr}_1, m)\colon M \times M \to M \times M, \qquad (x, y) \mapsto (x, xy)$$

is an isomorphism in C. We denote by

$$\mathrm{Grp}(C) \subseteq \mathrm{Mon}(C)$$

the full subcategory spanned by the groups in C.

Recall from Exercise 7.7 that for an ∞-category C with finite limits the loop object ΩX of a pointed object X admits a canonical structure of a group object in the homotopy category $\mathrm{Ho}(C)$. We will now show that this can be enhanced to a group object in C itself:

Construction 4.1.4 (Group structure on loop space)**.** Let C be an ∞-category with finite limits, and consider a pointed object X with base point $x\colon * \to X$. For an object $[n] \in \mathbf{\Delta}$, denote by I_n the following category:

$$I_n \quad := \quad \begin{tikzcd}[row sep=small] 0 \arrow[drr] & & \\ 1 \arrow[dr] & & \\ \vdots & \infty & \\ n-1 \arrow[ur] & & \\ n \arrow[uur] & & \end{tikzcd}$$

In other words, I_n consists of $n+1$ non-identity morphisms whose targets agree. We may consider for every n the functor $\tilde{X}_n\colon I_n \to C$ which sends each of the $n+1$ non-identity morphisms of I_n to the map $x\colon * \to X$, and we define

$$(\overline{\Omega}X)_n := \lim(\tilde{X}_n\colon I_n \to C) \quad \in \quad C.$$

Observe that the categories I_n are covariantly functorial in $[n] \in \mathbf{\Delta}$: for a morphism $\varphi\colon [n] \to [m]$ in $\mathbf{\Delta}$ we obtain a functor

$$I_\varphi\colon I_n \to I_m, \qquad I_\varphi(i) := \varphi(i), \qquad I_\varphi(\infty) = \infty.$$

It follows that $(\overline{\Omega}X)_n$ is contravariantly functorial in $[n] \in \mathbf{\Delta}$, and hence defines a functor $\overline{\Omega}X\colon \mathbf{\Delta}^{\mathrm{op}} \to C$. All in all, we obtain

$$\overline{\Omega}\colon C_* \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C).$$

Note that we have $(\overline{\Omega}_X)_1 \simeq * \times_X * = \Omega$, so the functor $\overline{\Omega}$ lifts the loop functor Ω.

Lemma 4.1.5. *For every pointed object X in C the simplicial object $\overline{\Omega}X$ is a group object in C, giving rise to a functor*

$$\overline{\Omega}\colon C_* \to \mathrm{Grp}(C)$$

which lifts the loop functor $\Omega\colon C_* \to C_*$.

Proof. To show that $\overline{\Omega}X$ is a monoid, we will show by induction that for every n the map

$$(e_i^*)_{i=1}^n\colon (\overline{\Omega}X)_n \to \prod_{i=1}^n \Omega X$$

is an isomorphism in C. For $n=0$, $(\overline{\Omega}X)_0$ is the limit of the diagram $*\to X$, but since the category $[1]$ has 0 as an initial object this limit is simply given by $*$, see Lemma A.2.19. For $n\geq 1$, note that we may write I_{n+1} as a pushout of I_n and I_1 along I_0:

$$\begin{tikzcd} I_0 \arrow[r] \arrow[d] & I_n \arrow[d] \\ I_1 \arrow[r, "I_{e_{n+1}}"'] & I_{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

By applying Theorem A.2.25(2) with $J=\ulcorner$, we thus obtain a pullback square of the form

$$\begin{tikzcd} \lim(\overline{X}_{n+1}\colon I_{n+1}\to C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim(\overline{X}_n\colon I_n\to C) \arrow[d] \\ \lim(\overline{X}_1\colon I_1\to C) \arrow[r] & \lim(\overline{X}_0\colon I_0\to C). \end{tikzcd}$$

By induction, this pullback square is isomorphic to the commutative square

$$\begin{tikzcd} (\overline{\Omega}X)_{n+1} \arrow[r, "(e_i^*)_{i=1^n}"] \arrow[d, "e_{n+1}^*"'] & \prod_{i=1}^n \Omega X \arrow[d] \\ \Omega X \arrow[r] & *, \end{tikzcd}$$

and so this being a pullback square means that the map $(\overline{\Omega}X)_{n+1}\xrightarrow{(e_i^*)_{i=1^{n+1}}}\prod_{i=1}^{n+1}\Omega X$ is an isomorphism, finishing the induction step. This shows that $\overline{\Omega}X$ is a monoid in C.

To see that it is even a group, we must show that the shear map $(\mathrm{id},m)\colon \overline{\Omega}X\times\overline{\Omega}X\to \overline{\Omega}X\times\overline{\Omega}X$ is an isomorphism in C, or equivalently that it is an isomorphism in the homotopy category $\mathrm{Ho}(C)$. For this we observe that the monoid in $\mathrm{Ho}(C)$ induced by the monoid $\overline{\Omega}X\in\mathrm{Mon}(C)$ is precisely the monoid structure constructed on ΩX in Exercise 7.7 (see the solution sheet on GRIPS), and it was shown there that ΩX is even a group object. This finishes the proof. □

Remark 4.1.6. In practice, the functor $\overline{\Omega}\colon C_*\to\mathrm{Grp}(C)$ is usually simply denoted again by Ω. We will temporarily use the unusual notation $\overline{\Omega}$ to distinguish the group object $\overline{\Omega}(X)$ from its underlying pointed object ΩX.

The loop functor admits a left adjoint:

Construction 4.1.7 (Classifying space of a monoid)**.** Let C be an ∞-category admitting geometric realizations (i.e. $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits). We define a functor[1]

$$\mathbf{B}\colon \mathrm{Mon}(C)\to C_*$$

[1] I try to use boldface for this functor, but now and then I may accidentally write BM instead of $\mathbf{B}M$.

by setting

$$\mathbf{B}M := |M| := \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n,$$

which comes equipped with the canonical basepoint $* = M_0 \to \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n$. We refer to $\mathbf{B}M$ as the *classifying space of M*.

Proposition 4.1.8. *Assume that C admits both finite limits and geometric realizations. Then the functors $\mathbf{B}$ and $\overline{\Omega}$ define an adjunction*

$$\mathbf{B}\colon \operatorname{Mon}(C) \rightleftarrows C_* : \overline{\Omega}.$$

Proof. Let $\mathbf{\Delta}_+$ denote the *augmented simplex category*, which in addition to the objects $[n]$ for $n \geq 0$ also has an object $[-1] = \emptyset$ that has a unique map to each $[n]$. There are fully faithful inclusions

$$i\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}_+ \qquad \text{and} \qquad j\colon [1] \hookrightarrow \mathbf{\Delta}_+, \quad j(0) = [-1], \quad j(1) = [0].$$

A functor $(\mathbf{\Delta}_+)^{\mathrm{op}} \to C$ is precisely the same data as functor $X\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ equipped with a cocone $X \to \mathrm{const}_W$. We may now consider the following two adjunctions:

$$\begin{tikzcd} \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \arrow[r, shift left=2, "i_!"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}((\mathbf{\Delta}_+)^{\mathrm{op}}, C) \arrow[l, shift left=2, "i^*"] \arrow[r, shift left=2, "j^*"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}([1]^{\mathrm{op}}, C). \arrow[l, shift left=2, "j_*"] \end{tikzcd}$$

The functors $i_!$ and j_* are called *left Kan extension along i* and *right Kan extension along j*; I have written down some additional explanations on Kan extensions in Appendix A.2.5. Unwinding the pointwise formulas for Kan extensions explained in that section, one observes that:

- The composite $j^* i_!$ sends a simplicial object X in C to the map $X_0 \to |X| := \operatorname{colim} X$;
- The composite $i^* j_*$ sends a morphism $f\colon Y \to X$ in C to its so-called *Čech nerve*:

  $$\check{C}(f)\colon \mathbf{\Delta}^{\mathrm{op}} \to C, \qquad \check{C}(f)_n := Y^{\times^n_X}.$$

  (Observe that the categories I_n used above are precisely the relative slice categories for the inclusion $[1]^{\mathrm{op}} \hookrightarrow {\mathbf{\Delta}_+}^{\mathrm{op}}$, and the pointwise limits computing the right Kan extension j_* are given by I_n-indexed limits, that just as in the proof of Lemma 4.1.5 can be computed to be an n-fold fiber product of Y with itself over X.)

So we see that the functors $\mathbf{B}$ and $\overline{\Omega}$ are precisely the restrictions of these functors to the subcategories

$$\operatorname{Mon}(C) \subseteq \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, C) \qquad \text{and} \qquad C_* \subseteq \operatorname{Fun}([1], C),$$

and the adjunction $j^* i_! \dashv i^* j_*$ accordingly restricts to an adjunction $\mathbf{B} \dashv \overline{\Omega}$. $\square$

4.2 The recognition principle for loop spaces

The goal of this section is to prove the following theorem:

Theorem 4.2.1 (Recognition principle for loop spaces, cf. Stasheff [Sta63], May [May72], Boardman and Vogt [BV73], Lurie [Lur09b, Theorem 7.2.2.11])**.** *Let M be a monoid in* An *and let X be a pointed anima.*

(1) The unit map $M \to \Omega\mathbf{B}M$ is an equivalence if and only if M is a group in An*;*

(2) The pointed anima $\mathbf{B}M$ is connected;

(3) The counit $\mathbf{B}\Omega X \to X$ is an inclusion of the connected component of the basepoint of X.

The formulation by May was in terms of 'operads': it says that if a topological space M comes equipped with certain algebraic structure making it a 'grouplike E_1-space', then it is equivalent to $\Omega\mathbf{B}M$. Our formulation above is a refined version of this theorem in the setting of ∞-categories, which has the additional advantage that it immediately provides us with an equivalence of ∞-categories between groups in An and connected pointed animae:

Corollary 4.2.2. *The adjunction $\mathbf{B} \dashv \Omega$ restricts to an equivalence of ∞-categories*

$$\mathbf{B}\colon \mathrm{Grp}(\mathrm{An}) \underset{\longleftarrow}{\overset{\simeq}{\longrightarrow}} \mathrm{An}_*^{\geq 1} :\Omega$$

between groups in An *and connected pointed animae.*

Proof. The adjunction from Proposition 4.1.8 restricts to groups and pointed connected animae since $\Omega(X)$ is always a group and $\mathbf{B}M$ is always connected. The unit $M \to \Omega\mathbf{B}M$ is always an isomorphism by part (1) of the theorem, while the counit $\mathbf{B}\Omega X \to X$ is always an isomorphism by part (3) of the theorem. This shows that the two functors are in fact inverse to one another. □

For the proof of the theorem we will need some preliminaries.

Extra degeneracies

Definition 4.2.3. We define the subcategory $\mathbf{\Delta}^{\mathrm{deg}} \subseteq \mathbf{\Delta}$ as the subcategory consisting of *all* objects $[n]$, but with only those morphisms $\varphi\colon [n] \to [m]$ satisfying $\varphi(n) = m$. Note that there exists a functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ given on objects by $[n] \mapsto [n+1]$, and on morphisms by sending $\varphi\colon [n] \to [m]$ to the map

$$[n+1] \to [m+1], \qquad i \mapsto \begin{cases} \varphi(i) & 0 \le i \le n \\ m+1 & i = n+1. \end{cases}$$

We wish to think of $\mathbf{\Delta}^{\mathrm{deg}}$ as an enlargement of the augmented simplex category $\mathbf{\Delta}_+$ in which there exist 'extra degeneracies'. Indeed, note that all the face maps that are allowed in $\mathbf{\Delta}^{\mathrm{deg}}$ are actually already in the image of the functor $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$, but the degeneracy map $s_n\colon [n+1] \to [n]$ is an example of a morphism in $\mathbf{\Delta}^{\mathrm{deg}}$ which does not lie in this image. This means that we may think of functors $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$ as *augmented simplicial objects with extra degeneracies*. We may draw them as follows, with the 'extra degeneracies' marked with dashed arrows:

$$\begin{tikzcd} \cdots \arrow[r, shift left=5] \arrow[r, shift left=1] \arrow[r, shift right=3] \arrow[r, leftarrow, shift left=3] \arrow[r, leftarrow, shift right=1] \arrow[r, leftarrow, dashed, shift right=5] & Y_2 \arrow[r, shift left=3] \arrow[r, shift right=1] \arrow[r, leftarrow, shift left=1] \arrow[r, leftarrow, dashed, shift right=3] & Y_1 \arrow[r, shift left=1] \arrow[r, leftarrow, dashed, shift right=1] & Y_0. \end{tikzcd}$$

The non-dashed arrows define its *underlying augmented simplicial object*, obtained by precomposition with $(+1)\colon \mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$. It is useful in practice to have extra degeneracies, since it leads to the following easy computation of the geometric realization:

Lemma 4.2.4 (Extra degeneracy argument)**.** *Consider an augmented simplicial object $X\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to C$, and assume that it admits 'extra degeneracies', in the sense that it is of the form $X_n = Y_{n+1}$ for some functor $Y\colon (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \to C$, i.e. X is given by the following composite:*

$${\mathbf{\Delta}_+}^{\mathrm{op}} \xrightarrow{(+1)} (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}} \xrightarrow{Y} C.$$

Then X is a colimit diagram, in the sense that the corresponding cocone $X|_{\mathbf{\Delta}^{\mathrm{op}}} \Rightarrow \mathrm{const}_{X_{-1}}$ is a colimit cocone:

$$\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n \xrightarrow{\sim} X_{-1}.$$

Proof. We need to show that the preferred map from $\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n$ to X_{-1} induced by the cocone is an isomorphism.

Observe that the inclusion functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ admits a right adjoint given by the inclusion $\mathbf{\Delta}^{\mathrm{deg}} \hookrightarrow \mathbf{\Delta}$: for natural numbers $n, m \geq 0$, providing a map $\varphi\colon [n] \to [m]$ of partially ordered sets is the same as providing a map $\overline{\varphi}\colon [n+1] \to [m]$ of posets satisfying the condition that $\overline{\varphi}(n+1) = m$. It follows from Corollary A.2.43 that the functor $(+1)\colon \mathbf{\Delta} \hookrightarrow \mathbf{\Delta}^{\mathrm{deg}}$ is an initial functor, and hence its dual $(+1)^{\mathrm{op}}\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow (\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ is a final functor. By Theorem A.2.38, this means that the geometric realization of X may be computed as

$$\mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n = \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} Y_{n+1} \simeq \mathrm{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m.$$

But since $\mathbf{\Delta}^{\mathrm{deg}}$ has an initial object given by $[0]$, $(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}$ has a terminal object, and so it follows from Lemma A.2.19 that $\mathrm{colim}_{[m]\in(\mathbf{\Delta}^{\mathrm{deg}})^{\mathrm{op}}} Y_m$ is isomorphic to $Y_0 = X_{-1}$ as desired. ◻

Descent

The ∞-category An has a special feature that distinguishes it from other ∞-categories: it satisfies *descent*. In the following statement, we use the notation $I^{\triangleright} := (I \times [1]) \times_{I\times\{1\}} *$ from Definition 2.6.27.

Theorem 4.2.5 (Descent for colimits)**.** *Let I be a small ∞-category, let $\overline{F}, \overline{G} \colon I^{\triangleright} \to \mathrm{An}$ be functors and let $\overline{\alpha} \colon \overline{F} \Rightarrow \overline{G}$ be a natural transformation such that the restriction $\alpha := \overline{\alpha}|_I \colon F \Rightarrow G$ is a cartesian transformation, in the sense that for every morphism $i \to j$ in I the commutative square*

$$\begin{tikzcd} F(i) \arrow[r] \arrow[d, "\alpha(i)"'] & F(j) \arrow[d, "\alpha(j)"] \\ G(i) \arrow[r] & G(j) \end{tikzcd} \tag{4.1}$$

is a pullback square. Assume that $\overline{G}$ is a colimit diagram. Then $\overline{F}$ is a colimit diagram is and only if $\overline{\alpha}$ is a cartesian transformation, i.e. also all the squares

$$\begin{tikzcd} F(i) \arrow[rr] \arrow[d, "\alpha(i)"'] & & \overline{F}(\infty) \arrow[d, "\overline{\alpha}(\infty)"] \\ G(i) \arrow[r] & \overline{G}(\infty) = \operatorname{colim}_I G. & \end{tikzcd}$$

are pullback squares.

Proof. This is [Lur09b, Theorem 6.1.3.9]. □

Let us unwind what the above result is saying:

- First assume that both $\overline{F}$ and $\overline{G}$ are colimit diagrams, i.e. $\overline{G}(\infty) = \operatorname{colim}_I G$ and $\overline{F}(\infty) = \operatorname{colim}_I F$. If we assume that each of the squares (4.1) is a pullback square, then we may recover the whole diagram $F(i)$ from just the map $\operatorname{colim}_i F(i) \to \operatorname{colim}_i G(i)$ on colimits induced by α.

- Conversely, if we only know that $\overline{G}$ is a colimit diagram, but we assume that $\overline{\alpha}$ is a cartesian transformation, then we may deduce that the value $\overline{F}$ must be the colimit of $F \colon I \to \mathrm{An}$.

We do not yet have enough theory to do this, but assume for a moment that we had a functor of the form

$$\mathrm{An}_{/-} \colon \mathrm{An}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \qquad X \mapsto \mathrm{An}_{/X}, \qquad (f \colon X \to Y) \mapsto (f^* \colon \mathrm{An}_{/Y} \to \mathrm{An}_{/X}).$$

Then the above result will say that this functor preserves small limits: for every functor $G\colon I \to \mathrm{An}$ we have an equivalence of ∞-categories

$$\mathrm{An}_{/\mathrm{colim}_{i\in I} G(i)} \xrightarrow{\sim} \lim_{i\in I^{\mathrm{op}}} \mathrm{An}_{/G(i)}.$$

In other words: the data of an anima X equipped with a map $X \to \mathrm{colim}_{i\in i} G(i)$ is the same as a compatible family of animae X_i equipped with maps $X_i \to G(i)$.

Example 4.2.6 (Universality of initial anima)**.** Consider $I = \emptyset$, so that $I^{\triangleright} = *$. In this case $\overline{\alpha}$ is just a morphism $X \to Y$ of animae, and the condition that $\overline{G}$ is a colimit diagram means that $Y = \emptyset$. Since $I^{\triangleright}$ has no morphisms, $\overline{\alpha}$ is always cartesian, hence by the theorem also $\overline{F}$ is a colimit diagram, i.e. we also have $X = \emptyset$. So descent for colimits in this case precisely boils down to the universality of the empty category from Axiom B.2'.

Example 4.2.7 (Universality of coproducts)**.** Consider $I = * \sqcup *$, so that $I^{\triangleright} = \lrcorner$. The transformation $\overline{\alpha}$ will thus be a diagram of the form

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "\alpha_0"] & X_2 \arrow[d, "\alpha_2"] & X_1 \arrow[l] \arrow[d, "\alpha_1"] \\
Y_0 \arrow[r] & Y_0 \sqcup Y_1 & Y_1, \arrow[l]
\end{tikzcd}$$

where in the bottom we get a coproduct, reflecting the condition that $\overline{G}$ is a colimit diagram. The theorem then expresses that the following two conditions are equivalent:

- The two squares in the diagram are both pullback squares;
- The functor $X_0 \sqcup X_1 \to X_2$ is an isomorphism.

But the fact that these two conditions are equivalent is precisely an instance of the universality of coproducts from Axiom B.4', with the two parts of the axiom corresponding to the two implications.

Whitehead's theorem

One final ingredient we will need is a version of Whitehead's Theorem for An.

Definition 4.2.8. For a pointed anima (X,x) and a natural number $n \geq 1$, we define the *n-th homotopy group of X* as

$$\pi_n(X,x) := [S^n, X]_* := \pi_0 \mathrm{Hom}_{\mathrm{An}_*}(S^n, X) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{An}_*)}(S^n, X).$$

Since $S^n = \Sigma^n(S^0)$, this may also be written as the set of path components of the n-fold loop space of X:

$$\pi_n(X,x) \cong \pi_0(\Omega^n X) = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{An}_*)}(S^0, \Omega^n(X)).$$

Since $\Omega^n X$ is a group object in $\mathrm{Ho}(\mathrm{An}_*)$ by Exercise 7.7, it follows that $\pi_n(X,x)$ is a group. It is abelian for $n \geq 2$ by the Eckmann-Hilton argument (see Lemma 1.3.10).

To prove Whitehead's theorem for animae, we will need to lift any morphism of animae to a morphism of CW-complexes. This will be achieved via the following proposition:

Proposition 4.2.9 (Cisinski [Cis19])**.** *For CW-complexes $\overline{X}$ and $\overline{Y}$ with underlying animae $X = \Pi_\infty(\overline{X})$ and $Y = \Pi_\infty(\overline{Y})$, there is an equivalence*

$$\mathrm{Hom}_{\mathrm{An}}(X,Y) \simeq \mathrm{colim}_{\overline{Y} \overset{\simeq}{\hookrightarrow} \overline{Z}} \mathrm{Hom}_{\mathrm{CW}}(\overline{X}, \overline{Z}),$$

where the colimit runs over the subcategory of $\mathrm{CW}_{\overline{Y}/}$ spanned by the trivial cofibrations $\overline{Y} \overset{\simeq}{\hookrightarrow} \overline{Z}$, i.e. those cofibrations that are homotopy equivalences.

Proof. This is the dual of [Cis19, Theorem 7.2.8], applied to left calculus of fractions coming from the class of trivial cofibrations via [Cis19, Corollary 7.2.18]. □

Corollary 4.2.10. *The localization functor $\Pi_\infty \colon \mathrm{CW} \to \mathrm{An}$ is essentially surjective on objects and on morphisms:*

- *Every object in An is isomorphic to $\Pi_\infty(X)$ for some $X \in \mathrm{CW}$;*
- *Every morphism in An is isomorphic to $\Pi_\infty(f)$ for some morphism $f \colon X \to Y$ of CW-complexes.*

Proof. For essential surjectivity, note that the essential image Π_∞ again satisfies the universal property of the localization, hence must be equivalent to An. For essential surjectivity on morphisms, consider a morphism $f \colon X \to Y$ of animae. We may write X and Y as the underlying animae of two CW-complexes $\overline{X}$ and $\overline{Y}$. By Proposition 4.2.9 there exists a trivial cofibration $\overline{Y} \hookrightarrow \overline{Z}$ and a continuous map $g \colon \overline{X} \to \overline{Z}$ that induces the map f after applying $\Pi_\infty(-)$. This finishes the proof. □

Recall also from Proposition B.2.8 that the localization functor $\Pi_\infty \colon \mathrm{CW} \to \mathrm{An}$ sends homotopy pushouts of CW-complexes to pushouts of animae.

Proposition 4.2.11 (Whitehead's theorem for An)**.** *A morphism of animae $f \colon X \to Y$ is an isomorphism if and only if it induces a bijection $\pi_0(f) \colon \pi_0(X) \to \pi_0(Y)$ on sets of path components, and if for every point x of X the map $\pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism of groups.*

Proof. By Corollary 4.2.10, we may write f as the map on animae induced by a continuous map $\overline{f}\colon \overline{X} \to \overline{Y}$ of CW-complexes. Note that the homotopy groups of $\overline{X}$ are the same as the homotopy groups of its underlying anima X: we have

$$\pi_n(\overline{X}) = \mathrm{Hom}_{\mathrm{Ho(CW_*)}}(S^n, \overline{X}) \cong \mathrm{Hom}_{\mathrm{Ho(An_*)}}(S^n, X) = \pi_n(X),$$

due to the equivalence $\mathrm{Ho(CW_*)} \simeq \mathrm{Ho(An_*)}$ from Corollary A.2.15. From the assumptions on f, it thus follows that the map $\overline{f}\colon \overline{X} \to \overline{Y}$ induces isomorphisms on all homotopy groups (and a bijection between sets of path components), and hence by the ordinary Whitehead theorem, Theorem 1.2.2, is a homotopy equivalence of CW-complexes. But since the localization functor $\Pi_\infty(-)\colon \mathrm{CW} \to \mathrm{An}$ inverts homotopy equivalences, this means that $f = \Pi_\infty(\overline{f})$ is an isomorphism of animae, as desired. □

Corollary 4.2.12. *Let $f\colon X \to Y$ be a morphism of connected pointed animae. Then f is an isomorphism if and only if the induced map $\Omega f\colon \Omega X \to \Omega Y$ is an isomorphism of animae.*

Proof. The 'only if'-direction is clear. For the converse, assume that Ωf is an isomorphism. To show that f is an isomorphism, it suffices by Proposition 4.2.11 to show that the map $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ is a bijection and that each $\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism of groups for $n \geq 0$. The former is clear, since $\pi_0(X)$ and $\pi_0(Y)$ are both assumed to be a point. For the letter we use that $\pi_n(X) \cong \pi_{n-1}(\Omega X)$, and similarly the map $\pi_n(f)$ is isomorphic to the map $\pi_{n-1}(\Omega f)\colon \pi_{n-1}(\Omega X) \to \pi_{n-1}(\Omega Y)$. But this is an isomorphism since Ωf is an isomorphism. □

Proof of the Recognition Principle

We are now ready to prove the recognition principle.

Proof of Theorem 4.2.1. We start with (2): the fact that $\mathbf{B}M$ is connected for any monoid object M in An. By definition, $\mathbf{B}M$ is connected if and only if its set of path components $\pi_0(\mathbf{B}M)$ is a single point. Since $\pi\colon \mathrm{An} \to \mathrm{Set}$ is a left adjoint, it preserves colimits by Lemma A.2.21, and thus $\pi_0(\mathbf{B}M)$ is a colimit in sets of the following simplicial diagram:

$$\cdots \quad \pi_0(M_2) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \pi_0(M_1) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \pi_0(M_0).$$

But $M_0 \cong *$ hence $\pi_0(M_0) = *$ and since the resulting map $* = \pi_0(M_0) \to \operatorname{colim}_{[n]} \pi_0(M_n) \cong \pi_0(\mathbf{B}M)$ is surjective it follows that $\pi_0(\mathbf{B}M)$ has a single element.

Let us assume (1) for a moment and use it to deduce (3), the fact that the counit $\varepsilon_X\colon \mathbf{B}\Omega X \to X$ is the inclusion of the path component X_0 of the basepoint of X. Since $\mathbf{B}\Omega X$ is connected, the counit map certainly lands in X_0 and we need to show that the resulting map $\mathbf{B}\Omega X \to X_0$ is an isomorphism of animae. Since both sides are connected pointed animae, it suffices by Corollary 4.2.12 to show that the map $\Omega(\varepsilon_X)\colon \Omega(\mathbf{B}\Omega X) \to \Omega(X_0) \simeq \Omega(X)$ is an isomorphism. By the triangle identities for the adjunction, there is a commutative triangle of the form

$$\begin{tikzcd}
& \Omega(\mathbf{B}\Omega X) \arrow[dr, "\Omega(\varepsilon_X)"] & \\
\Omega X \arrow[ur, "\eta_{\Omega X}"] \arrow[rr, equal] & & \Omega X,
\end{tikzcd}$$

and so $\Omega(\varepsilon_X)$ is an isomorphism if and only if $\eta_\Omega\colon \Omega X \to \Omega(\mathbf{B}\Omega X)$ is an isomorphism. But since ΩX is a group object, this is an instance of part (1).

It remains to prove part (1), i.e. that M is a group if and only if the map $\eta_M\colon M \to \Omega\mathbf{B}M$ is an isomorphism of monoids. We will deduce this as an application of Theorem 4.2.5. For this, we need to rephrase both conditions in terms of suitable pullback squares.

First, note that $\eta_M\colon M \to \Omega\mathbf{B}M$ is an isomorphism if and only if the map $(\eta_M)_1\colon M_1 \to (\Omega\mathbf{B}M)_1$ on underlying animae is an isomorphism: since $M_n \simeq M_1^{\times n}$ and $(\Omega\mathbf{B}M)_n \simeq ((\Omega\mathbf{B}M)_1)^{\times n}$ we then get that it is an isomorphism at every level. The map $M_1 \to \Omega\mathbf{B}M$ is induced by the commutative square

$$\begin{tikzcd}
M_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & M_0 \simeq * \arrow[d] \\
M_0 \simeq * \arrow[r] & \mathbf{B}M
\end{tikzcd}$$

and so is an isomorphism if and only if this is a pullback square of animae.

Second, note that M is a group if and only if the commutative square

$$\begin{tikzcd}
M_1^2 \arrow[r, "m"] \arrow[d, "\mathrm{pr}_1"'] & M_1 \arrow[d] \\
M_1 \arrow[r] & *
\end{tikzcd} \tag{4.2}$$

is a pullback square. Let us rephrase this condition a bit. Consider the functor

$$PM\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}, \qquad [n] \mapsto M_{n+1}$$

and consider the natural transformation $\alpha\colon PM \to M$ whose components are the maps $d_{n+1}\colon M_{n+1} \to M_n$.

Claim: M is a group if and only if α is a cartesian natural transformation, in the sense that for every morphism $\varphi\colon [n] \to [m]$ the square

$$\begin{tikzcd}
PM_m \arrow[r, "\varphi^*"] \arrow[d, "\alpha_m"'] & PM_n \arrow[d, "\alpha_n"] \\
M_m \arrow[r, "\varphi^*"] & M_n
\end{tikzcd} \tag{4.3}$$

is a pullback square.

Proof. One direction is clear: by taking φ to be the map $d_1\colon [0] \to [1]$, the square (4.3) reduces to (4.2). Conversely, assume (4.2) is a pullback square. We start by showing that (4.3) is a pullback square for $\varphi = d_i\colon [n-1] \to [n]$, where $n \geq 1$ and $0 \leq i \leq n$. In the case where $i = 0$, then under the isomorphisms $M_n \cong M_1^n$ coming from the Segal condition this square takes the form

$$\begin{tikzcd}
M_1^{n+2} \arrow[r, "{(m_1,\ldots,m_{n+2})\mapsto(m_2,\ldots,m_{n+2})}"] \arrow[d, "{(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})}"'] & M_1^{n+1} \arrow[d, "{(m_2,\ldots,m_{n+2})\mapsto(m_2,\ldots,m_{n+1})}"] \\
M_1^{n+1} \arrow[r, "{(m_1,\ldots,m_{n+1})\mapsto(m_2,\ldots,m_{n+1})}"'] & M^n,
\end{tikzcd}$$

which is always a pullback square. In the case $0 < i < n$, the square (4.3) takes the form

$$\begin{tikzcd}
M_1^{n+2} \arrow[r, "{(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_i m_{i+1},\ldots,m_{n+2})}"] \arrow[d, "{(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})}"'] & M_1^{n+1} \arrow[d, "{(x_1,\ldots,x_{n+1})\mapsto(x_1,\ldots,x_n)}"] \\
M_1^{n+1} \arrow[r, "{(m_1,\ldots,m_{n+1})\mapsto(m_1,\ldots,m_i m_{i+1},\ldots,m_{n+1})}"'] & M^n,
\end{tikzcd}$$

which is also always a pullback square. Finally, for $i = n$ the square takes the form

$$\begin{tikzcd}
M_1^{n+2} \arrow[r, "{(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_n,m_{n+1}m_{n+2})}"] \arrow[d, "{(m_1,\ldots,m_{n+2})\mapsto(m_1,\ldots,m_{n+1})}"'] & M_1^{n+1} \arrow[d, "{(x_1,\ldots,x_{n+1})\mapsto(x_1,\ldots,x_n)}"] \\
M_1^{n+1} \arrow[r, "{(m_1,\ldots,m_{n+1})\mapsto(m_1,\ldots,m_n)}"'] & M^n.
\end{tikzcd}$$

Note that none of the maps afffect the first n coordinates, and that for the last two coordinates m_{n+1} and m_{n+2} this is precisely the square (4.2). So this square is a product of a degenerate pullback square and the pullback square from (4.2) and hence is also a pullback square. This shows the claim for all maps φ of the form d_i.

Now, when φ is a degeneracy map of the form $s_i\colon [n+1] \to [n]$, then it will have a section of the form $d_i\colon [n] \to [n+1]$, and it follows from the pasting law of pullback squares that the square (4.3) is a pullback square:

$$\begin{tikzcd}
PM_n \arrow[r, "s_i^*"] \arrow[d, "\alpha_n"'] & PM_{n+1} \arrow[r, "d_i^*"] \arrow[d, "\alpha_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & PM_n \arrow[d, "\alpha_n"] \\
M_n \arrow[r, "s_i^*"] & M_{n+1} \arrow[r, "d_i^*"] & M_n.
\end{tikzcd}$$

This shows the claim when φ is either a face map d_i or a degeneracy map s_i. Since every map $\varphi\colon [n] \to [m]$ is an iterated composite of face maps and degeneracy maps, we then get the claim for general φ by using the pasting law of pullback squares again. $\square$

We may now finish the proof of the theorem. By taking the colimit of $M\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, we may extend this simplicial object to an augmented simplicial object $\overline{M}\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to \mathrm{An}$ with $\overline{M}_{-1} = \mathbf{B}M = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} M_n$. The simplicial object PM then inherits an extension to an augmented simplicial object $\overline{PM}\colon {\mathbf{\Delta}_+}^{\mathrm{op}} \to \mathrm{An}$ via $\overline{PM}_n := M_{n+1}$. Since $\overline{PM}$ has extra degeneracies, it is a colimit diagram by Lemma 4.2.4. The transformation $\alpha\colon PM \to M$ extends to a transformation $\overline{\alpha}\colon \overline{PM} \to \overline{M}$. We are now finally in a position to use descent for colimits from Theorem 4.2.5:

- First assume that M is a group. By the above claim, this means that $\alpha\colon PM \to M$ is a cartesian transformation. Since $\overline{PM}$ and $\overline{M}$ are colimit diagrams, Theorem 4.2.5 then implies that $\overline{\alpha}$ is also a cartesian transformation, hence in particular the square

$$\begin{tikzcd}
PM_0 = M_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & \overline{PM}_{-1} \simeq M_0 \simeq * \arrow[d, "d_0"] \\
M_0 \simeq * \arrow[r, "d_0"] & \overline{M}_{-1} \simeq \mathbf{B}M
\end{tikzcd}$$

  is a pullback square, expressing the fact that the map $M_1 \to \Omega\mathbf{B}M$ is an isomorphism.

- Conversely, if the previous square is a pullback square, then by the pasting law of pullback squares we get that each outer square

$$\begin{tikzcd}
PM_n = M_{n+1} \arrow[r, "d_0^*"] \arrow[d, "d_{n+1}^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & PM_0 = M_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & \overline{PM}_{-1} \simeq M_0 \simeq * \arrow[d, "d_0"] \\
M_n \arrow[r, "d_0^*"] & M_0 \simeq * \arrow[r, "d_0"] & \overline{M}_{-1} \simeq \mathbf{B}M
\end{tikzcd}$$

  is a pullback square, since the left-hand square is always a pullback square. But then another application of the pasting law of pullback squares applied to

$$\begin{tikzcd}
PM_m \arrow[r, "\varphi^*"] \arrow[d, "\alpha_m"'] & PM_n \arrow[r, "d_0"] \arrow[d, "\alpha_n"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{PM}_{-1} \arrow[d, "\overline{\alpha}_{-1}"] \\
M_m \arrow[r, "\varphi^*"] & M_n \arrow[r, "d_0"] & \overline{M}_{-1}
\end{tikzcd}$$

  shows that each of the squares (4.3) is a pullback square, hence M is a group by the Claim.

This finishes the proof of the Theorem. □

4.3 Commutative monoids and commutative groups

We now move to *commutative* monoids in an ∞-category. Just as for monoids, these will be defined as functors out of a suitable indexing category that encodes all the operations we have in a commutative monoid.

Construction 4.3.1 (Span category)**.** There is an ∞-category $\mathrm{Span}(\mathrm{Fin})$, called the *span category of finite sets*. Its objects are finite sets, and for finite sets S and T its Hom animae are given by the groupoids

$$\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(S,T) \qquad := \qquad \left\{ \begin{tikzcd} & U \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\cong"] & \\ S & & T \\ & U' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd} \right\}$$

of *spans from S to T*; so the objects of this groupoid are spans $S \leftarrow U \rightarrow T$ and the morphisms are isomorphisms of spans. The identity span is given by $S \xleftarrow{\mathrm{id}_S} S \xrightarrow{\mathrm{id}_S} S$, and composition of spans is defined by taking pullbacks:

$$\begin{tikzcd} & & U \times_T V \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] & & \\ & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\ S & & T & & R. \end{tikzcd}$$

Remark 4.3.2. Note that $\mathrm{Span}(\mathrm{Fin})$ is not an ordinary category: the Hom animae $\mathrm{Hom}_{\mathrm{Span}(\mathrm{Fin})}(S,T)$ are *groupoids* rather than *sets*. (Such ∞-categories are called $(2,1)$*-categories*.) Since we haven't explained yet how to associate an ∞-category to a $(2,1)$-category, the above definition is strictly speaking not complete.

There is a more general construction, due to Barwick [Bar17], of a *span ∞-category* $\mathrm{Span}(C)$ for an ∞-category with pullbacks C (which specializes to the above construction when $C = \mathrm{Fin}$ is the ordinary category of finite sets). The idea is to use Theorem A.5.4, which allows us to write down the simplicial anima $N(\mathrm{Span}(C)) \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ associated to $\mathrm{Span}(C)$:

- For $n \geq 0$, there is a partially ordered set $\mathrm{Tw}([n])$ called the *twisted arrow category of* $[n]$. The objects of $\mathrm{Tw}([n])$ are the morphisms in $[n]$, which are given by pairs (i,j) with $0 \leq i \leq j \leq n$. The ordering is given by demanding that $(i,j) \leq (i',j')$ if and only if either $i \leq i'$ or $j' \leq j$. We may display this poset as follows:

$$\begin{tikzcd}[column sep=tiny]
& & & & (0 \leq n) \arrow[dl] \arrow[dr] & & & & \\
& & & \dots \arrow[dl] \arrow[dr] & & (1 \leq n) \arrow[dl] \arrow[dr] & & & \\
& & (0 \leq 2) \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & \\
& (0 \leq 1) \arrow[dl] \arrow[dr] & & (1 \leq 2) \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & & \dots \arrow[dl] \arrow[dr] & \\
(0 \leq 0) & & (1 \leq 1) & & (2 \leq 2) & & \dots & & (n \leq n).
\end{tikzcd}$$

- These posets are functorial in $[n] \in \mathbf{\Delta}$: given a map $\varphi\colon [n] \to [m]$ of posets, we obtain an induced map of posets

  $$\mathrm{Tw}(\varphi)\colon \mathrm{Tw}([n]) \to \mathrm{Tw}([m]), \qquad (i \leq j) \mapsto (\varphi(i) \leq \varphi(j)).$$

  This determines a functor $\mathrm{Tw}\colon \mathbf{\Delta} \to \mathrm{Cat}_\infty$.

- We may now define $Q(X)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ by defining $Q(X)_n$ as the full subanima

  $$Q(X)_n \subseteq \mathrm{Map}(\mathrm{Tw}([n]), C)$$

  spanned by those diagrams $\mathrm{Tw}([n]) \to C$ in C such that each square

  $$\begin{tikzcd}[column sep=small]
  & (i \leq l) \arrow[dl] \arrow[dr] & \\
  (i \leq j) \arrow[dr] & & (k \leq l) \arrow[dl] \\
  & (k \leq j) &
  \end{tikzcd}$$

  is sent to a pullback square in C, for $i \leq k \leq j \leq l$.

- One can then prove that $Q(X)$ is a complete Segal anima, hence by Theorem A.5.4 is the nerve of some ∞-category $\mathrm{Span}(C)$, see [Bar17, Proposition 5.6] or [Hau+22, Lemma 2.17].

Lemma 4.3.3. *The ∞-category* $\mathrm{Span}(\mathrm{Fin})$ *is semiadditive, with biproducts given by disjoint unions of finite sets.*

Proof. Consider finite sets S_i for $i = 1, \ldots, n$. We have spans of the form

$$S_i \xleftarrow{\mathrm{id}_{S_i}} S_i \hookrightarrow \coprod_{i=1}^{n} S_i.$$

We claim that these morphisms exhibit $\bigsqcup_{i=1}^n S_i$ as a coproduct of the S_i in Span(Fin). Indeed, note that for every map $U \to \bigsqcup_{i=1}^n S_i$ of finite sets we may write $U = \bigsqcup_{i=1}^n U_i$, where U_i is the preimage of S_i. This gives the desired equivalence of groupoids

$$\mathrm{Hom}_{\mathrm{Span(Fin)}}(T, \bigsqcup_{i=1}^n S_i) = \left\{ \begin{tikzcd} & \bigsqcup_{i=1}^n U_i \arrow[dl, "f"'] \arrow[dd, "\cong"] \arrow[dr, "\bigsqcup_{i=1}^n g_i"] & \\ T & & \bigsqcup_{i=1}^n S_i \\ & \bigsqcup_{i=1}^n U'_i \arrow[ul, "f'"] \arrow[ur, "\bigsqcup_{i=1}^n g'_i"'] & \end{tikzcd} \right\} \simeq \prod_{i=1}^n \mathrm{Hom}_{\mathrm{Span}}(T, S_i).$$

By dualizing this argument (using that spans from S to T are the same as spans from T to S) we see that similarly the spans

$$\bigsqcup_{i=1}^n S_i \hookleftarrow S_i \xrightarrow{\mathrm{id}_{S_i}} S_i$$

exhibits $\bigsqcup_{i=1}^n S_i$ also as the *product* of the S_i in Span(Fin). This shows that Span(Fin) is semiadditive. □

Definition 4.3.4 (Commutative monoid)**.** Let C be an ∞-category with finite products. We define a *commutative monoid* in C to be a product-preserving functor

$$M\colon \mathrm{Span(Fin)} \to C.$$

We denote by

$$\mathrm{CMon}(C) \subseteq \mathrm{Fun}(\mathrm{Span(Fin)}, C)$$

the full subcategory spanned by the commutative monoids. We refer to the evaluation $M(*)$ at the one-point set as the *underlying object* of M. We will sometimes abuse notation and simply refer to $M(*)$ as M.

Remark 4.3.5 (Restriction and addition maps)**.** Let $M\colon \mathrm{Span(Fin)} \to C$ be a commutative monoid. Every morphism $f\colon S \to T$ of finite sets determines two 'half-degenerate' spans

$$f^* \quad = \quad (T \xleftarrow{f} S \xrightarrow{\mathrm{id}_S} S) \qquad \text{and} \qquad f_\oplus \quad = \quad (S \xleftarrow{\mathrm{id}_S} S \xrightarrow{f} T),$$

and hence the functoriality of M provides two maps

$$f^*\colon M(T) \to M(S) \qquad \text{and} \qquad f_\oplus\colon M(S) \to M(T).$$

Since products in Span(Fin) are computed as disjoint unions of finite sets, the condition that M preserves finite products means that for every finite set S the map

$$(e_s^*)_{s\in S}\colon M(S) \to M^S := \prod_{s\in S} M(*)$$

is an isomorphism, where $e_s\colon \{s\} \hookrightarrow S$ is the inclusion of the element s into S. We will use this identification from now on to identify $M(S)$ with the S-fold product of M in C. Under these identifications, we should think of the maps $f^*\colon M^T \to M^S$ and $f_\oplus\colon M^S \to M^T$ as the *restriction* and the *addition* maps, respectively:

$$f^*(x_t)_{t\in T} = (x_{f(s)})_{s\in S} \qquad \text{and} \qquad f_\oplus(x_s)_{s\in S} = (y_t)_{t\in T}, \quad y_t := \sum_{s\in f^{-1}(t)} x_s.$$

Specializing to the case $T = *$ and $S = * \sqcup *$, we obtain an addition map

$$+\colon M \times M = M^{*\sqcup *} \to M,$$

and similarly the map $\emptyset \to *$ in Fin gives rise to the zero map $0\colon * \to M$. Just as with non-commutative monoids, we are not just encoding the binary addition map $+\colon M \times M \to M$, but all n-fold addition maps $M^n \to M$ together will all possible compatibilities between them.

Remark 4.3.6 (Comparison to usual definition)**.** To state the condition that $M(S)$ is isomorphic to $\prod_{s\in S} M(*)$, one only needs the contravariant functoriality of M along the inclusion maps $e_s\colon \{s\} \hookrightarrow S$. Instead of working with the whole span category $\mathrm{Span}(\mathrm{Fin})$, we could then also have worked with its subcategory

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \subseteq \mathrm{Span}(\mathrm{Fin})$$

which contains all the objects but only those spans of the form $S \hookleftarrow U \to T$ where the left-pointing morphism is an injection. More precisely, if we say that a functor $M\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to C$ satisfies the *Segal condition* if the induced maps $(e_s^*)_{s\in S}\colon M(S) \to \prod_{s\in S} M(*)$ are isomorphisms, then one can show that restriction along the inclusion induces an equivalence

$$\mathrm{CMon}(C) = \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}), C) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C),$$

with inverse given by right Kan extension along the inclusion; see [BH21, Proposition C.1]. The category $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ may be thought of as the category of finite sets and *partially defined maps*: a morphism from S to T in this category is by definition a choice of subset U of S together with a map $U \to T$.

Note that the category $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$ is equivalent to the category Fin_* of finite *pointed* sets: there is a functor

$$\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{Fin}_*, \qquad S \mapsto S_+ := S \sqcup *$$

which on objects equips every finite set S with an additional basepoint, and which on morphisms sends the partially defined map from S to T to the pointed map $S_+ \to T_+$ which on U is just given by the given map $U \to T$ and on the complement of U sends everything to the basepoint of T_+. The inverse to this equivalence is given on objects by removing the basepoint, and on morphisms by taking the partial map defined on those elements of S_+ that are not sent to the basepoint in T_+.

Warning 4.3.7. The ∞-category $\mathrm{CMon}(C)$ is usually defined as $\mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Fin}_*, C)$ in the literature; this was also the original approach taken by Segal [Seg68] after which this condition is named. I find the approach with $\mathrm{Span}(\mathrm{Fin})$ a bit more conceptual, hence have chosen to present it this way.

Definition 4.3.8. A commutative monoid M is called a *commutative group* if the 'shear map' $(\mathrm{pr}_1, +)\colon M \times M \to M \times M$ is an isomorphism. We denote by

$$\mathrm{CGrp}(C) \subseteq \mathrm{CMon}(C)$$

the full subcategory spanned by the commutative groups in C.

4.3.1 The underlying monoid of a commutative monoid

As the name suggests, every commutative monoid has an underlying monoid.

Construction 4.3.9. We will construct a functor $\mathrm{Cut}\colon \Delta^{\mathrm{op}} \to \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \subseteq \mathrm{Span}(\mathrm{Fin})$:

- On objects, it sends $[n]$ to the set $\langle n \rangle := \{1, \ldots, n\}$.
- On morphisms, it sends $\varphi\colon [m] \to [n]$ to the span

  $$\langle n \rangle \longleftarrow \{\varphi(0)+1, \varphi(0)+2, \ldots, \varphi(m)-1, \varphi(m)\} \longrightarrow \langle m \rangle,$$

  where the left-pointing map is the inclusion and the right-pointing map sends i to the unique $1 \le j \le m$ such that $\varphi(j-1) < i \le \varphi(j)$.

More conceptually, we may think of the set $\mathrm{Cut}(P)$ for a poset P as the set of 'Dedekind cuts' of P, defined as pairs (P_0, P_1) of non-empty subposets $P_0, P_1 \subseteq P$ satisfying $P = P_0 \cup P_1$ and $p_0 \le p_1$ for $p_0 \in P_0$ and $p_1 \in P_1$. We want to think of the element $k \in \{1, \ldots, n\} = \mathrm{Cut}([n])$ as the following partition of $[n]$:

$$k \quad \leftrightsquigarrow \quad (\{0, \ldots, k-1\}, \{k, \ldots, n\}).$$

On morphisms, $\mathrm{Cut}(\varphi)$ is then simply given by taking preimages of the two subsets P_0 and P_1, which is only defined if both of these preimages are non-empty.

Alternatively, we may think of the set $\mathrm{Cut}([n])$ as labeling the n "inequalities" in the partially ordered set $[n]$ that we can use to cut $[n]$ into two pieces:

$$[n] \quad = \quad \{0 \overset{1}{\le} 1 \overset{2}{\le} 2 \overset{3}{\le} \ldots \overset{n}{\le} n\}.$$

Exercise 4.3.10. Let $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$ be a commutative monoid. Show that $M \circ \mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is a monoid. Show similarly that $M \circ \mathrm{Cut}$ is a group whenever M is a commutative group.

In particular, we obtain forgetful functors

$$U := \mathrm{Cut}^*\colon \mathrm{CMon}(C) \to \mathrm{Mon}(C) \qquad \text{and} \qquad U := \mathrm{Cut}^*\colon \mathrm{CGrp}(C) \to \mathrm{Grp}(C).$$

Exercise 4.3.11. Assume that C is an ordinary category with finite products. Show that the forgetful functor $U\colon \mathrm{CMon}(C) \to \mathrm{Mon}(C)$ is fully faithful.

Warning 4.3.12. For a general ∞-category, it is *not* true that the forgetful functor $\mathrm{CMon}(C) \to \mathrm{Mon}(C)$ is fully faithful! In other words: in the world of ∞-categories, it is no longer true that commutativity is a *property* of a monoid; it is additional *structure* one needs to provide.

4.3.2 Semiadditivity

Recall that the ordinary category of abelian monoids is semiadditive: the product $A \times B$ of two abelian monoids is also the *co*product, and the resulting biproduct is usually denoted $A \oplus B$. We will now show that $\mathrm{CMon}(C)$ is semiadditive for arbitrary C. We start with two preliminary lemmas:

Lemma 4.3.13. *Let C be an ∞-category with finite products. Then the ∞-category* $\mathrm{CMon}(C)$ *has finite products and the evaluation map* $\mathrm{ev}_T\colon \mathrm{CMon}(C) \to C$ *preserves finite products for each* $T \in \mathrm{Fin}$.

Proof. The functor category $\mathrm{Fun}(\mathrm{Span}(\mathrm{Fin}), C)$ admits finite products (Lemma A.2.23) which are computed pointwise, and the subcategory $\mathrm{CMon}(C)$ is closed under finite products. □

Lemma 4.3.14. *Let C be an ∞-category with finite products. Then there exists a cotensoring functor*

$$(-)^{(-)}\colon \mathrm{Span}(\mathrm{Fin}) \times \mathrm{CMon}(C) \to \mathrm{CMon}(C), \qquad (S, X) \mapsto X^S,$$

defined by $X^S(T) := X(S \times T)$, which preserves finite products in both variables and whose restriction to $\{*\} \times \mathrm{CMon}(C)$ *is the identity functor.*

Proof. For a finite set S and a commutative monoid X we define the cotensoring by $X^S(T) := X(S \times T)$ for $T \in \mathrm{Fin}$. The functoriality in S and T comes from the observation that the product functor $- \times -\colon \mathrm{Fin} \times \mathrm{Fin} \to \mathrm{Fin}$ induces a functor on span categories by taking products of spans:

$$- \times -\colon \mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin}) \to \mathrm{Span}(\mathrm{Fin}), \qquad (S, T) \mapsto S \times T.$$

(This is no longer the categorical product in $\mathrm{Span}(\mathrm{Fin})$!)

- Since we have a natural bijection $* \times T \cong T$, it is clear that the cotensoring is the identity on $\{*\} \times \mathrm{CMon}(C)$.

- Fixing $S \in \mathrm{Fin}$, the functor $(-)^S \colon \mathrm{CMon}(C) \to \mathrm{CMon}(C)$ preserves finite products: we may check this pointwise for every $T \in \mathrm{Fin}$, where it is by definition given by the evaluation $X \mapsto X(S \times T)$, which preserves finite products.

- Fixing $X \in \mathrm{CMon}(C)$, the functor $X^{(-)} \colon \mathrm{Span}(C) \to \mathrm{CMon}(C)$ preserves finite products: again we may check this pointwise for every $T \in \mathrm{Fin}$, where we may write it as the composite
$$\mathrm{Span}(\mathrm{Fin}) \xrightarrow{-\times T} \mathrm{Span}(\mathrm{Fin}) \xrightarrow{X} C.$$
But X preserves finite products by assumption and $-\times T$ preserves finite products since the functor $-\times T \colon \mathrm{Fin} \to \mathrm{Fin}$ preserves finite *co*products: $\bigsqcup_{i=1}^n S_i \times T \xrightarrow{\sim} (\bigsqcup_{i=1}^n S_i) \times T$.

In particular we see that X^S is the S-fold product of X in $\mathrm{CMon}(C)$, explaining the notation.

$\square$

Proposition 4.3.15. *Let C be an ∞-category with finite products. Then the ∞-category* $\mathrm{CMon}(C)$ *is semiadditive. The subcategory* $\mathrm{CGrp}(C)$ *is even additive.*

Proof. The main point is showing that $\mathrm{CMon}(C)$ is semiadditive; it is then essentially by definition true that the subcategory $\mathrm{CGrp}(C)$ is even additive.

We first show that $\mathrm{CMon}(C)$ is pointed by showing that the constant functor $\mathrm{const}_* \colon \mathrm{Span}(\mathrm{Fin}) \to C$ is a zero object. Note that const_* is a terminal object in the functor category (Lemma A.2.23) and hence also in $\mathrm{CMon}(C)$. To see it is also an initial object, consider any commutative monoid M. Since Span(Fin) has an initial object given by the empty set, limits over Span(Fin) are given by evaluation at the empty set (see Lemma A.2.19) and henc we get

$$\mathrm{Hom}_{\mathrm{CMon}(C)}(\mathrm{const}_*, M) \simeq \mathrm{Hom}_C(*, \lim_{S \in \mathrm{Span}(\mathrm{Fin})} M(S)) \simeq \mathrm{Hom}_C(*, M(\emptyset)) \simeq \mathrm{Hom}_C(*,*) \simeq *,$$

where we also use that $M(\emptyset) \simeq *$ because M preserves finite products.

To see that $\mathrm{CMon}(C)$ is semiadditive, consider two commutative monoids X and Y. We need to show that the maps $(\mathrm{id}_X, 0) \colon X \to X \times Y$ and $(0, \mathrm{id}_Y) \colon Y \to X \times Y$ exhibit the product $X \times Y$ also as a *co*product. In other words, we need to show that the corresponding natural transformation

$$((\mathrm{id}_X, 0), (0, \mathrm{id}_Y)) \colon (X, Y) \to (X \times Y, X \times Y) = \Delta(X \times Y)$$

is the unit of an adjunction between the product functor and the diagonal functor $\Delta \colon \mathrm{CMon}(C) \to \mathrm{CMon}(C) \times \mathrm{CMon}(C)$ (with the product being the *left* adjoint!). To this end, we have to produce for every third commutative monoid Z a compatible counit map

$$m_Z\colon Z\times Z\to Z$$

satisfying the triangle identities. Using the cotensoring map from Lemma 4.3.14, we may construct the map m_Z as the following composite:

$$m_Z\colon Z\times Z\cong Z^{*\sqcup *}\to Z^{*}\cong Z,$$

where the middle map uses functoriality of the cotensoring in the forward-ponting span associated to $*\sqcup *\to *$. The triangle identities take the following form:

$$\begin{tikzcd} & (Z\times Z, Z\times Z) \arrow[dr, "{(m_Z,m_Z)}"] & \\ (Z,Z) \arrow[ur, "{(\mathrm{id}_Z,0)\times(0,\mathrm{id}_Z)}"] \arrow[rr, equal] & & (Z,Z) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & (X\times Y)\times(X\times Y) \arrow[dr, "m_{X\times Y}"] & \\ X\times Y \arrow[ur, "{(\mathrm{id}_X,0)\times(0,\mathrm{id}_Y)}"] \arrow[rr, equal] & & X\times Y. \end{tikzcd}$$

For the right-hand triangle, we use that the cotensoring preserves products, so that the map $m_{X\times Y}$ may be rewritten as the product map

$$m_X\times m_Y\colon (X\times X)\times(Y\times Y)\to X\times Y.$$

We then see that to produce both triangles, it suffices to produce for every $X\in \mathrm{CMon}(C)$ two natural homotopies between the composites

$$X\xrightarrow{(\mathrm{id}_X,0)} X\times X\xrightarrow{m_X} X,\qquad X\xrightarrow{(0,\mathrm{id}_X)} X\times X\xrightarrow{m_X} X$$

and the identity on X. For this, observe that we may write the zero map $0\colon \mathrm{const}_*\to X$ as the map $X^{\emptyset}\to X^{*}$ induced on cotensorings by the inclusion $\emptyset\hookrightarrow *$ in Fin, so the claim boils down to functoriality of the cotensoring in the Span(Fin)-variable, together with the fact that the composites

$$*=*\sqcup\emptyset\hookrightarrow *\sqcup *\to * \qquad\text{and}\qquad *=\emptyset\sqcup *\hookrightarrow *\sqcup *\to *$$

are the identity maps in Fin. This finishes the proof. ◻

Remark 4.3.16. Observe that there are equivalences $\mathrm{CMon}(C_*)\simeq\mathrm{CMon}(C)_*$ and $\mathrm{CGrp}(C_*)\simeq\mathrm{CGrp}(C)_*$. Since $\mathrm{CMon}(C)$ and $\mathrm{CGrp}(C)$ are pointed, it follows from Lemma 3.1.4 that the forgetful functor $C_*\to C$ induces equivalences

$$\mathrm{CMon}(C_*)\xrightarrow{\sim}\mathrm{CMon}(C)\qquad\text{and}\qquad \mathrm{CGrp}(C_*)\xrightarrow{\sim}\mathrm{CGrp}(C).$$

Using a similar argument we find equivalences

$$\mathrm{Mon}(C_*)\xrightarrow{\sim}\mathrm{Mon}(C)\qquad\text{and}\qquad \mathrm{Grp}(C_*)\xrightarrow{\sim}\mathrm{Grp}(C).$$

4.3.3 Monoids and commutative monoids in semiadditive ∞-categories

For a semiadditive ∞-category C, it turns out every object admits a unique structure of a commutative monoid. If C is in fact *additive*, then each object is even a commutative *group*. More precisely:

Proposition 4.3.17. *Let C be a semiadditive ∞-category. Then the two forgeful functors*

$$\mathrm{CMon}(C) \to \mathrm{Mon}(C) \to C$$

are equivalences. If C is additive, then also the two forgetful functors

$$\mathrm{CGrp}(C) \to \mathrm{Grp}(C) \to C$$

are equivalences.

Proof sketch. Let us first do the case for commutative monoids. Consider the following subcategories of $\mathrm{Span}(\mathrm{Fin})$:

$$* \hookrightarrow \mathrm{Fin}^{\leq 1} \hookrightarrow \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}).$$

Here $\mathrm{Fin}^{\leq 1} \subseteq \mathrm{Fin}$ is the full subcategory on sets of cardinality at most 1, i.e. it has objects $\emptyset$ and $*$ and a single non-identity morphism $\emptyset \to *$. We claim that restriction along each of these inclusions defines equivalences

$$\begin{aligned} \mathrm{CMon}(C) = \mathrm{Fun}^{\times}(\mathrm{Span}(\mathrm{Fin}), C) &\xrightarrow{\sim} \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C) \\ &\xrightarrow{\sim} \mathrm{Fun}^{*}(\mathrm{Span}(\mathrm{Fin}^{\leq 1}), C) \xrightarrow{\sim} \mathrm{Fun}^{*}(\mathrm{Fin}^{\leq 1}, C) \xrightarrow{\sim} C, \end{aligned}$$

with inverses given by right, left, right and left Kan extension, respectively. Here $\mathrm{Fun}^{*}(-,-)$ denotes those functors that send the empty set to the terminal object of C. Each of these claims is proved in an analogous way, hence let us merely explain the general strategy and only looking at one case in more detail.

For the general case, one observes that each functor is conservative, that it admits a left/right adjoint given by Kan extension, and that this Kan extension does not change the value on the point so that the Kan extension of the restriction of a functor F must be isomorphic to F via the unit/counit.

Let us spell this out in case of the restriction functor

$$i^{*} \colon \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}), C) \to \mathrm{Fun}^{*}(\mathrm{Span}(\mathrm{Fin}^{\leq 1}), C)$$

along the inclusion $i \colon \mathrm{Span}(\mathrm{Fin}^{\leq 1}) \hookrightarrow \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$. Note that by semiadditivity of C, a functor $F \colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to C$ satisfies the Segal condition if and only if its restriction

$F|_{\mathrm{Fin}}\colon \mathrm{Fin}\to C$ is left Kan extended from the point. Similarly, a functor $G\colon \mathrm{Span}(\mathrm{Fin}^{\leq 1})\to C$ satisfies $F(\emptyset)=*$ if and only if its restriction $G|_{\mathrm{Fin}^{\leq 1}}\colon \mathrm{Fin}^{\leq 1}\to C$ is left Kan extended from the point. We now claim that left Kan extension along i restricts to a functor

$$i_!\colon \mathrm{Fun}^*(\mathrm{Span}(\mathrm{Fin}^{\leq 1}),C)\to \mathrm{Fun}^{\mathrm{Segal}}(\mathrm{Span}(\mathrm{Fin},\mathrm{inj},\mathrm{all}),C).$$

To see this, let $G\colon \mathrm{Span}(\mathrm{Fin}^{\leq 1})\to C$ be a functor satisfying $F(\emptyset)=*$. It will suffice to show that the restriction $i_!(G)|_{\mathrm{Fin}}$ of the left Kan extension of G is itself the left Kan extension of $G|_{\mathrm{Fin}^{\leq 1}}$, since then it must be Kan extended from the point. If we let $i'\colon \mathrm{Fin}^{\leq 1}\to \mathrm{Fin}$ denote the inclusion, there is a canonical map

$$i'_!(G|_{\mathrm{Fin}^{\leq 1}})\to i_!(G)|_{\mathrm{Fin}}.$$

Using the pointwise formula for Kan extensions, Theorem A.2.34, we may compute both sides as colimits of G over the relative slices of i' and i, and the previous comparison map is induced by the inclusion map $\mathrm{Fin}^{\leq 1}_{/S}\to \mathrm{Span}(\mathrm{Fin}^{\leq 1})_{/S}$ of relative slices, for $S\in\mathrm{Fin}$. Since this inclusion admits a left adjoint (given by taking only the forward map of a span into S, exercise!), this functor is final by Corollary A.2.43, and hence restriction along it does not change the value of the colimit.

Finally we observe that the Kan extension $i_!G$ has the same value on the point as G itself. From this one concludes that the unit $G\to i^*i_!G$ and counit $i_!i^*F\to F$ of the adjunction are both natural isomorphisms, and hence these two functors are inverse equivalences.

The claim for $\mathrm{Mon}(C)$ is proved similarly, instead using the subcategories

$$\{[1]\}\hookrightarrow \mathbf{\Delta}^{\leq 1}\hookrightarrow \mathbf{\Delta}_{\mathrm{int}}\hookrightarrow \mathbf{\Delta}.$$

The claim about groups and commutative groups is an immediate consequence: if C is semiadditive, the shear map of any (commutative) monoid M in C is simply given by the map $\begin{pmatrix}\mathrm{id}_X & \mathrm{id}_X\\ 0 & \mathrm{id}_X\end{pmatrix}\colon X^2\to X^2$, hence is by assumption an isomorphism if C is additive. □

With this at hand, we can now prove universal properties for the assignments $C\mapsto \mathrm{CMon}(C)$ and $C\mapsto \mathrm{CGrp}(C)$.

Notation 4.3.18. Let $\mathrm{Cat}_\infty^\times$ be the subcategory of Cat_∞ spanned by those ∞-categories that have finite products and the finite-product-preserving functors. We further denote by

$$\mathrm{Cat}_\infty^{\mathrm{sadd}}\subseteq \mathrm{Cat}_\infty^\times \qquad\text{and}\qquad \mathrm{Cat}_\infty^{\mathrm{add}}\subseteq \mathrm{Cat}_\infty^\times$$

the full subcategories spanned by the semiadditive and additive ∞-categories, respectively.

Proposition 4.3.19. *The two functors*

$$\mathrm{CMon}(-)\colon \mathrm{Cat}^\times\to \mathrm{Cat}^{\mathrm{sadd}} \qquad \textit{and} \qquad \mathrm{CGrp}(-)\colon \mathrm{Cat}^\times\to \mathrm{Cat}^{\mathrm{add}}$$

are right adjoint to the respective inclusion functors.

Proof. We will check the (dual version of the) criterion from Proposition A.2.63. We will treat the case for $\mathrm{CMon}(-)$; the case for $\mathrm{CGrp}(-)$ is identical. The forgetful functor defines a natural transformation $\varepsilon\colon \mathrm{CMon}(-) \to \mathrm{id}_{\mathrm{Cat}^{\times}}$, and we must check that for every ∞-category C with finite products the two forgetful functors

$$\varepsilon_{\mathrm{CMon}(C)}\colon \mathrm{CMon}(\mathrm{CMon}(C)) \to \mathrm{CMon}(C) \qquad \text{and} \qquad \mathrm{CMon}(\varepsilon_C)\colon \mathrm{CMon}(\mathrm{CMon}(C)) \to \mathrm{CMon}(C)$$

are equivalences. Since $\mathrm{CMon}(C)$ is semiadditive by Proposition 4.3.15, the claim for $\varepsilon_{\mathrm{CMon}(C)}$ is an instance of Proposition 4.3.17. But now the case for $\mathrm{CMon}(\varepsilon_C)$ follows immediately: the swap map $\mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin}) \xrightarrow{\sim} \mathrm{Span}(\mathrm{Fin}) \times \mathrm{Span}(\mathrm{Fin})$ determines an equivalence

$$\mathrm{CMon}(\mathrm{CMon}(C)) \xrightarrow{\sim} \mathrm{CMon}(\mathrm{CMon}(C)), \qquad (M_n)_m \mapsto (M_m)_n,$$

and under this equivalence the forgetful functor $\varepsilon_{\mathrm{CMon}(C)}((M_n)_m) = (M_n)_1$ corresponds to the forgetful functor $\mathrm{CMon}(\varepsilon_C)((M_m)_n) = (M_1)_n$. □

4.4 The recognition principle for connective spectra

In Section 4.2 we proved the recognition principle for loop spaces: the loop functor

$$\Omega\colon \mathrm{An}_*^{\geq 1} \xrightarrow{\sim} \mathrm{Grp}(\mathrm{An})$$

is an equivalence, with inverse given by the classifying space construction $\mathbf{B}\colon \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*^{\geq 1}$. In this section, we will show that this equivalence can be enhanced to an equivalence

$$\Omega^{\infty}(-)\colon \mathrm{Sp}^{\geq 0} \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An})$$

between the ∞-category of connective spectra and the ∞-category of commutative groups in An. Let us start by recalling the definition of $\mathrm{Sp}^{\geq 0}$:

Definition 4.4.1. An anima X is called *n-connected* for $n \geq 1$ if it is connected and the homotopy groups $\pi_k(X)$ are trivial whenever $k \leq n$. We write

$$\mathrm{An}^{\geq n} \subseteq \mathrm{An}$$

for the full subcategory spanned by the $(n-1)$-connected animae. Note that X is n-connected if and only if it is connected and ΩX is $(n-1)$-connected.

A spectrum X is called *connective* if its n-th space $X_n = \Omega^{\infty-n}X$ is $(n-1)$-connected for every n. We write

$$\mathrm{Sp}^{\geq 0} \subseteq \mathrm{Sp}$$

for the full subcategory spanned by the connective spectra. Connective spectra are also known as *infinite loop spaces*.

Remark 4.4.2. Since we have $\pi_i(X) \cong \pi_{n+i}(X_n)$ whenever $n+i \geq 0$, we see that a spectrum X is connective if $\pi_i(X) = 0$ for $i < 0$.

As an immediate corollary of the recognition principle for loop spaces we obtain a recognition principle for n-fold loop spaces. We need one auxiliary lemma:

Lemma 4.4.3. *The functors*

$$|-|\colon \mathrm{sAn} \to \mathrm{An} \qquad \textit{and} \qquad \mathbf{B}\colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$$

preserve finite products.

Proof. The claim for $\mathbf{B}$ immediately follows from the claim for $|-|$ since the inclusion $\mathrm{Mon}(\mathrm{An}) \hookrightarrow \mathrm{sAn} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An})$ preserves finite products. For $|-|$, consider simplicial animae X and Y. Then we compute

$$\begin{aligned} |X \times Y| = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}(X_n \times Y_n) &\overset{(1)}{\simeq} \operatorname{colim}_{([n],[m])\in\mathbf{\Delta}^{\mathrm{op}}\times\mathbf{\Delta}^{\mathrm{op}}}(X_n \times Y_m) \\ &\overset{(2)}{\simeq} \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \operatorname{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}}(X_n \times Y_m) \\ &\overset{(3)}{\simeq} \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}}(X_n \times \operatorname{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} Y_m) \\ &\overset{(4)}{\simeq} (\operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n) \times (\operatorname{colim}_{[m]\in\mathbf{\Delta}^{\mathrm{op}}} Y_m) \\ &= |X| \times |Y|. \end{aligned}$$

Here equivalence (1) holds because the diagonal functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$, $[n] \mapsto ([n],[n])$ is final, see the exercise below. The equivalence (2) is an instance of Lemma A.2.24. Finally, the equivalences (3) and (4) hold because for an anima X the functor $X \times -\colon \mathrm{An} \to \mathrm{An}$ admits a right adjoint $\mathrm{Hom}_{\mathrm{An}}(X,-)\colon \mathrm{An} \to \mathrm{An}$ and hence preserves all colimits. □

Exercise 4.4.4. Show that the diagonal functor $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$ is initial: for $[n],[m] \in \mathbf{\Delta}$ the relative slice category $\mathbf{\Delta} \times_{\mathbf{\Delta}\times\mathbf{\Delta}} (\mathbf{\Delta}\times\mathbf{\Delta})_{/([n],[m])}$ is weakly contractible. *Hints:*

- Show that this relative slice category is equivalent to the full subcategory $\mathbf{\Delta}_{/[n]\times[m]}$ of $\mathrm{Poset}_{/[n]\times[m]}$ consisting of morphisms of posets of the form $\varphi\colon [k] \to [n]\times[m]$.
- Consider the functor
$$f\colon \mathbf{\Delta}_{/[n]\times[m]} \to [n]\times[m]$$
that sends $\varphi\colon [k] \to [n]\times[m]$ to $\varphi(k)$. Consider also the functor
$$g\colon [n]\times[m] \to \mathbf{\Delta}_{/[n]\times[m]}$$
that sends a pair $(i,j) \in [n]\times[m]$ to the map $(i,j)\colon [0] \to [n]\times[m]$. Show that $fg = \mathrm{id}$ and define a natural transformation $gf \to \mathrm{id}$. Conclude that f and g induce equivalences on geometric realizations.

- Show that $|C \times D| \simeq |C| \times |D|$ for ∞-categories C and D;
- Show that $[n]$ and $[m]$ are both weakly contractible, cf. Exercise 6.4.

Definition 4.4.5. Let C be an ∞-category with finite products. For $n \geq 0$ we iteratively define the notion of an *E_n-monoid in C*:

- For $n = 0$ we define $\mathrm{Mon}_{E_0}(C) := C_*$;
- For $n \geq 1$ we define $\mathrm{Mon}_{E_n}(C) := \mathrm{Mon}(\mathrm{Mon}_{E_{n-1}}(C))$.

We similarly define $\mathrm{Grp}_{E_0}(C) = C_*$ and $\mathrm{Grp}_{E_n}(C) := \mathrm{Grp}(\mathrm{Grp}_{E_{n-1}}(C))$.

Remark 4.4.6. For $n = 1$ note that we get $\mathrm{Mon}_{E_1}(C) \xrightarrow{\sim} \mathrm{Mon}(C)$ and $\mathrm{Grp}_{E_1}(C) \xrightarrow{\sim} \mathrm{Grp}(C)$ by Remark 4.3.16.

For general n, an E_n-monoid has n different multiplication operations that all commute with each other. In light of the Eckmann-Hilton arument (Lemma 1.3.10) these operations are all identified in the homotopy category $\mathrm{Ho}(C)$ and are homotopy commutative for $n \geq 2$. We should think of higher n as 'more commutativity'.

Proposition 4.4.7 (Recognition principle for iterated loop spaces)**.** *For every $n \geq 0$ there is an adjunction*

$$\mathbf{B}^n \colon \mathrm{Mon}_{E_n}(\mathrm{An}) \rightleftarrows \mathrm{An}_* : \Omega^n$$

which restricts to an equivalence

$$\mathbf{B}^n \colon \mathrm{Grp}_{E_n}(\mathrm{An}) \underset{\simeq}{\rightleftarrows} \mathrm{An}_*^{\geq n} : \Omega^n.$$

Proof. Consider the adjunction $\mathbf{B} \colon \mathrm{Mon}(\mathrm{An}) \rightleftarrows \mathrm{An}_* : \Omega$. Since both functors preserve finite products, we may apply $\mathrm{Mon}_{E_{n-1}}(-)$ to get an adjunction

$$\mathbf{B} \colon \mathrm{Mon}_{E_n}(\mathrm{An}) \rightleftarrows \mathrm{Mon}_{E_{n-1}}(\mathrm{An}) : \Omega.$$

By the Recognition Principle, Theorem 4.2.1, this restricts to an equivalence $\mathrm{Grp}_{E_n}(\mathrm{An}_*^{\geq i}) \simeq \mathrm{Grp}_{E_{n-1}}(\mathrm{An}^{\geq i+1})$ for all $i \geq 0$. Pasting all these adjunctions together, this gives the claim. $\square$

Corollary 4.4.8. *The ∞-category $\mathrm{Sp}^{\geq 0}$ is equivalent to limit of the following diagram of ∞-categories:*

$$\cdots \to \mathrm{Grp}_{E_n}(\mathrm{An}) \to \cdots \to \mathrm{Grp}_{E_2}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*. \qquad \square$$

Proof. By definition, $\mathrm{Sp}^{\geq 0}$ is the full subcategory of Sp spanned by those spectra (X_n) such that the pointed anima X_n is $(n-1)$-connected for all n, hence is given by the limit of the following diagram of ∞-categories:

$$\cdots \xrightarrow{\Omega} \mathrm{An}_*^{\geq n} \xrightarrow{\Omega} \cdots \xrightarrow{\Omega} \mathrm{An}_*^{\geq 2} \xrightarrow{\Omega} \mathrm{An}_*^{\geq 1} \xrightarrow{\Omega} \mathrm{An}_*.$$

The claim thus follows from the fact that these two diagrams are equivalent to each other, in light of the following commutative squares:

$$\begin{tikzcd} \mathrm{An}_*^{\geq n} \arrow[r, "\Omega"] \arrow[d, "\Omega^n"', "\simeq"] & \mathrm{An}_*^{\geq n-1} \arrow[d, "\simeq"', "\Omega^{n-1}"] \\ \mathrm{Grp}_{E_n}(\mathrm{An}) \arrow[r, "\mathrm{fgt}"] & \mathrm{Grp}_{E_{n-1}}(\mathrm{An}). \end{tikzcd}$$

□

We will now relate this to the ∞-category CGrp(An) of *commutative* groups in An.

Proposition 4.4.9. *Applying* CMon$(-)$ *to the adjunction* $\mathbf{B} \dashv \Omega$ *produces an adjunction*

$$\mathbf{B}\colon \mathrm{CMon}(\mathrm{An}) \simeq \mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \rightleftarrows \mathrm{CMon}(\mathrm{An}_*) \overset{4.3.16}{\simeq} \mathrm{CMon}(\mathrm{An}) : \Omega.$$

Moreover, both functors actually land in CGrp(An) *and induce for all* $n \geq 0$ *an equivalence*

$$\mathbf{B}\colon \mathrm{CGrp}(\mathrm{An}^{\geq n}) \underset{}{\overset{\simeq}{\rightleftarrows}} \mathrm{CGrp}(\mathrm{An}^{\geq n+1}) : \Omega.$$

Proof. For the first claim, it remains to argue that the forgetful functor $\mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ induces an equivalence

$$\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \xrightarrow{\sim} \mathrm{CMon}(\mathrm{An}).$$

Since both CMon$(-)$ and Mon$(-)$ are defined as certain limit-preserving functors, there is an obvious equivalence $\mathrm{CMon}(\mathrm{Mon}(\mathrm{An})) \simeq \mathrm{Mon}(\mathrm{CMon}(\mathrm{An}))$, and under this equivalence the previous functor corresponds to the forgetful functor

$$\mathrm{Mon}(\mathrm{CMon}(\mathrm{An})) \to \mathrm{CMon}(\mathrm{An}).$$

But since CMon(An) is semiadditive by Proposition 4.3.15, this forgetful functor is an equivalence by Proposition 4.3.17.

One can show that a commutative monoid X is a commutative group if and only if its set of path components $\pi_0(X)$ is a group, see Exercise 10.4 on page 264. In particular, every connected commutative monoid is a commutative group, hence in particular so is $\mathbf{B}M$ for every M. So the adjunction restricts to commutative groups, and by Theorem 4.2.1 we get that it restricts to an equivalence

$$\mathbf{B}\colon \mathrm{CGrp}(\mathrm{An}) \simeq \mathrm{CMon}(\mathrm{Grp}(\mathrm{An})) \overset{\simeq}{\rightleftarrows} \mathrm{CMon}(\mathrm{An}^{\geq 0}) \simeq \mathrm{CGrp}(\mathrm{An}^{\geq 0}) : \Omega.$$

Since Ω shifts homotopy groups, this then gives the claim for arbitrary n as well. □

Corollary 4.4.10 (Group completion)**.** *The inclusion* $\mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{CMon}(\mathrm{An})$ *admits a left adjoint*

$$(-)^{\mathrm{grp}} \colon \mathrm{CMon}(\mathrm{An}) \to \mathrm{CGrp}(\mathrm{An})$$

called group-completion. It is given on objects by $M \mapsto \Omega\mathbf{B}M$.

Proof. It follows immediately from the proposition that for every commutative group $G \in \mathrm{CGrp}(\mathrm{An})$, precomposition with the unit $M \to \Omega\mathbf{B}M$ defines an equivalence

$$\mathrm{Hom}_{\mathrm{CGrp}(\mathrm{An})}(\Omega\mathbf{B}M, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CMon}(\mathrm{An})}(M, G),$$

since both sides are equivalent to $\mathrm{Hom}_{\mathrm{Grp}(\mathrm{An})}(\mathbf{B}M, BG)$. $\square$

After all this hard work, we are now able to prove the claim that connective spectra are commutative groups in An, as announced in the introduction to this course:

Theorem 4.4.11 (Recognition principle for connective spectra, cf. May [May72], Boardman and Vogt [BV73])**.** *The functor* $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{An}$ *uniquely lifts to a functor*

$$\Omega^\infty \colon \mathrm{Sp} \to \mathrm{CGrp}(\mathrm{An}).$$

This functor admits a fully faithful left adjoint

$$\mathbf{B}^\infty \colon \mathrm{CGrp}(\mathrm{An}) \hookrightarrow \mathrm{Sp}$$

whose image is precisely the subcategory $\mathrm{Sp}^{\geq 0}$ *of connective spectra.*

Proof. Since Ω^∞ preserves finite products and Sp is additive by Lemma 3.3.3, the fact that Ω^∞ uniquely lifts to $\mathrm{CGrp}(\mathrm{An})$ is an immediate consequence of the fact that $\mathrm{CGrp}(-) \colon \mathrm{Cat}_\infty^\times \to \mathrm{Cat}_\infty^{\mathrm{add}}$ is right adjoint to the inclusion functor, see Proposition 4.3.19.

For the left adjoint, we use the equivalence $\mathbf{B}^n \colon \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\sim} \mathrm{CGrp}(\mathrm{An}^{\geq n})$ from Proposition 4.4.7, with inverse given by the iterated loop space functor Ω^n. For a commutative group G, we may now define the spectrum $\mathbf{B}^\infty G$ as follows:

$$\mathbf{B}^\infty G \quad := \quad (\mathbf{B}^n G, \sigma_n \colon \mathbf{B}^n G \xrightarrow{\sim} \Omega\mathbf{B}^{n+1} G),$$

where the structure maps σ_n are the units of the adjunction $\mathbf{B} \dashv \Omega$. To see that this is a left adjoint to $\Omega^\infty \colon \mathrm{Sp} \to \mathrm{CGrp}(\mathrm{An})$, note that by additivity of Sp we get an equivalence $\mathrm{Sp} \simeq \mathrm{CGrp}(\mathrm{Sp}) \simeq \mathrm{Sp}(\mathrm{CGrp}(\mathrm{An}))$, so that we may write Sp as the following limit:

$$\mathrm{Sp} \quad \simeq \quad \lim(\cdots \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\Omega} \mathrm{CGrp}(\mathrm{An})).$$

We may then compute

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sp}}(\mathbf{B}^\infty G, X) &\simeq \lim_n \mathrm{Hom}_{\mathrm{CGrp(An)}}(\mathbf{B}^n G, X_n)\\ &\simeq \lim_n \mathrm{Hom}_{\mathrm{CGrp(An)}}(G, X_0)\\ &\simeq \mathrm{Hom}_{\mathrm{CGrp(An)}}(G, X_0).\end{aligned}$$

Since all these equivalences are natural in G and X, this provides the adjunction. From this computation it also follows immediately that the unit $G \to \Omega^\infty \mathbf{B}^\infty G$ of the adjunction is simply the identity map of G, hence is an isomorphism, showing that the functor $\mathbf{B}^\infty\colon \mathrm{CGrp(An)} \to \mathrm{Sp}$ is fully faithful. The essential image is the subcategory of the above limit given by

$$\mathrm{CGrp(An)} \quad\simeq\quad \lim(\cdots \xrightarrow[\simeq]{\Omega} \mathrm{CGrp}(\mathrm{An}^{\geq 2}) \xrightarrow[\simeq]{\Omega} \mathrm{CGrp}(\mathrm{An}^{\geq 1}) \xrightarrow[\simeq]{\Omega} \mathrm{CGrp(An)}),$$

which are precisely those spectra X such that X_n is $(n-1)$-connected, i.e. the connective spectra. $\square$

Corollary 4.4.12. *The inclusion* $\mathrm{Sp}^{\geq 0} \hookrightarrow \mathrm{Sp}$ *admits a right adjoint* $\tau_{\geq 0}\colon \mathrm{Sp} \to \mathrm{Sp}^{\geq 0}$ *given by* $\tau_{\geq 0}(X) = \mathbf{B}^\infty \Omega^\infty X$. $\square$

4.5 Symmetric monoidal ∞-categories

Let us briefly say some words about symmetric monoidal ∞-categories and commutative algebras in them.

Definition 4.5.1. A *symmetric monoidal ∞-category* is defined to be a commutative monoid in Cat_∞. In particular, it is an ∞-category which comes equipped with a *tensor product functor*

$$-\otimes -\colon C \times C \to C$$

and a *monoidal unit* $\mathbb{1} \in C$, and the tensor product is coherently unital, associative and commutative. We will often denote symmetric monoidal ∞-categories as $(C, \otimes, \mathbb{1})$, or often also just as C. Given symmetric monoidal ∞-categories C and D, a *symmetric monoidal functor* $F\colon C \to D$ is a morphism in $\mathrm{CMon}(\mathrm{Cat}_\infty)$. We write $\mathrm{Map}^\otimes(C, D)$ for the Hom-anima in $\mathrm{CMon}(\mathrm{Cat}_\infty)$.

Example 4.5.2. We will state without proof several examples of symmetric monoidal ∞-categories:

(1) Let C and D be ∞-categories and assume that D is symmetric monoidal. Since the functor $\mathrm{Fun}(C, -)\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ is a right adjoint, it preserves finite limits, and hence

induces a functor $\mathrm{CMon}(\mathrm{Cat}_\infty) \to \mathrm{CMon}(\mathrm{Cat}_\infty)$. It follows that the functor category $\mathrm{Fun}(C,D)$ admits a canonical symmetric monoidal structure, called the *pointwise symmetric monoidal structure*.

(2) Let C be an ∞-category with finite products. Then C admits a unique symmetric monoidal structure $\otimes$ with the following two properties:

- The monoidal unit $\mathbb{1}$ is the terminal object in C;
- For every pair of objects X and Y in C, the two maps $X \otimes Y \to X \otimes \mathbb{1} \simeq X$ and $X \otimes Y \to \mathbb{1} \otimes Y \simeq Y$ induce an isomorphism $X \otimes Y \xrightarrow{\sim} X \times Y$.

This symmetric monoidal structure is usually denoted by $(C, \times, *)$ or by $C^{\times}$ and is referred to as the *cartesian monoidal structure* on C.

(3) Dually, let C be an ∞-category with finite coproducts. Then C admits a unique symmetric monoidal structure $(C, \sqcup, \emptyset)$ (sometimes also written $C^{\sqcup}$) such that the monoidal unit is the initial object in C, and the tensor product is given by the coproduct in C. This is known as the *cocartesian monoidal structure* on C.

(4) The ∞-category Sp of spectra admits a symmetric monoidal structure $\otimes$ satisfying the property that or every spectrum X, the functor $X \otimes -\colon \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits. Furthermore, Sp is *universal* with this structure: If D is another symmetric monoidal stable ∞-category D which admits colimits and in which the tensor product preserves colimits in both variables, there is a unique colimit-preserving symmetric monoidal functor $\mathrm{Sp} \to D$, i.e. the subcategory of $\mathrm{Map}^{\otimes}(\mathrm{Sp}, D)$ spanned by the colimit-preserving functors is contractible.

(5) The ∞-categories An_*, $\mathrm{CMon}(\mathrm{An})$ and $\mathrm{CGrp}(\mathrm{An})$ similarly each admit symmetric monoidal structures which commute with colimits in both variables, and have a similar universal property where instead of requiring D to be stable on requires it to be pointed, semiadditive, or additive, respectively. We refer to Gepner, Groth, and Nikolaus [GGN15] for an excellent discussion. As a consequence, the functors

$$\mathrm{An} \xrightarrow{(-)_+} \mathrm{An}_* \to \mathrm{CMon}(\mathrm{An}) \xrightarrow{(-)^{\mathrm{grp}}} \mathrm{CGrp}(\mathrm{An}) \xhookrightarrow{\mathbf{B}^\infty} \mathrm{Sp}$$

all have unique enhancements to colimit-preserving symmetric monoidal functor.

(6) Let R be a commutative ring. Then the *derived ∞-category* of R, defined as the localization

$$D(R) := \mathrm{Ch}(R)[\{\text{quasi-isomorphisms}\}^{-1}]$$

of the category of chain complexes of R-modules at the quasi-isomorphisms, admits a symmetric monoidal structure $(D(R), \otimes, R[0])$ whose monoidal unit is the chain complex $R[0]$ given by putting R in degree 0. The tensor product corresponds to the *derived tensor product* of chain complexes.

(7) Recall the full subcategory $\mathrm{Cat}_1 \subseteq \mathrm{Cat}_\infty$ from Definition A.2.12 spanned by the ordinary categories. A commutative monoid in Cat_1 is precisely that of a symmetric monoidal category as defined clasically. Since the inclusion $\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty$ preserves products, this implies that any symmetric monoidal category defines a symmetric monoidal ∞-category.

Definition 4.5.3. Let C and D be symmetric monoidal ∞-categories. We define the ∞-category $\mathrm{Fun}^{\otimes}(C,D)$ of symmetric monoidal functors from C to D as the ∞-category whose associated complete Segal anima (see Appendix A.5) is given by

$$N(\mathrm{Fun}^{\otimes}(C,D))_n = \mathrm{Map}^{\otimes}(C, \mathrm{Fun}([n],D)).$$

Here we equip $\mathrm{Fun}([n],D)$ with the pointwise symmetric monoidal structure. The fact that the simplicial anima on the right-hand side is indeed a complete Segal anima follows immediately from the observation that the Hom-functor $\mathrm{Map}^{\otimes}(C,-)\colon \mathrm{CMon}(\mathrm{Cat}_\infty) \to \mathrm{An}$ preserves limits, see Theorem 2.6.43.

Recall from classical algebra that a *commutative algebra* in a symmetric monoical category C is an object A of C that comes equipped with a unit $1\colon\ \mathbb{1} \to A$ and a multiplication map $m\colon\ A\otimes A \to A$ such that the following diagrams commute:

$$\begin{tikzcd} A\otimes \mathbb{1} \arrow[r, "\mathrm{id}_A\otimes 1"] \arrow[dr, "\simeq"'] & A\otimes A \arrow[d, "m"] & A \arrow[l, "1\otimes \mathrm{id}_A"'] \arrow[dl, "\simeq"] \\ & A, & \end{tikzcd} \qquad \begin{tikzcd} A\otimes A\otimes A \arrow[r, "m\otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A\otimes m"'] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A, \end{tikzcd} \qquad \begin{tikzcd} A\otimes A \arrow[rr, "\tau", "\simeq"'] \arrow[dr, "m"'] & & A\otimes A \arrow[dl, "m"] \\ & A. & \end{tikzcd}$$

Just as in the situation for (commutative) monoids, the ∞-categorical notion of commutative algebra is not defined by writing down some relations by hand, but instead builds in all the required maps and coherences into a suitable indexing category:

Definition 4.5.4 (Commutative algebra)**.** Let $(C,\otimes,\mathbb{1})$ be a symmetric monoidal ∞-category. We define a *commutative algebra in* C to be a symmetric monoidal functor

$$A\colon \mathrm{Fin} \to C,$$

where we equip the category Fin of finite sets with the cocartesian monoidal structure from Example 4.5.2. We refer to the object $A(*)$ in C as the *underlying object of* A. We will frequently abuse notation and simply write A for $A(*)$.

We define the ∞-category of commutative algebras in C as

$$\mathrm{CAlg}(C) := \mathrm{Fun}^{\otimes}(\mathrm{Fin}, C).$$

Remark 4.5.5. If C has the cartesian monoidal structure, there is an equivalence $\mathrm{CMon}(C) \simeq \mathrm{CAlg}(C)$; see [Lur17, Proposition 2.4.2.5] for a more general statemet in the setting of ∞-operads.

Since for every natural number n the finite set $\{1,\dots,n\}$ is a disjoint union of the one-point-sets $\{i\}$, the fact that the map $A\colon \mathrm{Fin} \to C$ is symmetric monoidal implies that there are isomorphisms

$$A(\{1,\dots,n\}) \cong A^{\otimes n}.$$

In particular, the unique map $\{1,\dots,n\} \to *$ determines an n-fold multiplication map $A^{\otimes n} \to A$ for every n. These maps satisfy all kinds of coherence relations, obtained by taking a commutative diagrams in Fin and looking at its image in C. For example, unitality, associativity and commutativity follow from the following three commutative diagrams in Fin:

$$\begin{tikzcd} \{0\} \arrow[r, hook] \arrow[dr, "\cong"'] & \{0,1\} \arrow[d] & \{1\} \arrow[l, hook'] \arrow[dl, "\cong"] \\ & *, & \end{tikzcd} \qquad \begin{tikzcd} \{0,1,2\} \arrow[r, "{0,1\mapsto 01}", "{2\mapsto 2}"'] \arrow[d, "{0\mapsto 0}"', "{1,2\mapsto 12}"'] & \{01,2\} \arrow[d] \\ \{0,12\} \arrow[r] & *, \end{tikzcd} \qquad \begin{tikzcd} \{0,1\} \arrow[rr, "{0\mapsto 1}", "{1\mapsto 0}"'] \arrow[dr] & & \{0,1\} \arrow[dl] \\ & *. & \end{tikzcd}$$

In a similar way, we may talk about *modules* over commutative algebras. First recall the definition in classical algebra: a module over a commutative algebra A is an object M equipped with an action map $\mathrm{act}\colon A \otimes M \to M$ which is unital and associative, in the sense that the following two diagrams commute:

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "1\otimes \mathrm{id}_M"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\mathrm{act}"] \\ & M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{id}_A \otimes \mathrm{act}"] \arrow[d, "m\otimes \mathrm{id}_M"'] & A \otimes M \arrow[d, "\mathrm{act}"] \\ A \otimes M \arrow[r, "\mathrm{act}"] & M. \end{tikzcd}$$

To translate this to the setting of ∞-categories, we will again need to encode *all* the action maps $A^{\otimes n} \otimes M \to M$ for $n \geq 0$, together with all possible coherence relations between them. As before we will do this by defining a suitable indexing category that encodes all these compatibilities.

Definition 4.5.6. We define a category $\mathrm{Fin}^{\mathrm{Mod}}$ as follows:

- An object of $\mathrm{Fin}^{\mathrm{Mod}}$ is a pair (S_A, S_M) consisting of two finite sets S_A and S_M;
- A morphisms in $\mathrm{Fin}^{\mathrm{Mod}}$ from (S_A, S_M) to (T_A, T_M) is a morphism $f\colon S_A \sqcup S_M \to T_A \sqcup T_M$ of finite sets that restricts to a bijection $f|_{S_M}\colon S_M \xrightarrow{\cong} T_M$;
- Composition and identities are defined in the obvious way.

The category $\mathrm{Fin}^{\mathrm{Mod}}$ is symmetric monoidal via disjoint union: for objects (S_A, S_M) and (T_A, T_M), we obtain a new object $(S_A \sqcup T_A, S_M \sqcup T_M)$. The monoidal unit is $(\emptyset, \emptyset)$. Note that there is a symmetric monoidal inclusion functor

$$a\colon \mathrm{Fin} \hookrightarrow \mathrm{Fin}^{\mathrm{Mod}}, \qquad S \mapsto (S, \emptyset).$$

Definition 4.5.7. Let C be a symmetric monoidal ∞-category. We define the ∞-category of *modules in C* as

$$\mathrm{Mod}(C) := \mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{Mod}}, C).$$

Precomposition with $a\colon \mathrm{Fin} \hookrightarrow \mathrm{Fin}^{\mathrm{Mod}}$ defines a functor $a^*\colon \mathrm{Mod}(C) \to \mathrm{CAlg}(C)$ that we refer to as the *underlying commutative algebra functor*. If A is a commutative algebra in C, we define the ∞-category of *A-modules in C* as the following pullback:

$$\begin{tikzcd}
\mathrm{Mod}_A(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(C) \arrow[d, "a^*"] \\
* \arrow[r, "A"] & \mathrm{CAlg}(C).
\end{tikzcd}$$

For a module $\mathrm{Fin}^{\mathrm{Mod}} \to C$, we refer to its value at $(\emptyset, *)$ as the *underlying object*, which we will frequently denote by M.

Every algebra is canonically a module over itself:

Construction 4.5.8. Consider the symmetric monoidal functor $\mathrm{Fin}^{\mathrm{Mod}} \to \mathrm{Fin}$ defined by $(S_A, S_M) \mapsto S_A \sqcup S_M$ on objects, and $f \mapsto f$ on morphisms. Restriction along this functor produces a map

$$\mathrm{mod} : \mathrm{CAlg}(C) \to \mathrm{Mod}(C).$$

Observe that the underlying commutative algebra of $\mathrm{mod}(A)$ is A, while the underlying object is *also* A. The action map $A \otimes A \to A$ is simply given by multiplication in A.

Remark 4.5.9. It is possible to obtain versions of Definitions 4.5.4 and 4.5.7 for *not-necessarily-commutative* algebras:

- We let $\mathrm{Fin}^{\mathrm{Alg}}$ denote the category whose objects are finite sets and whose morphisms are maps $f\colon S \to T$ together with choices of linear orders on each of the fibers $f^{-1}(t)$ for $t \in T$. This is symmetric monoidal under disjoint union, which lets us define $\mathrm{Alg}(C) := \mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{Alg}}, C)$.

- We let $\mathrm{Fin}^{\mathrm{LMod}}$ denote the category whose objects are pairs (S_A, S_M) of finite sets and whose morphisms are maps $f\colon S_A \sqcup S_M \to T_A \sqcup T_M$ that restrict to bijections $f|_{S_M}\colon S_M \xrightarrow{\cong} T_M$ and that are equipped with linear orders on each preimage $f^{-1}(t) \cap S_A$ for $t \in T$. We define $\mathrm{LMod}(C) := \mathrm{Fun}^{\otimes}(\mathrm{Fin}^{\mathrm{LMod}}, C)$ and for an algebra A we write $\mathrm{LMod}_A(C)$ for the fiber over A of the forgetful functor $\mathrm{LMod}(C) \to \mathrm{Alg}(C)$.

We are now finally ready to introduce the homotopical analogues of rings and modules:

Definition 4.5.10 (Commutative ring spectrum)**.** A *commutative ring spectrum R* is a commutative algebra in the symmetric monoidal ∞-category $(\mathrm{Sp}, \otimes, \mathbb{S})$. We denote by

$$\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$$

the ∞-category of commutative ring spectra. Given a commutative ring spectrum R, we define the *∞-category of R-modules* as

$$\mathrm{Mod}_R := \mathrm{Mod}_R(\mathrm{Sp}).$$

Warning 4.5.11. While Definitions 4.5.4 and 4.5.7 are valid and concrete, they are slightly inflexible since they are formulated in terms of *symmetric monoidal* functors rather than arbitrary functors. In particular, it will not be easy to actually construct examples of commutative algebras and modules using this definition. A more flexible approach to defining commutative algebras and modules over them uses the theory of *∞-operads*, and is developed extensively in Chapters 2-4 in [Lur17]. Unfortunately we will not have the time to introduce this more flexible theory.

5 Complex K-theory

The goal of this chapter is to introduce a cohomology theory/spectrum known as *complex K-theory*. In fact, we will introduce two variants: the connective complex K-theory spectrum ku and the periodic compex K-theory spectrum KU.

5.1 Vector bundles and complex K-theory groups

Before introducing the spectra ku and KU, we will first introduce the associated zeroth cohomology group $\mathrm{ku}^0(X) \cong \mathrm{KU}^0(X)$ for a paracompact space X. This group is known as the *complex K-theory group* $K^0(X)$, and admits an explicit description in terms of vector bundles over X.

Definition 5.1.1. Let X be a topological space and let $k \geq 0$. A *complex vector bundle of rank* k over X is a continuous map $p\colon E \to X$ equipped with the structure of complex vector spaces on all its fibers $p^{-1}(x)$, satisfying *local triviality*: there is an open cover $\{U_i\}_{i \in I}$ of X together with homeomorphisms

$$\varphi_i\colon U_i \times \mathbb{C}^k \xrightarrow{\cong} p^{-1}(U_i)$$

satisfying $p(\varphi_i(x,v)) = x$ for all $x \in X$, and such that the induced map $\varphi_i(x,-)\colon \mathbb{C}^k \to p^{-1}(\{x\})$ on fibers over x is $\mathbb{C}$-linear, i.e. $\varphi_i(x, \lambda v + \mu w) = \lambda\varphi_i(x,v) + \mu\varphi_i(x,w)$ for all $v, w \in \mathbb{C}^k$ and $\lambda, \mu \in \mathbb{C}$.

Given another complex vector bundle $p'\colon E' \to X$, a *morphism of vector bundles from E to E'* is a continuous map $f\colon E \to E'$ over X such that the restriction $p^{-1}(x) \to p'^{-1}(x)$ is $\mathbb{C}$-linear. In particular, we get the notion of an *isomorphism* of vector bundles. We denote by $\pi_0\mathrm{Vect}_k(X)$ the set of isomorphism classes of rank k complex vector bundles over X.

Definition 5.1.2. A continuous map $p\colon E \to X$ is called a *complex vector bundle* of its restriction to each path component of X is a complex vector bundle of some specified rank. We let $\pi_0\mathrm{Vect}(X)$ denote the set of isomorphism classes of complex vector bundles over X.

Let X be a topological space, and let $p_1\colon E_1 \to X$ and $p_2\colon E_2 \to X$ be two complex vector bundles over X. Their *direct sum* is defined as the following fiber product:

$$E_1 \oplus E_2 \quad := \quad E_1 \times_X E_2 \quad = \quad \{(e_1, e_2) \mid p_1(e_1) = p_2(e_2)\}.$$

Exercise 5.1.3. Show that this construction satisfies the following properties:

(1) The map $E_1 \oplus E_2 \to X$ is a complex vector bundle over X;

(2) There are isomorphisms of vector bundles

$$\underline{0} \oplus E \cong E \cong E \oplus \underline{0}, \qquad E_1 \oplus E_2 \cong E_2 \oplus E_1, \qquad (E_1 \oplus E_2) \oplus E_3 \cong E_1 \oplus (E_1 \oplus E_3).$$

(3) The isomorphism class of $E_1 \oplus E_2$ only depends on the isomorphism classes of E_1 and E_2.

As a consequence of the exercise, we see that the direct sum construction defines an operation

$$- \oplus -\colon \pi_0\mathrm{Vect}(X) \times \pi_0\mathrm{Vect}(X) \to \pi_0\mathrm{Vect}(X)$$

on the set of isomorphism classes of vector bundles, turning it into an abelian monoid.

Definition 5.1.4 (Complex K-theory)**.** Let X be a compact Hausdorff space. We define the *complex K-theory group* $K^0(X)$ as the Grothendieck group of the abelian monoid $\pi_0\mathrm{Vect}(X)$:

$$K^0(X) := (\pi_0\mathrm{Vect}(X))^{\mathrm{grp}}.$$

In other words, $K^0(X)$ consists of equivalence classes formal differences $E - E'$ of complex vector bundles over X, where we say that $E_1 - E_1' = E_2 - E_2'$ whenever there is an isomorphism $E_1 \oplus E_2' \oplus E_3 \cong E_1' \oplus E_2 \oplus E_3$ of vector bundles over X for some $E_3 \in \pi_0\mathrm{Vect}(X)$.

Example 5.1.5. For $X = \mathrm{pt}$, a vector bundle is simply a complex vector space, which up to isomorphism is determined by its dimension. We thus see that the abelian monoid $\pi_0\mathrm{Vect}(\mathrm{pt})$ is isomorphic to $\mathbb{N}$, and hence $K^0(\mathrm{pt}) \cong \mathbb{Z}$.

Note that the abelian group $K^0(X)$ is contravariantly functorial in X: given a continuous map $f\colon X \to Y$, pullback along f determines a map

$$f^*\colon \pi_0\mathrm{Vect}(Y) \to \pi_0\mathrm{Vect}(X),$$

which induces a map $f^*\colon K^0(Y) \to K^0(X)$ on Grothendieck groups. In particular, the map $X \to \mathrm{pt}$ determines a group homomorphism

$$\mathbb{Z} \cong K^0(\mathrm{pt}) \to K^0(X).$$

Definition 5.1.6 (Reduced K-theory)**.** Let X be a compact Hausdorff space. We define the abelian group $\widetilde{K}^0(X)$ as the cokernel of the map $\mathbb{Z} \to K^0(X)$. This is called the *reduced K-theory group* of X.

The group $\widetilde{K}^0(X)$ admits an alternative description as a quotient of the abelian monoid $\pi_0\mathrm{Vect}(X)$. It relies on the following important result, whose proof will be delegated to Section 5.1.1.

Theorem 5.1.7. *Let X be a compact Hausdorff space. For every complex vector bundle $E \to X$ there exists another vector bundle $E' \to X$ such that the direct sum $E \oplus E'$ is isomorphic to a trivial vector bundle $\underline{\mathbb{C}}^N := X \times \mathbb{C}^N$ for some $N \in \mathbb{N}$.*

Definition 5.1.8. Let X be a topological space and let E and E' be two complex vector bundles over X. We say that E and E' are *stably isomorphic* if there are trivial vector bundles $\underline{\mathbb{C}}^n$ and $\underline{\mathbb{C}}^m$ and an isomorphism $E \oplus \underline{\mathbb{C}}^n \cong E' \oplus \underline{\mathbb{C}}^m$.

Exercise 5.1.9. Show that this notion defines an equivalence relation on the set $\pi_0\mathrm{Vect}(X)$ of isomorphism classes of complex vector bundles over X.

Proposition 5.1.10. *Let X be a compact Hausdorff space.*

(1) The monoid $\pi_0\mathrm{Vect}(X)/\!\sim$ obtained from $\pi_0\mathrm{Vect}(X)$ by identifying stably isomorphic vector bundles is a group.

(2) The map $\pi_0\mathrm{Vect}(X) \to K^0(X) \to \widetilde{K}^0(X)$ identifies stably isomorphic vector bundles.

(3) The resulting group homomorphism $\pi_0\mathrm{Vect}(X)/\!\sim \to \widetilde{K}^0(X)$ is an isomorphism.

Proof. For part (1), consider a complex vector bundle E. Then Theorem 5.1.7 provides another vector bundle E' such that $E \oplus E'$ is stably isomorphic to the trivial vector bundle $\underline{0}$, which is the neutral element of $\pi_0\mathrm{Vect}(X)$. This implies that $[E']$ becomes inverse to E in the quotient $\pi_0\mathrm{Vect}(X)/\!\sim$, showing that this quotient is a group.

For part (2), note that the quotient map $K^0(X) \to \widetilde{K}^0(X)$ by definition kills all trivial bundles. This implies that for stably isomorphic vector bundles E and E' we have the following equalities in $\widetilde{K}^0(X)$:

$$[E] = [E] + [\underline{\mathbb{C}}^n] = [E \oplus \underline{\mathbb{C}}^n] = [E' \oplus \underline{\mathbb{C}}^m] = [E'] + [\underline{\mathbb{C}}^m] = [E'].$$

Finally, for part (3), it will suffice to show that the group map $\pi_0\mathrm{Vect}(X)/\!\sim \to \widetilde{K}^0(X)$ is both injective and surjective. For surjectivity, let $[E_1] - [E_2]$ be a virtual vector bundle over X. By choosing some vector bundle E_2' such that $E_2 \oplus E_2' \cong \underline{\mathbb{C}}^N$, we get $[E_1] - [E_2] = [E_1] + [E_2'] = [E_1 \oplus E_2']$, showing surjectivity. For injectivity, let E_1 and E_2 be two vector bundles with the same class in $\widetilde{K}^0(X)$. Then their formal difference $[E_1] - [E_2]$ in $K^0(X)$

lies in the kernel of $K^0(X) \to \widetilde{K}^0(X)$, hence in the image of $K^0(\mathrm{pt}) \to K^0(X)$. It follows that there is an equality $[E_1] - [E_2] = [\underline{\mathbb{C}}^m] - [\underline{\mathbb{C}}^n]$ in $K^0(X)$, which means there is a third vector bundle E_3 and an isomorphism $E_1 \oplus \underline{\mathbb{C}}^n \oplus E_3 \cong E_2 \oplus \underline{\mathbb{C}}^m \oplus E_3$. By taking the direct sum with yet another vector bundle E_3' satisfying $E_3 \oplus E_3' \cong \underline{C}^N$, we deduce that E_1 and E_2 are stably isomorphic. □

Since the assignment $X \mapsto K^0(X)$ is supposed to be the zeroth degree of a cohomology theory, it should in particular be a homotopy invariant of paracompact Hausdorff spaces. The following result says that this is indeed the case:

Theorem 5.1.11. *Every homotopy equivalence $f\colon X \to Y$ between paracompact topological spaces induces a isomorphisms*

$$f^*\colon \pi_0\mathrm{Vect}(Y) \xrightarrow{\cong} \pi_0\mathrm{Vect}(X) \qquad \textit{and} \qquad f^*\colon K^0(Y) \xrightarrow{\cong} K^0(X)$$

of abelian monoids and abelian groups, respectively.

The proof of this result will be delegated to Section 5.1.2. The next two subsections may be skipped without loss of continuity.

5.1.1 Digression: embedding a vector bundle into a trivial one

The goal of this subsection is to provide a proof of Theorem 5.1.7. A main tool we will use is the notion of a partition of unity:

Definition 5.1.12. A topological space X is called *paracompact* if for every open cover $\{U_i\}_{i\in I}$ of X there exists a *partition of unity subordinate to this cover*, i.e. a collection of continuous maps $\rho_\alpha\colon X \to [0,1]$ satisfying the following three conditions:

(1) For each $x \in X$ only finitely many $\rho_\alpha(x)$ are non-zero;

(2) The resulting sum satisfies $\sum_\alpha \rho_\alpha(x) = 1$ for all x;

(3) For each α, the *support* $\rho_\alpha^{-1}((0,1])$ of ρ_α is contained in one of the open sets U_i.

Example 5.1.13. The following classes of topological spaces are paracompact:

- Every compact Hausdorff space is paracompact [Hat03, Proposition 1.18];
- Every union $\bigcup_i X_i$ of compact Hausdorff spaces $X_1 \subseteq X_2 \subseteq \ldots$ is paracompact [Hat03, Proposition 1.19];
- Every CW-complex is paracompact [Hat03, Proposition 1.20].

Lemma 5.1.14. *Let X be a paracompact Hausdorff space. Then every complex vector bundle $p\colon E \to X$ admits an inner product, i.e. a map $\langle -,- \rangle\colon E \oplus E \to \mathbb{C}$ such that the restriction $E_x \oplus E_x \to \mathbb{C}$ to each fiber $E_x := p^{-1}(x)$ is a (complex) inner product.*

Proof. Let $\{U_i\}_{i\in I}$ be a trivializing open cover of X with trivializations $\varphi_i\colon p^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^n$. For each $i \in I$, we obtain a local inner product $\langle -,- \rangle_i$ on $p^{-1}(U_i)$ by pulling back the standard inner product on $\mathbb{C}^n$ along φ_i. Picking a partition of unity $\{\rho_\alpha\}_{i\in I}$ subordinate to the cover $\{U_i\}$, with ρ_α supported in $U_{i(\alpha)}$, we may then define a global inner product on E by setting

$$\langle v, w \rangle := \sum_\alpha \rho_\alpha(p(v)) \langle v, w \rangle_{i(\alpha)}.$$

for all $v, w \in E$ with $p(v) = p(w)$. This sum is well-defined by local finiteness of the partition of unity. The resulting form is continuous since the ρ_i and local inner products are continuous. Moreover, since convex combinations of inner products are again inner products, the restriction of $\langle -,- \rangle$ to each fiber E_x defines a complex inner product, as desired. □

Lemma 5.1.15. *Let $p\colon E \to X$ and $p'\colon E' \to X$ be vector bundles over X and let $E \hookrightarrow E'$ be a morphism of vector bundles over X which is fiberwise injective. Then there exists a vector bundle $E^\perp \to X$ satisfying $E \oplus E^\perp \cong E'$.*

Proof. By the previous lemma, we may equip E' with an inner product $\langle -,- \rangle$. For each $x \in X$, let $E_x^\perp$ denote the orthogonal complement of E_x in E'_x with respect to this inner product. We claim that the union $E^\perp := \bigcup_{x\in X} E_x^\perp$ forms a vector bundle over X.

To see this, let $U \subseteq X$ be a trivializing neighborhood for both E and E', with trivializations $\varphi\colon E|_U \xrightarrow{\cong} U \times \mathbb{C}^n$ and $\varphi'\colon E'|_U \xrightarrow{\cong} U \times \mathbb{C}^m$. The injection $E \hookrightarrow E'$ is then given by a continuous family of injective linear maps $A(x)\colon \mathbb{C}^n \to \mathbb{C}^m$. The orthogonal complement $E^\perp|_U$ is locally described by the kernel of $A(x)^*$, which varies continuously with x. This shows that $E^\perp$ is indeed a vector bundle.

The natural map $E \oplus E^\perp \to E'$ sending (v, w) to $v + w$ is then a fiberwise isomorphism, providing the desired decomposition $E \oplus E^\perp \cong E'$. □

Lemma 5.1.16. *Let $p\colon E \to X$ be a vector bundle over a compact Hausdorff space X. Then there exists an embedding of vector bundles $E \hookrightarrow X \times \mathbb{C}^N$ for some N.*

Proof. Let $\{U_i\}_{i\in I}$ be a trivializing open cover of X with trivializations $\varphi_i\colon E|_{U_i} \xrightarrow{\cong} U_i \times \mathbb{C}^n$, and let $\{\rho_i\}_{i\in I}$ be a partition of unity subordinate to this cover. By compactness of X, we may take the indexing set I to be finite. For each $i \in I$, let $\pi_i\colon \mathbb{C}^n \to \mathbb{C}^n$ denote the projection onto the i-th copy of $\mathbb{C}^n$ in the direct sum $\bigoplus_{i\in I} \mathbb{C}^n$.

We define a bundle map $\Psi\colon E \to X \times (\bigoplus_{i \in I} \mathbb{C}^n)$ by setting

$$\Psi(v) := (p(v), (\rho_i(p(v)) \cdot \pi_i(\varphi_i(v)))_{i \in I})$$

for all $v \in E$. This map is well-defined by local finiteness of the partition of unity, and it is continuous since the ρ_i and φ_i are continuous. To see that Ψ is fiberwise injective, suppose $v \in E$ is non-zero. Then there exists some $i \in I$ such that $\rho_i(p(v)) \neq 0$ and $v \in p^{-1}(U_i)$. Since φ_i is injective on the fiber over $p(v)$, we see that the i-th component of $\Psi(v)$ is non-zero, hence $\Psi(v) \neq 0$. Taking $N := n \cdot |I|$ then gives the desired embedding. □

Combining all these results, we may now prove our main theorem:

Proof of Theorem 5.1.7. By Lemma 5.1.16, there exists an embedding of vector bundles $E \hookrightarrow X \times \mathbb{C}^N$ for some $N \in \mathbb{N}$. Applying Lemma 5.1.15 to this injection, we obtain a vector bundle E' over X together with an isomorphism $E \oplus E' \cong X \times \mathbb{C}^N$, as desired. □

5.1.2 Digression: Homotopy invariance of complex K-theory

The goal of this subsection is to provide a proof of Theorem 5.1.11, the homotopy invariance of the assignments $X \mapsto \pi_0\mathrm{Vect}(X)$ and $X \mapsto K^0(X)$.

Lemma 5.1.17. *Let X be a paracompact Hausdorff space and $\{U_i\}$ be an open cover. Then there exists a locally finite countable cover $\{V_n\}_{n\geq 0}$ such that every V_n is the disjoint union of open subsets each contained in some U_i, and there exists a partition of unity $\{\rho_n\}$ with ρ_n supported in V_n.*

Proof. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinated to the cover $\{U_i\}$. For a finite subset $S \subseteq A$, we define

$$V_S = \{x \in X \mid \text{For all } \alpha \in S \text{ and } \beta \notin S \text{ we have } \rho_\alpha(x) > \rho_\beta(x)\}.$$

Since in the neighborhood of any point only finitely many of the ρ_α are non-zero, the subset V_S is defined by only finitely many inequalities near x, hence V_S is open. Note that for every $x \in X$ we have $x \in V_S$ where $S = \{\alpha \in A \mid \rho_\alpha(x) > 0\}$, so the V_S form an open cover. Also note that V_S is contained in some U_i: pick some $\alpha \in S$ and pick some i such that the support of ρ_α is contained in U_i, then $\rho_\alpha(x) = 0$ for $x \notin U_i$.

We may now define V_n as the union of the open subsets V_S such that S has precisely n elements. To prove the claim, it remains to check that $V_S \cap V_T = \varnothing$ whenever S and T are distinct sets with the same cardinality. Indeed, in this case we can find $\alpha \in S \setminus T$ and $\beta \in T \setminus S$, which means that every $x \in V_S \cap V_T$ needs to satisfy both $\rho_\alpha(x) > \rho_\beta(x)$ and $\rho_\beta(x) > \rho_\alpha(x)$ simultaneously. This is impossible, hence $V_S \cap V_T = \emptyset$. We conclude that

$\{V_n\}_{n\geq 0}$ is the desired locally finite countable cover where each V_n is a disjoint union of open subsets contained in one of the U_i's.

For the final claim, let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the cover $\{V_n\}$. We may then define ρ_n as the some of those φ_α which are supported in V_n but not in V_k for $k < n$. $\square$

Proposition 5.1.18. *Let X be a paracompact Hausdorff space, and let $p\colon E \to X \times [0,1]$ be a rank k complex vector bundle. Let $p_0\colon E_0 \to X$ and $p_1\colon E_1 \to X$ denote the pullbacks of p to $X \times \{0\}$ and $X \times \{1\}$, respectively. Then there exists an isomorphism of vector bundles $E_0 \cong E_1$ over X.*

Proof. By local triviality of p, we may find for each $x \in X$ and $t \in T$ an open neighborhood of $(x,t) \in X \times [0,1]$ over which p trivializes. Since the interval $[0,1]$ is compact, we may for fixed x find open neighborhoods $U_{x,i} \subseteq X$ of x and a partition $0 = t_0 < t_1 < \cdots < t_{k_x} = 1$ such that E is trivial over $U_{x,i} \times [t_{i-1}, t_i]$ for each i. By setting $U_x := \bigcap_i U_{x,i}$, we thus get local trivializations on each subspace $U_x \times [0,1]$ of $X \times [0,1]$.

By Lemma 5.1.17, there exists a countable, locally finite open cover $\{V_n\}_{n\geq 1}$ of X subordinate to $\{U_x\}$ such that E is trivial over each $V_n \times [0,1]$, and there exists a partition of unity $\{\rho_n\}_{n\geq 1}$ subordinate to $\{V_n\}$. Consider the cumulative functions $\psi_n = \rho_1 + \cdots + \rho_n$ for $n \geq 0$, which are functions satisfying $0 \leq \psi_n \leq 1$, $\psi_0 = 0$ and $\psi_n \to 1$ as $n \to \infty$. For each $n \geq 1$, we may then consider the graph $X_n := \{(x, \psi_n(x)) \mid x \in X\}$ of ψ_n, and we let $E_n := p^{-1}(X_n)$ denote the restriction of E to X_n. Since the projection maps $X_n \to X$ are homeomorphisms, we may regard E_n as vector bundles over X.

We will now inductively construct isomorphisms $h_n\colon E_n \xrightarrow{\cong} E_{n-1}$ over X, starting with $E_0 = E|_{X\times\{0\}}$. Since $\psi_n = \psi_{n-1} + \rho_n$, these to maps only differ on the support of ρ_n, which is contained in V_n. Hence outside of V_n we may take h_n to be the identity map. Since the bundle E is trivial over $V_n \times [0,1]$ for every $n \geq 1$, there exists a fiber-preserving homeomorphism

$$\varphi_n\colon p^{-1}(V_n \times [0,1]) \xrightarrow{\cong} V_n \times [0,1] \times \mathbb{C}^k,$$

linear on fibers. Using this homeomorphism, we may identify $E_n|_{V_n}$ with $(V_n \times [0,1] \cap X_n) \times \mathbb{C}^k$ and $E_{n-1}|_{V_n}$ with $(V_n \times [0,1] \cap X_{n-1}) \times \mathbb{C}^k$. Under these identifications, we then define $h_n|_{V_n}\colon E_n|_{V_n} \to E_{n-1}|_{V_n}$ by

$$h_n(x, \psi_n(x), v) := (x, \psi_{n-1}(x), v).$$

The resulting map $h_n\colon E_n \to E_{n-1}$ is continuous, linear on fibers, and commutes with projections. The composition $h = \cdots \circ h_2 \circ h_1$ converges to a well-defined isomorphism $E_1 = E|_{X\times\{1\}} \to E_0$: due to local finiteness we see that around every point in X there is an open neighbhood such that only finitely many of the h_i are non-identity maps. This finishes the proof. $\square$

Proposition 5.1.19. *Let $p\colon E \to X$ be a vector bundle and let $f_0, f_1\colon Y \to X$ be two homotopic maps. Assume that Y is a paracompact Hausdorff space. Then the pullback bundles $f_0^*(E)$ and $f_1^*(E)$ over Y are isomorphic.*

Proof. Let $H\colon Y \times [0,1] \to X$ be a homotopy between f_0 and f_1. Consider the pullback bundle $H^*(E) \to Y \times [0,1]$. Since Y is paracompact Hausdorff, we may apply the previous proposition to obtain the desired isomorphism

$$f_0^*(E) = H^*(E)|_{Y\times\{0\}} \xrightarrow{\cong} H^*(E)|_{Y\times\{1\}} = f_1^*(E). \qquad \square$$

We are now finally ready to prove the result we are after.

Proof of Theorem 5.1.11. The homotopy invariance of $K^0(-)$ follows immediately from that of $\pi_0\mathrm{Vect}(-)$. For the homotopy invariance of $\pi_0\mathrm{Vect}(-)$, let $g\colon Y \to X$ be a homotopy inverse of f, with homotopies $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$. It them follows from Proposition 5.1.19 that for every vector bundle E over X we have

$$E \cong (\mathrm{id}_X)^*(E) \cong (gf)^*(E) \cong f^*g^*(E).$$

Similarly we have $E' \cong g^*f^*(E')$, showing that g^* is inverse to f^*. $\square$

5.2 Classifying vector bundles

The functor $\mathrm{hTop}^{\mathrm{op}} \to \mathrm{Set}$ given by $X \mapsto \pi_0\mathrm{Vect}(X)$ turns out to be representable: the data of an isomorphism class of rank k complex vector bundles over X is equivalent to the data of a homotopy class of continuous maps from X into a certain topological space $\mathrm{Gr}_k(\mathbb{C}^\infty)$, see Theorem 5.2.4. We will also show that it is equivalent to the classifying space $\mathbf{B}U(k)$ of the topological group $U(k)$.

Definition 5.2.1 (Infinite grassmannian)**.** Let $\mathbb{C}^\infty$ denote the union of the complex vector spaces $\mathbb{C}^n$ for $n \geq 0$, along the inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^n \oplus \mathbb{C} = \mathbb{C}^{n+1}$. We define the *infinite Grassmannian of k-planes* as

$$\mathrm{Gr}_k(\mathbb{C}^\infty) := \{V \subseteq \mathbb{C}^\infty \mid V \text{ rank } k \text{ subvector space}\}.$$

To equip $\mathrm{Gr}_k(\mathbb{C}^\infty)$ with a topology, consider the *infinite Stiefel manifold of orthonormal k-frames*

$$V_k(\mathbb{C}^\infty) := \mathrm{Emb}_{\mathbb{C}}(\mathbb{C}^k, \mathbb{C}^\infty) \cong \{(v_1, \dots, v_k) \in (\mathbb{C}^\infty)^k \mid |v_i| = 1, v_i \perp v_j \text{ for all } i \neq j\} \subseteq (\mathbb{C}^\infty)^k,$$

where $\mathrm{Emb}_{\mathbb{C}}$ denotes the space of linear isometric embeddings. We give $V_k(\mathbb{C}^\infty)$ the subspace topology from $(\mathbb{C}^\infty)^k$. There is surjective map

$$\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty), \qquad (\varphi\colon \mathbb{C}^k \hookrightarrow \mathbb{C}^\infty) \mapsto \varphi(\mathbb{C}^k),$$

and we equip $\mathrm{Gr}_k(\mathbb{C}^\infty)$ with the quotient topology.

Definition 5.2.2. We define the *canonical bundle* $p_{\mathrm{univ}}\colon E_{\mathrm{univ}} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ by defining

$$E_{\mathrm{univ}} := \{(V,x) \in \mathrm{Gr}_k(\mathbb{C}^\infty) \times \mathbb{C}^\infty \mid x \in V\},$$

and setting $p_{\mathrm{univ}}(V,x) := V$.

Exercise 5.2.3. Show that the map p_{univ} is a complex vector bundle of rank k. *Hint:* compare with the proof of Lemma 5.2.15 below.

Let X be a paracompact Hausdorff space. Given a continuous map $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$, we may pull back the bundle E_{univ} along f, providing a vector bundle

$$f^*(E_{\mathrm{univ}}) := \{(x,\varphi) \in X \times E_{\mathrm{univ}} \mid f(x) = p_{\mathrm{univ}}(\varphi)\}.$$

By Proposition 5.1.19, the isomorphism class of this vector bundle only depends on the homotopy class of f, providing a map

$$[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \to \pi_0\mathrm{Vect}_k(X), \qquad [f] \mapsto f^*(E_{\mathrm{univ}}). \tag{5.1}$$

The following classical theorem expresses that p_{univ} is in some sense the *universal* rank k complex vector bundle:

Theorem 5.2.4 (Hatcher [Hat03, Theorem 1.16])**.** *For a paracompact Hausdorff space X, the map* (5.1) *is an isomorphism.*

Because of this theorem, $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is known as the *classifying space for rank k complex vector bundles*.

Proof of Theorem 5.2.4. We follow Hatcher's proof. The key observation is as follows: given a complex vector bundle $p\colon E \to X$ of rank k, the following two types of data are equivalent:

(1) A pair (f,φ) consisting of a continuous map $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ together with an isomorphism $E \cong f^*(E_{\mathrm{univ}})$ of complex vector bundles;

(2) A continuous map $g\colon E \to \mathbb{C}^\infty$ that is a $\mathbb{C}$-linear injection on each fiber.

To see this, suppose first that we are given an isomorphism $E \xrightarrow{\sim} f^*(E_{\mathrm{univ}})$ of vector bundles over X. Then we may define $g\colon E \to \mathbb{C}^\infty$ to be the top composite in the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "\sim"] \arrow[dr, "p"'] & f^*(E_{\mathrm{univ}}) \arrow[r, "\tilde{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\mathrm{univ}} \arrow[r, "\pi"] \arrow[d, "\pi_{\mathrm{univ}}"] & \mathbb{C}^\infty \\
& X \arrow[r, "f"] & \mathrm{Gr}_k(\mathbb{C}^\infty) &
\end{tikzcd}$$

where $\pi(V, v) = v$. The resulting map g is a linear injection on each fiber, since both $\tilde{f}$ and π have this property. Conversely, given a map $g\colon E \to \mathbb{C}^\infty$ that is a linear injection on each fiber, we may define $f\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ by letting $f(x)$ be the image of the linear map $g|_{p^{-1}(x)}\colon p^{-1}(x) \hookrightarrow \mathbb{C}^\infty$. The map g then refines to a map $\tilde{f}\colon E \to E_{\mathrm{univ}}$ over f which induces isomorphisms on all fibers and thus defines an isomorphism $E \cong f^*(E_{\mathrm{univ}})$.

We now prove surjectivity of the map $[X, \mathrm{Gr}_k(\mathbb{C}^\infty)] \to \pi_0\mathrm{Vect}_k(X)$. Let $p\colon E \to X$ be a complex vector bundle of rank k. Let $\{U_\alpha\}$ be an open cover of X such that E is trivial over each U_α. By paracompactness, we can refine this to a locally finite countable cover $\{V_n\}_{n\geq 0}$ as in Lemma 5.1.17, with a partition of unity $\{\rho_n\}$ subordinate to it.

For each n, let $h_n\colon p^{-1}(V_n) \to V_n \times \mathbb{C}^k$ be a trivialization, and let $g_n\colon p^{-1}(V_n) \to \mathbb{C}^k$ be the composition of h_n with the projection onto $\mathbb{C}^k$. The map $(\rho_n \circ p) \cdot g_n$, which sends $v \mapsto \rho_n(p(v)) \cdot g_n(v)$, extends to a map $E \to \mathbb{C}^k$ that is zero outside $p^{-1}(V_n)$. By local finiteness of the cover, these extended maps define the coordinates of a map $g\colon E \to \bigoplus_{n\geq 0} \mathbb{C}^k \cong \mathbb{C}^\infty$ that is a linear injection on each fiber. By the observation above, it follows that $E \cong f^*(E_{\mathrm{univ}})$ for some f, as desired.

For injectivity, suppose we have isomorphisms $E \cong f_0^*(E_{\mathrm{univ}})$ and $E \cong f_1^*(E_{\mathrm{univ}})$ for two maps $f_0, f_1\colon X \to \mathrm{Gr}_k(\mathbb{C}^\infty)$. These give maps $g_0, g_1\colon E \to \mathbb{C}^\infty$ that are linear injections on fibers, as in the first paragraph of the proof. We claim g_0 and g_1 are homotopic through maps g_t that are linear injections on fibers. If this is so, then f_0 and f_1 will be homotopic via $f_t(x) = g_t(p^{-1}(x))$.

The first step in constructing a homotopy g_t is to compose g_0 with the homotopy $L_t\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ defined by $L_t(z_1, z_2, \ldots) = (1-t)(z_1, z_2, \ldots) + t(z_1, 0, z_2, 0, \ldots)$. For each t this is a linear map whose kernel is easily computed to be 0, so L_t is injective. Composing the homotopy L_t with g_0 moves the image of g_0 into the odd-numbered coordinates. Similarly, we can homotope g_1 into the even-numbered coordinates. If we abusively denote the modified maps again by g_0 and g_1, we may now define a homotopy between them by $g_t = (1-t)g_0 + tg_1$. This is linear and injective on fibers for each t since g_0 and g_1 have disjoint support in their coordinates and are linear and injective on fibers. $\square$

5.2.1 Classifying spaces revisited

Recall from Construction 4.1.7 that we have previously already introduced a notion of 'classifying space' $\mathbf{B}G$ for a group object $G \in \mathrm{Grp}(\mathrm{An})$. Our goal below will be to show that the infinite Grassmannian $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is an example of such. We start by giving another perspective on classifying spaces at the level of topological spaces.

Lemma 5.2.5. *The localization functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *preserves finite products, and* *in particular induces a functor*

$$\Pi_\infty(-)\colon \mathrm{Grp}(\mathrm{Top}) \to \mathrm{Grp}(\mathrm{An})$$

from the category of topological groups to the ∞-category of group objects in animae.

Proof. Recall from Theorem 2.6.38 that this localization functor is given as the composite

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \hookrightarrow \mathrm{sAn} \xrightarrow{|-|} \mathrm{An}.$$

The functor $\mathrm{Sing}(-)$ preserves all limits, since for every $[n] \in \mathbf{\Delta}$ the functor $X \mapsto \mathrm{Sing}(X)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ preserves limits. The inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ has a left adjoint $\pi_0(-)$, so also the inclusion $\mathrm{sSet} \hookrightarrow \mathrm{sAn}$ preserves limits. Finally, the geometric realization functor $|-|\colon \mathrm{sAn} \to \mathrm{An}$ preserves finite products by Lemma 4.4.3. □

In fact, we showed in Corollary B.3.7 that the localization functor $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ preserves all pullbacks along Serre fibrations. Since the map $X \to *$ is a Serre fibration for any topological space X, this provides an alternative proof of Lemma 5.2.5.

Definition 5.2.6. Let G be a topological group. By Lemma 5.2.5 its underlying anima $\Pi_\infty(G)$ is a group object in animae, and hence it has a classifying space

$$\mathbf{B}G \quad := \quad \mathbf{B}\Pi_\infty(G) \quad \in \quad \mathrm{An}.$$

We will now discuss a concrete way of recognizing topological spaces X whose underlying anima is isomorphic to $\mathbf{B}G$.

Definition 5.2.7 (Principal bundle)**.** Let G be a topological group and let B be a paracompact Hausdorff space.[1] The *trivial G-principal bundle* is the projection map $B \times G \to B$, where we equip $B \times G$ with the canonical right G-action given by $(b, g)g' := (b, gg')$.

A *G-principal bundle* is a continuous map $\pi\colon P \to B$ such that P is equipped with a continuous right G-action such that:

(1) The right G-action on P preserves the fibers of π: we have $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$;

(2) The map π is a *locally trivial bundle*: around every point $b \in B$ there is an open neighborhood $b \in U \subseteq B$ such that the restriction $\pi^{-1}(U) \to U$ of π is a trivial G-principal bundle, i.e. there exists a G-equivariant homeomorphism $\varphi\colon U \times G \xrightarrow{\cong} \pi^{-1}(U)$ over U:

$$\begin{tikzcd} U \times G \arrow[r, "\varphi", "\cong"'] \arrow[dr, "\mathrm{pr}_1"'] & \pi^{-1}(U) \arrow[r, hook] \arrow[d, "\pi|_U"] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, "\pi"] \\ & U \arrow[r, hook] & B. \end{tikzcd}$$

[1]Paracompactness means that for every open cover there exists a partition of unity subordinate to the cover.

We refer to P as the *total space*, to G as the *structure group*, and to B as the *base space*.

Exercise 5.2.8. Show that every principal $\mathrm{GL}_n(\mathbb{C})$-bundle $P \to B$ defines a complex vector bundle of rank k over B by taking

$$E := P \times_{\mathrm{GL}_n(\mathbb{C})} \mathbb{C}^n = (P \times \mathbb{C}^n)_{/(p,Av)\sim(pA,v)},$$

where the equivalence relation identifies the pair (p, Av) with (pA, v) for all $p \in P$, $v \in \mathbb{C}^n$ and $A \in \mathrm{GL}_k(\mathbb{C})$. Show that this defines a bijection between the set of isomorphism classes of principal $\mathrm{GL}_n(\mathbb{C})$-bundles and the set of isomorphism classes of rank k vector bundles.

Remark 5.2.9. Since $\pi\colon P \to B$ is locally trivial, hence locally a Serre fibration, it follows from Theorem B.3.2 that every principal bundle is a Serre fibration. In particular, pullbacks along principal bundles compute homotopy pullbacks by Corollary B.3.7.

Lemma 5.2.10. *Let $\pi\colon P \to B$ be a principal G-bundle. Then for every $n \geq 0$ the continuous map*

$$\Phi_n\colon P \times G^n \to P \times_B P \times_B \cdots \times_B P, \qquad (p, g_1, \dots, g_n) \mapsto (p, pg_1, \dots, pg_1 \dots g_n)$$

is a homeomorphism.

Proof. By induction we may immediately reduce to the case $n = 1$, i.e. we must show that the map $\Phi_1\colon P \times G \to P \times_B P$, $(p, g) \mapsto (p, pg)$ is a homeomorphism. Observe that this map fits in a commutative diagram of the form

$$\begin{tikzcd} P \times G \arrow[rr, "\Phi_1"] \arrow[dr, "\mathrm{pr}_1"'] & & P \times_B P \arrow[dl, "\mathrm{pr}_1"] \\ & P, & \end{tikzcd}$$

and so it suffices to find an open cover of P such that the given map restricts to a homeomorphism on the preimages. Since π is locally trivial, we may thus assume that it is of the form $\mathrm{pr}\colon B \times G \to B$. But in this case, the map Φ_1 takes the form $B \times G \times G \to B \times G \times G$, $(b, g, h) \mapsto (b, g, gh)$, which is indeed a homeomorphism with inverse $(b, g, h') \mapsto (b, g, g^{-1}h')$. $\square$

Observe that both sides of Lemma 5.2.10 define simplicial objects in Top: the simplicial structure on the right-hand side is that of the Čech nerve $\check{C}(\pi)$, see the proof of Proposition 4.1.8, a the simplicial structure on the left may be given for $\varphi\colon [n] \to [m]$ by the formula

$$\varphi^*\colon P \times G^m \to P \times G^n, \quad (p, g_1, \dots, g_n) \mapsto (p', g_1', \dots, g_m'),$$

where $p' := pg_0g_1 \dots g_{\varphi(0)}$ and $g_i' := \prod_{\varphi(i-1)<j\leq\varphi(i)} g_j$.

Lemma 5.2.11. *The homeomorphisms Φ_n from Lemma 5.2.10 define an isomorphism of simplicial objects in* Top*.*

Proof. We need to show that the maps Φ_n are compatible with the simplicial structure maps φ^* for every morphism $\varphi\colon [n] \to [m]$ in $\mathbf{\Delta}$. It will suffice to do this for the face maps d_i and for the degeneracy maps s_i:

- For $\varphi = d_0\colon [n-1] \to [n]$ we have

$$\begin{aligned}\Phi_{n-1}(d_0^*(p, g_1, \dots, g_n)) &= \Phi_{n-1}(pg_1, g_2, \dots, g_n)\\ &= (pg_1, pg_1g_2, \dots, pg_1g_2\dots g_n)\\ &= d_0^*(p, pg_1, pg_1g_2, \dots, pg_1g_2\dots g_n)\\ &= d_0^*\Phi_n(p, g_1, \dots, g_n);\end{aligned}$$

- For $\varphi = d_n\colon [n-1] \to [n]$ we have

$$\begin{aligned}\Phi_{n-1}(d_n^*(p, g_1, \dots, g_n)) &= \Phi_{n-1}(p, g_1, \dots, g_{n-1})\\ &= (p, pg_1, \dots, pg_1\dots g_{n-1})\\ &= d_{n+1}^*\Phi_n(p, g_1, \dots, g_n);\end{aligned}$$

- For $\varphi = d_i\colon [n-1] \to [n]$ we have

$$\begin{aligned}\Phi_{n-1}(d_i^*(p, g_1, \dots, g_n)) &= \Phi_{n-1}(p, g_1, \dots, g_ig_{i+1}, \dots, g_n)\\ &= (p, pg_1, \dots, pg_1\dots g_{i-1}, pg_1\dots g_{i+1}, \dots, pg_1\dots g_n)\\ &= d_i^*\Phi_{n+1}(p, g_1, \dots, g_{n+1});\end{aligned}$$

- Finally, for $\varphi = s_i\colon [n+1] \to [n]$ we have

$$\begin{aligned}\Phi_{n+1}(s_i^*(p, g_1, \dots, g_n)) &= \Phi_{n+1}(p, g_1, \dots, g_i, e, g_{i+1}, \dots, g_n)\\ &= (p, pg_1, \dots, pg_1\dots g_i, pg_1\dots g_ie, pg_1\dots g_ieg_{i+1}, \dots, pg_1\dots g_n)\\ &= s_i^*\Phi_n(p, g_1, \dots, g_n).\end{aligned}$$

This finishes the proof. □

We now get to the promised result about recognizing topological spaces of the form $\mathbf{B}G$:

Proposition 5.2.12. *Let $\pi\colon P \to B$ be a principal G-bundle such that P is contractible.*

(1) The Čech nerve of $\Pi_\infty(\pi)\colon \Pi_\infty(P) \to \Pi_\infty(B)$ *is isomorphic to* $\Pi_\infty(G) \in \mathrm{Grp}(\mathrm{An})$*;*

(2) In particular, passing to colimits provides an isomorphism of pointed animae

$$\Pi_\infty(B) \xrightarrow{\sim} \mathbf{B}G.$$

Proof. For part (1), recall that the Čech nerve $\check{C}(\Pi_\infty(\pi))$ is given by the following simplicial diagram in An:

$$\cdots \quad \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \Pi_\infty(P)\times_{\Pi_\infty(B)}\Pi_\infty(P)\times_{\Pi_\infty(B)}\Pi_\infty(P) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \Pi_\infty(P)\times_{\Pi_\infty(B)}\Pi_\infty(P) \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \Pi_\infty(P).$$

Since the functor $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ preserves pullbacks along Serre fibrations, we see that there are equivalences

$$\Pi_\infty(P\times_B\cdots\times_B P) \xrightarrow{\sim} \Pi_\infty(P)\times_{\Pi_\infty(B)}\cdots\times_{\Pi_\infty(B)}\Pi_\infty(P)$$

for all n, which assemble into an isomorphism of simplicial animae $\Pi_\infty(\check{C}(\pi)) \xrightarrow{\sim} \check{C}(\Pi_\infty(\pi))$. By Lemma 5.2.11, the simplicial anima $\Pi_\infty(\check{C}(\pi))$ is isomorphic to $[n]\mapsto \Pi_\infty(P\times G^n)$. By contractibility of P, the projection maps $\Pi_\infty(P\times G^n)\cong \Pi_\infty(P)\times\Pi_\infty(G^n)\to\Pi_\infty(G^n)$ are isomorphisms, which are compatible with the simplicial structure maps (see Exercise 8.5). This finishes the proof. □

Remark 5.2.13. There is up to homotopy equivalence a unique space B equipped with a principal G-bundle $\pi\colon P\to B$ with P contractible. This principal bundle is often called $EG\to BG$, and BG is referred to as the *classifying space* of G (as it classfies principal G-bundles). The previous proposition thus says that $\Pi_\infty(BG)\simeq \mathbf{B}\Pi_\infty(G)$, justifying the notation and terminology.

Let us record an immediate corollary of the (proof of the) proposition:

Corollary 5.2.14. *Let G be a topological group. Consider a commutative square*

$$\begin{tikzcd} P \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & P' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

of topological spaces, where π and π' are principal G-bundles, and $\tilde{f}$ is G-equivariant (i.e. $\tilde{f}(pg)=\tilde{f}(p)g$). If both P and P' are contractible, then the map $\Pi_\infty(f)\colon \Pi_\infty(B)\to\Pi_\infty(B')$ is an isomorphism of pointed animae.

Proof. It will suffice to show that the map $\check{C}(\Pi_\infty(\pi))\to\check{C}(\Pi_\infty(\pi'))$ of simplicial animae is an isomorphism. As in the proof of Proposition 5.2.12, these two simplicial animae are given by $[n]\mapsto\Pi_\infty(P\times G^n)$ and $[n]\mapsto\Pi_\infty(P'\times G^n)$. The claim thus follows from the fact that each map $\tilde{f}\times\mathrm{id}_{G^n}\colon P\times G^n\to P'\times G^n$ is a homotopy equivalence by contractibility of P and P'. □

5.2.2 The infinite Grassmannian as a classifying space

We now return to the infinite Grassmannian: we wish to apply Proposition 5.2.12 to the map $\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ to conclude that $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is a classifying space for the unitary group $U(k)$, the group of complex isometries $\mathbb{C}^k \xrightarrow{\cong} \mathbb{C}^k$.

Lemma 5.2.15. *The projection map $\pi\colon V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is a principal $U(k)$-bundle. Here $U(k)$ acts on $V_k(\mathbb{C}^\infty)$ by precomposition: given $A \in U(k)$ and $\varphi\colon \mathbb{C}^k \hookrightarrow \mathbb{C}^\infty$, we define $A\varphi$ as the composite $\mathbb{C}^k \xrightarrow{A} \mathbb{C}^k \xrightarrow{\varphi} \mathbb{C}^\infty$.*

Proof. Since we may write $\mathrm{Gr}_k(C^\infty)$ as a union of open subspaces $\mathrm{Gr}_k(C^n)$ for $n \geq 0$, it will suffice to show that the map $V_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ is a principal $U(k)$-bundle for every n. Let W be a k-dimensional subspace of $\mathbb{C}^n$ and define $U \subseteq \mathrm{Gr}_k(\mathbb{C}^n)$ as the open subspace consisting of those k-planes V such that the orthogonal projection $W \to V$ is an isomorphism of complex vector spaces. We claim that there exists a $U(k)$-equivariant homeomorphism

$$\varphi\colon U \times U(k) \xrightarrow{\cong} \pi^{-1}(U)$$

over U, which will suffice to finish the proof. To construct φ, fix a preferred k-frame $(w_1,\dots,w_k)$ for W. Given any k-plane $V \in U$, the projection $W \to V$ is an isomorphism, hence sends the k-frame $(w_1,\dots,w_k)$ to some basis for V. Using Gram-Schmidt orthonormalization (which is a continuous operation), we may turn this into a k-frame $(v_1,\dots,v_k)$ for V. All in all, we may then define

$$\varphi(V,A) := (v_1,\dots,v_k)\circ A.$$

Bijectivity of φ holds because two orthonormal frames of a k-plane differ by a unique unitary $(k \times k)$-matrix. We leave it to the reader to check that this is indeed a well-defined homeomorphism over U. □

Remark 5.2.16. Under the bijection of Exercise 5.2.8, the principal bundle $V_k(\mathbb{C}^\infty) \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ is in fact the principal associated to the universal rank k vector bundle E_{univ} from Definition 5.2.2.

Lemma 5.2.17. *The topological space $V_k(\mathbb{C}^\infty)$ is contractible.*

Proof. Consider the space $\widetilde{V}_k(C^\infty)$ of all linear injections $\mathbb{C}^k \hookrightarrow \mathbb{C}^\infty$ which are not necessarily isometries. Equivalently, $\widetilde{V}_k(C^\infty)$ is the topological space of k-tuples $(v_1,\dots,v_k)$ of linearly independent vectors in $\mathbb{C}^\infty$ which are not necessarily orthogonal to each other. Observe that the inclusion $V_k(\mathbb{C}^\infty) \hookrightarrow \widetilde{V}_k(C^\infty)$ is a homotopy equivalence: the Gram-Schmidt orthogonalization procedure provides a deformation retract $\widetilde{V}_k(C^\infty) \to V_k(\mathbb{C}^\infty)$.

It will thus suffice to show that $\widetilde{V}_k(C^\infty)$ is contractible. To this end, consider the 'shift map' $T^k\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ given on basis vectors by $T^k(e_n) := e_{n+k}$. This induces a map $T^k\colon \widetilde{V}_k(C^\infty) \to \widetilde{V}_k(C^\infty)$. Note that this map is homotopic to the constant map on $(e_1, \ldots, e_k)$ via the straight-line homotopy:

$$H(v,t) := (t-1)(e_1, \ldots, e_k) + tT^k(v).$$

This is well-defined since the first k components of the vectors $T^k(v_i)$ are all zero. Similarly, the map T^k is homotopic to the identity on $\widetilde{V}_k(C^\infty)$, again via the straight-line homotopy:

$$H(v,t) := (t-1)v + tT^k(v).$$

To see that this is well-defined, assume for contradiction that for some time t the vectors are no longer linearly dependent. Then there are scalars $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i(t-1)v_i + \lambda_i tT^k(v_i) = 0$. But then the vector $w := \lambda_i v_i$ satisfies $(t-1)w + tT^k(w) = 0$, which can only happen if $w = 0$. This contradicts linear independence of v.

It follows that the identity on $\widetilde{V}_k(C^\infty)$ is homotopic to a constant map, and hence this space is contractible. □

Writing $U(k)$ also for the underlying anima of the topological group $U(k)$, we get:

Corollary 5.2.18. *The underlying anima of the Grassmannian* $\mathrm{Gr}_k(\mathbb{C}^\infty)$ *is isomorphic to the classifying space* $\mathbf{B}U(k)$.

Proof. In light of Lemma 5.2.15 and Lemma 5.2.17, this is an instance of Proposition 5.2.12. □

Corollary 5.2.19. *There is a natural bijection*

$$[X, \bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)] \xrightarrow{\cong} \pi_0 \mathrm{Vect}(X).$$

Proof. By writing X as a disjoint union of its path components, we observe that both sides decompose as disjoint unions and it will suffice to show the claim when X is path-connected. In this case, every map $X \to \bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)$ factors through some $\mathbf{B}U(k)$ and similarly every vector bundle over X is of some fixed rank k, and hence see

$$[X, \bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)] \cong \bigsqcup_{k=0}^{\infty} [X, \mathbf{B}U(k)] \cong \bigsqcup_{k=0}^{\infty} \pi_0 \mathrm{Vect}_k(X) \cong \pi_0 \mathrm{Vect}(X),$$

using Corollary 5.2.18 and Theorem 5.2.4. □

5.3 Connective complex K-theory

In this section, we introduce the spectrum ku, called *connective complex K-theory*.

Definition 5.3.1. Let X be an anima. We define the anima $\mathrm{Vect}(X)$ of complex vector bundles over X as the follow mapping anima:

$$\mathrm{Vect}(X) := \mathrm{Map}(X, \bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)).$$

Remark 5.3.2. The previous definition is a slight abuse of notation, but is justified by the fact that the set of path components of $\mathrm{Vect}(X)$ is naturally isomorphic to the set of isomorphism classes of complex vector bundles on X, see Corollary 5.2.19.

We would like to upgrade the abelian monoid structure on the set $\pi_0\mathrm{Vect}(X)$ to a commutative monoid structure on the anima $\mathrm{Vect}(X)$. Since the functor $\mathrm{Vect}(-)\colon \mathrm{An} \to \mathrm{An}$ is represented by the anima $\bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)$, this is equivalent to defining a commutative monoid structure on $\bigsqcup_{k=0}^{\infty} \mathbf{B}U(k)$. We will do this by using the explicit geometric description of $\mathbf{B}U(k)$ as a Grassmannian manifold from Corollary 5.2.18.

Construction 5.3.3. Recall from Remark 4.3.6 the span category $\mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all})$, whose objects are finite sets and whose morphisms are spans of the form $S \hookleftarrow U \to T$ where the left-pointing morphism is an injection. We define a functor

$$\mathbf{Gr}\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{Top}.$$

To define it, we first consider the topological space

$$\mathbf{Gr}(*) := \bigsqcup_{k=0}^{\infty} \mathrm{Gr}_k(\mathbb{C}^\infty)$$

of linear subspaces ('planes') in $\mathbb{C}^\infty$. We then define $\mathbf{Gr}$ on objects by sending a finite set S to the topological space

$$\mathbf{Gr}(S) := \{(V_s) \in \mathbf{Gr}(*)^S \mid V_s \perp V_{s'} \text{ for } s \neq s'\} \subseteq \mathbf{Gr}(*)^S$$

of pairwise orthogonal S-tuples of planes in $\mathbb{C}^\infty$, equipped with the subspace topology. On morphisms, we set a span $S \overset{f}{\hookleftarrow} U \xrightarrow{g} T$ to the composite

$$\mathbf{Gr}(S) \xrightarrow{f^*} \mathbf{Gr}(U) \xrightarrow{g_\oplus} \mathbf{Gr}(T),$$

where $f^*(V_s) := (V_{f(u)})_{u \in U}$ and $g_\oplus(W_u) := (\bigoplus_{u \in g^{-1}(t)} W_u)_{t \in T}$.

Lemma 5.3.4. *The functor* $\mathbf{Gr}$ *satisfies the Segal condition: for every finite set S, the inclusion*

$$\mathbf{Gr}(S) \hookrightarrow \mathbf{Gr}(*)^S$$

is a homotopy equivalence.

Proof. Without loss of generality, we may assume that $S = \{1, \ldots, n\}$ for some $n \geq 0$. For an n-tuple $(k_1, \ldots, k_n)$ of natural numbers $k_i \geq 0$, define

$$\mathrm{Gr}_{k_1,\ldots,k_n}(\mathbb{C}^\infty) := \{(V_1, \ldots, V_n) \mid V_i \in \mathrm{Gr}_{k_i}(\mathbb{C}^\infty), V_i \perp V_j \text{ for } i \neq j\}.$$

Then we have that

$$\mathrm{Gr}(\{1, \ldots, n\}) \cong \bigsqcup_{k_1,\ldots,k_n} \mathrm{Gr}_{k_1,\ldots,k_n}(\mathbb{C}^\infty).$$

By similarly wwriting $\mathbf{Gr}(*)^S$ as a big disjoint union over all n-tuples $(k_1, \ldots, k_n)$, this reduces the problem to showing that the inclusion

$$\mathrm{Gr}_{k_1,\ldots,k_n}(\mathbb{C}^\infty) \hookrightarrow \mathrm{Gr}_{k_1}(\mathbb{C}^\infty) \times \cdots \times \mathrm{Gr}_{k_n}(\mathbb{C}^\infty)$$

is a homotopy equivalence. We will do this by showing that both sides are equivalent to the classifying space $B(U(k_1) \times \cdots \times U(k_n))$, as an instance of Proposition 5.2.12. Consider the following commutative diagram:

$$\begin{tikzcd}
V_{k_1+\cdots+k_n}(\mathbb{C}^\infty) \arrow[r, hook] \arrow[d] & V_{k_1}(\mathbb{C}^\infty) \times \cdots \times V_{k_n}(\mathbb{C}^\infty) \arrow[d] \\
\mathrm{Gr}_{k_1,\ldots,k_n}(\mathbb{C}^\infty) \arrow[r, hook] & \mathrm{Gr}_{k_1}(\mathbb{C}^\infty) \times \cdots \times \mathrm{Gr}_{k_n}(\mathbb{C}^\infty),
\end{tikzcd}$$

where the left vertical map sends an isometric embedding $\varphi\colon \mathbb{C}^{k_1} \oplus \cdots \oplus \mathbb{C}^{k_n} = \mathbb{C}^{k_1+\cdots+k_n} \hookrightarrow \mathbb{C}^\infty$ to the n-tuple $(\varphi(\mathbb{C}^{k_1}), \ldots, \varphi(\mathbb{C}^{k_n}))$ of orthogonal planes. It follows from Lemma 5.2.15 that the right vertical map is a principal $(U(k_1) \times \cdots \times U(k_n))$-bundle, and an analogous argument also shows that the left vertical map is a principal $(U(k_1) \times \cdots \times U(k_n))$-bundle. Furthermore, by Lemma 5.2.17 the two total spaces are contractible. It thus follows from Corollary 5.2.14 that both botom map is a homotopy equivalence, as desired. ◻

Definition 5.3.5. We define the functor $\mathbf{V}\mathrm{ect}(\mathrm{pt})\colon \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \to \mathrm{An}$ by

$$\mathbf{V}\mathrm{ect}(\mathrm{pt})(S) := \Pi_\infty(\mathbf{Gr}(S)).$$

By Lemma 5.3.4, this functor satisfies the Segal condition, hence by Remark 4.3.6 it corresponds to an object $\mathbf{V}\mathrm{ect}(\mathrm{pt}) \in \mathrm{CMon}(\mathrm{An})$. Note that its underlying anima is $\mathrm{Vect}(\mathrm{pt})$, justifying the notation:

$$\mathbf{V}\mathrm{ect}(\mathrm{pt})(*) = \Pi_\infty(\mathbf{Gr}(*)) = \Pi_\infty(\bigsqcup_{k=0}^{\infty} \mathrm{Gr}_k(\mathbb{C}^\infty)) \overset{5.2.18}{\simeq} \bigsqcup_{k=0}^{\infty} \mathbf{B}U(k) = \mathrm{Vect}(\mathrm{pt}).$$

Lemma 5.3.6. *The addition map*

$$\left(\coprod_{k=0}^{\infty}\mathbf{B}U(k)\right)\times\left(\coprod_{l=0}^{\infty}\mathbf{B}U(l)\right)\to\left(\coprod_{m=0}^{\infty}\mathbf{B}U(m)\right)$$

of $\mathbf{V}\mathrm{ect}(\mathrm{pt})$ *is given by the maps* $\mathbf{B}U(k)\times\mathbf{B}U(l)\to\mathbf{B}U(k+l)$ *induced by the inclusions of groups*

$$U(k)\times U(l)\hookrightarrow U(k+l), \qquad (A,B)\mapsto\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}.$$

In particular, the induced monoid structure on $\pi_0\mathrm{Vect}(*)\cong\mathbb{N}$ *is given by addition of natural numbers.*

Proof. The addition map is given by the maps $\mathrm{Gr}_{k,l}(\mathbb{C}^\infty)\to\mathrm{Gr}_{k+l}(\mathbb{C}^\infty)$, $(V,W)\mapsto V\oplus W$. The analogous map $V_{k,l}(\mathbb{C}^\infty)\to V_{k+l}(\mathbb{C}^\infty)$ covering this map induces the inclusion $U(k)\times U(l)\hookrightarrow U(k+l)$ on all fibers, and the claim follows. □

Recall from Corollary 4.4.10 the group completion functor $(-)^{\mathrm{grp}}\colon \mathrm{CMon}(\mathrm{An})\to\mathrm{CGrp}(\mathrm{An})$.

Definition 5.3.7 (Connective complex K-theory)**.** We define the *connective K-theory spectrum*

$$\mathrm{ku}\in\mathrm{Sp}^{\geq 0}.$$

as the connective spectrum which under the Recognition Principle $\mathrm{Sp}^{\geq 0}\simeq\mathrm{Grp}(\mathrm{An})$ corresponds to the group completion $\mathbf{V}\mathrm{ect}(\mathrm{pt})^{\mathrm{grp}}$ of the commutative monoid $\mathbf{V}\mathrm{ect}(\mathrm{pt})$.

5.4 Bott periodicity and periodic complex K-theory

The spectrum ku admits a variant KU called *(periodic) complex K-theory*, which we will introduce momentarily. The spectra ku and KU have the same underlying commutative group (namely the group completion $\mathrm{Vect}(*)^{\mathrm{grp}}$), but the spectrum KU, unlike ku, also has homotopy groups in *negative* degrees.

The main ingredient for the construction of KU is *Bott periodicity*. To state it, consider the *infinite unitary group*

$$U \quad := \quad \bigcup_{n\geq 0}U(n) \quad \cong \quad \{\varphi\colon \mathbb{C}^\infty\xrightarrow{\cong}\mathbb{C}^\infty \mid \varphi_{V^\perp}=\mathrm{id}_{V^\perp}\text{ for some fin.-dim. } V\subseteq\mathbb{C}^\infty\}.$$

Bott [Bot59] computed that the homotopy groups of this group are 2-periodic: there are isomorphisms

$$\pi_{k+2}(U)\cong\pi_k(U)$$

for all $k \geq 0$. Bott's proof of this result was geometric in nature, using Morse theory, and using similar methods he also proved that $\pi_{k+4}(O) \cong \pi_k(Sp)$ and $\pi_{k+4}(Sp) \cong \pi_k(O)$, providing an 8-fold periodicity of the homotopy groups of the infinite orthogonal group. Since then, many different proofs of Bott periodicity have been given.

From the purposes of stable homotopy theory, we want to think of Bott periodicity as the existence of a homotopy equivalence

$$U \simeq \Omega^2(U),$$

which then immediately provides a spectrum by considering the sequence $(\Omega U, U, \Omega U, U, \dots)$ of pointed animae. The goal of this section is to discuss a short proof of Bott periodicity due to Harris [Har80], which is very much homotopical in spirit. The proof uses three key ingredients:

- The *group completion theorem* due to McDuff and Segal [MS76], which in certain cases lets us compute the group completion M^{grp} of a commutative monoid in terms of a certain *telescope construction*;
- The *spectral theorem* which says that every unitary matrix can be diagonalized;
- The fact from Corollary 5.2.18 that the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^\infty)$ is a model for $\mathbf{B}U(k)$.

Let us start with a discussion of the group completion theorem. We will leave it as a black box and only mention a restricted form of the theorem that is easier to formulate and still suffices for our purposes. An excellent reference for the full version of the group completion theorem, written in modern language, is Nikolaus [Nik17].

Construction 5.4.1 (Telescope)**.** Let $M \in \mathrm{CMon}(\mathrm{An})$ be a commutative monoid. For an object $x \in M$ we define the *telescope of M at x* as the colimit

$$T(M,x) := \mathrm{colim}(\, M \xrightarrow{-\cdot x} M \xrightarrow{-\cdot x} M \xrightarrow{-\cdot x} \dots),$$

where $- \cdot x$ denotes multiplication with x in M. Now consider the map $M \to M^{\mathrm{grp}} = \Omega\mathbf{B}M$ to the group completion of M. Since this map sends the element x to an invertible element in M^{grp}, this map factors uniquely through a map $T(M,x) \to M^{\mathrm{grp}}$.

Theorem 5.4.2 (Special case of group completion theorem, Nikolaus [Nik17, Proposition 6], cf. McDuff and Segal [MS76])**.** *In the situation above, assume that the following two conditions are satisfied:*

(1) The commutative monoid $\pi_0(M)[x^{-1}]$ obtained from $\pi_0(M)$ by inverting x is a group;

(2) Each path component of $T(m,x)$ has an abelian fundamental group.

Then the map $T(m,x) \to M^{\mathrm{grp}}$ is an isomorphism.

For us, this theorem is of interest since it provides an alternative description of $\mathrm{Vect}(\mathrm{pt})^{\mathrm{grp}}$:

Corollary 5.4.3. *The underlying anima of the group completion of the commutative monoid* $\mathbf{V}\mathrm{ect}(\mathrm{pt}) \in \mathrm{CMon}(\mathrm{An})$ *is* $\mathbb{Z}\times\mathbf{B}U$*:*

$$\Omega\mathbf{B}\mathbf{V}\mathrm{ect}(\mathrm{pt}) \simeq \mathbf{V}\mathrm{ect}(\mathrm{pt})^{\mathrm{grp}} \simeq \mathbb{Z}\times\mathbf{B}U.$$

Proof. In $M = \mathrm{Vect}(\mathrm{pt}) = \bigsqcup_{k=0}^{\infty}\mathbf{B}U(k)$, consider the point $x \in \mathbf{B}U(1) \in \mathrm{Vect}(\mathrm{pt})$ corresponding to the identity matrix. The telescope will then be given by the colimit

$$T(M,x) = \mathrm{colim}\left(\bigsqcup_{k=0}^{\infty}\mathbf{B}U(k) \xrightarrow{-\cdot x} \bigsqcup_{k=0}^{\infty}\mathbf{B}U(k) \xrightarrow{-\cdot x} \bigsqcup_{k=0}^{\infty}\mathbf{B}U(k) \to \cdots\right).$$

Notice how this colimit is a disjoint union of countably infinitely many copies of the colimit

$$\mathbf{B}U = \mathrm{colim}(\mathbf{B}U(0) \to \mathbf{B}U(1) \to \mathbf{B}U(2) \to \cdots),$$

hence isomorphic to $\mathbb{Z}\times\mathbf{B}U$. We wish to show that the induced map $\mathbb{Z}\times\mathbf{B}U \to \mathbf{V}\mathrm{ect}(\mathrm{pt})^{\mathrm{grp}}$ is an equivalence, for which we check the two conditions from Theorem 5.4.2:

- We have $\pi_0(\mathrm{Vect}(\mathrm{pt})) \cong \mathbb{N}$, and inverting the element $1 \in \mathbb{N}$ gives the group $\mathbb{Z}$;
- The fundamental group of $\mathbf{B}U$ is isomorphic to the set of path components of the group U. By connectedness of U this is the trivial group, hence in particular abelian. □

As a consequence, we obtain equivalences $U \cong \Omega(\mathbf{B}U) \cong \Omega(\mathbb{Z}\times\mathbf{B}U) \cong \Omega^2(\mathbf{B}\mathbf{V}\mathrm{ect}(\mathrm{pt}))$. It thus remains to compare the classifying space $\mathbf{B}\mathbf{V}\mathrm{ect}(\mathrm{pt})$ with U. For this, we will recall the notion of the *geometric realization of a simplicial topological space*.

Definition 5.4.4. Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ be a simplicial topological space. We define the *geometric realization* of X as the quotient space

$$|X| \quad := \quad \left(\bigsqcup_{n\geq 0} X_n \times |\Delta^n|\right)/\sim \quad \in \quad \mathrm{Top},$$

where $|\Delta^n| = \{(y_i) \in [0,1]^{n+1} \mid \sum_i y_i = 1\}$ is the topological n-simplex from Example 2.5.10 and where the equivalence relation is generated by $(\varphi^*(x), y) \sim (x, |\Delta^{\varphi}|(y))$ for all $\varphi\colon [n] \to [m]$, $x \in X_m$ and $y \in |\Delta^n|$. In other words, $|S|$ is the following coequalizer in Top:

$$|X| \quad := \quad \mathrm{coeq}\left(\bigsqcup_{\varphi\colon [n]\to[m]} X_m \times |\Delta^n| \quad \rightrightarrows \quad \bigsqcup_{n\geq 0} X_n \times |\Delta^n|\right).$$

Exercise 5.4.5. Explicitly describe the resulting 'gluing relation' $(\varphi^*(x), y) \sim (x, |\Delta^\varphi|(y))$ in the following two special cases:

- When $\varphi = d_i\colon [n-1] \hookrightarrow [n]$;
- When $\varphi = s_i\colon [n+1] \twoheadrightarrow [n]$.

Definition 5.4.6. A simplicial topological space $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ is called *good*[2] if for every $n \geq 0$ and $0 \leq i \leq n$ the face map $d_i\colon X_{n-1} \to X_n$ is a relative cell complex in the sense of Definition B.2.3.

Proposition 5.4.7. *Let* $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ *be a good simplicial topological space. Then there exists an equivalence*

$$|\Pi_\infty(X)| = \operatorname{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Pi_\infty(X_n) \simeq \Pi_\infty(|X|)$$

between the geometric realization of the simplicial anima $\Pi_\infty \circ X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ *and the underlying anima of the geometric realization of* X.

Proof. See Proposition B.6.6. □

Assuming this result, the question of producing an equivalence $\mathbf{BV}\mathrm{ect}(\mathrm{pt}) \xrightarrow{\sim} U$ can be reduced to the question of producing a homotopy equivalence between U and the geometric realization of the simplicial topological space $\mathbf{Gr}^{\mathbf{\Delta}} = \mathbf{Gr} \circ \mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$. As we will argue now, it turns out these two topological spaces are in fact *homeomorphic*.

Construction 5.4.8. Let $\mathbf{Gr}^{\mathbf{\Delta}}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ denote the following composite

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Cut}} \mathrm{Span}(\mathrm{Fin}, \mathrm{inj}, \mathrm{all}) \xrightarrow{\mathbf{Gr}} \mathrm{Top},$$

where Cut is the functor from Construction 4.3.9. Explicitly, $\mathbf{Gr}^{\mathbf{\Delta}}([n])$ consists of n-tuples $(V_j)_{j=1}^n$ of pairwise orthogonal planes $V_j \subseteq \mathbb{C}^\infty$. For a morphism $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$, the map $\varphi^*\colon \mathbf{Gr}^{\mathbf{\Delta}}([n]) \to \mathbf{Gr}^{\mathbf{\Delta}}([m])$ is given by

$$\varphi^*(V_j) = (W_k), \qquad W_k := \bigoplus_{\varphi(k-1)<j\leq\varphi(k)} V_j.$$

We will construct a continuous map

$$\Phi\colon |\mathbf{Gr}^{\mathbf{\Delta}}| \to U.$$

To this end, note that sending a tuple $(x_0, \ldots, x_n) \in |\Delta^n|$ to the tuple $(t_j)_{j=1}^n$ given by $t_j := \sum_{i=1}^j x_i$ results in a homeomorphism

$$|\Delta^n| \cong \{(t_1, \ldots, t_n) \in [0,1]^n \mid 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1\}.$$

[2]This terminology is due to Segal [Seg74, Appendix A].

We then define a map $\Phi_n\colon \mathbf{Gr}([n]) \times |\Delta^n| \to U$ by sending a tuple $(V_1,\dots,V_n,t_1,\dots,t_n)$ to the unitary transformation

$$\Phi_n(V_1,\dots,V_n;t_1,\dots,t_n) := \sum_{j=1}^{n} e^{2\pi i t_j} P_{V_j} + P_{(\bigoplus_j V_j)^\perp},$$

where $P_W\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ denotes the orthogonal projection onto a subspace $W \subseteq \mathbb{C}^\infty$. In other words, this is the unitary transformation $\mathbb{C}^\infty \to \mathbb{C}^\infty$ which on V_j is given by multiplication by $\lambda_j := e^{2\pi i t_j}$ and on the orthogonal complement of all the V_j is the identity.

To check that the equivalence relation is preserved, note that:

- We have

$$\Phi_n(V_1,\dots,V_n;0,t_1,\dots,t_{n-1}) = \Phi_{n-1}(V_2,\dots,V_n;t_1,\dots,t_{n-1}),$$

  since $\exp^{2\pi i 0} P_{V_1} = \mathrm{id}$;

- We have

$$\Phi_n(V_1,\dots,V_n;t_1,\dots,t_{n-1},1) = \Phi_{n-1}(V_1,\dots,V_{n-1};t_1,\dots,t_{n-1}),$$

  since $\exp^{2\pi i 1} P_{V_n} = \mathrm{id}$;

- We have

$$\Phi_n(V_1,\dots,V_n;t_1,\dots,t_j,t_j,\dots,t_{n-1}) = \Phi_{n-1}(V_1,\dots,V_j \oplus V_{j+1},\dots,V_n;t_1,\dots,t_{n-1}),$$

  since $e^{2\pi i t_j} P_{V_j} \cdot e^{2\pi i t_j} P_{V_{j+1}} = e^{2\pi i t_j} P_{V_j \oplus V_{j+1}}$;

- Finally, we have

$$\Phi_{n+1}(V_1,\dots,V_j,0,V_{j+1},\dots,V_n;t_1,\dots,t_{n+1}) = \Phi_n(V_1,\dots,V_n;t_1,\dots,\widehat{t_{j+1}},\dots,t_{n+1}).$$

This ensures that the Φ_n assemble into a continuous map $\Phi\colon |\mathbf{Gr}^{\boldsymbol{\Delta}}| \to U$.

Proposition 5.4.9. *The map $\Phi\colon |\mathbf{Gr}^{\boldsymbol{\Delta}}| \to U$ is a homeomorphism.*

Proof. By writing $\mathbb{C}^\infty$ as a colimit of its finite-dimensional subspaces $\mathbb{C}^m$, we may also write both sides as unions of subspaces and it suffices to show that Φ restrict to homeomorphisms between these subspaces for every m:

$$|\mathbf{Gr}^{\boldsymbol{\Delta}}(-;\mathbb{C}^m)| \xrightarrow{\cong} U(m).$$

Since both sides are compact Hausdorff spaces, it in fact suffices to show that this map is a bijection. Note that every element of $|\mathbf{Gr}^{\boldsymbol{\Delta}}|$ can uniquely be represented by a tuple

$(V_1,\dots,V_n;t_1,\dots,t_n) \in |\mathbf{Gr}^{\boldsymbol{\Delta}}|$ such that all the V_j are non-zero and the t_j satisfy the strict inequalities $0 < t_1 < t_2 < \cdots < t_n < 1$. Similarly, every unitary matrix $A\colon \mathbb{C}^m \xrightarrow{\cong} \mathbb{C}^m$ can uniquely be written as a product $\prod_{j=1}^n e^{2\pi i t_j} P_{V_j}$, where V_j are the eigenspaces of A and $\lambda_j = e^{2\pi i t_j}$ are the corresponding eigenvalues. It follows that Φ is a bijection, finishing the proof. □

Corollary 5.4.10 (Bott periodicity)**.** *There is a homotopy equivalence $\Omega^2(U) \simeq U$ given by the following composite:*

$$\Omega^2(U) \overset{5.4.9}{\simeq} \Omega^2\Pi_\infty(|\mathbf{Gr}^{\boldsymbol{\Delta}}|) \overset{5.4.7}{\simeq} \Omega^2\mathbf{B}\mathbf{V}\mathrm{ect}(\mathrm{pt}) \overset{4.4.10}{\simeq} \Omega(\mathbf{V}\mathrm{ect}(\mathrm{pt})^{\mathrm{grp}}) \overset{5.4.3}{\simeq} \Omega(\mathbb{Z}\times\mathbf{B}U) \simeq \Omega(\mathbf{B}U) \simeq U.$$

Definition 5.4.11. We define the *periodic complex K-theory spectrum* $\mathrm{KU} \in \mathrm{Sp}$ by setting $\mathrm{KU}_{2n} := \mathbb{Z}\times\mathbf{B}U$ and $\mathrm{KU}_{2n+1} := U$ for all $n \in \mathbb{N}$. The structure maps are provided by the equivalences

$$\Omega(\mathbf{B}U\times\mathbb{Z}) \simeq U \qquad \text{and} \qquad \Omega(U) \simeq \mathbf{B}U\times\mathbb{Z}$$

as before.

6 Localization and completion

When studying mathematical objects, we often want to focus attention on certain features while systematically controlling how we handle others. In algebra, this insight led to two fundamental operations: *localization*, which makes chosen elements invertible, and *completion*, which adds limits of certain sequences. Both of these operations provide systematic ways to modify rings/abelian groups to better suit particular questions.

The ∞-category of spectra admits analogous operations with similar formal properties. A nice formalism to capture these kinds of operations is that of *Bousfield localizations*, which lets us zoom in at those aspects of spectra that are 'seen by' a given homology theory $E_*(-)$. Important examples of Bousfield localizations are rationalization, localization at a prime p, and completion at a prime p. Bousfield localizations are also fundamental in the field of chromatic homotopy theory. In this chapter, we will give a brief introduction to these constructions.

6.1 Bousfield localizations

Let E be a spectrum, taken to be fixed throughout this section. As discussed in Definition 3.7.1, this leads to a definition of the *E-homology groups* of an anima:

$$X \mapsto E_*(X) := \pi_*(E \otimes \mathbb{S}[X]).$$

For example, taking $E = HA$ the Eilenberg-MacLane spectrum of an abelian group A results in singular homology with coefficients in A, while taking $E = \mathbb{S}$ the sphere spectrum results in the stable homotopy groups.

The homology groups $E_*(X)$ will generally only see *some* aspects of the given anima X, and may completely ignore others. Consider for example the case of the projective space $X = \mathbb{R}\mathrm{P}^2$. If we took E to be the Eilenberg-MacLane spectrum of the rational numbers $\mathbb{Q}$, we would get trivial groups, hence this does not capture any information about $\mathbb{R}\mathrm{P}^2$. But taking E the Eilenberg-MacLane spectrum of $\mathbb{Z}/2$ *does* capture interesting information about $\mathbb{R}\mathrm{P}^2$. The point of Bousfield localizations is to focus precisely on those parts of an anima (or spectrum) which are "seen by E".

Definition 6.1.1. A morphism $f: X \to Y$ is called an *E-equivalence* if the induced map $E \otimes f: E \otimes X \to E \otimes Y$ is an equivalence. We say that a spectrum X is *E-acyclic* if the map $X \to 0$ is an E-equivalence, i.e. if $E \otimes X$ is the zero spectrum.

Recall that a morphism of spectra is an equivalence if and only if it induces isomorphisms on all homotopy groups. So if we write $E_*(X) := \pi_*(E \otimes X)$ for the E-homology of X, generalizing the definition for animae, we see that the E-equivalences are precisely those maps that induce isomorphisms on E-homology. Similarly, the E-acyclic spectra are those with trivial E-homology.

Example 6.1.2. For $E = \mathbb{Q}$, the E-equivalences are known as the *rational equivalences*: those that induce isomorphisms on rational homology.

We will now introduce a class of spectra that 'cannot distinguish between E-equivalent spectra':

Definition 6.1.3. A spectrum L is called *E-local* if for every E-equivalence $f: X \to Y$, the induced map $f^*: \operatorname{map}(Y, L) \to \operatorname{map}(X, L)$ on mapping spectra is an equivalence. We denote by

$$\mathrm{Sp}_E \subseteq \mathrm{Sp}$$

the full subcategory spanned by the E-local spectra.

Exercise 6.1.4. Show the following basic properties of E-local spectra:

(1) A spectrum L is E-local if and only if $\operatorname{map}(X, L) = 0$ for every E-acyclic spectrum X;

(2) Given an exact sequence $X \to Y \to Z$, if two of these three spectra are E-local then so is the third;

(3) In particular, if L is E-local then so are all its shifts $E[n]$;

(4) A limit $\lim_i L_i$ of E-local spectra is E-local;

(5) A retract of an E-local spectrum is E-local.

Exercise 6.1.5. Show that for every spectrum Y, the mapping spectrum $\operatorname{map}(E, Y)$ is E-local. Using this, show that for a morphism $f: X \to X'$ of spectra the following conditions are equivalent:

(1) The morphism $l: X \to X'$ is an E-equivalence;

(2) The induced map $l^*: \operatorname{map}(X', L) \to \operatorname{map}(X, L)$ is an equivalence for all E-local spectra L;

(3) The induced map $l^*\colon \mathrm{Hom}_{\mathrm{Sp}}(X', L) \to \mathrm{Hom}_{\mathrm{Sp}}(X, L)$ is an equivalence for all E-local spectra L.

Hint: use that we have $\pi_n \operatorname{map}(X,Y) \cong \pi_0 \operatorname{map}(X, Y[n]) \cong \pi_0 \mathrm{Hom}_{\mathrm{Sp}}(X, Y[n])$, and similarly for X'.

Lemma 6.1.6 ("E_*-Whitehead theorem")**.** *Let $f\colon L \to L'$ be a morphism between E-local spectra. Then f is an equivalence if and only if it is an E-equivalence.*

Proof. By passing to fibers, we may equivalently show that an E-local spectrum L is zero if and only if it is E-acyclic. One direction is clear. For the other direction, assume L is E-acyclic. Then by assumption we have $\operatorname{map}(L, L) = 0$, hence $\mathrm{Hom}_{\mathrm{Sp}}(L, L) = *$, hence the identity on L is the zero map. This implies that L itself is zero, as desired. □

We will now talk about the universal way of approximating an arbitrary spectrum by an E-local one:

Definition 6.1.7. A morphism $l\colon X \to X'$ of spectra is called an *E-localization* if X' is local and the morphism l is an E-equivalence.

Corollary 6.1.8. *Let X' be an E-local spectrum. A morphism $l\colon X \to X'$ is an E-localization if and only if for every E-local spectrum Y the induced map*

$$l^*\colon \mathrm{Hom}_{\mathrm{Sp}_E}(X', Y) \to \mathrm{Hom}_{\mathrm{Sp}}(X, Y)$$

is an equivalence.

Proof. This is immediate from Exercise 6.1.5. □

Note that the conclusion of the corollary says that X' is a *left adjoint object* of X under the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$, in the sense of Definition A.2.3. In particular, the E-localization of X is unique if it exists, and is then denoted by $L_E(X)$.

Theorem 6.1.9 (Bousfield [Bou79, Theorem 1.1])**.** *For every object X, there exists an E-localization $l\colon X \to L_E X$. In particular, the inclusion $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ admits a left adjoint*

$$L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$$

called the E-localization functor.

Very rough proof sketch. Step 1: As a first step, one shows that there exists some regular cardinal κ such that every E-acyclic spectrum X is a colimit of "κ-small"[1] E-acyclic spectra.

[1] A spectrum is called *κ-sma*

Step 2: One then shows that there exists an E-acyclic spectrum A with the property that a spectrum L is E-local if and only if $\mathrm{map}(A, L) = 0$: take A to be the direct sum of all the κ-small E-acyclic spectra (up to iso there is only a set worth of these, not a proper class).

Step 3: One now builds $L_E X$ out of X via a transfinite induction process, where in each step one "kills off all possible maps from A": we define $X_0 := X$, given X_α for an ordinal α we define $X_{\alpha+1}$ as the cofiber of the map

$$\bigoplus_{n\in\mathbb{N}, f\colon A[n]\to X_\alpha} A[n] \to X_\alpha,$$

and for limit ordinals λ we set $X_\lambda := \mathrm{colim}_{\alpha<\lambda} X_\alpha$. Then one can show that for a sufficiently large ordinal κ' (depending only on A but possibly larger than the previous κ) the map $X \to X_{\kappa'}$ is an E-localization. □

Remark 6.1.10. Historically, this important technical result by Bousfield provided the tools for systematic structure of phenomena in chromatic homotopy theory. Nowadays, the phrase 'Bousfield localization' is used not only for localizations of the form $L_E\colon \mathrm{Sp} \to \mathrm{Sp}_E$, as in Bousfield's work, but more generally for left adjoints to any fully faithful functor. The notion of Bousfield localization in this more general sense is discussed in Appendix A.2.8.

6.2 Localization at a prime

As a first example of Bousfield localizations, we discuss the inversion of primes.

Notation 6.2.1. Let $n \in \mathbb{N}$ be a natural number, and let X be a spectrum. We denote by $\cdot n\colon X \to X$ the map obtained by forming the n-fold addition of the map $\mathrm{id}_X\colon X \to X$ with itself in the commutative monoid $\mathrm{Hom}_{\mathrm{Sp}}(X, X)$. Explicitly, this map may be written as the following composite:

$$X \xrightarrow{\Delta} \bigoplus_n X \xrightarrow{\nabla} X.$$

We refer to this map as *multiplication by n*. We will sometimes also denote it by $n\colon X \to X$.

Construction 6.2.2. Let $P \subseteq \mathbb{N}$ be a set of prime numbers, and let X be a spectrum. We define a spectrum $X[P^{-1}]$ as follows:

- If P consists of finitely many primes, we let $N := \prod_{p\in P} p$, and we define $X[P^{-1}]$ as the colimit

$$X[P^{-1}] \quad := \quad \mathrm{colim}(X \xrightarrow{\cdot N} X \xrightarrow{\cdot N} X \xrightarrow{\cdot N} \dots) \quad \in \quad \mathrm{Sp}.$$

- If P consists of infinitely many primes, say $p_1, p_2, p_3, \ldots$, we define $X[P^{-1}]$ as the colimit

$$X[P^{-1}] \quad := \quad \operatorname{colim}(X \xrightarrow{\cdot p_1} X \xrightarrow{\cdot p_1 p_2} X \xrightarrow{\cdot p_1 p_2 p_3} \ldots) \quad \in \quad \mathrm{Sp}.$$

Observe that we have $X[P^{-1}] \simeq X \otimes \mathbb{S}[P^{-1}]$ for all X.

Example 6.2.3. We have the following three special cases:

- When $P = \{p\}$ consists of a single prime, we write $X[\frac{1}{p}]$ for $X[P^{-1}]$;
- When P consists of all primes except p, we write $X_{(p)}$ for $X[P^{-1}]$, and refer to it as the *p-localization of X*;
- When P consists of all primes, we write $X_{\mathbb{Q}}$ for $X[P^{-1}]$, and refer to it as the *rationalization of X*.

Remark 6.2.4. For an abelian group A, we may similarly consider the colimit

$$\begin{aligned} A[P^{-1}] \quad &:= \quad \operatorname{colim}(A \xrightarrow{\cdot p_1} A \xrightarrow{\cdot p_1 p_2} A \xrightarrow{\cdot p_1 p_2 p_3} \ldots) \quad \in \quad \mathrm{Ab} \\ &\cong \quad \operatorname{coeq}(\bigoplus_{n \in \mathbb{N}} A \rightrightarrows \bigoplus_{n \in \mathbb{N}} A). \end{aligned}$$

In what follows, we will make use of some basic properties of this construction, that we will now recall:

(1) For every $p \in P$ the multiplication by p map

$$p \colon A[P^{-1}] \to A[P^{-1}]$$

is an isomorphism: one can check by hand it is both injective and surjective.

(2) If B is an abelian group such that $p \colon B \to B$ is an isomorphism for all $p \in P$, then we get $B \xrightarrow{\cong} B[P^{-1}]$.

(3) In particular, $A[P^{-1}] \xrightarrow{\cong} A[P^{-1}][P^{-1}]$.

It follows that the map $A \to A[P^{-1}]$ is universal among morphisms of abelian groups $A \to B$ such that $p \colon B \to B$ is an isomorphism for all $p \in P$.

We wish to show that the analogous universal property is satisfied in the case of spectra. We start by identifying the $\mathbb{S}[P^{-1}]$-acyclic spectra:

Lemma 6.2.5. *Let X be a spectrum. Then there is an isomorphism*

$$\pi_*(X[P^{-1}]) \cong \pi_*(X)[P^{-1}].$$

In particular, X is $\mathbb{S}[P^{-1}]$-acycic if and only if the homotopy groups $\pi_*(X)$ are P-power torsion, in the sense that the order of any element $x \in \pi_*(X)$ is a product of primes in P.

Proof. For every $n \in \mathbb{Z}$, the functor $\pi_n(-)\colon \mathrm{Sp} \to \mathrm{Ab}$ preserves filtered colimits. To see this, note that it may be factored as the composite

$$\mathrm{Sp} \xrightarrow[\simeq]{[-n]} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathrm{CGrp}(\mathrm{An}) \xrightarrow{\pi_0} \mathrm{CGrp}(\mathrm{Set}) = \mathrm{Ab}.$$

The first functor is an equivalence so preserves all colimits, the third functor is a left adjoint so also preserves all colimits, and the middle functor preserves filtered colimits by Proposition 3.5.1. We thus conclude that

$$\pi_*(\mathbb{S}[P^{-1}] \otimes X) \simeq \mathrm{colim}(\pi_*(X) \xrightarrow{\cdot p_1} \pi_*(X) \xrightarrow{\cdot p_1 p_2} \pi_*(X) \xrightarrow{\cdot p_1 p_2 p_3} \dots) = \pi_*(X)[P^{-1}].$$

The 'in particular' follows since for an abelian group A we have $A[P^{-1}] = 0$ if and only if the order of every element of A is a product of primes in P. $\square$

Corollary 6.2.6. *Let X be a spectrum. For every prime $p \in P$, the map $\cdot p\colon X[P^{-1}] \to X[P^{-1}]$ is an isomorphism of spectra.*

Proof. We may check this on homotopy groups, where it becomes the statement that the map $p\colon \pi_*(X)[P^{-1}] \to \pi_*(X)[P^{-1}]$ is an isomorphism, see (1) in Remark 6.2.4. $\square$

Next we will identify the $\mathbb{S}[P^{-1}]$-local spectra:

Lemma 6.2.7. *A spectrum X is $\mathbb{S}[P^{-1}]$-local if and only if for each prime $p \in P$ the map $p\colon X \to X$ is an equivalence.*

Proof. First assume that $p\colon X \to X$ is an equivalence for every $p \in P$. To see that X is $\mathbb{S}[P^{-1}]$-local, consider an $\mathbb{S}[P^{-1}]$-acyclic spectrum Y. We must show that $\mathrm{map}(Y, X) = 0$. This follows from the following sequence of equivalences:

$$\begin{aligned} \mathrm{map}(Y,X) &\cong \lim(\cdots \xrightarrow[\cong]{p_1p_2p_3} \mathrm{map}(Y,X) \xrightarrow[\cong]{p_1p_2} \mathrm{map}(Y,X) \xrightarrow[\cong]{p_1} \mathrm{map}(Y,X)) \\ &\cong \mathrm{map}(\mathrm{colim}\Big(Y \xrightarrow{p_1} Y \xrightarrow{p_1p_2} Y \xrightarrow[\cong]{p_1p_2p_3} \dots\Big), X) \\ &= \mathrm{map}(Y[P^{-1}], X) \cong \mathrm{map}(0, X) \cong 0; \end{aligned}$$

here the first equivalence holds because each map $p\colon \mathrm{map}(Y,X) \to \mathrm{map}(Y,X)$ is an equivalence, as it is induced by the equivalence $p\colon X \xrightarrow{\sim} X$.

Conversely, assume that X is $\mathbb{S}[P^{-1}]$-local. To show that $p\colon X \to X$ is an equivalence, it suffices by Lemma 6.1.6 to show that the map $p\colon X[P^{-1}] = X \otimes \mathbb{S}[P^{-1}] \to X \otimes \mathbb{S}[P^{-1}] = X[P^{-1}]$ is an equivalence, which is the content of Corollary 6.2.6. $\square$

Theorem 6.2.8. *For every spectrum X, the map $X \to X[P^{-1}]$ is an $\mathbb{S}[P^{-1}]$-localization of X.*

Proof. The spectrum $X[P^{-1}]$ is $\mathbb{S}[P^{-1}]$-local by Lemma 6.2.7, since every $p \in P$ acts invertibly by Corollary 6.2.6. To see the map $X \to X[P^{-1}]$ is an $\mathbb{S}[P^{-1}]$-equivalence, we must show that the induced map $X[P^{-1}] \to X[P^{-1}][P^{-1}]$ is an isomorphism. We may check this on homotopy groups, where by Lemma 6.2.5 it becomes the statement that the map $\pi_*(X)[P^{-1}] \to \pi_*(X)[P^{-1}][P^{-1}]$ is an isomorphism, which is part (3) in Remark 6.2.4. □

Remark 6.2.9. For $E = \mathbb{S}[P^{-1}]$, the previous theorem says that $L_E(X) \cong L_E(\mathbb{S}) \otimes X$. Localizations of this form are called *smashing localizations*.

Proposition 6.2.10. *Let X be a spectrum, let p be a prime number and let $n \geq 1$. Then the spectrum $X/p^n := \mathrm{cofib}(X \xrightarrow{\cdot p^n} X)$ is p-local.*

Proof. Let q be a prime number different from p. We must show that the map $q\colon X/p^n \to X/p^n$ is an isomorphism, which we may check on homotopy groups. Let $k \geq 1$ be a natural number such that $q^k \equiv 1 \bmod p^n$; such a k exists since q is invertible mod p^n and $(\mathbb{Z}/p^n)^\times$ has finite order. Let us write $q^k = 1 + ap^n$ for some $a \in \mathbb{Z}$. We will show that the map $q^k : \pi_m(X/p^n) \to \pi_m(X/p^n)$ is both injective and surjective for all m, proving the claim.

We may consider the long exact sequence on homotopy groups associated to the exact sequence

$$X \xrightarrow{\cdot p^n} X \xrightarrow{g} X/p^n.$$

This provides a commutative diagram of the following form, where the top and botton rows are exact:

$$\begin{tikzcd}
\pi_m(X) \arrow[r, "\cdot p^n"] \arrow[d, "\cdot q^k"'] & \pi_m(X) \arrow[r, "g"] \arrow[d, "\cdot q^k"'] & \pi_m(X/p^n) \arrow[r, "\partial"] \arrow[d, "\cdot q^k"] & \pi_{m-1}(X) \arrow[r, "\cdot p^n"] \arrow[d, "\cdot q^k"] & \pi_{m-1}(X) \arrow[d, "\cdot q^k"] \\
\pi_m(X) \arrow[r, "\cdot p^n"] & \pi_m(X) \arrow[r, "g"] & \pi_m(X/p^n) \arrow[r, "\partial"] & \pi_{m-1}(X) \arrow[r, "\cdot p^n"] & \pi_{m-1}(X).
\end{tikzcd}$$

Surjectivity: Given some $x \in \pi_m(X/p^n)$, we wish to show that $x = q^k z$ for some z. We have $p^n \cdot \partial(x) = 0$, hence $x = 1 \cdot x + 0 = (1 + ap^n)x = q^k x$. We conclude that $\partial(x - q^k x) = 0$, hence there exists some $y \in \pi_m(X)$ such that $g(y) = x - q^k x$. Note that $g(y) = g(y) + ag(p^n \cdot y) = g(q^k \cdot y) = q^k g(y)$. We conclude that $x = q^k(x + g(y))$, showing that x is divisible by q^k as desired.

Injectivity: Consider some $x \in \pi_m(X/p^n)$ such that $q^k x = 0$. We wish to show that $x = 0$. First note that we have

$$\partial(x) = \partial(x) + a0 = \partial(x) + ap^n\partial(x) = \partial(q^k x) = 0.$$

In particular, we may write $x = g(y)$ for some $y \in \pi_m(X)$. But then we have

$$x = g(y) = g(q^k y - ap^n y) = q^k g(y) - ag(p^n y) = 0 - 0 = 0,$$

showing injectivity. $\square$

Theorem 6.2.11 (Serre)**.** *The rational sphere $\mathbb{S}_{\mathbb{Q}}$ is equivalent to the Eilenberg MacLane spectrum $H\mathbb{Q}$ of the rational numbers.*

Proof. Consider the map $\mathbb{S} \to H\mathbb{Q}$ corresponding to $1 \in \mathbb{Q} \cong \pi_0(H\mathbb{Q})$. Since the homotopy groups of $H\mathbb{Q}$ are rational, we see that $H\mathbb{Q}$ is $\mathbb{S}_{\mathbb{Q}}$-local, hence this map uniquely refines to a map $\mathbb{S}_{\mathbb{Q}} \to H\mathbb{Q}$. We have to show that this map induces isomorphisms on homotopy groups. On π_0 we have $\pi_0(\mathbb{S}_{\mathbb{Q}}) = \pi_0(\mathbb{S})_{\mathbb{Q}} = \mathbb{Z}_{\mathbb{Q}} \cong \mathbb{Q}$. It thus remains to show that $\pi_n(\mathbb{S}_{\mathbb{Q}}) = 0$ for all $n \neq 0$. But by a theorem of Serre [Ser53], the stable homotopy groups $\pi_n(\mathbb{S}) = \pi_n^{\mathrm{st}}$ are all finite, hence are torsion, and thus get killed by rationalization. $\square$

Definition 6.2.12. We write $\mathrm{Sp}_{\mathbb{Q}} \subseteq \mathrm{Sp}$ for the full subcategory of *rational* spectra, i.e. the $\mathbb{Q}$-local spectra, or equivalently the spectra on which all primes act invertibly.

Proposition 6.2.13. *For every rational spectrum X, there is an isomorphism*

$$\bigoplus_{n\in\mathbb{Z}} H(\pi_n X)[n] \xrightarrow{\sim} X.$$

Proof. For every integer $n \in \mathbb{Z}$, we pick a basis of $\pi_n(X)$ as a rational vector space. An element of $\pi_n(X)$ corresponds to a map of spectra $\mathbb{S}[n] \to X$. Since X is rational, this uniquely extends to a map $H\mathbb{Q}[n] = \mathbb{S}_{\mathbb{Q}}[n] \to X$. By taking a direct sum over all basis vectors, this induces a map $H(\pi_n X)[n] \cong \bigoplus H\mathbb{Q}[n] \to X$ which by construction induces an isomorphism on $\pi_n(X)$. Taking the direct sum over all n, this provides the isomorphism $\bigoplus_{n\in\mathbb{Z}} H(\pi_n X)[n] \xrightarrow{\sim} X$. $\square$

Corollary 6.2.14. *There is an equivalence* $\mathrm{KU}_{\mathbb{Q}} \simeq \bigoplus_n \mathbb{Q}[2n]$.

Proof. This follows from the fact that $\pi_k(\mathrm{KU}_{\mathbb{Q}}) = \pi_k(\mathrm{KU})_{\mathbb{Q}}$ is $\mathbb{Q}$ for k even and 0 for k odd, see **??**. $\square$

Remark 6.2.15. The morphism $\mathrm{KU} \to \mathrm{KU}_{\mathbb{Q}} \simeq \bigoplus_n \mathbb{Q}[2n]$ induces maps on cohomology: for every anima X we obtain a map

$$\mathrm{ch}\colon K(X) \cong \mathrm{KU}^0(X) \to \prod_n H^{2n}(X;\mathbb{Q}).$$

This map is known as the *Chern character*. It is closely related to the notion of *Chern classes*: for a complex vector bundle E over X, the Chern character may be written as

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + \frac{1}{6}(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)) + \cdots,$$

where $c_i(E) \in H^{2i}(E;\mathbb{Z})$ is the i-th Chern class. We refer to Milnor and Stasheff [MS74] for an excellent introduction to the theory of characteristic classes.

Corollary 6.2.16. *Let X be a finite anima. Then the Chern character induces an isomorphism*

$$\mathrm{KU}^0(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_n H^{2n}(X;\mathbb{Q}).$$

Proof. Since X is finite, the functor $\mathrm{map}(\mathbb{S}[X],-)\colon \mathrm{Sp} \to \mathrm{Sp}$ commutes with filtered colimits, hence with rationalization, hence

$$\mathrm{map}(\mathbb{S}[X],\mathrm{KU})_{\mathbb{Q}} \simeq \mathrm{map}(\mathbb{S}[X],\mathrm{KU}_{\mathbb{Q}}) \simeq \mathrm{map}(\mathbb{S}[X],\bigoplus_n \mathbb{Q}[2n]) \simeq \bigoplus_n \mathrm{map}(\mathbb{S}[X],\mathbb{Q}[2n]).$$

Passing to π_0 on both sides then gives the claim. □

6.3 Completion at a prime

In the previous section we introduced the p-localization $X_{(p)}$ of a spectrum. We will now introduce another useful operation, known as *p-completion*. We will also discuss the *arithmetic fracture square*.

Definition 6.3.1. Consider the *mod p Moore spectrum* $\mathbb{S}/p$, defined as the cofiber

$$\mathbb{S}/p \quad := \quad \mathrm{cofib}(p\colon\ \mathbb{S} \to \mathbb{S}).$$

We refer to an $\mathbb{S}/p$-local spectrum as *p-complete*. We write

$$\mathrm{Sp}_p^{\wedge} \subseteq \mathrm{Sp}$$

for the full subcategory of p-complete spectra. The localization functor will be denoted by

$$(-)_p^{\wedge} := L_{\mathbb{S}/p}\colon \mathrm{Sp} \to \mathrm{Sp}_p^{\wedge}.$$

Observation 6.3.2. Note that $X \otimes \mathbb{S}/p \simeq X/p$ for every X. In particular, a spectrum X is $\mathbb{S}/p$-acyclic if and only if the map $p\colon\ X \to X$ is an equivalence, i.e. X is $\mathbb{S}[\frac{1}{p}]$-local.

We would like to get a more explicit description of p-completeness that makes its connection with p-completion in ordinary algebra clear.

Definition 6.3.3. We define the spectrum $\mathbb{S}/p^\infty$ as the cofiber of the map $\mathbb{S} \to \mathbb{S}[p^{-1}]$:

$$\mathbb{S}/p^\infty \quad := \quad \mathrm{cofib}(\mathbb{S} \to \mathbb{S}[p^{-1}] = \mathrm{colim}(\mathbb{S} \xrightarrow{\cdot p} \mathbb{S} \xrightarrow{\cdot p} \mathbb{S} \xrightarrow{\cdot p} \dots)).$$

By writing $\mathbb{S}$ as the colimit over the identity maps $\mathbb{S} \xrightarrow{\mathrm{id}} \mathbb{S}$, we may alternatively write $\mathbb{S}/p^\infty$ as the colimit over the bottom row of the following commutative diagram, where all vertical sequences are cofiber sequences:

$$\begin{tikzcd}
\mathbb{S} \arrow[r, equal] \arrow[d, "\cdot p"'] & \mathbb{S} \arrow[r, equal] \arrow[d, "\cdot p^2"] & \mathbb{S} \arrow[r, equal] \arrow[d, "\cdot p^3"] & \dots \\
\mathbb{S} \arrow[r, "\cdot p"] \arrow[d] & \mathbb{S} \arrow[r, "\cdot p"] \arrow[d] & \mathbb{S} \arrow[r, "\cdot p"] \arrow[d] & \dots \\
\mathbb{S}/p \arrow[r, "\cdot p"] & \mathbb{S}/p^2 \arrow[r, "\cdot p"] & \mathbb{S}/p^3 \arrow[r, "\cdot p"] & \dots.
\end{tikzcd}$$

(This also justifies the notation $\mathbb{S}/p^\infty$.)

Theorem 6.3.4. *For a spectrum X, the map*

$$X = \mathrm{map}(\mathbb{S}, X) \to \mathrm{map}(\mathbb{S}/p^\infty[-1], X)$$

induced by the map $\mathbb{S}/p^\infty[-1] \to \mathbb{S}$ exhibits its target as a p-completion of X, i.e.

$$X_p^\wedge \xrightarrow{\sim} \mathrm{map}(\mathbb{S}/p^\infty[-1], X).$$

Proof. We have to show that $\mathrm{map}(\mathbb{S}/p^\infty[-1], X)$ is p-complete and that the map $X \to \mathrm{map}(\mathbb{S}/p^\infty[-1], X)$ is an $\mathbb{S}/p$-equivalence. For the latter, we may equivalently show that its fiber is $\mathbb{S}/p$-acyclic, or equivalently that its fiber is $\mathbb{S}[\frac{1}{p}]$-local by Observation 6.3.2. Because of the exact sequence $\mathbb{S}/p^\infty[-1] \to \mathbb{S} \to \mathbb{S}[\frac{1}{p}]$, this fiber is equivalent to $\mathrm{map}(\mathbb{S}[\frac{1}{p}], X)$. But this is clearly $\mathbb{S}[\frac{1}{p}]$-local: for any $\mathbb{S}[\frac{1}{p}]$-acyclic spectrum Y we have $\mathrm{map}(Y, \mathrm{map}(\mathbb{S}[\frac{1}{p}], X)) \simeq \mathrm{map}(Y \otimes \mathbb{S}[\frac{1}{p}], X) \simeq \mathrm{map}(0, X) = 0$.

We will now show that $\mathrm{map}(\mathbb{S}/p^\infty[-1], X)$ is p-complete, i.e. $\mathbb{S}/p$-local. To this end, let A be an $\mathbb{S}/p$-acyclic spectrum, i.e. the map $p\colon A \to A$ is an equivalence. We need to show that $0 = \mathrm{map}(A, \mathrm{map}(\mathbb{S}/p^\infty[-1], X)) \simeq \mathrm{map}(A \otimes \mathbb{S}/p^\infty[-1], X)$. For this, it will suffice to show that $A \otimes \mathbb{S}/p^\infty = 0$. Since we may write $\mathbb{S}/p^\infty$ as a colimit of $\mathbb{S}/p^n$, we may similarly write

$$A \otimes \mathbb{S}/p^\infty \simeq \mathrm{colim}_n(A \otimes \mathbb{S}/p^n) \simeq \mathrm{colim}_n(A/p^n).$$

But as A is $\mathbb{S}/p$-acyclic, the map $p\colon A \to A$ and hence also $p^n\colon A \to A$ is an equivalence, and so $A/p^n = 0$. This shows that $A \otimes \mathbb{S}/p^\infty \simeq 0$, as desired. □

As a corollary of the theorem, we may now describe the p-completion of X as a familiar limit:

Corollary 6.3.5. *For a spectrum X, its p-completion is given by the following limit:*

$$X_p^\wedge \quad \simeq \quad \lim(\cdots \to X/p^3 \to X/p^2 \to X/p \to X).$$

Proof. The exact sequence $\mathbb{S}/p^n[-1] \to \mathbb{S} \xrightarrow{\cdot p^n} \mathbb{S}$ induces an exact sequence

$$\mathrm{map}(\mathbb{S}, X) \xrightarrow{\cdot p^n} \mathrm{map}(\mathbb{S}, X) \to \mathrm{map}(\mathbb{S}/p^n[-1], X).$$

Since $\mathrm{map}(\mathbb{S}, X) \simeq X$, it follows that $\mathrm{map}(\mathbb{S}/p^n[-1], X) \simeq X/p^n$ for all n. Furthermore, under these equivalences the maps $\mathbb{S}/p^n \xrightarrow{\cdot p} \mathbb{S}/p^{n+1}$ induce the canonical maps $X/p^{n+1} \to X/p^n$, as these are the ones induced on vertical cofibers in the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\cdot p"] \arrow[d, "\cdot p^{n+1}"'] & X \arrow[d, "\cdot p^n"] \\ X \arrow[r, equal] & X. \end{tikzcd}$$

We may then compute that

$$\begin{aligned} X_p^\wedge &\overset{6.3.4}{\simeq} \mathrm{map}(\mathbb{S}/p^\infty[-1], X) \simeq \mathrm{map}(\mathrm{colim}_n\, \mathbb{S}/p^n[-1], X) \\ &\simeq \lim_n \mathrm{map}(\mathbb{S}/p^n[-1], X) \simeq \lim_n X/p^n, \end{aligned}$$

as desired. ◻

Corollary 6.3.6. *Let p be a prime number. Then every p-complete spectrum is p-local.*

Proof. By assumption, the map $X \to X_p^\wedge = \lim_n(X/p^n)$ is an equivalence. Since p-local spectra are closed under limits by Exercise 6.1.4, it will suffice to show that each spectrum X/p^n is p-local. This is the content of Proposition 6.2.10. ◻

Corollary 6.3.7. *Let p and q be two distinct primes. If X is both p-complete and q-complete, then X is the zero spectrum.*

Proof. The q-completeness of X means that X is $\mathbb{S}/q$-local. The p-completeness of X implies, via Corollary 6.3.6, that the map $q\colon X \to X$ is invertible, hence that X is $\mathbb{S}/q$-acyclic. It then follows from Lemma 6.1.6 that $X = 0$. ◻

Lemma 6.3.8. *Let p be a prime and let X be a spectrum satisfying $X[\frac{1}{p}] = 0$ and $X/p = 0$. Then $X = 0$.*

Proof. The condition that $X/p = 0$ means that the map $p\colon X \to X$ is an equivalence, hence that X is $\mathbb{S}[\frac{1}{p}]$-local. Since X is also $\mathbb{S}[\frac{1}{p}]$-acyclic, it follows from Lemma 6.1.6 that $X = 0$. ◻

Theorem 6.3.9 (The p-adic arithmetic fracture square)**.** *Let X be a spectrum. Then the following commutative square is a pullback square:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X_p^\wedge \arrow[d] \\ X[\frac{1}{p}] \arrow[r] & X_p^\wedge[\frac{1}{p}]. \end{tikzcd}$$

Proof. We have to show that the map $X \to X[\frac{1}{p}] \times_{X_p^\wedge[\frac{1}{p}]} X_p^\wedge$ is an equivalence, or equivalently that its fiber F is the zero spectrum. Note that $F[\frac{1}{p}] = 0$, since the square

$$\begin{tikzcd} X[\frac{1}{p}] \arrow[r] \arrow[d] & X_p^\wedge[\frac{1}{p}] \arrow[d] \\ X[\frac{1}{p}][\frac{1}{p}] \arrow[r] & X_p^\wedge[\frac{1}{p}][\frac{1}{p}] \end{tikzcd}$$

is a pullback square (as the two vertical maps are equivalences). We claim that also $F/p = 0$, so that F by Lemma 6.3.8. To see this, we need to show that the commutative square

$$\begin{tikzcd} X/p \arrow[r] \arrow[d] & X_p^\wedge/p \arrow[d] \\ X[\frac{1}{p}]/p \arrow[r] & X_p^\wedge[\frac{1}{p}]/p \end{tikzcd}$$

is a pullback square. The two objects at the bottom are zero, since the maps $p\colon X[\frac{1}{p}] \to X[\frac{1}{p}]$ and $p\colon X_p^\wedge[\frac{1}{p}] \to X_p^\wedge[\frac{1}{p}]$ are equivalences by Corollary 6.2.6. In particular the bottom map is an equivalence. But the top map is also an equivalence, since the map $X \to X_p^\wedge$ is by definition an $\mathbb{S}/p$-equivalence. This finishes the proof. $\square$

Exercise 6.3.10 (Arithmetic fracture square)**.** Show that for every spectrum X, the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

is a pullback square. Here the product is taken over all prime numbers p.

7 Thom spectra

This chapter is not part of the exam.

In this chapter, we briefly introduce a class of examples of spectra called *Thom spectra*, following the modern approach of Ando et al. [And+14]. To motivate this approach, we will start in Section 7.1 by recalling the classical formulation of a Thom space of a vector bundle, and providing an alternative description as an ∞-categorical colimit. We define abstract Thom spectra in Section 7.2, and show that in various prominent examples, like the *real/complex bordism spectra* MO and MU, they can be written as colimits of shifts of Thom spaces, justifying their name. In Section 7.3 we discuss *orientations* and in Section 7.4 we introduce multiplicative structures on Thom spectra.

7.1 Thom spaces and Thom animae

7.1.1 Thom spaces

We start by recalling the classical notion of a Thom space.

Construction 7.1.1 (Fiberwise one-point compactification)**.** Let $p\colon E \to X$ be a real vector bundle of rank n. We define its *fiberwise one-point compactification* S^E as the topological space obtained from E by forming the one-point compactification fiberwise, replacing $\mathbb{R}^n$ with $S^n = \mathbb{R}^n \cup \{\infty\}$ in each fiber. More precisely, given a local trivialization $\{\varphi_i\colon U_i \times \mathbb{R}^n \to p^{-1}(U_i)\}_{i\in I}$ of E, we define S^E by gluing together the spaces $U_i \times S^n$ along the transition functions: for $x \in U_i \cap U_j$, the map $\varphi_j^{-1} \circ \varphi_i\colon \{x\} \times \mathbb{R}^n \to \{x\} \times \mathbb{R}^n$ extends uniquely to a homeomorphism $\{x\} \times S^n \to \{x\} \times S^n$ fixing the point at infinity.

The space S^E comes equipped with a map $S^E \to X$. The 'points at infinity' assemble into a continuous section $s_\infty\colon X \to S^E$.

Remark 7.1.2. The previous construction is independent of the choice of local trivializations of $E \to X$.

Definition 7.1.3 (Thom space)**.** Let $p\colon E \to X$ be a real vector bundle. We define the *Thom space* $\mathrm{Th}(E)$ as the quotient of S^E by the points at infinity:

$$\mathrm{Th}(E) := S^E/s_\infty(X).$$

Remark 7.1.4. Given a metric on E, one obtains a homeomorphic description of $\mathrm{Th}(E)$ as the quotient $D(E)/S(E)$, where $D(E)$ and $S(E)$ denote the unit disk and sphere bundles with respect to the metric. The homeomorphism is given by sending a vector v to $v/(1+\|v\|)$ in each fiber. This is often taken as the definition, but has the disadvantage of requiring an auxiliary choice of metric.

Remark 7.1.5. If X is a compact Hausdorff space, the inclusion $E \hookrightarrow S^E \to \mathrm{Th}(E)$ induces a homeomorphism $E^+ \xrightarrow{\cong} \mathrm{Th}(E)$ between the Thom space and the one-point compactification of the total space E: both sides are compact Hausdorff spaces and the map is a bijection.

Example 7.1.6. Let $E = X \times \mathbb{R}^n$ be the trivial rank n vector bundle over X. Then $\mathrm{Th}(E) \cong X_+ \wedge S^n \cong \Sigma^n(X_+)$.

Example 7.1.7. Let $E \to X$ and $E' \to X'$ be vector bundles. Then their product $E \times E' \to X \times X'$ is again a vector bundle (called their *external direct sum*) and there is an isomorphism of pointed spaces

$$\mathrm{Th}(E \times E') \cong \mathrm{Th}(E) \wedge \mathrm{Th}(E').$$

To see this, equip $E \times E'$ with the metric $\|(v,w)\| = \max(|v|,|w|)$. Then the disk bundle $D(E \times E')$ is homeomorphic to the product $D(E) \times D(E')$, and under this homeomorphism the sphere bundle $S(E \times E')$ corresponds to the subspace $D(E) \times S(E) \cup S(E) \times D(E)$ of the product. Passing to qoutient spaces then proves the claim.

Example 7.1.8. Let us consider the Grassmannian $\mathrm{Gr}_n(\mathbb{R}^\infty)$ of n-dimensional subspaces of $\mathbb{R}^\infty$. Arguing as for the complex case in Section 5.1, this space is a model for the classifying space $BO(n)$. The universal bundle γ_n over $\mathrm{Gr}_n(\mathbb{R}^\infty)$ has as total space

$$E_n = \{(V,v) \in \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \mid v \in V\}$$

with the projection map sending (V,v) to V. The fiber over a point $V \in \mathrm{Gr}_n(\mathbb{R}^\infty)$ is precisely the n-dimensional vector space V itself.

The Thom space of this universal bundle γ_n is particularly important - it will show up later as one of the building blocks for the spectrum MO. In this case, we are looking at pairs (V,v) where $v \in V$ has length ≤ 1, modulo those pairs where v has length exactly 1.

Construction 7.1.9 (Functoriality of Thom spaces)**.** Consider a fiberwise injective morphism of rank n vector bundles, i.e. a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\widetilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

in which $\widetilde{f}$ induces injective $\mathbb{R}$-linear maps $E_x \hookrightarrow E'_{f(x)}$ on fibers. Then the map $\widetilde{f}$ induces a map $S^E \to S^{E'}$ on fiberwise one-point compactifications, which preserves the sections at infinity and hence induces a continuous map

$$\mathrm{Th}(\widetilde{f})\colon \mathrm{Th}(E) \to \mathrm{Th}(E').$$

7.1.2 Thom animae

The underlying anima of a Thom space admits a purely homotopical description, which we will discuss now.

Definition 7.1.10 (Spherical fibration)**.** Let B be an anima and let $n \geq 0$. A *rank n spherical fibration over* B is a functor $\xi\colon B \to \mathrm{An}_*$ which lands in the full subcategory spanned by the n-sphere S^n.

Definition 7.1.11 (Thom anima)**.** We define the *Thom anima* of a spherical fibration $\xi\colon B \to \mathrm{An}_*$ as its colimit:

$$\mathrm{Th}(\xi) \quad := \quad \mathrm{colim}(\xi\colon B \to \mathrm{An}_*).$$

Recall from Theorem A.3.5 the straightening-unstraightening equivalence $\mathrm{An}_{/B} \simeq \mathrm{Fun}(B, \mathrm{An})$. Passing to pointed objects, this provides an equivalence

$$\mathrm{Fun}(B, \mathrm{An}_*) \simeq \mathrm{Fun}(B, \mathrm{An})_* \simeq (\mathrm{An}_{/B})_*,$$

so that a rank n spherical fibration over B is the same as a pair (p, s) of a morphism $p\colon A \to B$ of animae and a section $s\colon B \to A$ of p such that each fiber A_b of p is isomorphic to the n-sphere S^n.

Example 7.1.12 (Sphere bundle)**.** Let X be a topological space and consider an n-dimensional sphere bundle $Y \to X$ over X, i.e. a fiber bundle whose generic fiber is S^n and whose structure group is $\mathrm{Homeo}_*(S^n)$. Then the underlying map of animae $\Pi_\infty(Y) \to \Pi_\infty(X)$ is a spherical fibration: by Corollary B.3.7 and Corollary B.2.11 the fibers of this map are the spheres S^n.

Example 7.1.13 (Spherical fibration of vector bundle)**.** Given a rank n vector bundle $p\colon E \to X$, we may apply the previous example to the fiberwise one-point compactification $S^E \to X$ from Construction 7.1.1. We denote the resulting spherical fibration by

$$\xi_E \colon \Pi_\infty(X) \to \mathrm{An}_*.$$

Informally, ξ_E is the functor which sends a point $x \in X$ to $\Pi_\infty(S^{E_x})$, where S^{E_x} is the one-point compactification of the fiber E_x of p over x.

We wish to show that, in the setting of Example 7.1.13, the underlying anima of the Thom space $\mathrm{Th}(E)$ is the Thom anima $\mathrm{Th}(\xi_E)$. We start with an auxiliary lemma.

Lemma 7.1.14. *Consider a real vector bundle $p\colon E \to X$ of rank n, equipped with a fiberwise metric, and assume that X is a cell complex. Then the inclusion $S(E) \hookrightarrow D(E)$ is a relative cell complex, in the sense of Definition B.2.3.*

Proof. The approach will be to construct $D(E)$ from $S(E)$ via an inductive attachment of cells over the k-skeleta X^k of X.

For $k = 0$, X^0 is a disjoint union of points, hence $S(E)|_{X^0}$ is a disjoint union of $(n-1)$-spheres and $D(E)|_{X_0}$ is a disjoint union of n-disks. In particular, we obtain $D(E)|_{X_0}$ from $S(E)|_{X^0}$ by attaching an individual n-cell for every 0-cell of X.

For $k \geq 1$, assume we have exhibited the inclusion $S(E)|_{X^{k-1}} \hookrightarrow D(E)|_{X^{k-1}}$ as a relative cell complex. For each k-cell $e^k \subset X$, the trivialization $D(E)|_{e^k} \cong e^k \times D^n$ implies that $D(E)|_{e^k} \cong D^k \times D^n$, while $S(E)|_{e^k} \cong D^k \times S^{n-1}$. We thus see that we obtain $D(E)|_{e^k}$ from $D(E)|_{\partial e^k} \cup S(E)|_{e^k}$ by attaching a $(k+n)$-cell. The boundary map is constructed by considering the homeomorphism $S^{k+n-1} \cong \partial(D^k \times D^n) = (\partial D^k \times D^n) \cup (D^k \times S^{n-1})$, and observing that $\partial D^k \times D^n$ is already attached to $D(E)|_{X^{k-1}}$, and that $D^k \times S^{n-1} \cong S(E)|_{e^k}$. Proceeding like this for every k-cell of X, this shows that also the inclusion $S(E)|_{X^k} \hookrightarrow D(E)|_{X^k}$ as a relative cell complex.

Letting k go to infinity then proves the claim. ◻

Proposition 7.1.15. *For a vector bundle $p\colon E \to X$ over a cell complex X, the underlying anima of its Thom space is equivalent to the Thom anima of ξ_E:*

$$\Pi_\infty(\mathrm{Th}(E)) \simeq \mathrm{Th}(\xi_E).$$

Proof. Choose a metric on E, so that $\mathrm{Th}(E)$ is homeomorphic to the quotient $D(E)/S(E)$ by Remark 7.1.4. By Lemma 7.1.14, the inclusion $S(E) \hookrightarrow D(E)$ is a relative cell complex, hence by Corollary B.2.7 we obtain a cofiber sequence

$$\Pi_\infty(S(E)) \to \Pi_\infty(D(E)) \to \Pi_\infty(\mathrm{Th}(E)).$$

Now, recall that under the straightening-unstraightening equivalence $\mathrm{An}_{/\Pi_\infty(X)} \xrightarrow{\sim} \mathrm{Fun}(\Pi_\infty(X), \mathrm{An})$ from Theorem A.3.5, the forgetful functor $\mathrm{An}_{/\Pi_\infty(X)} \to \mathrm{An}$ corresponds to the colimit functor

$$\mathrm{colim}_{\Pi_\infty(X)} \colon \mathrm{Fun}(\Pi_\infty(X), \mathrm{An}) \to \mathrm{An}.$$

This provides equivalences

$$\Pi_\infty(S(E)) \simeq \mathrm{colim}_{x\in\Pi_\infty(X)} S(E_x) \qquad \text{and} \qquad \Pi_\infty(D(E)) \simeq \mathrm{colim}_{x\in\Pi_\infty(X)} D(E_x).$$

Since colimits commute with cofibers, the cofiber of the map $\Pi_\infty(S(E)) \to \Pi_\infty(D(E))$ may thus be computed as

$$\mathrm{colim}_{x\in\Pi_\infty(X)} \mathrm{cofib}(S(E_x) \to D(E_x)) = \mathrm{colim}_{x\in\Pi_\infty(X)} S^{E_x} = \mathrm{Th}(\xi_E).$$

This finishes the proof. □

For later purposes, we will record an alternative description of the functor $\xi_E \colon \Pi_\infty(X) \to \mathrm{An}_*$.

Construction 7.1.16 (Unstable J-homomorphism)**.** We construct for every $n \geq 0$ a functor $J_n \colon BO(n) \to \mathrm{An}_*$. Consider the topological n-sphere S^n, and identify it with the one-point compactification of $\mathbb{R}^n$. The canonical action of the orthogonal group $O(n)$ on $\mathbb{R}^n$ uniquely extends to a continuous $O(n)$-action on S^n. Note that this action preserves the point at infinity, so that this defines an action in Top_*. By applying the localization functor $\Pi_\infty \colon \mathrm{Top}_* \to \mathrm{An}_*$, this provides an action of $O(n)$ of S^n in An_*, which we may encode as a functor $J_n \colon BO(n) \to \mathrm{An}_*$.

Lemma 7.1.17. *Let X be a cell complex and let $f \colon X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ be a continuous map. Let $p \colon E \to X$ be the rank n vector bundle classified by f. Then the map $\xi_E \colon \Pi_\infty(X) \to \mathrm{An}_*$ is given by the composite*

$$\Pi_\infty(X) \xrightarrow{\Pi_\infty(f)} \Pi_\infty(\mathrm{Gr}_n(\mathbb{C}^\infty)) \overset{5.2.12}{\simeq} BO(n) \xrightarrow{J_n} \mathrm{An}_*.$$

Proof. By definition, $p \colon E \to X$ is the pullback of the universal bundle $p_{\mathrm{univ}} \colon E_{\mathrm{univ}} \to \mathrm{Gr}_n(\mathbb{R}^\infty)$. Under straightening-unstraightening, pullback of fibrations corresponds to precomposition, and so it will suffice t prove the claim in this special case of p_{univ}. TO DO. □

7.2 Thom spectra

The definition of a Thom anima may immediately be generalized to that of a Thom *spectrum*.

Definition 7.2.1. Let $\mathrm{Pic}(\mathbb{S}) \subseteq \mathrm{Sp}^{\simeq}$ denote the full subcategory spanned by shifts $\mathbb{S}[n]$ of the sphere spectrum. For an anima A, we define a *stable spherical fibration on A* to be a functor $\xi\colon A \to \mathrm{Sp}$ which lands in $\mathrm{Pic}(\mathbb{S})$. For an integer $n \in \mathbb{Z}$ we say that ξ has *rank n* if it in fact lands in the full subanima of $\mathrm{Pic}(\mathbb{S})$ spanned by $\mathbb{S}[n]$.

Example 7.2.2. If $\xi\colon A \to \mathrm{An}_*$ is a rank n spherical fibration, then its suspension spectrum $\Sigma^\infty(\xi)\colon A \to \mathrm{Sp}$ is a rank n stable spherical fibration.

Example 7.2.3. Let X be a topological space, and let $E \to X$ be a rank n vector bundle. Recall that in Example 7.1.13 we assigned to E a rank n spherical fibration $\xi_E\colon \Pi_\infty(X) \to \mathrm{An}_*$, hence by the previous example also a stable spherical fibration $\Sigma^\infty(\xi_E)$.

Definition 7.2.4 (Thom spectrum)**.** Given a stable spherical fibration $\xi\colon A \to \mathrm{Sp}$, we define its *Thom spectrum* as its colimit:

$$\mathrm{Th}(\xi) \quad := \quad \mathrm{colim}(\xi\colon A \to \mathrm{Sp}).$$

Since the suspension spectrum functor $\Sigma^\infty(-)\colon \mathrm{An}_* \to \mathrm{Sp}$ preserves colimits, it is clear that the suspension spectrum $\Sigma^\infty(\mathrm{Th}(\xi))$ of any Thom anima is a Thom spectrum. Let us now construct an example of a Thom spectrum which is not a suspension spectrum.

Construction 7.2.5 (J-homomorphism)**.** Consider the infinite orthogonal group $O := \bigcup_n O(n)$, where the union is taken along the canonical inclusions $O(n) \hookrightarrow O(n+1)$ sending A to $\left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We will construct a functor $J\colon BO \to \mathrm{Sp}$ called the *J-homomorphism*. Since $BO \simeq \mathrm{colim}_n BO(n)$, it will suffice to provide a sequence of functors $J_n\colon BO(n) \to \mathrm{Sp}$ for all n, together with identifications between J_n and $J_{n+1}|_{\mathrm{BO}(n)}$. We proceed as follows:

(1) Consider the topological n-sphere S^n, and identify it with the one-point compactification of $\mathbb{R}^n$. The canonical action of $O(n)$ on $\mathbb{R}^n$ uniquely extends to a continuous $O(n)$-action on S^n. Note that this action preserves the point at infinity, so that this defines an action in Top_*. By applying the localization functor $\Pi_\infty\colon \mathrm{Top}_* \to \mathrm{An}_*$, this provides an action of $O(n)$ of S^n in An_*, which we may encode as a functor $\xi_n\colon BO(n) \to \mathrm{An}_*$. We then define J_n as the composite

$$BO(n) \xrightarrow{\xi_n} \mathrm{An}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp} \xrightarrow{[-n]} \mathrm{Sp}.$$

Observe that J_n sends the unique object to $\Sigma^\infty(S^n)[-n]$, which is indeed isomorphic to $\mathbb{S}$.

(2) For $n \geq 0$, we may consider the functor $\xi_{n+1}\colon BO(n+1) \to \mathrm{An}_*$, obtained from the $O(n+1)$-action on $\mathbb{R}^{n+1}$. Its restriction $\xi_{n+1}|_{\mathrm{BO}(n)}$ is similarly obtained from the restricted $O(n)$-action on $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$. Note that the one-point compactification of $\mathbb{R}^n \times \mathbb{R}$ is

$S^n \wedge S^1 = \Sigma(S^n)$, and under this identification the $O(n)$-action on S^{n+1} is the suspension of the $O(n)$-action on S^n. We conclude that $\xi_{n+1}|_{\mathrm{BO}(n)} \cong \Sigma(\xi_n)$, and hence that

$$J_{n+1}|_{\mathrm{BO}(n)} = \Sigma^\infty(\xi_{n+1}|_{\mathrm{BO}(n)})[-(n+1)] \cong \Sigma^\infty(\Sigma(\xi_n))[-(n+1)] \cong \Sigma^\infty(\xi_n)[-n] = J_n.$$

This provides the desired functor $J\colon BO \to \mathrm{Sp}$.

Definition 7.2.6 (Real cobordism spectrum)**.** We define the *real cobordism spectrum* MO as the Thom spectrum of the J-homomorphism:

$$\mathrm{MO} \quad := \quad \mathrm{Th}(J) \quad = \quad \mathrm{colim}(J\colon \mathrm{BO} \to \mathrm{Sp}).$$

Let us now discuss the relation between Definition 7.2.6 and the classical definition of MO, going back all the way to Thom [Tho54].

Construction 7.2.7. Recall from Example 7.1.8 the universal bundle $\gamma_n\colon E_n \to \mathrm{Gr}_n(\mathbb{R}^\infty)$, of which we may form its Thom space $\mathrm{Th}(\gamma_n)$. Notice that there is a canonical inclusion

$$\mathrm{Gr}_n(\mathbb{R}^\infty) \hookrightarrow \mathrm{Gr}_{n+1}(\mathbb{R}\times\mathbb{R}^\infty) \cong \mathrm{Gr}_{n+1}(\mathbb{R}^\infty), \quad V \mapsto \mathbb{R}\oplus V.$$

The pullback of γ_{n+1} along this inclusion is the product bundle $\gamma_n \times \underline{\mathbb{R}}$, and in particular we obtain a pointed continuous map

$$\Sigma(\mathrm{Th}(\gamma_n)) \cong \mathrm{Th}(\gamma_n \times \mathbb{R}) \to \mathrm{Th}(\gamma_{n+1}).$$

By adjunction, this defines a map $\sigma_n\colon \mathrm{Th}(\gamma_n) \to \Omega(\mathrm{Th}(\gamma_{n+1}))$, turning the sequence of pointed spaces $(\mathrm{Th}(\gamma_n))$ into a prespectrum, which we will denote by $\mathrm{MO}^{\mathrm{pre}}$.

Lemma 7.2.8. *The spectrum* MO *is the spectrum associated to the prespectrum* $\mathrm{MO}^{\mathrm{pre}}$ *via Construction 3.5.8:*

$$\mathrm{MO} \cong (\mathrm{MO}^{\mathrm{pre}})^{\mathrm{sp}}$$

Proof. Step 1: We start by providing an alternative description of the prespectrum $\mathrm{MO}^{\mathrm{pre}}$. Consider the functor $\xi_{E_n}\colon \Pi_\infty(\mathrm{Gr}_n(\mathbb{R}^\infty)) \to \mathrm{An}_*$ associated to γ_n via Example 7.1.13. We claim that ξ_{E_n} is isomorphic to ξ_n. TO DO. In particular, Proposition 7.1.15 provides an equivalence

$$\mathrm{MO}^{\mathrm{pre}}_n = \Pi_\infty(\mathrm{Th}(\gamma_n)) \cong \mathrm{Th}(\zeta_n)$$

of animae. For the structure maps, note that the identification $E_{n+1}|_{\mathrm{Gr}_n(\mathbb{R}^\infty)} \cong E_n \times \mathbb{R}$ corresponds to the identification $\xi_{n+1}|_{BO(n)} \cong \Sigma(\xi_n)$ from Construction 7.2.5. The structure maps of $\mathrm{MO}^{\mathrm{pre}}$ thus correspond to the following canonical comparison maps:

$$\Sigma(\mathrm{Th}(\xi_n)) \cong \mathrm{Th}(\Sigma(\xi_n)) \cong \mathrm{Th}(\xi_{n+1}|_{BO(n)}) = \mathrm{colim}_{BO(n)}\,\xi_{n+1}|_{BO(n)} \to \mathrm{colim}_{BO(n+1)}\,\xi_{n+1} = \mathrm{Th}(\xi_{n+1}).$$

Step 2: We now show that the associated spectrum is MO. To see this, we apply Theorem A.2.25 with $I = BO = \mathrm{colim}_{n\in\mathbb{N}} BO(n)$. It follows that the colimit $\mathrm{MO} = \mathrm{colim}_{BO}(J)$ may be written as an iterated colimit

$$\mathrm{MO} \cong \mathrm{colim}_n \mathrm{colim}_{BO(n)} J_n \cong \mathrm{colim}_n \Sigma^\infty(\mathrm{colim}_{BO(n)} \xi_n)[-n] \cong \mathrm{colim}_n \Sigma^\infty(\mathrm{Th}(\xi_n))[-n].$$

Unwinding definitions, one observes that the structure maps in this colimit diagram are precisely the ones coming from the prespectrum $\mathrm{MO}^{\mathrm{pre}}$, so that the right-hand side is precisely $(\mathrm{MO}^{\mathrm{pre}})^{\mathrm{st}}$. □

The spectrum MO has many variations, based on other topological groups.

Definition 7.2.9. Let G be a topological group equipped with a continuous group homomorphism $G \to O$. We define the corresponding Thom spectrum MG as the Thom spectrum of $J|_{BG}$:

$$MG \quad := \quad \mathrm{Th}(J|_{BG}) \quad = \quad \mathrm{colim}(BG \to BO \xrightarrow{J} \mathrm{Sp}).$$

This in particular defines the following Thom spectra:

$$\mathrm{MSO}, \quad \mathrm{MU}, \quad \mathrm{MSU}, \quad \mathrm{MSp}, \quad \text{and} \quad \mathrm{MSpin}.$$

The spectrum MU is known as the *complex cobordism spectrum*, and plays an important role in the field of *chromatic homotopy theory*.

Exercise 7.2.10. Construct the group homomorphism $G \to O$ for each of the five Thom spectra mentioned above.

7.3 Orientations and the Thom isomorphism

Throughout this section, we fix a ring spectrum $R \in \mathrm{Alg}(\mathrm{Sp})$.

Definition 7.3.1 (Orientation)**.** Let $p\colon E \to X$ be a rank n vector bundle. An *R-orientation* of p is a cohomology class $\theta \in R^n(\mathrm{Th}(E))$ such that for every point $x \in X$, the image $\theta_x \in \pi_0(R)$ of θ under the composite map

$$R^n(\mathrm{Th}(E)) \xrightarrow{i^*} R^n(S^{E_x}) \cong R^n(S^n) \cong R^0(S^0) \cong \pi_0(R)$$

is an invertible element of the ring $\pi_0(R)$; here the first map is induced by the inclusion $i\colon S^{E_x} \hookrightarrow \mathrm{Th}(E)$. The class θ is also called an *R-Thom class*.

We may rephrase the notion of orientation more conveniently in terms of trivializations of local systems, as we will explain now.

Definition 7.3.2 (R-lines)**.** Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be a ring spectrum. We denote by

$$\mathrm{Line}_R \subseteq \mathrm{Mod}_R^{\simeq}$$

the full subanima spanned by those R-modules that are isomorphic to R. An object of Line_R is called an *R-line*. Note that Line_R is not a full subcategory of Mod_R: the morphisms of R-lines are by definition R-linear *isomorphisms*.

Remark 7.3.3. The anima Line_R is pointed via the object $R \in \mathrm{Line}_R$. It is also connected by definition, and so by Theorem 4.2.1 we have an equivalence of animae

$$\mathrm{Line}_R \simeq \mathbf{B}\Omega(\mathrm{Line}_R)$$

Observe that the underlying anima of $\Omega(\mathrm{Line}_R)$ is the anima $\mathrm{Aut}_R(R)$ of R-linear automorphisms of R; the group operation corresponds to composition of automorphisms. This means that we may idenfity Line_R with the classifying space of the group of automorphisms of R.

Example 7.3.4. If R is a discrete ring, i.e. in the image of the inclusion $\mathrm{Alg}(\mathrm{Ab}) \hookrightarrow \mathrm{Alg}(\mathrm{Sp})$, then the group $\mathrm{Aut}_R(R)$ is also discrete and is simply the classical group of R-linear automorphisms of R. For example:

- When $R = \mathbb{F}_2$, the only automorphism of $\mathbb{F}_2$ is the identity, i.e. $\mathrm{Aut}_{\mathbb{F}_2}(\mathbb{F}_2) \cong *$. In particular
$$\mathrm{Line}_{\mathbb{F}_2} \simeq *.$$
- When $R = \mathbb{Z}$, there are precisely two automorphisms of $\mathbb{Z}$, i.e. we have $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}) \cong C_2$. In particular,
$$\mathrm{Line}_{\mathbb{Z}} \simeq BC_2.$$

Example 7.3.5. Consider the ring $R = \mathbb{Z}$ of integers. We claim that there is an

Definition 7.3.6 (Local sytem of R-lines)**.** Given an anima A, we define a *local system of R-lines on A* to be a functor $\xi\colon A \to \mathrm{Line}_R$.

Note that every functor $A \to \mathrm{Sp}$ which is pointwise the sphere spectrum induces a local system of R-lines on A for every R by composing with the functor $- \otimes R\colon \mathrm{Sp} \to \mathrm{Mod}_R$.

Definition 7.3.7 (Trivialization)**.** A *trivialization* of a local system of R-lines $\xi\colon A \to \mathrm{Mod}_R$ is an isomorphism $\xi \cong R_A$, where $R_A\colon A \to \mathrm{Mod}_R$ is the contant functor with value R.

Alternatively, we say that an R-line L is *trivial* if it comes equipped with an isomorphism $L \cong R$ of R-modules. We denote by

$$\mathrm{Triv}_R \quad := \quad (\mathrm{Line}_R)_{/R}$$

the anima of trivial R-lines. Then a trivialization of $\xi\colon A \to \mathrm{Mod}_R$ is the same as a lift of ξ to Triv_R:

$$\begin{tikzcd} & \mathrm{Triv}_R \arrow[d] \\ A \arrow[r, "\xi"'] \arrow[ur, dashed] & \mathrm{Line}_R\,. \end{tikzcd}$$

Proposition 7.3.8. *Let $p\colon E \to X$ be a rank n vector bundle. Then an R-orientation $\theta \in R^n(\mathrm{Th}(E))$ is the same as a trivialization $R[\xi_E][-n] \cong R_X$, where $\xi_{E,R}$ is the local system of R-lines given by the composite*

$$\Pi_\infty(X) \xrightarrow{\xi_E} \mathrm{An}_* \xrightarrow{\Sigma^\infty(-)[-n]} \mathrm{Line}_{\mathbb{S}} \xrightarrow{-\otimes R} \mathrm{Line}_R\,.$$

Proof. By definition, a cohomology class $\theta \in R^n(\mathrm{Th}(E))$ is a morphism of spectra $\mathrm{Th}(E)[-n] \to R$, or equivalently a morphism $\mathrm{Th}(E)[-n] \otimes R \to R$ of R-modules. Observe that by definition of $\mathrm{Th}(E)$ there is an equivalence

$$\mathrm{Th}(E)[-n] \otimes R \simeq \mathrm{colim}(\xi_{E,R}\colon \Pi_\infty(X) \to \mathrm{Line}_R \hookrightarrow \mathrm{Mod}_R),$$

and so by adjunction such a morphism corresponds to a morphism $\alpha\colon \xi_{E,R} \to R_X$ of local systems of R-modules.

It remains to show that θ is an R-orientation if and only if α is an isomorphism. By Theorem 2.6.10 the latter is the case if and only if for every $x \in X$ the map $\alpha_x\colon (\xi_{E,R})_x = \Sigma^\infty_+(\xi_E)_x[-n] \otimes R \to R$ is an isomorphism of R-modules. This map is given by multiplication by the element $\theta_x \in \pi_0(R)$, and so it is an isomorphism if and only if θ_x is invertible in $\pi_0(R)$. This finishes the proof. □

Example 7.3.9. For $R = \mathbb{F}_2$, the anima $\mathrm{Line}_{\mathbb{F}_2}$ is contractible, and so every vector bundle $E \to X$ is $\mathbb{F}_2$-orientable in a unique way.

Example 7.3.10. For $R = \mathbb{Z}$, there is an equivalence $\mathrm{Line}_{\mathbb{Z}} \simeq BC_2$. A $\mathbb{Z}$-orientation of a vector bundle classified by $\xi\colon X \to BO(n)$ is the same data as a null-homotopy of the composite

$$X \xrightarrow{\xi} BO(n) \xrightarrow{\det} BC_2,$$

where the second map is induced by the determinant $\det\colon O(n) \to \{\pm 1\} = C_2$.

Theorem 7.3.11 (Thom isomorphism)**.** *Let $E \to X$ be a rank n vector bundle and let $\theta \in R^n(\mathrm{Th}(E))$ be an R-orientation. Then there are isomorphisms*

$$\mathrm{Th}(E) \otimes R \cong (\mathbb{S}[X] \otimes R)[n] \qquad \textit{and} \qquad \mathrm{map}(\mathrm{Th}(E), R) \cong \mathrm{map}(\mathbb{S}[X][n], R),$$

in Mod_R*, hence in particular*

$$R_m(\mathrm{Th}(E)) \cong R_{m+n}(X) \qquad \textit{and} \qquad R^m(\mathrm{Th}(E)) \cong R^{m-n}(X).$$

Proof. By Proposition 7.3.8, the R-orientation provides an isomorphism $R[\xi_E][-n] \cong R_X$ of local systems of R-line bundles. Passing to colimits over X on both sides, we thus obtain the isomorphism

$$\mathrm{Th}(E) \otimes R = \operatorname{colim}_X \mathbb{S}[\xi_E(-)][-n] \otimes R \cong \operatorname{colim}_X R[\xi_E(-)][-n] \cong \operatorname{colim}_X R_X \cong R[X].$$

This proves the first isomorphism. For the second isomorphism, we observe that adjunction $\mathrm{Sp} \rightleftarrows \mathrm{Mod}_R$ produces natural isomorphisms

$$\mathrm{map}(X, Y) \cong \mathrm{map}_{\mathrm{Mod}_R}(X \otimes R, Y)$$

for $X \in \mathrm{Sp}$ and $Y \in \mathrm{Mod}_R$, hence in particular

$$\mathrm{map}(\mathrm{Th}(E), R) \cong \mathrm{map}_{\mathrm{Mod}_R}(\mathrm{Th}(E) \otimes R, R)$$

and

$$\mathrm{map}(\mathbb{S}[X][n], R) \cong \mathrm{map}(\mathbb{S}[X][n] \otimes R, R) \cong \mathrm{map}(R[X][n], R).$$

The second isomorphism thus follows from the first one. The third and fourth isomorphisms are immediate by passing to homotopy groups on both sides. □

7.4 Multiplicative structures

TO DO. *(The course has finished, hence it might take a while before I get to this, if ever.)*

8 Duality phenomena

This chapter is not part of the exam.

An important phenomenon in algebraic topology is the concept of *duality*: we have Poincaré duality, Alexander duality, Lefschetz duality, Spanier–Whitehead duality, and Atiyah duality, to name a few. The goal of this chapter is to give an introduction to duality phenomena in algebraic topology. We start in Section 8.1 with the categorical notion of duality in a symmetric monoidal ∞-category C; in the case $C = \mathrm{Sp}$ this is also known as *Spanier-Whitehead duality*. In Section 8.2, we will provide an explicit description of the dual of (the suspension spectrum of) a smooth manifold M: it is given by the Thom spectrum $\mathrm{Th}(-T_M)$ of its negative virtual tangent bundle; this result is known as *Atiyah duality* as it was first proved by Atiyah [Ati61]. Combining this with the Thom isomorphism for an orientable manifold, we then recover *Poincaré duality*; this will be explained in Section 8.3.

We refer to Becker and Gottlieb [BG99] for a historical overview of duality phenomena in algebraic topology.

8.1 Symmetric monoidal duality

Throughout this section, we let $(C, \otimes, \mathbb{1})$ be a symmetric monoidal ∞-category.

Definition 8.1.1 (Dualizable object)**.** Consider objects X and Y of C. We say that a morphism $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *exhibits Y as a dual to X* if there exists another morphism $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ such that the triangle identities are satisfied: there exist commutative diagrams

$$\begin{tikzcd} & X \otimes Y \otimes X \arrow[dr, "\mathrm{id}\otimes\mathrm{ev}"] & \\ X \arrow[ur, "\mathrm{coev}\otimes\mathrm{id}"] \arrow[rr, equal] & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Y \otimes X \otimes Y \arrow[dr, "\mathrm{ev}\otimes\mathrm{id}"] & \\ Y \arrow[ur, "\mathrm{id}\otimes\mathrm{coev}"] \arrow[rr, equal] & & Y. \end{tikzcd}$$

In this case we refer to ev as the *evaluation map* and as coev as the *coevaluation map*. We say that an object X is *dualizable* if it admits a duality datum (Y, ev).

The following result guarantees that the duality data (Y, ev) is unique if it exists:

Proposition 8.1.2. *Consider objects X and Y of C and consider a morphism* $\mathrm{ev}\colon Y \to X \to \mathbb{1}$ *in C. The following two conditions are equivalent:*

(1) The morphism $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *exhibits Y as a dual to X;*

(2) For all object Z and W of C, the composite

$$\mathrm{Hom}_C(W, Z \otimes Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, Z \otimes Y \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence.

(3) The map in (2) is an equivalence for $(W,Z) = (\mathbb{1}, X)$ and for $(W,Z) = (Y, \mathbb{1})$.

Proof. We first show that (1) implies (2). Let $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ be the corresponding coevaluation map. It then follows directly from the triangle identities that an inverse to the map in (2) is given by the composite

$$\mathrm{Hom}_C(W \otimes X, Z) \xrightarrow{-\otimes Y} \mathrm{Hom}_C(W \otimes X \otimes Y, Z \otimes Y) \xrightarrow{-\circ\mathrm{coev}} \mathrm{Hom}_C(W, Z \otimes Y).$$

It is clear that (2) implies (3). Finally, assume that (3) is satisfied. Taking $W = \mathbb{1}$ and $Z = X$, we get that the composite

$$\mathrm{Hom}_C(\mathbb{1}, X \otimes Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(X, X \otimes Y \otimes X) \xrightarrow{\mathrm{ev}\otimes -} \mathrm{Hom}_C(X, X)$$

is an equivalence. In particular, there exists a morphism $\mathrm{coev}\colon \mathbb{1} \to X \otimes Y$ that is mapped to id_X under this equivalence. In particular, the composite

$$X \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} X \otimes Y \otimes X \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} X$$

is homotopic to id_X, showing one of the two triangle identities. To show that the other triangle identity is also satisfied, consider $W = Y$ and $Z = \mathbb{1}$, so that the composite

$$\mathrm{Hom}_C(Y, Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(Y \otimes X, Y \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(Y \otimes X, \mathbb{1})$$

is an equivalence. We claim that both the map $\mathrm{id}_Y\colon Y \to Y$ as well as the composite $Y \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} Y \otimes X \otimes Y \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} Y$ are sent to $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ under this equivalence. This is clear for id_Y. For $(\mathrm{ev}\otimes\mathrm{id}) \circ (\mathrm{id}\otimes\mathrm{coev})$ this follows from the following commutative diagram:

$$\begin{tikzcd}
Y \otimes X \arrow[r, "\mathrm{id}\otimes\mathrm{coev}\otimes\mathrm{id}"] \arrow[dr, equal] & Y \otimes X \otimes Y \otimes X \arrow[r, "\mathrm{ev}\otimes\mathrm{id}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}"] & Y \otimes X \arrow[d, "\mathrm{ev}"] \\
& Y \otimes X \arrow[r, "\mathrm{ev}"] & \mathbb{1}.
\end{tikzcd}$$

This shows that also the second triangle identity is satisfied, finishing the proof. $\square$

Convention 8.1.3. Let X be a dualizable object of C. By the previous proposition, a dual for X is the same as an object of C representing the functor $\mathrm{Hom}_C(-\otimes X, \mathbb{1})\colon C^{\mathrm{op}} \to \mathrm{An}$. In particular, the dual is unique if it exists. In this case we denote it by $X^\vee$.

Definition 8.1.4. An object X of C is called *invertible* if there exists another object X^{-1} and an isomorphism $X \otimes X^{-1} \cong \mathbb{1}$. Equivalently, X is invertible if the functor $X \otimes -\colon C \to C$ is an equivalence.

Lemma 8.1.5. *Every invertible object is dualizable.*

Proof. Let X be an invertible object and let X^{-1} be an inverse with isomorphism $\mathrm{ev}\colon X^{-1} \otimes X \xrightarrow{\cong} \mathbb{1}$. We claim that ev exhibits X^{-1} as a dual to X. Using Proposition 8.1.2 we must show that for all objects Z and W of C the composite

$$\mathrm{Hom}_C(W, Z \otimes Y) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, Z \otimes Y \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence. But both of these maps are already equivalences before composing, since the functor $-\colon X\colon C \to C$ is an equivalence and the morphism ev is an isomorphism. □

Corollary 8.1.6. *The unit object $\mathbb{1} \in C$ is dualizable.*

Proof. It is invertible, using $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$. □

The notion of duality is closely related to that of *internal homs*:

Definition 8.1.7. Given two objects $X, Z \in C$, an object $Z^X \in C$ equipped with a map $\mathrm{ev}\colon Z^X \otimes X \to Z$ is said to *exhibit Z^X as an internal hom-object from X to Z* if for every third object W the composite

$$\mathrm{Hom}_C(W, Z^X) \xrightarrow{-\otimes X} \mathrm{Hom}_C(W \otimes X, Z^X \otimes X) \xrightarrow{\mathrm{ev}\circ -} \mathrm{Hom}_C(W \otimes X, Z)$$

is an equivalence.

Remark 8.1.8. Note that Z^X is an internal hom-object if and only if the map ev exhibits it as a right adjoint object to Z under the functor $-\otimes X\colon C \to C$. In particular, internal hom-objects are unique if they exist. We will often use $\underline{\mathrm{Hom}}(X, Y)$ as alternative notation for Y^X.

Lemma 8.1.9. *Let X and Y be objects and let* $\mathrm{ev}\colon Y \otimes X \to \mathbb{1}$ *be a morphism in C. Then the following conditions are equivalent:*

(1) The morphism ev *exhibits Y as a dual to X.*

(2) For every object Z of C, the morphism $\mathrm{id}_Z \otimes \mathrm{ev}\colon Z \otimes Y \otimes X \to Z$ *exhibits $Z \otimes Y$ as an internal hom-object Y^X.*

(3) *The functor $X \otimes - \colon C \to C$ admits a right adjoint $\underline{\mathrm{Hom}}(X,-)\colon C \to C$, and for every object Z of C the map*

$$Z \otimes Y \to \underline{\mathrm{Hom}}(X,Z)$$

induced by the morphism $\mathrm{id}\otimes\mathrm{ev}\colon Z \otimes Y \otimes X \to Z$ is an isomorphism in C.

Proof. The equivalence between (1) and (2) is a reformulation of Proposition 8.1.2. The equivalence between (2) and (3) is clear. □

We record some basic facts about duals.

Lemma 8.1.10. *Let $F\colon C \to D$ be a symmetric monoidal functor. For every dualizable object $X \in C$, the object $F(X)$ is again dualizable with dual $F(X^\vee)$.*

Proof. The symmetric monoidal functor F preserves the duality data. □

Lemma 8.1.11. *If $X, Y \in C$ are dualizable, then so is their tensor product $X \otimes Y$, with dual $X^\vee \otimes Y^\vee$.*

Proof. It is easy to verify that the composites

$$(X^\vee \otimes Y^\vee) \otimes (X \otimes Y) \cong (X^\vee \otimes X) \otimes (Y^\vee) \otimes Y \xrightarrow{\mathrm{ev}_X \otimes \mathrm{ev}_Y} \mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$$

and

$$\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} \xrightarrow{\mathrm{coev}_X \otimes \mathrm{coev}_Y} (X \otimes X^\vee) \otimes (Y \otimes Y^\vee) \cong (X \otimes Y) \otimes (X^\vee \otimes Y^\vee)$$

satisfy the triangle identities, and thus provide the desired duality data. □

There is often an a priori candidate for the dual of X, sometimes called the *weak dual*:

Definition 8.1.12. Assume that C admits internal hom-objects. We define the *internal duality functor* $D\colon C^{\mathrm{op}} \to C$ as

$$D(-) := \underline{\mathrm{Hom}}(-, \mathbb{1}_C)\colon C^{\mathrm{op}} \to C.$$

If X is dualizable, the evaluation map $\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1}$ induces an isomorphism $X^\vee \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, \mathbb{1}) = D(X)$.

Corollary 8.1.13. *Let X be an object of C for which the internal hom functor $\underline{\mathrm{Hom}}(X,-)\colon C \to C$ exists. Then X is dualizable if and only if for every $Z \in C$ the natural map $Z \otimes D(X) \to \underline{\mathrm{Hom}}(X,Z)$ adjoint to the composite $Z \otimes D(X) \otimes X \xrightarrow{Z\otimes\mathrm{ev}} Z \otimes \mathbb{1}_C \simeq Z$ is an equivalence.*

Corollary 8.1.14. *If $X \in C$ is dualizable, then for all $Y \in C$ the map $D(Y) \otimes D(X) \to D(X \otimes Y)$ adjoint to $D(Y) \otimes Y \otimes D(X) \otimes X \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{ev}_X} \mathbb{1}$ is an equivalence.*

Proof. This follows from the previous corollary by taking $Z = D(Y)$. □

There are other possible characterizations of strongly dualizable objects that are sometimes used in the literature, for example by Lewis, May, and Steinberger [LMS86, Definition 1.1].

Lemma 8.1.15. *For an object $X \in C$, let $\nu\colon X \otimes D(X) \to \underline{\mathrm{Hom}}(X,X)$ be the map adjoint to the composite $X \otimes D(X) \otimes X \xrightarrow{X \otimes \mathrm{ev}} X \otimes \mathbb{1} \simeq X$. Then the following are equivalent:*

(1) The object X is dualizable;

(2) There exists a coevaluation map $\mathrm{coev} : \mathbb{1} \to X \otimes D(X)$ *such that the diagram*

$$\begin{tikzcd} \mathbb{1}_C \arrow[r, "\mathrm{coev}"] \arrow[d, "\widetilde{\mathrm{id}}"'] & X \otimes D(X) \arrow[dl, "\nu"] \\ \underline{\mathrm{Hom}}(X,X) & \end{tikzcd}$$

commutes, where the left vertical map is adjoint to the identity $X \to X$.

Proof. First assume that (1) is satisfied. By Proposition 8.1.2, the dual of X is equivalent to $D(X)$, with evaluation map given by $\mathrm{ev}\colon D(X) \otimes X \to \mathbb{1}$. Let $\eta\colon \mathbb{1} \to X \otimes D(X)$ denote the corresponding coevaluation map. Then the diagram in part (2) translates to the first triangle identity of Definition 8.1.1, hence is commutative.

Conversely, assume that (2) is satisfied. We will prove that the maps $\mathrm{ev}\colon D(X) \otimes X \to \mathbb{1}$ and $\eta\colon \mathbb{1} \to X \otimes D(X)$ exhibit $D(X)$ as a dual to X. As before, the first triangle identity is equivalent to the diagram from (2), hence it remains to prove the second triangle identity:

$$\begin{tikzcd} & D(X) \otimes X \otimes D(X) \arrow[dr, "\mathrm{ev}_X \otimes D(X)"] & \\ D(X) \arrow[ur, "D(X) \otimes \eta"] \arrow[rr, equal] & & D(X). \end{tikzcd}$$

By adjunction, this is the same as a homotopy between $\mathrm{ev}\colon D(X) \otimes X \to \mathbb{1}$ and the composite

$$D(X) \otimes X \xrightarrow{D(X) \otimes \eta \otimes X} D(X) \otimes X \otimes D(X) \otimes X \xrightarrow{\mathrm{ev} \otimes D(X) \otimes X} D(X) \otimes X \xrightarrow{\mathrm{ev}} X,$$

which follows from the first triangle identity. □

8.1.1 More on internal homs

Lemma 8.1.16. *Let $Y \in C$ and assume that an internal hom-object $\underline{\mathrm{Hom}}(X,Y)$ exists for every $X \in C$. Then the functor $\underline{\mathrm{Hom}}(-,Y)\colon C^{\mathrm{op}} \to C$ admits a left adjoint given by $\underline{\mathrm{Hom}}(-,Y)\colon C \to C^{\mathrm{op}}$.*

Proof. For an object $X \in C$, consider the map

$$\eta \colon X \to \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X,Y),Y)$$

adjoint to the evaluation map $X \otimes \underline{\mathrm{Hom}}(X,Y) \simeq \underline{\mathrm{Hom}}(X,Y) \otimes X \to Y$. This map is natural in X. We claim that it is the unit transformation of an adjunction between $\underline{\mathrm{Hom}}(-,Y) \colon C^{\mathrm{op}} \to C$ and $\underline{\mathrm{Hom}}(-,Y) \colon C \to C^{\mathrm{op}}$. To see this, we need to show that for every object $Z \in C$, the composite

$$\mathrm{Hom}_C(Z,\underline{\mathrm{Hom}}(X,Y)) \xrightarrow{\underline{\mathrm{Hom}}(-,Y)} \mathrm{Hom}_C(\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(X,Y),Y),\underline{\mathrm{Hom}}(Z,Y)) \xrightarrow{-\circ\eta} \mathrm{Hom}_C(X,\underline{\mathrm{Hom}}(Z,Y))$$

is an equivalence. But this composite can be seen to be homotopic to the composite

$$\mathrm{Hom}_C(Z,\underline{\mathrm{Hom}}(X,Y)) \simeq \mathrm{Hom}_C(Z\otimes X,Y) \simeq \mathrm{Hom}_C(X\otimes Z,Y) \simeq \mathrm{Hom}_C(X,\underline{\mathrm{Hom}}(Z,Y)),$$

finishing the proof. □

Corollary 8.1.17. *If C admits internal homs, the functor $\underline{\mathrm{Hom}}(-,Y) \colon C^{\mathrm{op}} \to C$ sends colimits in C to limits.*

Lemma 8.1.18. *Assume that C admits internal hom-objects. Then for every object $X \in C$, the adjunction between $-\otimes X$ and $\underline{\mathrm{Hom}}(X,-)$ leads to an internal adjunction: for $Y,Z \in C$ there is a natural equivalence*

$$\underline{\mathrm{Hom}}(X\otimes Y,Z) \simeq \underline{\mathrm{Hom}}(X,\underline{\mathrm{Hom}}(Y,Z)).$$

Proof. This follows from the Yoneda-lemma, and the observation that for every fourth object $W \in C$ there are natural equivalences

$$\begin{aligned}\mathrm{Hom}_C(W,\underline{\mathrm{Hom}}(X\otimes Y,Z)) &\simeq \mathrm{Hom}_C(W\otimes X\otimes Y,Z)\\ &\simeq \mathrm{Hom}_C(W\otimes X,\underline{\mathrm{Hom}}(Y,Z))\\ &\simeq \mathrm{Hom}_C(W,\underline{\mathrm{Hom}}(X,\underline{\mathrm{Hom}}(Y,Z))).\end{aligned}$$ □

Remark 8.1.19. In a similar fashion, one can show that the composition maps $\circ\colon \mathrm{Hom}_C(Y,Z)\times \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_C(X,Z)$ admit internal versions.

8.1.2 Spanier-Whitehead duality

We will now be interested in duality for spectra, known as *Spanier-Whitehead duality*.

Lemma 8.1.20. *Let X and E be spectra and assume that X is dualizable. Then there are isomorphisms*

$$E_*(X^\vee) \cong E^{-*}(X).$$

Proof. This follows from the sequence

$$E_*(X^\vee) \cong \pi_*(E \otimes X^\vee) \cong \pi_*(\mathrm{map}(X, E)) \cong E^{-*}(X). \qquad \square$$

Definition 8.1.21. A *stably symmetric monoidal ∞-category* is a symmetric monoidal ∞-category C which is stable and whose tensor product $- \otimes -\colon C \to C \to C$ is exact in both variables.

Lemma 8.1.22. *Let C be a stably symmetric monoidal ∞-category. Then the collection of dualizable objects forms a thick subcategory of C, i.e. it is a stable subcategory closed under retracts.*

Proof. We start by showing that the collection of objects $X \in C$ for which the internal hom $\underline{\mathrm{Hom}}(X, -)\colon C \to C$ exists is a thick subcategory of C. To see that it is closed under finite colimits, consider a colimit $X \simeq \mathrm{colim}_i X_i$. We then observe that the functor $\underline{\mathrm{Hom}}(X, -) \simeq \lim_{i \in I^{\mathrm{op}}} \underline{\mathrm{Hom}}(X_i, -)$ is the required right adjoint. It is clear that this subcategory is closed under shifts and retracts, showing the claim.

Restricting attention to objects X for which $\underline{\mathrm{Hom}}(X, -)$ exists, recall from Corollary 8.1.13 that such an object X is dualizable if and only if the natural map

$$Z \otimes D(X) \to \underline{\mathrm{Hom}}(X, Z)$$

is an equivalence for every $Z \in C$. Since both sides are exact functors in X, it follows that the collection of objects X for which it is an equivalence forms a stable subcategory of C. It is also closed under retracts since equivalences in C are closed under retracts. $\square$

Corollary 8.1.23. *The subcategory of* Sp *spanned by the dualizable spectra contains the sphere spectrum $\mathbb{S}$ and is closed under finite colimits. In particular every finite spectrum is dualizable.* $\square$

Corollary 8.1.24. *Let X be a finite anima. Then the spectrum $\mathbb{S}[X]$ is dualizable.*

Proof. The collection of animae X for which $\mathbb{S}[X]$ is dualizable is closed under finite colimits and contains $* \in \mathrm{An}$; consequently it contains all finite animae. $\square$

8.2 Atiyah duality

Let M be a compact smooth manifold of dimension n. Since M can be given the structure of a finite CW-complex, it follows from Corollary 8.1.24 that its suspension spectrum $\mathbb{S}[X]$ is dualizable. In this section we will give an explicit description of its dual, due to Atiyah [Ati61]: there is an equivalence

$$D(\mathbb{S}[M]) \simeq \mathrm{Th}(-T_M)$$

between the dual of $\mathbb{S}[M]$ and the Thom spectrum of the *negative stable tangent bundle* $-T_M$. Let us start by explaining how to assign a Thom spectrum to every virtual vector bundle over M.

Construction 8.2.1. Consider a virtual vector bundle $E = [E_1] - [E_2]$ over M. Recall from Example 7.2.3 that the vector bundles E_1 and E_2 give rise to stable spherical fibrations $\Sigma^\infty(\xi_{E_1})$ and $\Sigma^\infty(\xi_{E_2})$. Since $\Sigma^\infty(\xi_{E_2})$ is pointwise given by a shift of the sphere spectrum, it is in particular an invertible object in Sp, and hence we may form its inverse $\Sigma^\infty(\xi_{E_2})^{-1}$. We then define

$$\xi_E := \Sigma^\infty(\xi_{E_1}) \otimes \Sigma^\infty(\xi_{E_2})^{-1} \quad \in \quad \mathrm{Fun}(\Pi_\infty(M), \mathrm{Sp}).$$

This is again a stable spherical fibration, and in particular we may consider its Thom spectrum

$$\mathrm{Th}(E) \quad := \quad \mathrm{Th}(\xi_E) \quad = \quad \mathrm{colim}(\xi_E \colon \Pi_\infty(M) \to \mathrm{Sp}).$$

To get a better understanding of the Thom spectrum $\mathrm{Th}(-T_M)$, we will now show that we can write it as $\Sigma^{\infty-N}(\mathrm{Th}(\nu_\varphi))$ for a suitable vector bundle ν_φ over M. The main ingredient is the following simple observation:

Lemma 8.2.2. *Let M be a compact smooth n-manifold. Then there exists a natural number N and a smooth embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$.*

Proof. By compactness, we may cover M by finitely many open subsets $U_i \subseteq M$ each of which are diffeomorphic to $\mathbb{R}^n$. Using bump functions, this leads to smooth maps $\varphi_i\colon M \to \mathbb{R}^n$ whose restriction to U_i is an embedding. The resulting map $(\varphi_i)_{i=1}^k\colon M \to (\mathbb{R}^n)^k = \mathbb{R}^{nk}$ is then the desired smooth embedding. □

Remark 8.2.3. It is the content of the *Whitney embedding theorem* that N can be chosen to be $2n$ in the previous lemma.

For the remainder of this section, we will fix once and for all a smooth embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$. We will denote by $\eta := \nu_\varphi \to M$ the normal bundle of this embedding, i.e., the cokernel of the map $T_M \to \varphi^*(T_{\mathbb{R}^N}) \cong \underline{\mathbb{R}^N}$ of vector bundles over M:

$$T_M \to \underline{\mathbb{R}^N} \to \nu.$$

By choosing a metric on the vector bundle $\underline{\mathbb{R}^N}$ one obtains an orthogonal decomposition

$$\underline{\mathbb{R}^N} \cong T_M \oplus \nu.$$

In particular, the virtual vector bundle $-T_M$ can be rewritten as the virtual vector bundle $[\nu] - [\underline{\mathbb{R}^N}]$, and in particular we get

$$\mathrm{Th}(-T_M) \cong \mathrm{Th}(\Sigma^{\infty-N}(\nu)) \cong \Sigma^{\infty-N}(\mathrm{Th}(\nu)) \tag{8.1}$$

We will now construct the duality data exhibiting $\mathrm{Th}(-T_M)$ as dual to $\mathbb{S}[M]$, i.e. we will construct maps

$$\mathrm{ev}\colon\ \mathbb{S}[M]\otimes\mathrm{Th}(-T_M)\to\mathbb{S} \qquad \text{and} \qquad \mathrm{coev}\colon\ \mathbb{S}\to\mathrm{Th}(-T_M)\otimes\mathbb{S}[M]$$

satisfying the triangle identities. In light of the equivalences $\mathbb{S}[M]=\Sigma^\infty(M_+)$ and $\mathrm{Th}(-T_M)\simeq\Sigma^{\infty-N}(\mathrm{Th}(\nu))$, such maps are in fact equivalently described by maps of the form

$$\Sigma^\infty(M_+)\otimes\Sigma^\infty(\mathrm{Th}(\nu))\to\Sigma^\infty(S^N) \qquad \text{and} \qquad \Sigma^\infty(S^N)\to\Sigma^\infty(\nu)\otimes\Sigma^\infty(M_+).$$

We will in fact be able to produce such maps already on the level of topological spaces:

$$M_+\wedge\mathrm{Th}(\nu)\to S^N \qquad \text{and} \qquad S^N\to\mathrm{Th}(\nu)\wedge M_+.$$

For this, we will consider two types of maps between Thom spaces: one type obtained from the functoriality of Thom spaces, and another type obtained from the so-called *Pontryagin-Thom collapse map*.

Construction 8.2.4. Let $f\colon X\to Y$ be a continuous map of topological spaces. Let $p\colon E\to Y$ be a vector bundle over Y, and let $f^*(E)\to X$ be its pullback bundle. By applying functoriality of the Thom space construction (Construction 7.1.9) to the commutative diagram

$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

we obtain a continuous map

$$f_*\colon\ \mathrm{Th}(f^*E)\to\mathrm{Th}(E).$$

Example 8.2.5. Consider the terminal map $p\colon M\to\mathrm{pt}$, and consider the trivial vector bundle $E=\mathbb{R}^N\to\mathrm{pt}$. Then the pullback bundle $p^*(\mathbb{R}^N)$ is the trivial vector bundle $\underline{\mathbb{R}^N}$ over M, and we obtain a map

$$p_*\colon\ \Sigma^N(M_+)\overset{7.1.6}{\cong}\mathrm{Th}(\underline{\mathbb{R}^N})\xrightarrow{p_*}\mathrm{Th}(\mathbb{R}^N)\overset{7.1.6}{\cong}S^N.$$

Example 8.2.6. Consider the diagonal map $\Delta\colon\ M\to M\times M$, and let $E=\eta\times\underline{0}$ be the product bundle over $M\times M$, where ν is the normal bundle of our fixed embedding $\varphi\colon\ M\hookrightarrow\mathbb{R}^N$ and $\underline{0}$ is the zero bundle. Observe that the pullback bundle $\Delta^*(E)=\Delta^*(\nu\times\underline{0})$ is isomorphic to the direct sum $\nu\oplus\underline{0}$ of these two bundles, and hence is isomorphic to ν itself. The above construction thus produces a map

$$\Delta_*\colon\ \mathrm{Th}(\nu)\to\mathrm{Th}(\nu\times\underline{0})\overset{7.1.7}{\cong}\mathrm{Th}(\nu)\wedge\mathrm{Th}(\underline{0})\cong\mathrm{Th}(\nu)\wedge M_+$$

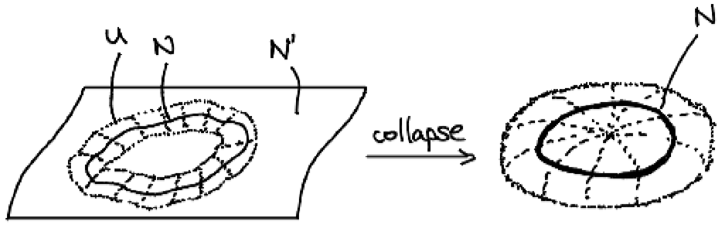


Figure 8.1: Illustration of the Pontryagin-Thom collapse map.

Construction 8.2.7 (Pontryagin-Thom collapse map)**.** Let $i\colon N \hookrightarrow N'$ be a smooth embedding of smooth manifolds, and let $\nu(i)$ denote its normal bundle. Recall that a *tubular neighborhood* of N in N' is an open submanifold $U \subseteq N'$ containing N together with a diffeomorphism $\varphi\colon \nu(i) \xrightarrow{\cong} U$ which on the zero section $s_0\colon N \hookrightarrow \nu(i)$ restricts to the inclusion $N \hookrightarrow U$. If N is compact, there always exists a tubular neighborhood.

Given a tubular neighborhood U for i, φ induces a homeomorphism

$$\mathrm{Th}(\nu(i)) = D(\nu(i))/S(\nu(i)) \cong \varphi(D(\nu(i)))/\varphi(D^{\circ}(\nu(i))) \cong N'/(N' \setminus \varphi(D^{\circ}(\nu(i)))).$$

We define the *Pontryagin-Thom collapse map* to be the composite

$$\mathrm{PT}(i)\colon N' \to N'/(N' \setminus \varphi(D^{\circ}(\nu(i)))) \cong \mathrm{Th}(\nu(i)),$$

where the first map is the quotient map.

Construction 8.2.8 (Twisted Pontryagin-Thom collapse map)**.** The Pontryagin-Thom collapse map also has a 'twisted' version, where in addition to the tubular neighborhood U of N in N' we are also given a vector bundle E over N'. It takes the form

$$\mathrm{PT}(i,E)\colon \mathrm{Th}(E) \to \mathrm{Th}(\nu(i) \oplus i^*E).$$

To define it, we consider the composite embedding $N \overset{i}{\hookrightarrow} N' \overset{s_0}{\hookrightarrow} E$, where the second map is the zero-section. The normal bundle of this embedding is the direct sum $\nu(i) \oplus E|_N$, hence we obtain a Pontryagin-Thom collapse map $\mathrm{PT}(s_0 i)\colon E \to \mathrm{Th}(\nu(i) \oplus i^*E)$. Since this map collapses everything outside of an open neighborhood of U in E, this map uniquely extends to the Thom space of E, providing the desired map $\mathrm{PT}(i,E)$.

Example 8.2.9. Consider our fixed embedding $\varphi\colon M \hookrightarrow \mathbb{R}^N$. Since M is compact, it is contained in a compact subspace of $\mathbb{R}^N$, and hence the composite $M \hookrightarrow \mathbb{R}^N \hookrightarrow S^N$ into the N-sphere is still a smooth embedding. The Pontryagin-Thom construction thus provides a map

$$\mathrm{PT}(\varphi)\colon S^N \to \mathrm{Th}(\nu).$$

Example 8.2.10. Consider the diagonal embedding $\Delta\colon M \hookrightarrow M\times M$. Its normal bundle is isomorphic to the tangent bundle T_M of M. As a twist, we use the external direct sum $E = \underline{0}\times\nu$ over $M\times M$, where $\nu = \nu(\varphi)$ as before. The pullback of E along Δ is isomorphic to ν, as in Example 8.2.6, and we see that there is a trivialization $\nu(\Delta)\oplus\Delta^*(E)\cong T_M\oplus\nu\cong\underline{\mathbb{R}^N}$. The twisted Pontryagin-Thom map in this case takes the form

$$\mathrm{PT}(\Delta,\underline{0}\times\nu)\colon M_+\wedge\mathrm{Th}(\nu)\cong\mathrm{Th}(\underline{0})\wedge\mathrm{Th}(\nu)\cong\mathrm{Th}(\underline{0}\oplus\nu)\to\mathrm{Th}(\underline{\mathbb{R}^N})\cong\Sigma^N(M_+).$$

Combining the four examples, we obtain our evaluation and coevaluation maps:

$$\mathrm{ev}\colon M_+\wedge\mathrm{Th}(\nu)\xrightarrow{\mathrm{PT}(\Delta,\underline{0}\times\nu)}\Sigma^N(M_+)\xrightarrow{p_*}\Sigma^N(S^0)=S^N$$
$$\mathrm{coev}\colon S^N\xrightarrow{\mathrm{PT}(\varphi)}\mathrm{Th}(\nu)\xrightarrow{\Delta_*}\mathrm{Th}(\nu)\wedge M_+.$$

To finish the proof of Atiyah duality, it remains to show that these maps satisfy the (space-level analogues of the) triangle identities. The main ingredient or this will be the following observation, which essentially says that under some transversality assumptions the Pontryagin-Thom construction behaves well under pullback, and that in this case we 'commute' the Pontryagin-Thom construction with the pushforward functoriality.

Observation 8.2.11. Let $f\colon M_1\to M_2$ be a smooth map between smooth manifolds. Let $i_2\colon N_2\hookrightarrow M_2$ be a smooth embedding, for convenience thought of as an inclusion $N_2\subseteq M_2$. Assume that f is *transversal* to N_2, in the sense that for all $x\in f^{-1}(N_2)$ we have

$$df(T_xM_1)+T_{f(x)}N_2=T_{f(x)}M_2.$$

Then the preimage $N_1 := f^{-1}(N_2)\subseteq M_1$ is again a smooth submanifold, by the implicit function theorem, and the normal bundle of the inclusion $i_1\colon N_1\hookrightarrow M_1$ is the pullback along $f\colon N_1\to N_2$ of the normal bundle of $i_2\colon N_2\hookrightarrow M_2$:

$$\nu(i_1)\cong f^*\nu(i_2).$$

Furthermore, there exists a small enough tubular neighborhood U_2 of N_2 such that its preimage $f^{-1}(U_2)$ is a tubular neighborhood of N_1. With these choices, we see that the following diagram commutes:

$$\begin{tikzcd}
M_1 \arrow[rr, "f"] \arrow[d, "\mathrm{PT}(i_1)"'] & & M_2 \arrow[d, "\mathrm{PT}(i_2)"] \\
\mathrm{Th}(\nu(i_1)) \arrow[r, "\cong"] & \mathrm{Th}(f^*\nu(i_2)) \arrow[r, "f_*"] & \mathrm{Th}(\nu(i_2)).
\end{tikzcd}$$

More generally, if we are also given a vector bundle E over M_2, then the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Th}(f^*E) \arrow[rr, "f_*"] \arrow[d, "\mathrm{PT}(i_1,f^*E)"'] & & \mathrm{Th}(E) \arrow[d, "\mathrm{PT}(i_2,E)"] \\
\mathrm{Th}(\nu(i_1)\oplus f^*E) \arrow[r, "\cong"] & \mathrm{Th}(f^*(\nu(i_2)\oplus E)) \arrow[r, "f_*"] & \mathrm{Th}(\nu(i_2)\oplus E).
\end{tikzcd}$$

We are finally ready to prove Atiyah duality.

Theorem 8.2.12 (Atiyah duality). *Let M be a compact smooth manifold, and let $\varphi\colon M \hookrightarrow \mathbb{R}^N$ be a smooth embedding with chosen tubular neighborhood U. Then the composite*

$$\begin{aligned}\mathbb{S}[M] \otimes \mathrm{Th}(-T_M) &\overset{(8.1)}{\cong} \mathbb{S}[M] \otimes \Sigma^{\infty-N}\mathrm{Th}(\nu) \cong \Sigma^{\infty-N}(M_+ \wedge \mathrm{Th}(\nu)) \\ &\xrightarrow{\Sigma^{\infty-N}\mathrm{PT}(\Delta,\underline{0}\times\nu)} \Sigma^{\infty-N}\Sigma^N(M_+) \\ &\xrightarrow{p_*} \Sigma^{\infty-N}\Sigma^N(S^0) \cong \mathbb{S}\end{aligned}$$

exhibits the Thom spectrum $\mathrm{Th}(-T_M)$ of the negative tangent bundle of M as the dual of the unreduced suspension spectrum of M.

Proof. Observe that the map we wrote is obtained by applying $\Sigma^{\infty-N}(-)\colon \mathrm{Top}_* \to \mathrm{An}_* \to \mathrm{Sp}$ to the following composite:

$$\mathrm{ev}\colon M_+ \wedge \mathrm{Th}(\nu) \xrightarrow{\mathrm{PT}(\Delta,\underline{0}\times\nu)} \Sigma^N(M_+) \xrightarrow{p_*} \Sigma^N(S^0) = S^N.$$

For the associated coevaluation map, we similarly apply $\Sigma^{\infty-N}(-)$ to the following composite:

$$\mathrm{coev}\colon S^N \xrightarrow{\mathrm{PT}(\varphi)} \mathrm{Th}(\nu) \xrightarrow{\Delta_*} \mathrm{Th}(\nu) \wedge M_+.$$

To see that the triangle identities are satisfied, we may do so already at the level of topological spaces: we must show that the two composites

$$S^n \wedge \mathrm{Th}(\nu) \xrightarrow{\mathrm{coev}\wedge\mathrm{id}} \mathrm{Th}(\nu) \wedge M_+ \wedge \mathrm{Th}(\nu) \xrightarrow{\mathrm{id}\wedge\mathrm{ev}} \mathrm{Th}(\nu) \wedge S^N$$

and

$$M_+ \wedge S^N \xrightarrow{\mathrm{id}\wedge\mathrm{coev}} M_+ \wedge \mathrm{Th}(\nu) \wedge M_+ \xrightarrow{\mathrm{ev}\wedge\mathrm{id}} S^N \wedge M_+$$

are homotopic to the respective identities. We will start with the former. Consider the following pullback square:

$$\begin{tikzcd} M \arrow[r, hook, "\Delta"] \arrow[d, hook, "\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \times M \arrow[d, hook, "\mathrm{id}\times\Delta"] \\ M \times M \arrow[r, hook, "\Delta\times\mathrm{id}"] & M \times M \times M. \end{tikzcd}$$

Applying Observation 8.2.11 to this square (i.e. we take $M_2 := M \times M \times M$, $N_2 := M \times M$, $i_2 = \mathrm{id}\times\Delta$, $M_1 := M \times M$, $N_1 := M$, $f = \Delta \times \mathrm{id}$ and $E = \underline{0} \times \underline{0} \times \nu$) gives the following commutative diagram:

$$\begin{tikzcd} \mathrm{Th}(\nu) \wedge \mathrm{Th}(\nu) \arrow[r, "(\Delta\times\mathrm{id})_*"] \arrow[d, "\mathrm{PT}(\Delta,\nu\times\nu)"'] & \mathrm{Th}(\nu) \wedge M_+ \wedge \mathrm{Th}(\nu) \arrow[d, "\mathrm{PT}(\mathrm{id}\times\Delta,\nu\times\underline{0}\times\nu)"] \\ \Sigma^N \mathrm{Th}(\nu) \arrow[r, "\Delta_*"] & \mathrm{Th}(\nu) \wedge \Sigma^N(M_+). \end{tikzcd}$$

Unwinding the definitions of $\mathrm{coev} \wedge \mathrm{id}$ and $\mathrm{id} \wedge \mathrm{ev}$, we then get the following commutative diagram:

$$\begin{tikzcd}
S^N \wedge \mathrm{Th}(\nu) \arrow[rr, "\mathrm{coev}\wedge\mathrm{id}"] \arrow[dr, "\mathrm{PT}(\varphi\times\mathrm{id})\,\mathrm{PT}(\varphi)\wedge\mathrm{id}"'] & & \mathrm{Th}(\nu) \wedge M_+ \wedge \mathrm{Th}(\nu) \arrow[rr, "\mathrm{id}\wedge\mathrm{ev}"] \arrow[dr, "\mathrm{id}\wedge\mathrm{PT}(\Delta,\underline{0}\times\nu)=\mathrm{PT}(\mathrm{id}\times\Delta,\nu\times\underline{0}\times\nu)"] & & \mathrm{Th}(\nu) \wedge S^N \\
& \mathrm{Th}(\nu) \wedge \mathrm{Th}(\nu) \arrow[ur, "\Delta_*\wedge\mathrm{id}=(\Delta\times\mathrm{id})_*"] \arrow[dr, "\mathrm{PT}(\Delta,\nu\times\nu)"'] & & \mathrm{Th}(\nu) \wedge \Sigma^N(M_+) \arrow[ur, "\mathrm{id}\wedge p_*=(\mathrm{id}\times p)_*"'] & \\
& & \Sigma^N \mathrm{Th}(\nu). \arrow[ur, "\Delta_*"'] & &
\end{tikzcd}$$

Now, the left diagonal composite is the Pontryagin-Thom collapse map associated to the composite

$$M \overset{\Delta}{\hookrightarrow} M \times M \xhookrightarrow{\varphi\times\mathrm{id}} S^N \times M.$$

But this embedding is isotopic, via the straight-line isotopy in $\mathbb{R}^N \subseteq S^N$, to the embedding $(0,\mathrm{id})\colon M \hookrightarrow \mathbb{R}^N \times M \subseteq S^N \times M$. It follows that the Pontryagin Thom collapse map $\mathrm{PT}((\varphi \times \mathrm{id}) \times \Delta)$ is homotopic to the Pontryagin-Thom collapse map for $(0,\mathrm{id})$. Using $U = \mathbb{R}^N \times M$ as tubular neighbhorhood for this inclusion, we see that the latter is simply the identity on $S^N \wedge \mathrm{Th}(\nu)$.

The right diagonal is the pushforward functoriality of Thom spaces applied to the composite

$$M \overset{\Delta}{\hookrightarrow} M \times M \xrightarrow{\mathrm{id}\times p} M \times \mathrm{pt} = M.$$

But this composite is simply the identity on M, hence it induces the identity on $S^N \wedge \mathrm{Th}(\nu)$. We conclude that the composite $(\mathrm{id} \wedge \mathrm{ev}) \circ (\mathrm{coev} \wedge \mathrm{id})$ is homotopic to the identity on $S^N \wedge \mathrm{Th}(\nu)$, proving the first triangle identity.

For the second triangle identity, we apply Observation 8.2.11 to the same commutative square as before (that is, the same choices of M_1, M_2, N_1 and N_2) but this time using $E =$??. We then obtain a commutative diagram as follows:

$$\begin{tikzcd}
M_+ \wedge S^N \arrow[rr, "\mathrm{id}\wedge\mathrm{coev}"] \arrow[dr, "\mathrm{PT}(\mathrm{id}\times\varphi)"'] & & M_+ \wedge \mathrm{Th}(\nu) \wedge M_+ \arrow[rr, "\mathrm{ev}\wedge\mathrm{id}"] \arrow[dr, "\mathrm{PT}(\Delta\times\mathrm{id})"] & & S^N \wedge M_+ \\
& M_+ \wedge \mathrm{Th}(\nu) \arrow[ur, "(\mathrm{id}\times\Delta)_*"] \arrow[dr, "\mathrm{PT}(\Delta,\underline{0}\times\nu)"'] & & \Sigma^N(M_+) \wedge M_+ \arrow[ur, "(p\times\mathrm{id})_*"] & \\
& & \Sigma^N(M_+). \arrow[ur, "\Delta_*"] & &
\end{tikzcd}$$

One argues as before that both the two diagonal composites are homotopic to the identity, so that also the top composite $(\mathrm{ev} \wedge \mathrm{id}) \circ (\mathrm{id} \wedge \mathrm{coev})$ is homotopic to the identity. This finishes the proof. $\square$

8.3 Poincaré duality

Poincaré duality for compact smooth manifolds is an immediate consequence of Atiyah duality. Recall that Poincaré duality says that when M is a compact orientable smooth n-manifold (without boundary), then for each $k \in \mathbb{Z}$ there is an isomorphism

$$- \cap [M] \colon H^{n-k}(M) \xrightarrow{\cong} H_k(M).$$

Many readers will have enountered this statement in their algebraic topology courses, where it is usually proved in quite a down-to-earth way, based on manipulations of simplices in the singular complex of M and using a couple of reduction arguments. This proof makes it look like Poincaré duality is intrinsically a statement about singular (co)homology. However, this is not the case: there is a more general version of Poincaré duality that works for an arbitrary cohomology theory E^*:

Theorem 8.3.1 (Poincaré duality)**.** *Let E be a ring spectrum, and let M be a compact smooth n-manifold which is E-orientable, in the sense that its tangent bundle T_M comes equipped with an E-orientation. Then there is an isomorphism of spectra*

$$\mathbb{S}[M] \otimes E \cong \mathrm{map}(\mathbb{S}[M], E[n]).$$

In particular, for every $k \in \mathbb{Z}$ there is an isomorphism

$$E_k(M) \cong E^{n-k}(M).$$

Proof. By Atiyah duality, the dual of the spectrum $\mathbb{S}[M]$ in Sp is $\mathrm{Th}(-T_M)$, and in particular we obtain an isomorphism of spectra

$$\mathbb{S}[M] \otimes E \cong \mathrm{map}(\mathrm{Th}(-T_M), E).$$

Since T_M is E-orientable, we have by Theorem 7.3.11 an isomorphism $\mathrm{Th}(-T_M) \otimes E \cong (\mathbb{S}[M] \otimes E)[-n]$ in Mod_E, and it follows that

$$\begin{aligned}
\mathrm{map}(\mathrm{Th}(-T_M), E) &\cong \mathrm{map}_E(\mathrm{Th}(-T_M) \otimes E, E) \\
&\cong \mathrm{map}_E((\mathbb{S}[M] \otimes E)[-n], E) \\
&\cong \mathrm{map}(\mathbb{S}[M][-n], E) \\
&\cong \mathrm{map}(\mathbb{S}[M], E[n]).
\end{aligned}$$

Combining these two isomorphisms thus gives the first claim. The second claim immediately follows by applying $\pi_k(-)$ to both sides. $\square$

A Even more ∞-category theory

When I wrote in Section 2.6 that it would cover all the ‘remaining constructions of ∞-category theory that we need’, this turned out to be too optimistic: at various points during the rest of the lectures, we needed additional basic ∞-categorical concepts that I had not explicitly introduced. To keep these notes well organized, I have collected those extra definitions and results here in this Appendix. Due to time constraints, we did not discuss the contents of this appendix in full detail during the lectures, but I have included additional details here for those of you who want to dive deeper into the theory.

Remark. A note about the exam: you are expected to have a working understanding of ∞-category theory, but you do not need to learn all the material in this appendix in detail. What’s most important is that you develop an intuition for which statements from ordinary category theory naturally carry over to ∞-categories, and that you’re comfortable with the universal properties of all the ∞-categorical constructions we introduced.

Warning. Various arguments presented in this appendix depend on statements taken as black boxes in Section 2.6, most notably the Yoneda lemma, Theorem 2.6.42. Proving those statements would require some of the notions introduced in this appendix, giving rise to some circularity. I made this choice for pragmatic and pedagogical considerations, prioritizing accessibility of the theory’s applications over a strictly linear development of its foundations.

A.1 (Full) Subcategories

In Section 2.6.5 we introduced a way of associating a subcategory $\langle M\rangle_C$ to a given collection of morphisms M in an ∞-category C which is closed under composition. In this section we will see that a functor $F\colon D \to C$ is of the form $\langle M\rangle_C \hookrightarrow C$ for some M if and only if F is a monomorphism (Corollary A.1.2). Furthermore, we will see that F exhibits D as a *full* subcategory if and only if F is fully faithful (Corollary A.1.3).

Proposition A.1.1. *A functor $F\colon C \to D$ is an equivalence if and only if the induced functor*

$$F_*\colon \operatorname{Map}([1],C) \to \operatorname{Map}([1],D)$$

is an equivalence.

Proof. If F is an equivalence, it is clear that F_* is an equivalence. Conversely, assume that F_* is an equivalence. By Theorem 2.6.9, it will suffice to show that F is essentially surjective and fully faithful. Consider the following commutative diagram:

$$\begin{tikzcd} C^{\simeq} \arrow[r, "p_{[1]}^*"] \arrow[d, "F^{\simeq}"'] & \operatorname{Map}([1],C) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "F_*"] & C^{\simeq} \arrow[d, "F^{\simeq}"] \\ D^{\simeq} \arrow[r, "p_{[1]}^*"] & \operatorname{Map}([1],D) \arrow[r, "\mathrm{ev}_0"] & D^{\simeq}. \end{tikzcd}$$

Since the horizontal composites are the identity, it follows that $F^{\simeq}$ is a retract of F_*, and thus in particular it is an equivalence. In particular, f is essentially surjective. Now consider the following commutative diagram:

$$\begin{tikzcd} \operatorname{Map}([1],C) \arrow[r, "f_*"] \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] & \operatorname{Map}([1],D) \arrow[d, "(\mathrm{ev}_0{,}\mathrm{ev}_1)"'] \\ C^{\simeq} \times C^{\simeq} \arrow[r, "f^{\simeq}\times f^{\simeq}"] & D^{\simeq} \times D^{\simeq}. \end{tikzcd}$$

Since the top and bottom maps are equivalences, it follows that the square is a pullback square. In particular, the induced maps on fibers over all objects $(x,y) \in C^{\simeq} \times C^{\simeq}$ is an equivalence. But this induced map on fibers is precisely the map

$$\operatorname{Hom}_C(x,y) \to \operatorname{Hom}_D(Fx,Fy),$$

showing that F is fully faithful. □

Corollary A.1.2. *A functor $F\colon C \to D$ is a monomorphism if and only if the induced functor*

$$F_*\colon M := \operatorname{Map}([1],C) \to \operatorname{Map}([1],D)$$

is an embedding and F induces an equivalence $C \xrightarrow{\sim} \langle M \rangle_D$.

Proof. If F_* is an embedding, then the functor

$$\operatorname{Map}([1],C) \to \operatorname{Map}([1],C) \times_{\operatorname{Map}([1],D)} \operatorname{Map}([1],C) \simeq \operatorname{Map}([1],C \times_D C)$$

is an equivalence, hence by Proposition A.1.1 so is $C \to C \times_D C$, showing that F is an embedding. Conversely, if F is an embedding, so is F_*. To see that $C \to \langle M \rangle_D$ is an equivalence, it suffices by Proposition A.1.1 to show that $\operatorname{Map}([1],C) \to \operatorname{Map}([1],\langle M \rangle_D)$ is an equivalence, which is true since both sides are M. □

Corollary A.1.3. *A functor $f\colon C \to D$ is fully faithful if and only if the induced functor*

$$f^{\simeq}\colon \Gamma := C^{\simeq} \to D^{\simeq}$$

is an embedding and f induces an equivalence $C \xrightarrow{\sim} \langle M_\Gamma \rangle_D$.

Proof. See [Cis+24, Proposition 6.4.2]. □

Definition A.1.4. A *subcategory* of an ∞-category D is an ∞-category C equipped with a monomorphism $C \hookrightarrow D$. It is a *full subcategory* if this monomorphism is fully faithful.

Warning A.1.5. In the literature this is often called a *replete* subcategory: if an object X is contained in the subcategory, and $X \xrightarrow{\sim} Y$ is an isomorphism in D, then also Y and the isomorphism are contained in the subcategory. When working 'model-independently' as we are doing in this course, it does not really make sense to talk about non-replete subcategories, and it would go against the fundamental principle of homotopy theory. So for this reason I'm dropping the word 'replete'.

A.2 Adjunctions

In this section we will introduce the important notion of adjunctions in the setting of ∞-categories. We start in Appendix A.2.1 with the definition and various equivalent characterizations. In Appendix A.2.2 we discuss some important examples of adjunctions: the path component functor $\pi_0\colon \mathrm{An} \to \mathrm{Set}$ as a left adjoint to the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ and the homotopy category functor $\mathrm{h}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}$ left adjoint to the inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$. In Appendix A.2.3 we discuss the relation between (co)limits and adjunctions, and deduce various useful consequences about limits and colimits. In Appendix A.2.5 we discuss Kan extensions and their pointwise formulas. In Appendix A.2.6 we briefly introduce the notion of cofinality in ∞-category theory.

A.2.1 Definition and characterizations

Recall from Definition 2.6.44 the definition of an adjunction between ∞-categories:

Definition A.2.1. Let $F\colon C \to D$ and $G\colon D \to C$ be two functors. An *adjunction* between C and D consists of a natural isomorphism

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}} \times D \to \mathrm{An}$. We often write $F \dashv G$ to indicate that such an isomorphism has been given.

Proposition A.2.2. *Consider two functors $F\colon C \to D$ and $G\colon D \to C$. The following data are equivalent:*

(1) The data of an ajunction $F \dashv G$;

(2) The data of a natural transformation $\eta\colon \mathrm{id}_C \to GF$, called the unit, such that for all objects X of C and Y of D the induced map

$$\mathrm{Hom}_D(FX, Y) \xrightarrow{G} \mathrm{Hom}_C(GFX, GY) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X, GY)$$

is an equivalence;

(3) The data of a natural transformation $\varepsilon\colon FG \to \mathrm{id}_D$, called the counit, such that for all objects X of C and Y of D the induced map

$$\mathrm{Hom}_C(X, GY) \xrightarrow{F} \mathrm{Hom}_D(FX, FGY) \xrightarrow{\varepsilon_Y\circ-} \mathrm{Hom}_D(FX, Y)$$

is an equivalence;

(4) The data of both a unit $\eta\colon \mathrm{id} \to GF$ and a counit $\varepsilon\colon FG \to \mathrm{id}$ together with commutative triangles as follows, called the triangle identities:[1]

$$\begin{tikzcd} & FGF \arrow[dr, "\varepsilon F"] & \\ F \arrow[ur, "F\eta"] \arrow[rr, equal] & & F \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} & GFG \arrow[dr, "G\varepsilon"] & \\ G \arrow[ur, "\eta G"] \arrow[rr, equal] & & G. \end{tikzcd}$$

Proof sketch. First assume (1), so that there is a natural equivalence $\mathrm{Hom}_D(FX, Y) \simeq \mathrm{Hom}_C(X, GY)$. If we fix $X \in C$, then by the Yoneda lemma the natural transformation $\mathrm{Hom}_D(FX, -) \xrightarrow{\sim} \mathrm{Hom}_D(X, G(-))$ corresponds to a map $\eta_X\colon X \to GFX$: we obtain this map by taking $Y = FX$ and letting η_X be the map that under this equivalence corresponds to the identity map $\mathrm{id}_{FX}\colon FX \to FX$. Conversely, the map η_X determines the transformation as the composite

$$\mathrm{Hom}_D(FX, -) \xrightarrow{G} \mathrm{Hom}_C(GFX, G(-)) \xrightarrow{-\circ\eta_X} \mathrm{Hom}_C(X, -).$$

The naturality of the equivalence in X corresponds to the naturality of $\eta\colon \mathrm{id}_C \to GF$, showing the equivalence between (1) and (2).

The equivalence between (1) and (3) is proved similarly, where now the counit map $\varepsilon_Y\colon FGY \to Y$ corresponds to the identity map $\mathrm{id}_{GY}\colon GY \to GY$.

[1]The details aren't super relevant here, but strictly speaking the data of an adjunction should only contain either the first or the second triangle identity, while for the other triangle we only demand the *existence*; see e.g. [Lur24, Tag 02F4].

To show that these conditions are also equivalent to (4), assume first that (1)-(3) are satisfied. For the first triangle identity, observe that by definition the map $\eta_X\colon X \to GFX$ is a preimage of $\mathrm{id}_{FX}\colon FX \to FX$ under the equivalence

$$\mathrm{Hom}_C(X, GFX) \xrightarrow{F} \mathrm{Hom}_D(FX, FGFX) \xrightarrow{\varepsilon_{FX}\circ -} \mathrm{Hom}_D(FX, FX).$$

This precisely says that the composite $\varepsilon_{FX} \circ F\eta_X$ is equivalent to id_{FX}, naturally in X, which is the first triangle identity. The second triangle identity similarly follows from the fact that the morphism $\varepsilon_Y\colon FGY \to Y$ is a preimage of $\mathrm{id}_{GY}\colon GY \to GY$ under the equivalence

$$\mathrm{Hom}_D(FGY, Y) \xrightarrow{G} \mathrm{Hom}_C(GFGY, GY) \xrightarrow{-\circ\eta_{GY}} \mathrm{Hom}_C(GY, GY).$$

Finally, assume that (4) is satisfied. An easy check shows that in this case the map defined in (2) is inverse to the map defined in (3), producing the desired natural equivalence $\mathrm{Hom}_D(FX, Y) \simeq \mathrm{Hom}_C(X, GY)$. □

It turns out that we may detect left/right functors 'objectwise':

Definition A.2.3. Let $F\colon C \to D$ be a functor and let Y be an object of D.

(1) An object Z of C is called a *right adjoint object to Y under F* if it comes equipped with a map $\varepsilon_Y\colon FZ \to Y$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(X, Z) \xrightarrow{F} \mathrm{Hom}_D(FX, FZ) \xrightarrow{\varepsilon_Y\circ -} \mathrm{Hom}_D(FX, Y)$$

is an equivalence of animae.

(2) Dually, Z is called a *left adjoint object to Y under F* if it comes equipped with a map $\eta_Y\colon Y \to FZ$ such that for every other object X of C the induced map

$$\mathrm{Hom}_C(Z, X) \xrightarrow{F} \mathrm{Hom}_D(FZ, FX) \xrightarrow{-\circ\eta_Y} \mathrm{Hom}_D(Y, FZ)$$

is an equivalence of animae.

Lemma A.2.4. *Consider a functor $F\colon C \to D$ and let $D_R \subseteq D$ be a full subcategory. Assume that for every $Y \in D_R$ there exists a right adjoint object Z to Y under F. Then these right adjoint objects assemble into a functor $G\colon D \to C$ which comes equipped with a natural isomorphism*

$$\mathrm{Hom}_D(F(-), -) \cong \mathrm{Hom}_C(-, G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$.

Dually, if $D_L \subseteq D$ is a subcategory such that every $Y \in D_L$ admits a left adjoint object under F, then these left adjoint objects assemble into a functor $L\colon D_L \to C$ that comes equipped with a natural isomorphism

$$\mathrm{Hom}_C(L(-),-) \cong \mathrm{Hom}_D(-,F(-))$$

of functors $D_L^{\mathrm{op}} \times C \to \mathrm{An}$.

Proof. We will prove the claim for right adjoints; the claim for left adjoints is dual. The assumption on F is that for every Y in D_R the functor $\mathrm{Hom}_D(F(-),Y)\colon C^{\mathrm{op}} \to \mathrm{An}$ is representable, in the sense that it lies in the image of the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{Fun}(C^{\mathrm{op}},\mathrm{An})$. Since the Yoneda embedding is fully faithful, this means there exists a functor

$$G\colon D_R \to C$$

making the following diagram commute:

$$\begin{tikzcd} & C \arrow[d, hook, "Y"] \\ D_R \arrow[ur, dashed, "G"] \arrow[r, "Y \mapsto \mathrm{Hom}_D(F(-),Y)"'] & \mathrm{Fun}(C^{\mathrm{op}},\mathrm{An}). \end{tikzcd}$$

Here the bottom functor is the one obtained from currying the functor $\mathrm{Hom}_D(F(-),-)\colon C^{\mathrm{op}} \times D_R \to \mathrm{An}$. Since $Y(X) = \mathrm{Hom}_C(-,X)$, this commutative triangle precisely encodes a natural isomorphism

$$\mathrm{Hom}_D(F(-),-) \cong \mathrm{Hom}_C(-,G(-))$$

of functors $C^{\mathrm{op}} \times D_R \to \mathrm{An}$, as desired. □

Corollary A.2.5. *A functor $F\colon C \to D$ admits a right adjoint if and only if every object of D admits a right adjoint object under F. Dually, F admits a left adjoint if and only if every object of D admits a left adjoint object under F.*

Proof. We again only prove the claim about right adjoints. If a right adjoint $G\colon D \to C$ exists, then by Proposition A.2.2 we see that the object $Z := GY$ together with the counit map $\varepsilon_Y\colon FGY \to Y$ provides a right adjoint object to Y under F. Conversely, if all the right adjoint objects exists, then we may apply Lemma A.2.4 with $D = D_R$ to obtain a functor $G\colon D \to C$ together with a natural isomorphism

$$\mathrm{Hom}_C(F(-),-) \cong \mathrm{Hom}_D(-,G(-)),$$

which means that G is a right adjoint to F. □

Every adjunction automatically induces new adjunctions at the level of functor categories:

Lemma A.2.6. *Let $F \dashv G$ be an adjunction of ∞-categories. Then:*

(1) For every ∞-category E the functors

$$G^*\colon \operatorname{Fun}(C,E) \rightleftarrows \operatorname{Fun}(D,E) : F^*$$

form an adjunction, with unit and counit given by

$$G^*F^* = (FG)^* \xrightarrow{\varepsilon^*} \mathrm{id}_D^* = \mathrm{id}_{\operatorname{Fun}(D,E)} \qquad \textit{and} \qquad \mathrm{id}_{\operatorname{Fun}(C,E)} = \mathrm{id}_C^* \xrightarrow{\eta^*} (GF)^* = F^*G^*.$$

(2) For every ∞-category E the functors

$$F_*\colon \operatorname{Fun}(E,C) \rightleftarrows \operatorname{Fun}(E,D) : G_*$$

form an adjunction, with unit and counit given by

$$F_*G_* = (FG)_* \xrightarrow{\varepsilon_*} (\mathrm{id}_D)_* = \mathrm{id}_{\operatorname{Fun}(D,E)} \qquad \textit{and} \qquad \mathrm{id}_{\operatorname{Fun}(C,E)} = (\mathrm{id}_C)_* \xrightarrow{\eta_*} (GF)_* = G_*F_*.$$

Proof. The triangle identities for ε^* and η^*, resp. ε_* and η_*, follow immediately from the triangle identities for ε and η. We leave the details to the reader. □

Every adjunction induces an equivalence on geometric realizations:

Lemma A.2.7. *Let $F\colon C \rightleftarrows G : G$ be an adjunction of ∞-categories. Then the induced maps $|F|\colon |C| \to |D|$ and $|G|\colon |D| \to |C|$ are inverse equivalences.*

Proof. Thinking of the counit $\varepsilon\colon FG \to \mathrm{id}_D$ as a functor $[1] \times D \to D$, it induces another functor

$$[1] \times |D| \to |[1]| \times |D| \overset{2.6.23}{\simeq} |[1] \times D| \xrightarrow{|\varepsilon|} |D|,$$

i.e. a natural transformation $|F||G| = |FG| \to |\mathrm{id}_D| = \mathrm{id}_{|D|}$. Since $|D|$ is an anima, this natural transformation is automatically a natural isomorphism by Theorem 2.6.10. Similarly, the unit $\eta\colon \mathrm{id}_C \to GF$ induces a natural transformation $|\eta|\colon \mathrm{id}_{|C|} \to |G||F|$. This shows that $|G|$ is inverse to $|F|$, as desired. □

A.2.2 Path components and homotopy categories

I want to record some relations between sets and animae. Because these lie in the intersection between the axiomatic approach and the classical approach I will refer to them as ‘facts’.

Fact A.2.8. Every set may be regarded as a discrete groupoid and hence as a discrete ∞-groupoid/anima. The inclusion

$$\mathrm{Set} \hookrightarrow \mathrm{An}$$

is fully faithful and admits a left adjoint

$$\pi_0\colon \mathrm{An} \to \mathrm{Set}.$$

We call $\pi_0(X)$ the *set of path components* of the anima X.

Definition A.2.9. We say an anima X is *connected* if the set $\pi_0(X)$ has a single element.

We will often abuse terminology and refer to an anima as a *set* if it is in the essential image of the inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$.

Definition A.2.10 (Path components of an anima)**.** For an anima X, the unit of the adjunction from Fact A.2.8 gives a map of animae

$$X \to \pi_0(X).$$

Since $\pi_0(X)$ is a disjoint union of points, this decomposes X as a disjoint union of individual path componens: for $[x] \in \pi_0(X)$ we define the *path component of X at* $[x]$ as the pullback

$$\begin{tikzcd} X_{[x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ * \arrow[r, "{[x]}"] & \pi_0(X). \end{tikzcd}$$

Fact A.2.11. An ∞-category C is equivalent to an ordinary category if and only if the Hom animae $\mathrm{Hom}_C(X,Y)$ is a set for all objects X and Y of C.

From now on we will drop the distinction between ordinary categories and ∞-categories whose Hom animae are sets.

Definition A.2.12. We denote by

$$\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty$$

the full subcategory spanned by the ordinary categories. The objects and morphisms of Cat are ordinary categories and ordinary functors. For categories C and D, the Hom anima $\mathrm{Hom}_{\mathrm{Cat}_1}(C,D)$ is the groupoid $\mathrm{Fun}(C,D)^{\simeq}$ whose objects are the functors $C \to D$ and whose isomorphisms are the natural isomorphisms $F \cong G$.

Fact A.2.13. The inclusion $\mathrm{Cat}_1 \hookrightarrow \mathrm{Cat}_\infty$ admits a left adjoint

$$\mathrm{h}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_1$$

called the *homotopy category functor*. Given an ∞-category C, the homotopy category $\mathrm{h}C$ has the same objects as C, but has Hom sets given by the set of path components of the Hom anima of C:

$$\mathrm{Hom}_{\mathrm{h}C}(X,Y) \cong \pi_0(\mathrm{Hom}_C(X,Y)).$$

Lemma A.2.14. *Let C be an ordinary category and let W be a collection of morphisms in C. Then the homotopy category* $\mathrm{h}(C[W^{-1}])$ *of the ∞-categorical localization $C[W^{-1}]$ of C at the morphisms in W has the universal property of the 1-categorical localization.*

Proof. For any ordinary category D we have an equivalence $\mathrm{Fun}(\mathrm{h}(C[W^{-1}]),D) \xrightarrow{\sim} \mathrm{Fun}(C[W^{-1}],D)$, so $\mathrm{h}(C[W^{-1}])$ simply inherits its universal property from $C[W^{-1}]$. □

Corollary A.2.15. *The homotopy category* hAn *of the ∞-category of animae is equivalent to the homotopy category* $\mathrm{h}\mathcal{S}$ *of spaces, defined in Convention 1.2.3:*

$$\mathrm{hAn} \simeq \mathrm{h}\mathcal{S}.$$

Similarly we have an equivalence $\mathrm{hAn}_* \simeq \mathrm{h}\mathcal{S}_*$.

Proof. By Theorem 2.6.38 we have $\mathrm{An} \simeq \mathrm{CW}[\{\text{homotopy equivalences}\}^{-1}]$, and hence hAn is the 1-categorical localization of the category of CW-complexes at the homotopy equivalences. But this is in essence precisely how we defined $\mathrm{h}\mathcal{S}$. □

A.2.3 (Co)limits and adjunctions

Just like for ordinary categories, we may describe the existence of (co)limits in terms of adjunctions:

Lemma A.2.16. *Let I and C be ∞-categories and let $F\colon I \to C$ be a functor.*

(1) An object W in C equipped with a cone $\varepsilon_F\colon \mathrm{const}_W \to F$ is a limit of F in C, in the sense of Definition 2.6.24, if and only if ε_F exhibits W as a right adjoint object to F under the functor $\mathrm{const}\colon C \to \mathrm{Fun}(I,C)$, in the sense of Definition A.2.3.

(2) Dually, an object W equipped with a cocone $\eta_F\colon F \to \mathrm{const}_W$ is a colimit of F in C if and only if it exhibits W as a left adjoint object to F under $\mathrm{const}\colon C \to \mathrm{Fun}(I,C)$.

Proof. This is simply a matter of unwinding definitions: the condition in (1) that W is a right adjoint object to F under const means that for every other object X of C the composite

$$\mathrm{Hom}_C(X,W) \xrightarrow{\mathrm{const}} \mathrm{Nat}(\mathrm{const}_X,\mathrm{const}_W) \xrightarrow{\varepsilon} \mathrm{Nat}(\mathrm{const}_X,F)$$

is an equivalence, where we recall that we write $\mathrm{Nat}(-,-)$ for the Hom anima in $\mathrm{Fun}(I,C)$. But this is precisely the universal property of the limit from Definition 2.6.24! □

Corollary A.2.17. *Let I and C be ∞-categories. Then C admits all I-indexed limits if and only if the functor*

$$\mathrm{const}\colon C \to \mathrm{Fun}(I,C), \qquad W \mapsto \mathrm{const}_W$$

admits a right adjoint

$$\lim_I \colon \operatorname{Fun}(I, C) \to C.$$

Dually, C admits all I-indexed colimits if and only if const: $C \to \operatorname{Fun}(I,C)$ *admits a left adjoint*

$$\operatorname{colim}_I \colon \operatorname{Fun}(I, C) \to C.$$

Proof. In light of Lemma A.2.16 this is an immediate consequence of the pointwise criterion for adjoints from Corollary A.2.5. □

As a particular consequence of the previous corollary, we see that limit cones and colimit cocones are fully functorial in the functor $F \colon I \to C$: the transformations

$$\mathrm{const}_{\lim F} \to F \qquad \text{and} \qquad F \to \mathrm{const}_{\operatorname{colim} F}$$

are given by the counit and unit, respectively, of the adjunctions const $\dashv \lim_I$ and $\operatorname{colim}_I \dashv$ const.

Remark A.2.18. For an object x of C, it follows from Corollary A.2.17 that X is terminal if and only if the functor

$$x \colon * \to C$$

is right adjoint to the functor $p_C \colon C \to *$. Dually, x is initial if and only if it is *left* adjoint to $p_C \colon C \to *$.

Lemma A.2.19. *Let i be an initial object of an ∞-category I. Then every ∞-category C admits I-indexed limits, and the limit functor is given by evaluation at i:*

$$\lim_I = \mathrm{ev}_i \colon \operatorname{Fun}(I, C) \to C.$$

Dually, if i is terminal in I then C admits I-indexed colimits which are given by evaluating at i.

Proof. By Corollary A.2.17 it suffices to show that ev_i defines a right adjoint to const: $C \to \operatorname{Fun}(I, C)$. This adjunction may be obtained by considering the adjunction $p_I \colon I \rightleftarrows * : i$ and passing to functor categories using Lemma A.2.6.

(An alternative proof would be to cite Lemma A.2.41.) □

Lemma A.2.20. *Let $F \colon C \to D$ be a functor and let I be an ∞-category such that both C and D admit I-indexed limits. Then F preserves I-indexed limits if and only if the 'canonical' natural transformation*

$$F(\lim_I(-)) \to \lim_I(F_*(-))$$

of functors $\operatorname{Fun}(I, C) \to D$ *is a natural isomorphism.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} C \arrow[r, "F"] \arrow[d, "\mathrm{const}"'] & D \arrow[d, "\mathrm{const}"] \\ \mathrm{Fun}(I,C) \arrow[r, "F_*"] & \mathrm{Fun}(I,D). \end{tikzcd}$$

The 'canonical' transformation above is the one that under the adjunction const $\dashv \lim_I$ corresponds to the transformation

$$\mathrm{const}_{F(\lim_I(-))} \cong F_*(\mathrm{const}_{\lim_I(-)}) \to F_*(-)$$

induced by the counit $\mathrm{const}_{\lim_I(X)} \to X$. By Theorem 2.6.10 we may check that this map is an isomorphism for every fixed diagram $X\colon I \to C$. But now we observe that the map $F(\lim_I(X)) \to \lim_I(F_*(-))$ is an isomorphism if and only if the associated cone $\mathrm{const}_{F(\lim_I(X))} \to F_*(X) = F \circ X$ is a limit cone, giving the claim. □

Lemma A.2.21. *Let $F\colon C \rightleftarrows D : G$ be an adjunction of ∞-categories. Then F preserves all colimits that exist in C and G preserves all limits that exist in D.*

Proof. We prove that F preserves colimits; the proof for G preserving limits is dual. Let $X\colon I \to C$ be a diagram that admits a colimit in C, and let $\eta\colon X \to \mathrm{const}_{\mathrm{colim}_I X}$ denote the associated colimit cone. We need to show that the cocone

$$F(\eta)\colon F \circ X \to F(\mathrm{const}_{\mathrm{colim}_I X}) = \mathrm{const}_{F(\mathrm{colim}_I X)}$$

is a colimit cone in D. By definition, this means that for every other object W in D the map

$$\mathrm{Hom}_D(F(\mathrm{const}_{\mathrm{colim}_I X}), W) \to \mathrm{Nat}(F \circ X, \mathrm{const}_W)$$

is an equivalence of animae. But under the adjunction $F \dashv G$ and the induced adjunction $F \circ - \dashv G \circ -$ from Lemma A.2.6, this map is equivalent to the map

$$\mathrm{Hom}_C(\mathrm{const}_{\mathrm{colim}_I X}, G(W)) \to \mathrm{Nat}(X, G \circ \mathrm{const}_W) = \mathrm{Nat}(X, \mathrm{const}_{G(W)}).$$

Since the latter map is an equivalence by the universal property of the colimit $\mathrm{colim}_I X$, this finishes the proof. □

As a consequence, we see that limits commute with limits and that colimits commute with colimits.

Corollary A.2.22. *If C admits I-indexed limits, then* $\lim_I\colon \mathrm{Fun}(I,C) \to C$ *preserves all limits that exist in* $\mathrm{Fun}(I,C)$*. If C admits I-indexed colimits, then* $\mathrm{colim}_I\colon \mathrm{Fun}(I,C) \to C$ *preserves all colimits that exist in* $\mathrm{Fun}(I,C)$.

Lemma A.2.23. *Consider ∞-categories C and I, and assume that C admits all I-indexed limits (resp. colimits).*

(1) For every ∞-category E, the functor category $\mathrm{Fun}(E,C)$ admits I-indexed limits (resp. colimits).

(2) For every functor $E' \to E$, the restriction functor $\mathrm{Fun}(E,C) \to \mathrm{Fun}(E',C)$ preserves I-indexed limits (resp. colimits).

(3) In particular, for every object X of E the functor

$$\mathrm{ev}_X \colon \mathrm{Fun}(E,C) \to C$$

preserves I-indexed limits (resp. colimits).

Proof. We treat the case for limits; the case for colimits is dual. By Corollary A.2.17 there is an adjunction

$$\mathrm{const}\colon C \rightleftarrows \mathrm{Fun}(I,C) : \lim_I .$$

By Lemma A.2.6 this adjunction induces an adjunction on functor categories of the form

$$\mathrm{Fun}(E,C) \rightleftarrows \mathrm{Fun}(E,\mathrm{Fun}(I,C)) \simeq \mathrm{Fun}(I,\mathrm{Fun}(E,C)).$$

By applying Corollary A.2.17 another time this says that $\mathrm{Fun}(E,C)$ admits I-indexed limits. Observe that the limit-functor for $\mathrm{Fun}(E,C)$ is given as the composite

$$\mathrm{Fun}(I,\mathrm{Fun}(E,C)) \simeq \mathrm{Fun}(E,\mathrm{Fun}(I,C)) \xrightarrow{(\lim_I)_*} \mathrm{Fun}(E,C).$$

This description makes it clear that precomposition with any functor $E' \to E$ preserves I-limits. $\square$

As a consequence, we get the following result on colimits along product categories:

Lemma A.2.24. *Let I, J and C be ∞-categories such that C admits all I-indexed and J-indexed colimits. Then C admits all $(I\times J)$-indexed colimits, and for a functor $F\colon I\times J \to C$ we have*

$$\mathrm{colim}_{(i,j)\in I\times J} F(i,j) \cong \mathrm{colim}_{i\in I}\, \mathrm{colim}_{j\in J} F(i,j).$$

Proof. The adjunction $\mathrm{colim}_J\colon \mathrm{Fun}(J,C) \rightleftarrows C : \mathrm{const}$ induces an adjunction

$$\mathrm{Fun}(I\times J) \simeq \mathrm{Fun}(I,\mathrm{Fun}(J,C)) \rightleftarrows \mathrm{Fun}(I,C)$$

by applying Lemma A.2.6. Composing this with the adjunction $\mathrm{colim}_I\colon \mathrm{Fun}(I,C) \rightleftarrows C : \mathrm{const}$ then gives the result. $\square$

A fact which will frequently be useful is that colimits along an indexing category I which itself may be written as a colimit may be computed as an iterated colimit:

Theorem A.2.25. *Let J be a small ∞-category and consider a J-indexed family of small ∞-categories $I_\bullet\colon J \to \mathrm{Cat}_\infty$. Define $I := \mathrm{colim}_j I_j$, and let $\varepsilon\colon I_\bullet \to \mathrm{const}_I$ denote the colimit cone, giving functors $\varepsilon_j\colon I_j \to I$. Let C be an ∞-category and let $F\colon I \to C$ be a functor.*

(1) Assume that the restricted diagram

$$F|_{I_j}\colon I_j \xrightarrow{\varepsilon_j} I \xrightarrow{F} C$$

admits a colimit for every $j \in J$. Then these colimits assemble into a functor

$$F'\colon J \to C, \qquad j \mapsto \mathrm{colim}(F_{I_j}\colon I_j \to C).$$

(2) In this case, the functor F admits a colimit in C if and only if the functor F' admits a colimit in C, and then we have

$$\mathrm{colim}_{i\in I} F(i) \cong \mathrm{colim}_{j\in J}\,\mathrm{colim}(F_{I_j}).$$

Proof. We do not have the techniques available to prove this result. Perhaps a proof will be given in Appendix A.3 at some point, after having introduced straightening/unstraightening. □

A.2.4 Slice categories

A useful categorical construction, which we will need in the next two subsections, is that of the *(relative) slice category*.

Definition A.2.26 (Slice category)**.** Let C be an ∞-category and let x be an object in C. We define the *slice categories $C_{/x}$* and $C_{x/}$, also known as the *over category* and the *under category*, respectively, via the following two pullback squares:

$$\begin{tikzcd} C_{/x} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_1"] \\ * \arrow[r, "x"] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C_{x/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}([1],C) \arrow[d, "\mathrm{ev}_0"] \\ * \arrow[r, "x"] & C. \end{tikzcd}$$

Note that an object of $C_{/x}$ is a pair $(y, f\colon y \to x)$ of an object y equipped with a morphism to x, and dually objects of $C_{x/}$ are pairs $(y, f\colon x \to y)$. As a result of Exercise 5.2 morphisms $(y,f) \to (y',f')$ in $C_{/x}$ may equivalently be encoded by commutative triangles of the form

$$\begin{tikzcd} y \arrow[rr, "g"] \arrow[dr, "f"'] & & y' \arrow[dl, "f'"] \\ & x, & \end{tikzcd}$$

and dually for morphisms in $C_{x/}$.

We denote by

$$s\colon C_{/x} \to C \qquad \text{and} \qquad t\colon C_{x/} \to C$$

the assignments $s(y, f\colon y \to x) = y$ and $t(y, f\colon x \to y) = y$, i.e. we first include the slice into the arrow category $\mathrm{Fun}([1], C)$ and then apply the source/target functor.

Lemma A.2.27. *Consider an object x of C and consider two objects (y, f) and (y', f') of the slice category $C_{/x}$. Then the Hom anima in $C_{/x}$ sits in a pullback square of the form*

$$\begin{tikzcd}
\mathrm{Hom}_{C_{/x}}((y,f),(y',f')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_C(y,y') \arrow[d, "f'\circ -"] \\
* \arrow[r, "f"] & \mathrm{Hom}_C(y,x).
\end{tikzcd}$$

A dual formula holds for the Hom animae of $C_{x/}$.

Proof. By Exercise 2.3.30, the Hom anima in $C_{/x}$ may be computed as the following pullback:

$$\begin{tikzcd}
\mathrm{Hom}_{C_{/x}}((y,f),(y',f')) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Nat}(f\colon y \to x, f'\colon y \to x) \arrow[d] \\
* \arrow[r, "\mathrm{id}_x"] & \mathrm{Hom}_C(x,x).
\end{tikzcd}$$

Using the equivalence $\mathrm{Fun}([1]\times[1], C) \simeq \mathrm{Fun}([2], C) \times_{\mathrm{Fun}([1],C)} \mathrm{Fun}([2], C)$ from the commutative square axiom, the top right Hom anima in $\mathrm{Fun}([1], C)$ sits in a pullback square as follows:

$$\begin{tikzcd}
\mathrm{Nat}(f\colon y \to x, f'\colon y \to x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_C(y,y') \arrow[d, "f'\circ -"] \\
\mathrm{Hom}_C(x,x) \arrow[r, "-\circ f"] & \mathrm{Hom}(y,x).
\end{tikzcd}$$

The claim thus follows by pasting pullback squares. □

Corollary A.2.28. *Let x be an object of C. Then the slice category $C_{/x}$ admits a terminal object (x, id_x), while $C_{x/}$ admits an initial object (x, id_x).*

Proof. The claim for $C_{x/}$ is immediate from Lemma A.2.27, since for every $(y, f) \in C_{/x}$ the induced map $\mathrm{id}_x \circ -\colon \mathrm{Hom}_C(y,x) \to \mathrm{Hom}_C(y,x)$ is an equivalence, hence its fiber over f is contractible. The claim for $C_{/x}$ is dual. □

Lemma A.2.29. *Let x be an object in an ∞-category C.*

(1) The object x is terminal in C if and only if the forgetful functor $s\colon C_{/x} \to C$ is an equivalence;

(2) *The object x is initial in C if and only if the forgetful functor $t\colon C_{x/} \to C$ is an equivalence.*

Proof. We only prove (1); the case of (2) is dual. If $C_{/x} \to C$ is an equivalence, then it follows from Corollary A.2.28 that x is terminal in x, as it is the image under this equivalence of the terminal object $(x, \mathrm{id}_x) \in C_{/x}$. So assume that x is a terminal object. By Remark A.2.18, this means that the functor $x\colon * \to C$ is a right adjoint to $p_C\colon * \to C$, and in particular there is a natural transformation $\varepsilon\colon \mathrm{id}_C \to \mathrm{const}_x$ satisfying the triangle identity. We may now construct a functor

$$\varphi\colon C \to C_{/x}, \qquad y \mapsto (y, \varepsilon_y\colon y \to x).$$

More precisely, we may think of ε as a functor $\overline{\varepsilon}\colon C \to \mathrm{Fun}([1], C)$, and since the target of ε is const_x we see that $\overline{\varepsilon}$ canonically factors through the slice $C_{/x}$. It is also clear from this definition that we have $s \circ \varphi \cong \mathrm{id}_C$. It remains to show that we also have $\varphi \circ s \cong \mathrm{id}_{C_{/x}}$.

The functor $\overline{\varepsilon}\colon C \to \mathrm{Fun}([1], C)$ induces a functor $\overline{\varepsilon}_*\colon \mathrm{Fun}([1], C) \to \mathrm{Fun}([1] \times [1], C)$, which informally speaking sends a morphism $f\colon y \to z$ to the 'naturality square' induced by ε:

$$\begin{tikzcd} y \arrow[r, "\varepsilon_y"] \arrow[d, "f"'] & x \arrow[d, equal] \\ z \arrow[r, "\varepsilon_z"'] & x. \end{tikzcd}$$

Note that for $z = x$, it follows from the triangle identities for the adjunction that the map $\varepsilon_x\colon x \to x$ is the identity map. So restricting the naturality square to objects of the form $(y, f) \in C_{/x}$, this square takes the form

$$\begin{tikzcd} y \arrow[r, "\varepsilon_y"] \arrow[d, "f"'] & x \arrow[d, equal] \\ x \arrow[r, equal] & x. \end{tikzcd}$$

This shows that the object (y, f) of $C_{/x}$ is naturally isomorphic to $(y, \varepsilon_y\colon y \to x)$, producing the desired natural isomorphism $\varphi \circ s \cong \mathrm{id}_{C/x}$. □

Definition A.2.30 (Relative slice category)**.** For a functor $F\colon C \to D$ and an object d of D, we define the *relative slice categories* $C_{/d}$ and $C_{d/}$ via the following two pullback squares:

$$\begin{tikzcd} C_{/d} \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_{/d} \arrow[d, "s"] \\ C \arrow[r, "F"] & D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C_{d/} \arrow[r] \arrow[d, "t"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_{d/} \arrow[d, "t"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

Objects of $C_{/d}$ consist of pairs $(y, f\colon F(y) \to d)$, and objects of $C_{d/}$ consist of pairs $(y, f\colon d \to F(y))$.

Remark A.2.31. These relative slice categories are sometimes also denoted by F/d and d/F, which makes their dependencies on F more transparent.

A.2.5 Kan extensions

Definition A.2.32. Let I, J and C be ∞-categories and let $\varphi\colon I \to J$ be a functor.

(1) Given a functor $F\colon I \to C$, a *left Kan extension of F along φ* is a left adjoint object to F under the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$. More explicitly, it is a functor $\varphi_!(F)\colon J \to C$ equipped with a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ such that for every other functor $G\colon J \to C$ the composite

$$\mathrm{Nat}(\varphi_!(F), G) \xrightarrow{\varphi^*} \mathrm{Nat}(\varphi^*\varphi_!(F), \varphi^*G) \xrightarrow{-\circ\eta_F} \mathrm{Nat}(F, \varphi^*G)$$

is an equivalence.

$$\begin{tikzcd} I \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\alpha"] & C \\ & J \arrow[ur, "G"'] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} I \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\Downarrow\eta_F"] & C \\ & J \arrow[ur, "\varphi_!(F)"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=15, "\overline{\alpha}"] & \end{tikzcd}$$

(2) Dually, a *right Kan extension of F along φ* is a right adjoint object to F under φ^*, i.e. a functor $\varphi_*(F)\colon J \to C$ equipped with a transformation $\eta_F\colon \varphi^*\varphi_*(F) \to F$ inducing equivalences

$$\mathrm{Nat}(G, \varphi_*(F)) \xrightarrow{\sim} \mathrm{Nat}(\varphi^*, G)$$

for all $G\colon J \to C$.

If every functor $F\colon I \to C$ admits a left Kan extension along φ, then by Corollary A.2.5 these left Kan extensions assemble into a left adjoint

$$\varphi_!\colon \mathrm{Fun}(I,C) \to \mathrm{Fun}(J,C)$$

to the restriction functor $\varphi^*\colon \mathrm{Fun}(J,C) \to \mathrm{Fun}(I,C)$, called the functor of *left Kan extension along φ*. Dually, if every functor F admits a *right* Kan extension along φ then φ^* admits a right adjoint

$$\varphi_*\colon \mathrm{Fun}(I,C) \to \mathrm{Fun}(J,C)$$

called *right Kan extension along φ*.

Observation A.2.33. Note that the left Kan extension of a functor $F\colon I \to C$ along the functor $p_C\colon C \to *$ is precisely a colimit of F. Dually, a right Kan extension of F along p_C is a limit of F.

The most useful and common form of Kan extensions are the *pointwise Kan extensions*: those that can be computed using explicit pointwise formulas. In practice, the only Kan extensions one cares about are the pointwise Kan extensions. To discuss these, we first need to introduce slice categories:

With this in place, we can come to the pointwise formulas for Kan extensions:

Theorem A.2.34. *Let $F\colon I \to C$ and $\varphi\colon I \to J$ be functors.*

(1) Assume that for every object j of J the composite functor

$$I_{/j} \xrightarrow{s} I \xrightarrow{F} C$$

admits a colimit in C. Then these colimits assemble into a functor

$$\varphi_!(F)\colon J \to C.$$

Furthermore, the maps

$$F(i) \simeq \mathrm{colim}_{i' \in I_{/i}} F(i') \to \mathrm{colim}_{i' \in I_{/\varphi(i)}} F(i') = \varphi_!(F)(i)$$

assemble into a natural transformation $\eta_F\colon F \to \varphi^*\varphi_!(F)$ exhibiting $\varphi_!(F)$ it as a left Kan extension of F along φ.

(2) Dually, if for every j in J the composite

$$I_{j/} \xrightarrow{t} I \xrightarrow{F} C$$

admits a limit in C, then these limits assemble into functor

$$\varphi_*(F)\colon J \to C$$

which is a right Kan extension to F.

Proof. Unfortunately we do not have the techniques yet to be able to prove this result. I may or may not type up a proof of this result in Appendix A.3 in the future. □

Definition A.2.35. If the condition in (1) is satisfied, we say that *F admits a pointwise left Kan extension along φ*, and if the condition in (2) is satisfied we say that *F admits a pointwise right Kan extension along φ*.

In general, the Kan extension of F might not actually be an 'extension', in the sense that the unit map $F \to \varphi^*(\varphi_! F)$ and counit map $\varphi^*(\varphi_* F) \to F$ may not be invertible. However, this *is* the case whenever φ is fully faithful, explaining the terminology. For this we need the following alternative characterization of fully faithful functors:

Lemma A.2.36. *Let $F\colon C \to D$ be a functor. Then the following conditions are equivalent:*

(1) The functor F is fully faithful;

(2) For every object y of C the commutative square

$$\begin{tikzcd} C_{/y} \arrow[r, "F"] \arrow[d, "s"'] & C_{/Fy} \arrow[d, "s"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

is a pullback square;

(3) For every object x of C the commutative square

$$\begin{tikzcd} C_{y/} \arrow[r, "F"] \arrow[d, "t"'] & C_{Fx/} \arrow[d, "t"] \\ C \arrow[r, "F"] & D \end{tikzcd}$$

is a pullback square.

Proof sketch. We prove the equivalence between (1) and (2); the equivalence between (1) and (3) is dual. The two vertical functors in the commutative square in (2) are *right fibrations*, in a sense to be explained below in Appendix A.3. One can prove that a functor between two right fibrations is an equivalence if and only if it induces equivalences on all fibers. Since the induced map on fibers is precisely the map $\mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(Fx,Fy)$, it follows that condition (2) is equivalent to fully faithfulness of F. □

Lemma A.2.37. *Assume that $\varphi\colon I \hookrightarrow J$ is a fully faithful functor, and let $F\colon I \to C$ be a functor.*

(1) Assume that F admits a pointwise left Kan extension $\varphi_!(F)$ along φ. Then the unit map $\eta_F\colon F \to \varphi^*\varphi_!(F)$ is a natural isomorphism.

(2) Dually, if F admits a pointwise right Kan extension $\varphi_*(F)$ along φ then the counit $\varepsilon_F\colon \varphi^*\varphi_*(F) \to F$ is a natural isomorphism.

Proof. We prove (1); the proof of (2) is dual. By Theorem 2.6.10 we may check that η_F is a pointwise isomorphism. For an object i of I, the map $F(i) \to \varphi^*\varphi_!(F)$ is induced by the map of slices

$$I_{/i} \to I_{/\varphi(i)}, \quad (i', f\colon i' \to i) \mapsto (i', F(f)\colon F(i') \to F(i)).$$

But since φ is fully faithful, this functor is an equivalence by Lemma A.2.36, and it follows that $\eta_F(i)$ is an isomorphism. □

A.2.6 Cofinal functors

We will briefly discuss the useful notion of *initial/final functors*:

Theorem A.2.38 (Joyal's version of Quillen's Theorem A)**.** *For a functor $\alpha\colon I \to J$, the following two conditions are equivalent:*

(1) A functor $F\colon J \to C$ has a colimit if and only if the functor $F \circ \alpha\colon I \to C$ has a colimit, and in this case the canonical map

$$\operatorname{colim}_I(F \circ \alpha) \to \operatorname{colim}_J F$$

is an isomorphism in C;

(2) For every object j of J, the relative slice category $I_{j/} = I \times_J J_{j/}$ is weakly contractible, meaning that its geometric realization $|I_{j/}|$ is contractible.

The dual result is also true, where in (1) we use limits and in (2) we use the under categories $I_{/j}$.

Proof. For a proof, see [Lur09b, Theorem 4.1.3.1]. □

Definition A.2.39. A functor α satisfying the conditions of the theorem is called *final*. We say that α is *initial* if $\alpha^{\mathrm{op}}\colon I^{\mathrm{op}} \to J^{\mathrm{op}}$ is final (i.e. each relative slice $I_{j/}$ is contractible, or equivalently limits of functors $F\colon J \to C$ may be computed as limits of $F \circ \alpha\colon I \to C$).

Warning A.2.40. The terminology regarding final functors is completely messed up in the literature: because of historic reasons, final functors are also called *co*final functors, where the 'co' roughly means 'jointly final' and has nothing to do with the usual meaning of 'co' in category theory. Nevertheless, some authors use the words 'final' and 'cofinal' for the two notions from Definition A.2.39, and there is no consensus as to which of the two words should refer to which of the two notions.

I think that the convention in Definition A.2.39 is easy to remember in light of the following lemma:

Lemma A.2.41. *Consider an object j of an ∞-category J. Then $j\colon * \to C$ is an initial functor if and only if j is an initial object, and it is a final functor if and only if j is a final (a.k.a. terminal) object of J.*

Proof. Given an object x of C, unwinding definitions reveals that the relative slice category of the functor $j\colon * \to C$ is equivalent to the Hom anima $\mathrm{Hom}_C(j,x)$. Since this is already an anima, it is equivalent to its geometric realization, and hence it is weakly contractible if and only if it is contractible. So we see that j is an initial functor if and only if the Hom anima $\mathrm{Hom}_C(j,x)$ is contractible for every object x of C, which by is the case by definition precisely if j is an initial object. $\square$

A common way of showing that the relative slice categories $I_{j/}$ or $I_{/j}$ are weakly contractible is by showing they admit an initial or terminal object:

Lemma A.2.42. *Let C be an ∞-category that admits an initial (resp. terminal) object $x\colon * \to C$. Then C is weakly contractible: $|C| \simeq *$.*

Proof. By Remark A.2.18 we have an adjunction $x\colon * \rightleftarrows C : p_C$, which by Lemma A.2.7 induces an equivalence $* \simeq |*| \simeq |C|$. $\square$

Corollary A.2.43. *If a functor $\alpha\colon I \to J$ admits a left adjoint, then α is a final functor. Dually, if it admits a right adjoint it is an initial functor.*

Proof. We need to show that for every object j the relative slice category $I_{j/}$ is weakly contractible. Left $\beta^L\colon J \to I$ be a left adjoint to α. Then the relative slice category $I_{j/}$ is equivalent to the slice category $I_{\beta(j)/}$ of I. But this admits an initial object given by $(\beta(j), \mathrm{id}_{\beta(j)})$, hence is weakly contractible by Lemma A.2.42. $\square$

A.2.7 Filtered and sifted ∞-categories

We introduce the notions of filtered and sifted ∞-categories, and record their basic properties.

Definition A.2.44. An ∞-category K is called *filtered* if for every finite diagram $F\colon I \to K$ in K there exists an object $W \in K$ with a cocone $F \to \mathrm{const}_W$; equivalently, F extends to a functor $\widetilde{F}\colon I^{\triangleright} \to K$.

Example A.2.45. Every directed poset is a filtered ∞-category. In particular, the poset $(\mathbb{N}, \leq)$ of natural numbers is filtered.

More generally, any category which is filtered in the ordinary sense is filtered as an ∞-category.

The following important characterization of filtered ∞-categories will be treated as a black box:

Theorem A.2.46 ([Lur09b, Proposition 5.3.3.3])**.** *An ∞-category K is filtered if and only if the functor* $\mathrm{colim}_K\colon \mathrm{Fun}(K, \mathrm{An}) \to \mathrm{An}$ *preserves finite limits.*

Let us list various consequences of this characterization:

Corollary A.2.47. *A filtered colimit* $\mathrm{colim}_K *$ *of terminal animae is again terminal.*

Proof. A terminal anima is a limit over the finite category $\emptyset$. □

Corollary A.2.48. *The functor* $\Omega\colon \mathrm{An}_* \to \mathrm{An}_*$ *preserves filtered colimits.*

Proof. By construction, this functor is given as the composite

$$\mathrm{An}_* \simeq \mathrm{An}_* \times_{\mathrm{An}_*} \mathrm{An}_* \hookrightarrow \mathrm{Fun}([1], \mathrm{An}_*) \times_{\mathrm{An}_*} \mathrm{Fun}([1], \mathrm{An}_*) \simeq \mathrm{Fun}(\lrcorner, \mathrm{An}_*) \xrightarrow{\lim} \mathrm{An}_*.$$

The last functor preserves filtered colimits by Theorem A.2.46. The inclusion $\mathrm{An}_* \simeq (\mathrm{An}_*)_* \to \mathrm{Fun}([1], \mathrm{An}_*)$ preserves filtered colimits by Corollary A.2.47. □

Corollary A.2.49. *For every* $n \geq 0$, *the functor* $\pi_n\colon \mathrm{An}_* \to \mathrm{Set}$ *preserves filtered colimits.*

Proof. We may factor this functor as follows:

$$\mathrm{An}_* \xrightarrow{\Omega^n} \mathrm{An}_* \xrightarrow{\mathrm{fgt}} \mathrm{An} \xrightarrow{\pi_0} \mathrm{Set}.$$

Since each of these functors preserves filtered colimits, the claim follows. □

Lemma A.2.50. *Every filtered* ∞*-category* K *is weakly contractible.*

Proof. Recall from [ref] that we may write K as a colimit $\mathrm{colim}_{k\in K} K_{/k}$ of its slice categories. Since $|-|\colon \mathrm{Cat}_\infty \to \mathrm{An}$ is a left adjoint and thus preserves colimits, we see that

$$|K| \simeq |\mathrm{colim}_{k\in K} K_{/k}| \simeq \mathrm{colim}_{k\in K} |K_{/k}| \overset{A.2.42}{\simeq} \mathrm{colim}_{k\in K} * \overset{A.2.47}{\simeq} *.$$

Here the second-to-last equivalence uses the fact that for each $k \in K$ the slice category $K_{/k}$ admits a terminal object by Corollary A.2.28, so that it is weakly contractible by Lemma A.2.42. □

Definition A.2.51. An ∞-category I is called *sifted* if it is nonempty and the diagonal functor $\Delta\colon I \to I \times I$ is final.

Example A.2.52. The opposite $\mathbf{\Delta}^{\mathrm{op}}$ of the simplex category is sifted by Exercise 4.4.4.

Lemma A.2.53. *Every filtered* ∞*-category is sifted.*

Lemma A.2.54. *Let* K *be a filtered* ∞*-category. We need to show that for every object* $(k,l) \in K \times K$ *the relative slice category* $K \times_{K\times K} (K\times K)_{(k,l)/}$ *is weakly contractible. We will show that this relative slice category is in fact a filtered* ∞*-category, so that Lemma A.2.50 implies the claim. To this end, consider a finite* ∞*-category* I *and a functor* $F\colon I \to$

$K \times_{K \times K} (K \times K)_{(k,l)/}$. *We may equivalently encode this diagram as a functor $P \to I$, where P is defined as the following pushout of ∞-categories:*

$$\begin{tikzcd} I \sqcup I \arrow[r, hook] \arrow[d, "\nabla"] & (I \sqcup I)^{\triangleleft} \arrow[d] \\ I \arrow[r] & P. \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note that P is again finite, so that by filteredness of K we obtain an extension $P^{\triangleright} \to K$. This extension then translates back into an extension $\widetilde{F}$ of F, finishing the proof.

Lemma A.2.55. *Let C be an ∞-category that admits finite products, and assume that for every object $X \in C$ the functor $X \times -\colon C \to C$ preserves colimits. Then the product functor $- \times -\colon C \times C \to C$ preserves sifted colimits.*

Proof. The proof is identical to that of Lemma 4.4.3 and will be omitted. □

A.2.8 Bousfield localizations

We introduce the notion of a Bousfield localization, and prove a useful characterization of them in Proposition A.2.63.

Definition A.2.56. A functor $L\colon C \to D$ is called a *Bousfield localization* if it admits a fully faithful right adjoint $R\colon D \hookrightarrow C$. In this case, we also call D a *reflective subcategory of C*.

A *Bousfield colocalization* is instead a functor $C \to D$ that admits a fully faithful *left* adjoint.

Example A.2.57 (Rationalization)**.** The inclusion $\mathrm{Vect}_{\mathbb{Q}} \hookrightarrow \mathrm{Ab}$ of rational vector spaces into abelian groups admits a left adjoint

$$\mathbb{Q} \otimes_{\mathbb{Z}} -\colon \mathrm{Ab} \to \mathrm{Vect}_{\mathbb{Q}},$$

called the *rationalization functor*. It is a Bousfield localization.

Example A.2.58 (Abelianization)**.** The inclusion $\mathrm{Ab} \hookrightarrow \mathrm{Grp}$ of abelian groups into all groups admits a left adjoint

$$(-)^{\mathrm{ab}}\colon \mathrm{Grp} \to \mathrm{Ab},$$

given by the *abelianization functor*.

Example A.2.59 (Group completion)**.** The inclusion $\mathrm{Grp} \hookrightarrow \mathrm{Mon}$ of groups into monoids admits a left adjoint

$$(-)^{\mathrm{grp}}\colon \mathrm{Mon} \to \mathrm{Grp}$$

called the (discrete) *group completion functor*.

Example A.2.60 (Path components)**.** The inclusion $\mathrm{Set} \hookrightarrow \mathrm{An}$ admits a left adjoint

$$\pi_0 \colon \mathrm{An} \to \mathrm{Set}$$

given by the set of path components.

Example A.2.61 (Geometric realization)**.** The inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$ has a left adjoint

$$|-| \colon \mathrm{Cat}_\infty \to \mathrm{An}$$

given by the geometric realization functor. It also admits a right adjoint

$$(-)^{\simeq} \colon \mathrm{Cat}_\infty \to \mathrm{An}$$

given by the groupoid core functor.

Example A.2.62 (Homotopy category)**.** The inclusion $\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\infty$ has a left adjoint

$$\mathrm{Ho}(-) \colon \mathrm{Cat}_\infty \to \mathrm{Cat}_1$$

given by the homotopy category functor.

Here is a useful criterion for checking that a full subcategory is a reflective subcategory:

Proposition A.2.63 (cf. [Lur09b, Proposition 5.2.7.4])**.** *Let $L \colon C \to C$ be a functor equipped with a natural transformation $\eta \colon \mathrm{id}_C \Rightarrow L$ such that both maps*

$$\eta_{Lx} \colon Lx \to LLx \qquad \textit{and} \qquad L\eta_x \colon Lx \to LLx$$

are isomorphisms for all $x \in C$. Then the functor $L \colon C \to \mathrm{im}(L)$ is left adjoint to the inclusion $\mathrm{im}(L) \hookrightarrow C$, with unit η. In particular, $L \colon C \to \mathrm{im}(L)$ is a Bousfield localisation.

Proof. We must show that for objects x and y in C, precomposition with $\eta_x \colon x \to Lx$ induces an equivalence of animae

$$- \circ \eta_x \colon \mathrm{Hom}_C(Lx, Ly) \xrightarrow{\sim} \mathrm{Hom}_C(x, Ly).$$

For this, it suffices to show that this map admits both a left and a right inverse:

- To obtain a left inverse, we use that the map $\eta_{Ly} \colon Ly \to LLy$ is an isomorphism by assumption on η. We claim that a left inverse is given by the composite

$$\mathrm{Hom}_C(x, Ly) \xrightarrow{L} \mathrm{Hom}_C(Lx, LLy) \xrightarrow{\eta_{Ly}^{-1} \circ -} \mathrm{Hom}_C(Lx, Ly).$$

Indeed, this follows from the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_C(x, Ly) \arrow[rr, "L"] \arrow[dr, "\eta_{Ly}\circ -"'] \arrow[ddrr, equal, bend right] & & \mathrm{Hom}_C(Lx, LLy) \arrow[dl, "-\circ\eta_x"] \arrow[dr, "\eta_{Ly}^{-1}\circ -"] & \\
& \mathrm{Hom}_C(x, LLy) \arrow[dr, "\eta_{Ly}^{-1}\circ -"] & & \mathrm{Hom}_C(Lx, Ly) \arrow[dl, "-\circ\eta_x"] \\
& & \mathrm{Hom}_C(x, Ly). &
\end{tikzcd}$$

Here the upper left triangle commutes since for any morphism $f\colon x \to Ly$ the transformation η provides a naturality square

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "\eta_x"'] & Ly \arrow[d, "\eta_{Ly}"] \\
Lx \arrow[r, "Lf"] & LLy
\end{tikzcd}$$

which is fully functorial in $f \in \mathrm{Hom}_C(x, Ly)$.

- To obtain a right inverse, we use that the map $L\eta_y\colon Ly \to LLy$ is an isomorphism. We claim that a right inverse is given by the composite

$$\mathrm{Hom}_C(x, Ly) \xrightarrow{L} \mathrm{Hom}_C(Lx, LLy) \xrightarrow{(L\eta_y)^{-1}\circ -} \mathrm{Hom}_C(Lx, Ly).$$

To see this, consider the following diagram:

$$\begin{tikzcd}
& \mathrm{Hom}_C(x, Ly) \arrow[dr, "L"] & \\
\mathrm{Hom}_C(Lx, Ly) \arrow[ur, "-\circ\eta_X"] \arrow[r, "L"] \arrow[dr, equal] & \mathrm{Hom}_C(LLx, LLy) \arrow[r, "-\circ L(\eta_X)"] & \mathrm{Hom}_C(Lx, LLy) \arrow[dd, "(L\eta_y)^{-1}\circ -"] \\
& \mathrm{Hom}_C(Lx, Ly) \arrow[ur, "L\eta_y\circ -"] \arrow[dr, equal] & \\
& & \mathrm{Hom}_C(Lx, Ly).
\end{tikzcd}$$

The top triangle commutes since the functor L preserves composition. The triangle in the middle commutes since for any morphism $g\colon Lx \to Ly$, the transformation η provides a naturality square

$$\begin{tikzcd}
Lx \arrow[r, "g"] \arrow[d, "L(\eta_x)"'] & Ly \arrow[d, "L(\eta_y)"] \\
LLx \arrow[r, "Lg"] & LLy
\end{tikzcd}$$

which is fully functorial in $g \in \mathrm{Hom}_C(Lx, Ly)$.

This finishes the proof. $\square$

A.3 Straightening/unstraightening

Consider an ordinary category C and a functor $F\colon C \to \mathrm{Cat}_1$ into the $(2,1)$-category of ordinary categories from Definition A.2.12. From this data, one may construct a new category denoted $\int F$, called the *Grothendieck construction*, which can be described as follows:

- An object in $\int F$ is a pair (x,e), where x is an object of C and e is an object of $F(x)$;
- A morphism in $\int F$ from (x,e) to (y,e') is a pair (f,φ), where $f\colon x \to y$ is a moprhism in C and $\varphi\colon f_!(x) \to y$ is a morphism in $F(y)$. Here we write $f_!$ for the functor $F(f)\colon F(x) \to F(y)$ induced by f using the functoriality of F.
- Composition of $(f,\varphi)\colon (x,e) \to (y,e')$ and $(g,\psi)\colon (y,e') \to (z,e'')$ is defined as $(g \circ f, \psi \circ g_!(\varphi))$.

The category $\int F$ comes equipped with a forgetful functor $p_F\colon \int F \to C$ defined by $p_F(x,e) := x$ and $p_F(f,\varphi) = f$. This construction is functorial in F, and hence determines a functor

$$\int\colon \mathrm{Fun}(C, \mathrm{Cat}_1) \to \mathrm{Cat}_{/C}, \qquad F \mapsto p_F.$$

It turns out that the functor F is uniquely determined by the functor $p_F\colon \int F \to C$. This provides a powerful way to construct functors of the form $F\colon C \to \mathrm{Cat}_1$:

- First one constructs a category E together with a functor $p\colon E \to C$;
- Then one shows that it is of the form p_F for some functor F (which is then uniquely determined by p).

The functors $p\colon E \to C$ that are of the form p_F for some F admit a concrete characterization, and are known as *Grothendieck opfibrations* or *cocartesian fibrations*.

In the setting of ∞-categories, there is an analogous correspondence between functors $C \to \mathrm{Cat}_\infty$ and cocartesian fibrations over C, which is known as the *straightening/unstraightening correspondence*. This correspondence is even more fundamental than the one in ordinary category: it is frequently not feasible to construct functors $C \to \mathrm{Cat}_\infty$ by hand, and one is forced to instead write down the associated cocartesian fibration $\int F \to C$. The goal of this section is to explain the ∞-categorical notion of cocartesian fibrations and the corresponding straightening/unstraightening correspondence. We will also use it to provide proofs of various statements left as black boxes in the previous section.

To motivate the definition of a cocartesian fibration, let us record some special features of the forgetful functor $p_F\colon \int F \to C$ for a given functor $F\colon C \to \mathrm{Cat}_1$:

(a) Consider a morphism $f\colon f\colon x \to y$ in C, and let $e \in C(x)$ be an object. Then we obtain a morphism $\mathrm{lift}(f,e) := (f, \mathrm{id}_{f_!(e)})\colon (x,e) \to (y, f_!(e))$ in the Grothendieck construction $\int F$.

(b) The morphism $\mathrm{lift}(f,e)$ is in a precise sense the *initial* morphism in $\int F$ whose source is e and whose image in C is f: any other morphism $(f,\varphi)\colon (x,e) \to (y,e'')$ in $\int F$ living over f may be factored as the following composite:

$$(x,e) \xrightarrow{\mathrm{lift}(f,e)} (y, f_!(e)) \xrightarrow{(\mathrm{id}_y,\varphi)} (y,e'').$$

(c) More generally, if $g\colon y \to z$ is any other morphism in C, then for every object composing with $\mathrm{lift}(f,e)$ determines a one-to-one correspondence between morphisms $(y, f_!(e)) \to (z,e'')$ in $\int F$ which live over g and morphisms $(x,e) \to (z,e'')$ which live over gf: in both cases we are specifying a morphism $g_!(f_!(e)) = (gf)_!(e) \to e''$ in $C(z)$.

The following definition tries to abstract this property:

Definition A.3.1. Let $p\colon E \to C$ be a functor between ∞-categories.

(1) A morphism $\overline{f}\colon \overline{e} \to \overline{e}'$ is called *p-cocartesian* if for every third object $\overline{e}''$ in C the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_E(\overline{e}', \overline{e}'') \arrow[r, "-\circ\overline{f}"] \arrow[d, "p"'] & \mathrm{Hom}_E(\overline{e}, \overline{e}'') \arrow[d, "p"] \\
\mathrm{Hom}_C(p\overline{e}', p\overline{e}'') \arrow[r, "-\circ p(\overline{f})"] & \mathrm{Hom}_C(p\overline{e}, p\overline{e}'')
\end{tikzcd}$$

is a pullback square.

(2) We say that p is a *cocartesian fibration* if for every morphism $f\colon x \to y$ in C and every object $\overline{e} \in E_x$ there exists a p-cocartesian morphism $\overline{f}\colon \overline{e} \to \overline{e}'$ satisfying $p(\overline{f}) = f$. We refer to such a morphism $\overline{f}$ as a *p-cocartesian lift of f*.

(3) Let $p\colon E \to C$ and $p'\colon E' \to C$ be two cocartesian fibrations. A functor $F\colon E \to E'$ over C, i.e., a commutative triangle

$$\begin{tikzcd}
E \arrow[rr, "F"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& C, &
\end{tikzcd}$$

is called *cocartesian over* C if it sends p-cocartesian morphisms in E to p'-cocartesian morphisms in E'.

(4) If C is a small ∞-category, we denote by

$$\mathrm{Cocart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}$$

the non-full subcategory spanned by the cocartesian fibrations $p\colon E \to C$ and the cocartesian functors between them.

Definition A.3.2. A functor $p\colon E \to C$ is called a *cartesian fibration* if $p^{\mathrm{op}}\colon E^{\mathrm{op}} \to C^{\mathrm{op}}$ is a cocartesian fibration. One may explicitly describe this condition in terms of the existence of *p-cartesian morphisms*, defined dually to the p-cocartesian ones. We similarly obtain a notion of a *cocartesian functor over* C, and this leads to a (non-full) subcategory

$$\mathrm{Cart}(C) \subseteq (\mathrm{Cat}_\infty)_{/C}.$$

Definition A.3.3. Let $p\colon E \to C$ be a functor of ∞-categories. We call p a *left fibration* if p is a cocartesian fibration and the following equivalent conditions are satisfied:

(a) p is conservative, i.e. a morphism f in E is an isomorphism as soon as $p(f)$ is an isomorphism in C;

(b) For each $X \in C$, the fiber $\{X\} \times_C E$ is an anima;

(c) Every morphism in E is cocartesian.

Similarly, p is called a *right fibration* if p is a cartesian fibration satisfying the equivalent conditions (a), (b) and

(c') Every morphism in E is cartesian.

With these definitions at hand, we may now go back to the example of the Grothendieck construction $p_F\colon \int F \to C$. The morphisms $\overline{f} = \mathrm{lift}(f, e)$ in $\int F$ considered before are p_F-cocartesian lifts, showing that the functor $p_F\colon \int F \to C$ is a cocartesian fibration. The point of the straightening/unstraightening equivalence is to go the other way: given a cocartesian fibration $p\colon E \to C$, we wish obtain a functor $F_p\colon C \to \mathrm{Cat}$. Let us start by describing this functor informally:

- On objects, F_p sends $x \in C$ to the fiber E_x;
- On morphisms, F_p sends a morphism $f\colon x \to y$ to the functor $f_!\colon E_x \to E_y$ determined as follows: for $e \in E_x$, the object $f_!(e) \in E_y$ is the target of a chosen cocartesian lift $e \to f_!(e)$ of f.

Morally speaking, the fact that the objects $f_!(e)$ assemble into a functor is because $f_!(e)$ corepresents the functor

$$E_y \to \mathrm{An}, \qquad e'' \mapsto \mathrm{Hom}_E(e,-) \times_{\mathrm{Hom}_C(pe,p(-))} \{f\}.$$

and is therefore uniquely determined by the Yoneda lemma. However, since the construction of the Hom functor $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$ relies on straightening/unstraighening, this is not a formal argument.

Theorem A.3.4 (Straightening/Unstraightening, Lurie [Lur09b])**.** *Let C be an ∞-category. Then there exists an equivalence of ∞-categories*

$$\mathrm{Str}^{\mathrm{cocart}}\,\mathrm{Cocart}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{Cat}_\infty),$$

which restricts to an equivalence

$$\mathrm{LFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C, \mathrm{An}).$$

Dually, there is an equivalence

$$\mathrm{Str}^{\mathrm{cart}}\,\mathrm{Cart}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Cat}_\infty)$$

which restricts to an equivalence

$$\mathrm{RFib}(C) \xrightarrow{\sim} \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Note that if C is an anima, then every functor $E \to C$ is both a cocartesian and a cartesian fibration. It is a left/right fibration if and only if also E is an anima. This gives the following immediate consequence:

Theorem A.3.5. *Let X be an anima. Then there is an equivalence of ∞-categories*

$$\mathrm{Str} \colon \mathrm{An}_{/X} \xrightarrow{\sim} \mathrm{Fun}(X, \mathrm{An}).$$

For more information about cocartesian fibrations and straightening/unstraightening, we recommend Appendix A.2 of Heyer and Mann [HM24].

A.4 Presheaf ∞-categories

Definition A.4.1 (Presheaf category)**.** Let C be an ∞-category. Its *presheaf category* is defined as

$$\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{An}).$$

Functors $F \colon C^{\mathrm{op}} \to \mathrm{An}$ are called *presheaves on C*.

Lemma A.4.2. *The presheaf category* $\mathrm{PSh}(C)$ *admits all small limits and colimits. The evaluation functors* $\mathrm{ev}_x \colon \mathrm{PSh}(C) \to C$ *preserve all small limits and colimits.*

Proof. Since An admits all small limits and colimits by Theorem 2.6.38, the same follows for $\mathrm{PSh}(C)$ by Lemma A.2.23. □

Recall that the Hom-functor $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{An}$ may be curried to get the *Yoneda embedding*

$$Y \colon C \hookrightarrow \mathrm{PSh}(C), \qquad x \mapsto \mathrm{Hom}_C(-,x),$$

which is fully faithful by Theorem 2.6.42. The presheaves in the (essential) image of this functor are called *representable presheaves*.

Lemma A.4.3. *The Yoneda embedding* $Y \colon C \hookrightarrow \mathrm{PSh}(C)$ *preserves all limits that exist in* C.

Proof. Let $F \colon I \to C$ be a diagram. We need to show that the canonical map $\mathrm{Hom}_C(-,\lim_i F(i)) \to \lim_i \mathrm{Hom}_C(-,F(i))$ in $\mathrm{PSh}(C)$ is an isomorphism. By Theorem 2.6.10 this may be checked objectwise: we must show that for every object x of C the map

$$\mathrm{Hom}_C(x,\lim_i F(i)) \to \lim_i \mathrm{Hom}_C(x,F(i))$$

is an isomorphism of animae. This is an instance of Theorem 2.6.43. □

While the inclusion of C into $\mathrm{PSh}(C)$ preserves all limits, it does not preserve any colimits: on the contrary, it *freely adds colimits* to C, in the following sense:

Theorem A.4.4 (Lurie [Lur09b, Theorem 5.1.5.6])**.** *Let* C *be a small* ∞*-category and let* D *be a large* ∞*-category admitting all small colimits. Then restriction along the Yoneda embedding* $Y \colon C \hookrightarrow \mathrm{PSh}(C)$ *induces an equivalence*

$$Y^* \colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{PSh}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C,D),$$

where the left-hand side denotes the full subcategory of colimit-preserving functors $\mathrm{PSh}(C) \to D$*. The inverse sends a functor* $u \colon C \to D$ *to the left Kan extension* $Y_!(u) \colon \mathrm{PSh}(C) \to D$ *of* u *along* Y*.*

More informally, this theorem says that every functor $u \colon C \to D$ admits a unique colimit-preserving extension $Y_!(u) \colon \mathrm{PSh}(C) \to D$.

There is a more general version of this theorem, in which we only add *some* colimits to C but not necessarily all of them:

Definition A.4.5. Let $\mathcal{K}$ be a collection of small ∞-categories and let C be a small ∞-category. We denote by

$$\mathrm{PSh}^{\mathcal{K}}(C) \subseteq \mathrm{PSh}(C)$$

the smallest full subcategory of $\mathrm{PSh}(C)$ that contains the representable presheaves and is closed under I-indexed colimits for all $I \in \mathcal{K}$. (For example, $\mathrm{PSh}^{\mathcal{K}}(C)$ could be taken to be the intersection of all such subcategories of $\mathrm{PSh}(C)$.)

Proposition A.4.6 (Lurie [Lur09b, Proposition 5.3.6.2])**.** *Let $\mathcal{K}$ be a collection of small ∞-categories. For every small ∞-category C and every (possibly large) ∞-category D with I-indexed colimits for all $I \in \mathcal{K}$, restriction along the Yoneda embedding $Y\colon C \hookrightarrow \mathrm{PSh}^{\mathcal{K}}(C)$ induces an equivalence*

$$\mathrm{Fun}_{\mathcal{K}}(\mathrm{PSh}^{\mathcal{K}}(C), D) \xrightarrow{\sim} \mathrm{Fun}(C, D),$$

where the left-hand side denotes the full subcategory of $\mathrm{Fun}(\mathrm{PSh}^{\mathcal{K}}(C), D)$ *spanned by the functors $F\colon \mathrm{PSh}^{\mathcal{K}}(C) \to D$ that preserve I-indexed colimits for all $I \in \mathcal{K}$.*

A.5 Complete Segal animae

Let C be an ∞-category. For every $n \geq 0$, we may consider the anima $\mathrm{Map}([n], C)$ of functors $[n] \to C$, which we may think of as 'strings of n composable morphisms in C, equipped with choices for their composites':

- For $n = 0$ we have $\mathrm{Map}([0], C) = \mathrm{Fun}(*, C)^{\simeq} \simeq C^{\simeq}$ is just the groupoid core of C;
- For $n = 1$, $\mathrm{Map}([1], C)$ is the anima of morphisms in C;
- For $n = 2$, $\mathrm{Map}([2], C)$ is the anima of commutative triangles in C, which by the Segal axiom, Axiom E, is equivalent to $\mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C)$;
- More generally, the 'higher version' of the Segal condition implies that there is a pushout square of the form

  $$\begin{tikzcd} {[0]} \arrow[r] \arrow[d] & {[1]} \arrow[d, "e_{n+1}"] \\ {[n]} \arrow[r, "d_{n+1}"] \arrow[ur, phantom, "\ulcorner", very near end] & {[n+1]} \end{tikzcd}$$

  which expresses that $[n+1]$ is 'glued' from $[n]$ and the interval $\{n \leq n+1\}$ along $\{n\}$. By induction we then get that

  $$\mathrm{Map}([n], C) \xrightarrow{\sim} \mathrm{Map}([1], C) \times_{C^{\simeq}} \mathrm{Map}([1], C) \times_{C^{\simeq}} \cdots \times_{C^{\simeq}} \mathrm{Map}([1], C)$$

  for all n.

As we will now make precise, these animae $\mathrm{Map}([n],C)$ completely determine the ∞-category C.

Definition A.5.1 (Nerve)**.** The *nerve* of an ∞-category C is the simplicial anima $N(C)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ defined by $N(C)_n := \mathrm{Map}([n],C)$, i.e. as the composite

$$\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathrm{Cat}_\infty}(-,C)} \mathrm{An}.$$

This construction is functorial in C and thus defines a functor $N\colon \mathrm{Cat}_\infty \to \mathrm{sAn}$.

Lemma A.5.2. *The nerve functor admits a left adjoint* $\mathrm{ac}\colon \mathrm{sAn} \to \mathrm{Cat}_\infty$.

Proof. Since Cat_∞ admits all colimits, Theorem A.4.4 guarantees that the inclusion $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}_\infty$ uniquely extends to a colimit-preserving functor $\mathrm{ac}\colon \mathrm{sAn} = \mathrm{PSh}(\mathbf{\Delta}) \to \mathrm{Cat}_\infty$. To show that this is a left adjoint to the nerve functor, we must produce a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{ac}(X),C) \cong \mathrm{Hom}_{\mathrm{sAn}}(X,N(C))$$

for every simplicial anima X and every ∞-category C. Regarding both sides as functors $\mathrm{sAn}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Cat}_\infty,\mathrm{An})$, it follows from Theorem 2.6.43 that both are limit-preserving, hence by another application of Theorem A.4.4 it suffices to construct such a natural isomorphism after restriction along the Yoneda embedding $\Delta^\bullet\colon \mathbf{\Delta} \hookrightarrow \mathrm{sAn}$. But in this case we may use the composite

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}(\mathrm{ac}(\Delta^n),C) \cong \mathrm{Hom}_{\mathrm{Cat}_\infty}([n],C) = N(C)_n \cong \mathrm{Hom}_{\mathrm{sAn}}(\Delta^n,N(C)),$$

where the last step uses the Yoneda lemma. □

Definition A.5.3. Given a simplicial anima X, we refer to the ∞-category $\mathrm{ac}(X)$ as the *associated category*.

Theorem A.5.4 ([JT07, Section 4], [Lur09a, Corollary 4.3.16], [HS23])**.** *The nerve functor*

$$N\colon \mathrm{Cat}_\infty \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{An})$$

is fully faithful. The essential image consists of the ***complete Segal animae****, i.e. those simplicial animae* $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ *satisfying the following two conditions:*

(1) (Segal condition) For every $n \geq 1$*, the inclusion maps* $e_i\colon [1] \cong \{i-1 \leq i\} \hookrightarrow [n]$ *induce an isomorphism of animae*

$$(e_1^*,\dots,e_n^*)\colon X_n \xrightarrow{\sim} X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1;$$

(2) *(Completeness/Rezk condition) The anima X_0 of 0-simplices is equivalent to the subanima of X_1 spanned by the invertible 1-simplices: the square*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_3 \arrow[d] \\ X_0 \times X_0 \arrow[r] & X_1 \times X_1 \end{tikzcd}$$

induced by the square

$$\begin{tikzcd} \{0 \le 2\} \sqcup \{1 \le 3\} \arrow[r, hook] \arrow[d] & {[3]} \arrow[d] \\ {[0] \sqcup [0]} \arrow[r] & {[0]} \end{tikzcd}$$

is a pullback square.

Observe that under the assumption of (1) we have $X_3 \simeq X_2 \times_{X_1} X_2$ and the second condition becomes an alternative formulation of the Rezk axiom, Axiom F.

Notation A.5.5. We write $\mathrm{CS}(\mathrm{An}) \subseteq \mathrm{sAn}$ for the full subcategory of complete Segal animae. The previous theorem thus produces an equivalence

$$N\colon \mathrm{Cat}_\infty \underset{\simeq}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{sAn} : \mathrm{ac}\,.$$

The counit $\mathrm{ac}(N(C)) \to C$ of the adjunction is always an equivalence, while the unit $X \to N(\mathrm{ac}(X))$ is an equivalence if and only if X is a complete Segal anima.

A.5.1 Geometric realizations

With the equivalence between ∞-categories and complete Segal animae at hand, we are now finally in a position to address the terminology regarding the phrase 'geometric realization'. Recall that we have previously used this phrase for two a priori unrelated constructions:

- For an ∞-category C, the geometric realization $|C|$ is obtained from C by inverting all the morphisms in C:

$$|C| := C[W^{-1}], \qquad W := \mathrm{Map}([1], C);$$

- For a simplicial anima $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$, its geometric realization is its colimit:

$$|X| := \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} X_n.$$

The relation between these two notions is explained by Theorem A.5.4:

Lemma A.5.6. *The geometric realization $|C|$ of an ∞-category C is naturally equivalent to the geometric realization $|N(C)|$ of its nerve.*

Proof. The colimit-functor is by definition a left adjoint to the constant simplicial anima functor:

$$|-|\colon \mathrm{sAn} \rightleftarrows \mathrm{An} : \mathrm{const}.$$

For any anima X, the constant simplicial anima $\mathrm{const}_X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is a complete Segal anima (equivalent to the nerve $N(X)$ of X), and hence this adjunction restricts to an adjunction

$$|-|\colon \mathrm{CS}(\mathrm{An}) \rightleftarrows \mathrm{An} : \mathrm{const}.$$

Under the equivalence $\mathrm{Cat}_\infty \simeq \mathrm{CS}(\mathrm{An})$ from Theorem A.5.4, the constant functor corresponds to the inclusion $\mathrm{An} \hookrightarrow \mathrm{Cat}_\infty$, and by uniqueness of adjoints this implies the claim. $\square$

B Homotopy (co)limits as ∞-categorical (co)limits

Recall from Theorem 2.6.38 that there is a functor $\Pi_\infty \colon \mathrm{Top} \to \mathrm{An}$ which exhibits An as the localization of Top at the weak homotopy equivalences. It is now natural to ask how the following two constructions relate to each other:

(1) The notions of homotopy pullbacks and homotopy pushouts of topological spaces, as introduced in Definitions 1.1.1 and 1.1.4;

(2) The notions of pullbacks and pushouts of animae, using the ∞-categorical notions of pushouts and pullbacks introduced in Section 2.6.7.

This question is an instance of a more general question:

Question. Given an ∞-category C and a collection of morphisms W in C, how do (homotopy) limits and colimits in C correspond to limits and colimits in the localization $C[W^{-1}]$?

In this chapter, we state a powerful theorem from Cisinski [Cis19, Chapter 7] that provides concrete conditions under which pullbacks in the localization $C[W^{-1}]$ can be computed directly in terms of pullbacks in C. As an application, we will deduce that the localization functor $\Pi_\infty \colon \mathrm{Top} \to \mathrm{An}$ sends homotopy pushouts and homotopy pullbacks of spaces to pushouts and pullbacks of animae, respectively. In Appendix B.5 we will discuss the stronger notion of *homotopy completeness*, which we will use in Appendix B.6 to show that geometric realizations of topological spaces compute $\boldsymbol{\Delta}^{\mathrm{op}}$-indexed colimits in An.

B.1 Left exact localizations

We start by formulating the main result from Cisinski [Cis19, Chapter 7] that provides us control over pullbacks in a localization $C[W^{-1}]$.

Definition B.1.1. Let C be an ∞-category with a terminal object $*$. A collection of morphisms $F \subseteq \mathrm{Map}([1], C)$ is called a *class of fibrations* if the following conditions are satisfied:

(1) It contains all the isomorphisms;

(2) It is closed under composition;

(3) It is closed under base change, in the sense that for all morphisms $f\colon X \to Y$ in F and $v\colon Y' \to Y$ in C, the pullback

$$\begin{tikzcd} Y' \times_Y X \arrow[r, "u"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

exists, and the morphism f' is again in F.

A morphism in F is called a *fibration*. An object X in C is called *fibrant* if the unique map $X \to *$ is a fibration.

Dually, a collection of morphisms I is called a *class of cofibrations* if the corresponding class I^{op} in C^{op} defines a class of fibrations in C^{op}.

Definition B.1.2. An ∞*-category with weak equivalences and fibrations* is a triple (C, W, F), where C is an ∞-category with a terminal object, and W and F are collections of morphisms satisfying the following conditions:

(1) The class W satisfies the *2-out-of-3 property*: for morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in C, if two of the morphisms f, g, gf are in W, then so is the third.

(2) The class F is a class of fibrations;

(3) For any pullback square in C of the form

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

in which p is a fibration between fibrant objects, and Y' is fibrant, if p belongs to W then so does p'.

(4) For any morphism $f\colon X \to Y$ in C with fibrant codomain, there exists a morphism $w\colon X \to X'$ in W and a fibration $p\colon X' \to Y$ such that f is a composite of p and w.

The morphisms in W will be called the *weak equivalences*. Morphisms in C that are both weak equivalences and fibrations will be called *trivial fibrations*.

We similarly get a notion of an *∞-category with weak equivalences and cofibrations* by dualizing.

Example B.1.3. Let C be an ∞-category that admits a terminal object and admits all pullbacks. Then we may turn C into an ∞-category with weak equivalences and fibrations by taking $W = \mathrm{Iso}(C)$ to consist only the isomorphisms and $F = \mathrm{Map}([1], C)$ to consist of all morphisms in C.

Definition B.1.4. Let C and D be ∞-categories with weak equivalences and fibrations. A functor $F\colon C \to D$ is called *left exact* if the following conditions are satisfied:

(1) The functor F preserves terminal objects;

(2) The functor F sens fibrations between fibrant objects to fibrations, and sends trivial fibrations between fibrant objects to trivial fibrations;

(3) For any pullback square in C of the form

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\ Y' \arrow[r, "v"] & Y \end{tikzcd}$$

in which p is a fibration and Y and Y' are fibrant, the square

$$\begin{tikzcd} F(X') \arrow[r, "F(u)"] \arrow[d, "F(p')"'] \arrow[dr, phantom, "\lrcorner", very near start] & F(X) \arrow[d, "F(p)"] \\ F(Y') \arrow[r, "F(v)"] & F(Y) \end{tikzcd}$$

is a pullback square in D.

If C and D are ∞-categories with weak equivalences and *co*fibrations, we say $F\colon C \to D$ is *right exact* if $F^{\mathrm{op}}\colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is left exact.

We will state without proof the following theorem:

Theorem B.1.5 (Cisinski [Cis19, Theorem 7.5.18])**.** *Let C be an ∞-category with weak equivalences and fibrations.*

(1) The localization $C[W^{-1}]$ has pullbacks and a terminal object, and the localization functor $\gamma\colon C \to C[W^{-1}]$ is left exact.

(2) *For any ∞-category D with a terminal object and pullbacks, and for any left exact functor $F\colon C \to D$, the induced functor $\overline{F}\colon C[W^{-1}] \to D$ is left exact.*

Corollary B.1.6. *Let C be an ∞-category with weak equivalences and cofibrations. Then the localization $C[W^{-1}]$ has pushouts and an initial object, and the localization functor $\gamma\colon C \to C[W^{-1}]$ is right exact.*

Proof. We apply Theorem B.1.5 to C^{op}. □

Our main interest for this theorem is in the case of the localization functor $\Pi_\infty\colon \mathrm{Top} \to \mathrm{An}$ from Theorem 2.6.38. Below we will see that it preserves both homotopy pushouts and homotopy pullbacks of spaces.

Remark B.1.7 (Model categories)**.** Before the framework of ∞-categories was developed and gained popularity, the most common framework for working with homotopical objects was that of *model categories*, introduced by Quillen [Qui67]. A model category is a category C equipped with three classes of morphisms, denoted W, F and I, and referred to as *weak equivalences*, *fibrations* and *cofibrations*, respectively, which are required to satisfy a variety of axioms which we won't recall here. For us, only the following two facts are relevant:

- The triple (C, W, F) is a category with weak equivalences and fibrations;
- Dually, the triple (C, W, I) is a category with weak equivalences and cofibrations.

We may think of a model category C as providing a 'model' for the ∞-categorical localization $C[W^{-1}]$ (and indeed this is essentially why Quillen chose the term). The presence of the fibrations and cofibrations is explained by Theorem B.1.5: it gives us a means to compute the 'homotopically correct' pullbacks and pushouts (i.e. the pullbacks and pushouts in the ∞-category $C[W^{-1}]$) in terms of the 'strict' pullbacks and pushouts in the category C.

Functorial factorizations and functor categories

Let C be an ∞-category with weak equivalences and fibrations. For an indexing ∞-category J, it is not necessarily the case that the functor category $\mathrm{Fun}(J, C)$ inherits the structure of an ∞-category with weak equivalences and fibrations by taking the *pointwise* weak equivalences and fibrations: the factorization property may no longer be satisfied. This problem disappears if we demand the following slightly stronger condition on C:

Definition B.1.8 (Functorial factorization)**.** Let C be an ∞-category with weak equivalences and fibrations. We say that C *has functorial factorizations* if there exists a functor

$$(w,p)\colon\ \mathrm{Fun}([1],C) \to \mathrm{Fun}([2],C) \xrightarrow{\sim} \mathrm{Fun}([1],C) \times_C \mathrm{Fun}([1],C)$$
$$(f\colon X \to Y) \mapsto (X \xrightarrow{w(f)} T(f) \xrightarrow{p(f)} Y),$$

where for each morphism f we have that $w(f)$ is a weak equivalence and $p(f)$ is a fibration.

Lemma B.1.9. *Assume that C has functorial factorizations. Then for every ∞-category J, the functor category* $\mathrm{Fun}(J,C)$ *becomes an ∞-category with weak equivalences and fibrations, in which a morphism $f\colon X \to Y$ in* $\mathrm{Fun}(J,C)$ *is a weak equivalence (resp. a fibration) if and only if the morphism $f(j)\colon X(j) \to Y(j)$ is a weak equivalence (resp. fibration) in C.*

Proof. Conditions (1)-(3) are easy to verify since limits in $\mathrm{Fun}(J,C)$ are computed pointwise, see Lemma A.2.23. For condition (4) we note that the functorial factorization for C immediately produces a functorial factorization for $\mathrm{Fun}(J,C)$ by currying/uncurrying:

$$\mathrm{Fun}([1],\mathrm{Fun}(J,C)) \simeq \mathrm{Fun}(J,\mathrm{Fun}([1],C)) \xrightarrow{(w,p)_*} \mathrm{Fun}(J,\mathrm{Fun}([2],C)) \simeq \mathrm{Fun}([2],\mathrm{Fun}(J,C)).$$

□

B.2 Homotopy pushouts of spaces

We will now apply Theorem B.1.5 to relate homotopy pushouts of spaces to pushouts of animae. In doing so, we will equip the category Top with a structure of a category with weak equivalences and cofibrations in three different ways:

- (Hurewicz) Using homotopy equivalences and h-cofibrations;
- (Quillen) Using weak homotopy equivalences and relative cell complexes;
- (Mixed) Using weak homotopy equivalences and m-cofibrations. To do.

The mixed structure is the one that best fits with our discussion of homotopy pushouts from Chapter 1: its cofibrant objects are precisely the topological spaces that are homotopy equivalent to CW-complexes, i.e., the 'spaces' from Convention 1.2.3.

B.2.1 The Hurewicz cofibration structure

We start with the most well-known cofibration structure in Top. Since the weak equivalences are the actual homotopy equivalences, we will ultimately not be very interested in this cofibration structure.

Definition B.2.1. Let $i\colon A \to X$ be a continuous map of topological spaces. We say that i is a *Hurewicz cofibration*, or *h-cofibration* for short, if it satisfies the homotopy extension property: for every solid commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "i"'] & Y^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ X \arrow[r, "f"] \arrow[ur, dashed, "\overline{h}"] & Y \end{tikzcd}$$

there exists a dashed lift $\overline{h}$ making the diagram commute.

Lemma B.2.2. *Consider the category* Top *of topological spaces, let I denote the collection of h-cofibrations and let W denote the collection of homotopy equivalences. Then the triple* (Top, W, I) *is an ∞-category with weak equivalences and cofibrations. It has functorial factorizations.*

Proof. We check the four criteria (1)-(4):

(1) It is clear that homotopy equivalences satisfy the 2-out-of-3 property.

(2) Note that all homeomorphisms are h-cofibrations, and that h-cofibrations are closed under composition. Also note that pushouts of h-cofibrations are again h-cofibrations. In particular, the h-cofibrations form a class of cofibrations in Top.

(4) Every continuous map $f\colon X \to Y$ factors through its *mapping cylinder*:

$$X \xrightarrow{x \mapsto (x,0)} M(f) := (X \times [0,1]) \times_{X \times \{1\}} Y \xrightarrow[y \mapsto y]{(x,t) \mapsto f(x)} Y,$$

were the first map is an h-cofibration and the second is a homotopy equivalence. Note that this construction is functorial in f.

(3) Consider a pushout square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ X \arrow[r, "g"] & X'., \end{tikzcd}$$

where i is both an h-cofibration and a homotopy equivalence, with homotopy inverse $r\colon X \to A$. Using the homotopy extension property for i, we may assume that $ri = \mathrm{id}_A$ and that we are given a homotopy $H\colon ir \sim \mathrm{id}_X$ whose restriction to A is the constant homotopy. We then define the map $r'\colon X' \to A'$ as the unique map satisfying $r' \circ g = f \circ r$ and $r' \circ i' = \mathrm{id}_{A'}$; his is well-defined since on A we have $f \circ r \circ i = f \circ \mathrm{id}_A = f$. We claim r' is a homotopy inverse to i'. To this end, we construct a homotopy $H'\colon X' \times [0,1] \to X'$ as the map satisfying $H' \circ (g \times \mathrm{id}_{[0,1]}) = g \circ H$ and $H' \circ (i' \times \mathrm{id}_{[0,1]}) = i' \circ \mathrm{pr}_{A'}$; this is well-defined using the assumption on H. $\square$

B.2.2 The Quillen cofibration structure

A more relevant cofibration structure is the one corresponding to the Quillen model structure on Top.

Definition B.2.3. A continuous map $i\colon A \to X$ is called a *relative cell complex* if X can be built from A by iteratively attaching cells: there exists an ordinal number λ and a sequence of spaces $(X_\alpha)_{\alpha\le\lambda}$ such that

- $X_0 = A$;
- $X_\lambda = X$;
- For each $\alpha < \lambda$ the space $X_{\alpha+1}$ is formed from X_α by attaching n-cells for some fixed $n \ge 0$:

$$\begin{tikzcd} \coprod_{i\in I_\alpha} S^{n-1} \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\ \coprod_{i\in I_\alpha} D^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_{\alpha+1}. \end{tikzcd}$$

- For limit ordinals $\beta \le \lambda$, X_β is the colimit of the sequence X_α for $\alpha < \beta$.

A topological space X is called a *cell complex* if the inclusion $\emptyset \to X$ is a relative cell complex.

Remark B.2.4. Cell complexes are more general than CW-complexes, since the cells do not need to be attached in order of increasing dimension. That being said, every cell complex is homotopy equivalent to a CW-complex: by cellular approximation, we may push the attaching maps for the n-dimensional cells into the $(n-1)$-skeleton.

Exercise B.2.5 (Pushout-product lemma)**.** Let $i\colon A \hookrightarrow X$ and $j\colon B \hookrightarrow Y$ be relative cell complexes. Show that the induced map

$$A \times Y \coprod_{A\times B} X \times B \hookrightarrow X \times Y$$

is again a relative cell complex. *Hint: reduce to the case where both i and j are inclusions $S^{n-1} \hookrightarrow D^n$ for some n.*

Lemma B.2.6. *Consider the category* Top *of topological spaces, let I denote the collection of relative cell complexes, and let W denote the collection of weak homotopy equivalences. Then the triple* (Top, W, I) *is an ∞-category with weak equivalences and cofibrations. It has functorial factorizations.*

Proof. Again it is clear that weak homotopy equivalences satisfy 2-out-of-3 and that the relative cell complexes form a class of cofibrations. The factorization of a map $f\colon X \to Y$ into a relative cell complex and a weak homotopy equivalence proceeds via a construction known as the *relative CW-approximation* of Y with respect to X, in which one iteratively adds cells to X to make its homotopy groups look more and more like those of Y. This construction can be made functorial in f, see for example [Hir15] for a detailed proof. (There is in fact a much more general machinery for producing functorial factorizations, known as the 'small object argument', see for example Hovey [Hov99, Section 2.1].) Finally, the trivial cofibrations are closed under pushouts: every trivial cofibration is a retract of a map obtained just as in Definition B.2.3 but with the inclusion $S^{n-1} \hookrightarrow D^n$ replaced by the inclusion $D^n \times \{0\} \hookrightarrow D^n \times [0,1]$. Since such maps are closed under pushouts and are all weak homotopy equivalences, this finishes the proof. □

Corollary B.2.7. *Let* Cell $\subseteq$ Top *denote the full subcategory spanned by the cell complexes. Then the localization functor* $\Pi_\infty(-)\colon \mathrm{Cell} \to \mathrm{An}$ *preserves all pushouts of the form*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ X \arrow[r, "g"] & X', \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

where i is a relative cell complex and A and A' are cell complexes.

Proof. In light of Lemma B.2.6 this is a special case of Theorem B.1.5. □

As our main corollary, we may now deduce that homotopy pushouts of spaces, as defined in Definition 1.1.4, do indeed correspond to pushouts in An. (Recall our convention regarding the word 'space' from Convention 1.2.3).

Proposition B.2.8. *The functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *sends homotopy pushouts of spaces to pushouts of animae.*

Proof. Consider continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$. We need to show that commutative square of animae

$$\begin{tikzcd} \Pi_\infty(Z) \arrow[r] \arrow[d] & \Pi_\infty(Y) \arrow[d] \\ \Pi_\infty(X) \arrow[r] & \Pi_\infty(X \sqcup^h_Z Y) \end{tikzcd}$$

is a pushout square. By (relative) CW-approximation, we may find CW-complexes Z', X' and Y' with weak homotopy equivalences $Z' \xrightarrow{\sim} Z$, $X' \xrightarrow{\sim} X$ and $Y' \xrightarrow{\sim} Y$ and relative cell complexes $f'\colon Z' \to X'$ and $g'\colon Z' \to Y'$ that make the obvious squares commute. Since

Z, X and Y are homotopy equivalent to CW-complexes by assumption, it follows from Whitehead's theorem, Theorem 1.2.2, that these three weak homotopy equivalences are actual homotopy equivalences, and in particular the induced map $X' \sqcup^h_{Z'} Y' \to X \sqcup^h_Z Y$ is also a homotopy equivalence by Exercise 1.2.5. Since $\Pi_\infty(-)$ inverts homotopy equivalences, we may assume that the three spaces involved are CW-complexes and that f and g are relative cell complexes.

The claim now follows from Corollary B.2.7: since the inclusion $Z \hookrightarrow M(f)$ is again a relative cell complex (see Exercise B.2.9), the pushout square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ M(f) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \sqcup^h_Z Y \end{tikzcd}$$

in Cell is sent under the localization functor $\Pi_\infty\colon \mathrm{Cell} \to \mathrm{An}$ to a pushout square. □

Exercise B.2.9. Let $f\colon Z \hookrightarrow X$ be a relative cell complex. Show that the inclusion $Z \hookrightarrow M(f)$ is again a relative cell complex.

Corollary B.2.10. *The funcor $\Pi_\infty(-)\colon \mathrm{Cell}_* \to \mathrm{An}_*$ sends the suspension $\Sigma(X)$ of a pointed cell complex (Example 1.1.8) to the suspension $\Sigma(\Pi_\infty(X))$ of its underlying anima (Definition 3.2.3).* □

Corollary B.2.11. *The functor $\Pi_\infty(-)\colon \mathrm{Cell}_* \to \mathrm{An}_*$ sends the topological n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ to the homotopical n-sphere $S^n = \Sigma^n(* \sqcup *)$.*

B.3 Homotopy pullbacks of spaces

We now move to a discussion of homotopy *pullbacks* of spaces, and show that these are sent to pullbacks of animae under the localization functor $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$.

Definition B.3.1. Let $p\colon E \to B$ be a continuous map of topological spaces. We say that p is a *Serre fibration* if it satisfies the homotopy lifting property with respect to all disks: for every $n \geq 0$ and every solid commutative diagram of the form

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "f"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\overline{H}"] & B. \end{tikzcd}$$

We will prove below that the triple $(\mathrm{Top}, \text{weak homotopy equivalences}, \text{Serre fibrations})$ defines a category with weak equivalences and fibrations, in the sense of Definition B.1.2. Let us start by recalling some standard facts about Serre fibrations:

Theorem B.3.2 (Locality for Serre fibrations, Dieck [Die08, Theorem 6.3.3])**.** *Let $p\colon E \to B$ be a continuous map, and assume that there exists an open cover $\{U_i\}_{i\in I}$ such that the restriction $p|_{U_i}\colon p^{-1}(U_i) \to U_i$ is a Serre fibration for each $i \in I$. Then p is itself a Serre fibration.*

In particular, locally trivial fiber bundles are Serre fibrations.

Theorem B.3.3 (Switzer [Swi75, Theorem 2.59])**.** *Let $p\colon E \to B$ be a Serre fibration and consider a point $e \in E$ with image $b := p(e)$. Define the fiber F_b as the following pullback of topological spaces:*

$$\begin{tikzcd} F_b \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ \mathrm{pt} \arrow[r, hook, "b"] & B. \end{tikzcd}$$

Then there is a long exact sequence of homotopy groups of the form

$$\cdots \to \pi_{n+1}(B,b) \to \pi_n(F_b,e) \xrightarrow{i_*} \pi_n(E,e) \xrightarrow{p_*} \pi_n(B,b) \to \pi_{n-1}(F_b,e) \to \ldots$$

Corollary B.3.4. *A Serre fibration $p\colon E \to B$ is a weak homotopy equivalence if and only if all the fiber F_b is weakly contractible for every $b \in B$.*

Proof. By working separately over each path component of B, we may assume that B is path-connected. Picking a preferred base point of b_0 of B, it then suffices to show that p is a weak homotopy equivalence if and only if the fiber $F := F_{b_0}$ is weakly contractible. This follows immediately from the long exact sequence from Theorem B.3.3. □

Lemma B.3.5. *Let $f\colon E \to B$ be a continuous map. Define the mapping path space $P(f)$ of f as*

$$P(f) := \{(e,\gamma) \in E \times B^{[0,1]} \mid f(e) = \gamma(0)\}.$$

Then the projection $p\colon P(f) \to B$ given by $(e,\gamma) \mapsto \gamma(1)$ is a Serre fibration.

Proof. Consider a solid commutative diagram of the form

$$\begin{tikzcd} D^n \times \{0\} \arrow[r, "g"] \arrow[d, hook, "i"'] & P(f) \arrow[d, "p"] \\ D^n \times [0,1] \arrow[r, "H"] \arrow[ur, dashed, "\overline{H}"] & B. \end{tikzcd}$$

We may define the dashed map $\overline{H}$ as $\overline{H}(x,s) := (e(x), K(x,s,-))$, where

- The map $e\colon D^n \to E$ is the first component of g;

- The map $K\colon D^n \times [0,1] \times [0,1] \to Y$ is a continuous map satisfying the following three constraints:
  - To guarantee that $\overline{H}(x,s) \in P(f)$ we need $K(x,s,0) = f(e(x))$;
  - To guarantee commutativity of the top triangle, we need $K(x,0,t) = \gamma(x)(t)$, where $\gamma(x)$ is the second component of $g(x)$;
  - To guarantee commutativity of the top triangle, we need $K(x,s,1) = H(x,s)$.

  Such a continous map exists, since the inclusion $[0,1] \times \{0,1\} \cup \{0\} \times [0,1] \hookrightarrow [0,1] \times [0,1]$ has a continuous retract. □

Proposition B.3.6. *Consider the category* Top *of topological spaces, let F denote the collection of Serre fibrations and let W denote the collection of weak homotopy equivalences. Then the triple* (Top, W, F) *is a category with weak equivalences and fibrations. It has functorial factorizations.*

Proof. It is clear that the weak homotopy equivalences satisfy 2-out-of-3, and it is also straightforward to check that Serre fibrations are closed under pullbacks and composition and contain all homeomorphisms. This gives conditions (1) and (2) from Definition B.1.2.

To see condition (3), consider a pullback square

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

of topological spaces, where p is both a Serre fibration and a weak homotopy equivalence. We need to show that also p' is a weak homotopy equivalence. By Corollary B.3.4, it suffices to show that for every point $b' \in B'$ the fiber $F'_{b'} := p'^{-1}(b')$ is weakly contractible. But since p' is a pullback of p, this fiber is homeomorphic to the fiber $F_b = p^{-1}(b)$, which in turn is weakly contractible by Corollary B.3.4.

Finally, for condition (4), let $f\colon E \to B$ be a continuous map. We may then consider the *mapping path space* $P(f)$, defined as

$$P(f) := E^{[0,1]} \times_E B.$$

The map f then factors as

$$E \xrightarrow{i} P(f) \xrightarrow{p} B,$$

where i maps e to $(e, \mathrm{const}_{f(e)})$. The first map is a homotopy equivalence, hence in particular a weak homotopy equivalence, and the second map is a Serre fibration by Lemma B.3.5. This finishes the proof. □

Corollary B.3.7. *The localization functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *preserves all pullbacks along Serre fibrations.*

Proof. In light of Proposition B.3.6 this is an instance of Theorem B.1.5. □

We are now ready to prove our main application concerning homotopy pullbacks of topological spaces:

Proposition B.3.8. *The localization functor* $\Pi_\infty(-)\colon \mathrm{Top} \to \mathrm{An}$ *sends homotopy pullbacks of topological spaces to pullbacks of animae.*

Proof. Consider continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, and consider the pullback square

$$\begin{tikzcd} X \times^h_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P(f) \arrow[d, "p"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

defining $X \times^h_Z Y$. The path fibration p is a Serre fibration by Lemma B.3.5, hence this square is sent to a pullback square in An by Corollary B.3.7. This finishes the proof. □

B.4 Homotopy pushouts and pullbacks of chain complexes

TO BE WRITTEN.

B.5 Homotopy completeness

In the previous subsections, we have discussed the interaction of localizations with pushouts and pullbacks. In this section, we will extend this discussion to arbitrary limits and colimits, following Cisinski [Cis19, Section 7.7]

Definition B.5.1. Let C be an ∞-category with weak equivalences and fibrations. We say that C is *homotopy complete* if the following two conditions are satisfied:

(1) The product $\Pi_{i\in I} X_i$ of any small collection of fibrant objects $(X_i)_{i\in I}$ exists in C and is again fibrant;

(2) Given a small collection $(f_i\colon X_i \to Y_i)$ of morphisms between fibrant objects, if each f_i is a fibration (resp. trivial fibration), then also $\prod_i f_i$ is a fibration (resp. trivial fibration).

A functor between homotopy complete ∞-categories with weak equivalences and fibrations is called *homotopy continuous* if it is left exact and preserves small products of fibrant objects.

Dually, we say an ∞-category with weak equivalences and cofibrations C is *homotopy cocomplete* if C^{op} is homotopy complete; we obtain the analogous notion of a *homotopy cocontinuous* functor.

Example B.5.2. Consider the category $C = \mathrm{Top}$.

- Top is homotopy complete with respect to the weak homotopy equivalences and Serre fibrations from Proposition B.3.6;
- Top is homotopy cocomplete with respect to the homotopy equivalences and cofibrations from Lemma B.2.2;
- Top is homotopy cocomplete with respect to the weak homotopy equivalences and relative cell complexes from Lemma B.2.6.

Theorem B.5.3 (Cisinski [Cis19, Proposition 7.7.4, Theorem 7.7.6])**.** *Let C be a homotopy complete ∞-category with weak equivalences and fibrations.*

(1) The localization $L(C)$ has small limits, and the localization functor $\gamma\colon C \to L(C)$ is homotopy continuous;

(2) If all objects of C are fibrant, then $L(C)$ is in fact universal with this property: for any ∞-category E with small limits, the functor γ induces an equivalence of ∞-categories

$$\gamma^*\colon \mathrm{Fun}_*(L(C), E) \xrightarrow{\sim} \mathrm{Fun}_{h*}(C, E),$$

where $\mathrm{Fun}_{h*}(-,-)$ denotes the full subcategory of $\mathrm{Fun}(-,-)$ spanned by the homotopy continuous functors, while $\mathrm{Fun}_*(-,-)$ denotes the full subcategory spanned by the limit-preserving functors.

(3) Given a homotopy continuous functor $F\colon C \to D$, the induced functor $L(f)\colon L(C) \to L(D)$ preserves small limits;

Homotopy completeness is also useful for understanding the interaction between localization and the formation of functor categories. Recall from Lemma B.1.9 when C has functorial factorizations, the functor category $\mathrm{Fun}(I, C)$ can be equipped with the pointwise weak equivalences and fibrations for every ∞-category I. The following result relates the localization of $\mathrm{Fun}(I, C)$ with the functor category $\mathrm{Fun}(I, L(C))$:

Theorem B.5.4 (Cisinski [Cis19, Theorem 7.9.8]). *Let C be a homotopy complete ∞-category with weak equivalences and fibrations. Assume that C has functorial factorizations, and assume that a morphism in C is a weak equivalence if and only if it becomes an isomorphism in $L(C)$. Then for every small ordinary category I, the functor $\gamma_*\colon \mathrm{Fun}(I,C) \to \mathrm{Fun}(I,L(C))$ exhibits its target as a localization of its source at the pointwise weak equivalences, i.e. it induces an*

$$L(\mathrm{Fun}(I,C)) \xrightarrow{\sim} \mathrm{Fun}(I,L(C)).$$

B.6 Application 3: Geometric realizations

As an application of the previous discussion of homotopy (co)completeness, we will now show that the *geometric realization* $|X|$ of a simplicial topological space $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ provides a concrete model for the ∞-categorical colimit of the composite $\Pi_\infty \circ X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{An}$:

$$\Pi_\infty(|X|) \simeq \mathrm{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} \Pi_\infty(X_n).$$

(Because of this, the colimit functor $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}\colon \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{An}) \to \mathrm{An}$ is usually denoted by $|-|$ as well and is also referred to as 'geometric realization'.)

I want to explain Denis-Charles Cisinski for explaining the proof strategy to me.

Let us start by recalling the definition of $|X|$:

Definition B.6.1. Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ be a simplicial topological space. We define the *geometric realization* of X as the quotient space

$$|X| \quad := \quad \left(\coprod_{n\geq 0} X_n \times |\Delta^n|\right)/\!\sim \quad \in \quad \mathrm{Top},$$

where $|\Delta^n| = \{(y_i) \in [0,1]^{n+1} \mid \sum_i y_i = 1\}$ is the topological n-simplex from Example 2.5.10 and where the equivalence relation is generated by $(\varphi^*(x), y) \sim (x, |\Delta^\varphi|(y))$ for all $\varphi\colon [n] \to [m]$, $x \in X_m$ and $y \in |\Delta^n|$. In other words, $|X|$ is the following coequalizer in Top:

$$|X| \quad := \quad \mathrm{coeq}\left(\coprod_{\varphi\colon [n]\to[m]} X_m \times |\Delta^n| \quad \rightrightarrows \quad \coprod_{n\geq 0} X_n \times |\Delta^n|\right).$$

This construction is functorial in X, and so defines a functor

$$|-|\colon \mathrm{sTop} \to \mathrm{Top}.$$

Exercise B.6.2. Explicitly describe the resulting 'gluing relation' $(\varphi^*(x), y) \sim (x, |\Delta^\varphi|(y))$ in the following two special cases:

- When $\varphi = d_i\colon [n-1] \hookrightarrow [n]$;

- When $\varphi = s_i\colon [n+1] \twoheadrightarrow [n]$.

The essential feature of the geometric realization functor $|-|\colon \mathrm{sTop} \to \mathrm{Top}$ is that it is homotopy cocontinuous, at least when we restrict it to a suitable subcategory of sTop spanned by the 'Reedy cofibrant' simplicial spaces:

Definition B.6.3 (Reedy cofibrations)**.** Let $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ be a simplicial topological space. For $n \geq 0$, we define the *n-th latching object* L_nX as the union

$$L_nX \quad := \quad \bigcup_{i=0}^{n-1} s_i(X_{n-1}) \quad \subseteq \quad X_n$$

of all the degenerate n-simplices of X (hence $L_0X = \emptyset$). We say that X is *Reedy cofibrant* if the inclusion $L_nX \hookrightarrow X_n$ is a relative cell complex for every n. More generally, we say that a morphism $f\colon X \to Y$ of simplicial topological spaces is a *Reedy cofibration* if the induced map

$$X_n \bigsqcup_{L_nX} L_nY \quad \hookrightarrow \quad Y_n$$

is a relative cell complex for every n.

We let

$$\mathrm{sTop}^{\mathrm{Reedy}} \subseteq \mathrm{sTop}$$

denote the full subcategory spanned by the Reedy cofibrant simplicial topological spaces.

Theorem B.6.4 (Hovey [Hov99, Theorem 5.2.5], Cisinski [Cis19, Theorem 7.4.20])**.** *The category* sTop *of simplicial topological spaces becomes a category with weak equivalences and cofibrations, where:*

- *The weak equivalences are the pointwise weak homotopy equivalences;*

- *The cofibrations are the Reedy cofibrations.*

The following result contains the essential features of the geometric realization functor that allow us to relate it to ∞-categorical colimits:

Proposition B.6.5. *The functor* $|-|\colon \mathrm{sTop}^{\mathrm{Reedy}} \to \mathrm{Top}$ *is homotopy cocontinuous.*

Proof. It is clear that $|-|$ preserves small coproducts: if $X = \bigsqcup_{i\in I} X^i$, then we get $X_n \times |\Delta^n| \cong \bigsqcup_{i\in I} (X^i)_n \times |\Delta^n|$ for all n, and the relations that are being quotiented out all respect the disjoint components. It thus remains to show that $|-|$ is right exact.

We will now show that if $f\colon X \to Y$ is a Reedy cofibration, then $|f|\colon |X| \to |Y|$ is a relative cell complex. We proceed by constructing a filtration of $|f|$ and showing that each successive map in the filtration is a cell attachment.

For $n \geq -1$, let $\mathrm{sk}_n|X|$ denote the image of $\bigsqcup_{k\leq n} X_k \times |\Delta^k| \to |X|$ under the quotient map (so that $\mathrm{sk}_{-1}|X| = \emptyset$). This gives filtrations of $|X|$ and $|Y|$, and $|f|$ preserves these filtrations. We claim that for each $n \geq 0$, the induced map

$$\mathrm{sk}_n|X| \bigsqcup_{\mathrm{sk}_{n-1}|X|} \mathrm{sk}_{n-1}|Y| \to \mathrm{sk}_n|Y|$$

is a relative cell complex. Since f may be written as the transfinite composite

$$|X| = |X| \sqcup_{\mathrm{sk}_{-1}|X|} \mathrm{sk}_{-1}|Y| \to |X| \sqcup_{\mathrm{sk}_0|X|} \mathrm{sk}_0|Y| \to |X| \sqcup_{\mathrm{sk}_1|X|} \mathrm{sk}_1|Y| \to \cdots \to |X| \sqcup_{|X|} |Y| = |Y|,$$

this implies that also $|f|$ is a relative cell complex. To prove the claim, observe that the n-skeleton of $|X|$ is obtained from the $(n-1)$-skeleton via the following pushout:

$$\begin{tikzcd}
L_nX \times |\Delta^n| \bigsqcup_{L_nX \times \partial|\Delta^n|} X_n \times \partial|\Delta^n| \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}|X| \arrow[d] \\
X_n \times |\Delta^n| \arrow[r] & \mathrm{sk}_n|X|.
\end{tikzcd}$$

Using the analogous expression for the map $\mathrm{sk}_{n-1}|Y| \to \mathrm{sk}_n|Y|$, it will thus suffice to show that the inclusion

$$(X_n \bigsqcup_{L_nX} L_nY) \times |\Delta^n| \bigsqcup_{(X_n \bigsqcup_{L_nX} L_nY) \times \partial|\Delta^n|} Y_n \times \partial|\Delta^n| \to Y_n \times |\Delta^n|$$

is a relative cell complex. Since f is a Reedy cofibration, the map $X_n \bigsqcup_{L_nX} L_nY \to Y_n$ is a relative cell complex. Moreover, the inclusion $\partial|\Delta^n| \hookrightarrow |\Delta^n|$ is a cell attachment. The claim thus follows from the pushout-product lemma, Exercise B.2.5.

If $f\colon X \to Y$ is a Reedby cofibration such that each map $f_n\colon X_n \to Y_n$ is a weak homotopy equivalence, then one may deduce that each map $X_n \bigsqcup_{L_nX} L_nY \hookrightarrow Y_n$ is a trivial cofibration, and arguing as before it follows that $|f|$ is a transfinite composition of trivial cofibrations and hence itself a trivial cofibration.

It remains to show that $|-|\colon \mathrm{sTop}^{\mathrm{Reedy}} \to \mathrm{Top}$ preserves pushouts along Reedy cofibrations. But an easy skeletal induction argument shows that in fact $|-|\colon \mathrm{sTop} \to \mathrm{Top}$ preserves all pushouts. This finishes the proof. □

Using this result, we may now prove the main result of this section:

Proposition B.6.6. *Geometric realizations of Reedy-cofibrant simplicial topological spaces compute $\mathbf{\Delta}^{\mathrm{op}}$-indexed colimits in* An*: there is a commutative square of the form*

$$\begin{tikzcd} \mathrm{sTop}^{\mathrm{Reedy}} \arrow[r, "|-|"] \arrow[d, "(\Pi_\infty)_*"'] & \mathrm{Top} \arrow[d, "\Pi_\infty"] \\ \mathrm{sAn} \arrow[r, "\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}"] & \mathrm{An}. \end{tikzcd}$$

Proof. As a special case of Theorem B.5.4, we see that the functor $(\Pi_\infty)_*\colon \mathrm{sTop}^{\mathrm{Reedy}} \to \mathrm{sAn}$ exhibits sAn as the localization of the category $\mathrm{sTop}^{\mathrm{Reedy}}$ at the pointwise weak homotopy equivalences. It then follows from Theorem B.5.3 that the two vertical functors are homotopy cocontinuous. The top functor is homotopy cocontinuous by the previous proposition, and the bottom functor preserves small colimits hence is also homotopy cocontinuous. To show that the two composites are naturally isomorphic, it thus suffices by part (3) of Theorem B.5.3 to show that their induced functors $\mathrm{sAn} \to \mathrm{An}$ are naturally isomorphic.

Since both these functors preserve small colimits, it follows from Theorem A.4.4 that both of these functors are the left Kan extension of their restriction along the Yoneda embedding $Y\colon \mathbf{\Delta} \hookrightarrow \mathrm{sAn}$. It thus suffices to show that these two restrictions are naturally isomorphic. Indeed, we claim that in both cases the canonical transformation to the constant functor $\mathrm{const}_*\colon \mathbf{\Delta} \to \mathrm{An}$ is a natural isomorphism:

- The composite
$$\mathbf{\Delta} \xrightarrow{[n]\mapsto\Delta^n} \mathrm{sSet} \hookrightarrow \mathrm{sTop} \xrightarrow{|-|} \mathrm{Top} \xrightarrow{\Pi_\infty(-)} \mathrm{An}$$
sends $[n]$ to $\Pi_\infty(|\Delta^n|)$, which is contractible.

- The composite
$$\mathbf{\Delta} \xrightarrow{Y} \mathrm{sAn} \xrightarrow{\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}}} \mathrm{An}$$
sends $[n]$ to $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} Y([n])$. This is contractible: for any other anima X we have natural equivalences
$$\begin{aligned}\mathrm{Hom}_{\mathrm{An}}(\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} Y([n]), X) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathrm{An})}(Y[n], \mathrm{const}_X) &\overset{2.6.42}{\simeq} \mathrm{const}_X([n]) \\ &\simeq X \simeq \mathrm{Hom}_{\mathrm{An}}(*, X),\end{aligned}$$
hence we get $\mathrm{colim}_{\mathbf{\Delta}^{\mathrm{op}}} Y([n]) \simeq *$ by the Yoneda lemma.

This finishes the proof. □

Exercise sheets

Exercise sheet 1

Exercise 1.1. Consider continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, and let T be a topological space. Make precise the bijection between the set of continuous maps $T \to X \times^h_Z Y$ and the set of triples (t_X, t_Y, H) with $t_X\colon T \to X$, $t_Y\colon T \to Y$ and $H\colon T \times [0,1] \to Z$ satisfying $H_0 = f \circ t_X$ and $H_1 = g \circ t_Y$, i.e. the set of homotopy commutative diagrams

$$\begin{tikzcd} T \arrow[r, "t_X"] \arrow[d, "t_Y"'] & X \arrow[d, "f"] \arrow[dl, phantom, "\overset{H}{\Leftarrow}"] \\ Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

Formulate and prove the dual analogue for homotopy pushouts.

Exercise 1.2. Consider a diagram of topological spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Z \arrow[d, "\gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta"] \\ X' \arrow[r, "f'"] & Z' & Y' \arrow[l, "g'"'] \end{tikzcd}$$

which commutes up to homotopy, in the sense that we are given homotopies $H\colon \gamma \circ f \sim f' \circ \alpha$ and $K\colon \gamma \circ g \sim g' \circ \beta$.

- Construct a map $\alpha \times_\gamma \beta\colon X \times^h_Z Y \to X' \times^h_{Z'} Y'$. (This map will depend on the choices of homotopies H and K, though this is not reflected in our notation.)

- Given a second diagram of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\alpha'"'] & Z' \arrow[d, "\gamma'"] & Y' \arrow[l, "g'"'] \arrow[d, "\beta'"] \\ X'' \arrow[r, "f''"] & Z'' & Y'' \arrow[l, "g''"'] \end{tikzcd}$$

which commutes up to homotopy, with homotopies $H'\colon \gamma'\circ f' \sim f''\circ\alpha'$ and $K'\colon \gamma'\circ g' \sim g''\circ\beta'$, show that the pasted diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha'\circ\alpha"'] & Z \arrow[d, "\gamma'\circ\gamma"] & Y \arrow[l, "g"'] \arrow[d, "\beta'\circ\beta"] \\
X'' \arrow[r, "f''"] & Z'' & Y'' \arrow[l, "g''"']
\end{tikzcd}$$

again commutes up to homotopy.

- Show that there is a homotopy between $(\alpha'\times_{\gamma'}\beta')\circ(\alpha\times_\gamma\beta)$ and $(\alpha'\circ\alpha)\times_{\gamma'\circ\gamma}(\beta'\circ\beta)$.
- Show that $\alpha\times_\gamma\beta$ is a homotopy equivalence whenever α, β and γ are homotopy equivalences.

Exercise 1.3. Formulate and prove the analogous statement for homotopy pushouts.

Recall that we call a topological space a *space* if it is homotopy equivalent to a CW-complex.

Exercise 1.4. Show that if $f\colon Z\to X$ and $g\colon Z\to Y$ are maps of spaces, then the homotopy pushout $X\sqcup^h_Z Y$ is again a space.

Hint: use Exercise 1.3 and the cellular approximation theorem.

Exercise 1.5 (Bonus exercise)**.** Using the theorem by Milnor stated below, show that for maps of spaces $f\colon X\to Z$ and $g\colon Y\to Z$ the homotopy pullback $X\times^h_Z Y$ is again a space.

Theorem (Milnor)**.** Let X and Y be CW-complexes, and let $A_1, A_2\subseteq X$ and $B_1, B_2\subseteq Y$ be subcomplexes. Let $\mathrm{Map}((X,A_1,A_2),(Y,B_1,B_2))$ be the subspace of $\mathrm{Map}(X,Y)$ (equipped with the open-compact topology) consisting of those continuous maps $f\colon X\to Y$ satisfying $f(A_1)\subseteq B_1$ and $f(A_2)\subseteq B_2$). If X is a finite CW-complex, then $\mathrm{Map}((X,A_1,A_2),(Y,B_1,B_2))$ is homotopy-equivalent to a CW-complex.

Exercise sheet 2

Exercise 2.1 (Mapping telescope as homotopy pushout)**.** Recall that the *mapping telescope* of a sequence

$$Z_0 \xrightarrow{f_0} Z_1 \xrightarrow{f_1} Z_2 \xrightarrow{f_2} \dots$$

of pointed spaces is defined as

$$\mathrm{hocolim}_n Z_n := \left(\bigvee_{n\in\mathbb{N}} Z_n\wedge[n,n+1]_+\right)/\sim,$$

where the equivalence relation $\sim$ is given by $(x_n, n+1)\sim(f(x_n), n+1)$.

(1) For a space T, show that a map $t\colon \operatorname{hocolim}_n Z_n \to T$ corresponds to a family of pointed maps $t_n\colon Z_n \to T$ together with pointed homotopies $H_n\colon t_n \sim t_{n+1} \circ f_n$.

(2) Let $t'\colon \operatorname{hocolim}_n Z_n \to T$ be another such map, corresponding to the family (t'_n, H'_n). Show that a homotopy $t \sim t'$ corresponds to a family of homotopies $K_n\colon t_n \sim t'_n$ together with a homotopy of homotopies $Z_n \times [0,1] \times [0,1] \to T$ which restricts to H_n and H'_n on $Z_n \times [0,1] \times \{0,1\}$ and to K_n and $K_{n+1} \circ (f_n \times \mathrm{id}_{[0,1]})$ on $Z_n \times \{0,1\} \times [0,1]$.

(3) Define Z as the following homotopy pushout:

$$\begin{tikzcd} \bigvee_{n\geq 0} Z_n \arrow[r] \arrow[d] & \bigvee_{k\geq 0} Z_{2k} \arrow[d] \\ \bigvee_{k\geq 0} Z_{2k+1} \arrow[r] & Z. \end{tikzcd}$$

Here the top horizontal map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k}$ for $n = 2k$ even, and the map $f_{2k+1}\colon Z_{2k+1}Z_{2k+2}$ for $n = 2k+1$ odd. The left vertical map uses the identity maps $\mathrm{id}\colon Z_n \to Z_{2k+1}$ for $n = 2k+1$ odd, an the map $f_{2k}\colon Z_{2k} \to Z_{2k+1}$ for $n = 2k$ even.

Show that one can describe maps $t\colon Z \to T$ with the same data as in (1). Show that one describe homotopies between maps $t, t'\colon Z \to T$ with the same data as in (2).

(4) Conclude that there is a homotopy equivalence $Z \simeq \operatorname{hocolim}_n Z_n$.

Exercise 2.2 (Pasting law for homotopy pushout squares)**.** Consider a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_X"'] & Y \arrow[r, "g"] \arrow[d, dashed, "\varphi_Y"] \arrow[dl, phantom, "\swarrow\sim"] & Z \arrow[d, "\varphi_Z"] \arrow[dl, phantom, "\swarrow\sim"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z'. \end{tikzcd}$$

(1) Use the universal properties of the homotopy pushouts to construct maps

$$Z \sqcup^h_Y (Y \sqcup^h_X X') \to Z \sqcup^h_X X' \qquad \text{and} \qquad Z \sqcup^h_X X' \to Z \sqcup^h_Y (Y \sqcup^h_X X')$$

and show that they are (homotopy) inverse to each other.

(2) Assume that the left square is a homotopy pushout square (i.e. the map $Y \sqcup^h_X X' \to Y'$ induced by the universal property of $Y \sqcup^h_X X'$ is a homotopy equivalence.) Show that the right square is a homotopy pushout square if and only if the outer composite rectangle is a homotopy pushout square.

Hint: Use part (1), and use Exercise 1.3 from last sheet to obtain a homotopy equivalence $Z \sqcup^h_Y (Y \sqcup^h_X X') \xrightarrow{\sim} Z \sqcup^h_Y Y'$.

(3) Formulate the pasting law for homotopy *pullback* squares by dualizing the statement in (2). Give a brief proof sketch by indicating how to dualize parts (1) and (2).

Exercise 2.3 (Puppe sequence)**.** Consider a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of pointed spaces.

(1) Using Exercise 2.2[1], show that the (reduced[2]) cofiber of g is homotopy equivalent to ΣX.

(2) Show similarly that the cofiber of the resulting map $Z \to \Sigma X$ is homotopy equivalent to ΣY.

(3) Explain how to obtain from this the *Puppe sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X \xrightarrow{\Sigma f} \Sigma Y \xrightarrow{\Sigma g} \Sigma Z \to \Sigma^2 X \to \dots,$$

in which every pair of composable maps that appear form a cofiber sequence.

(4) Deduce that every cohomology theory E^*, as defined in the lecture, gives rise to a long exact sequence of the form

$$\dots \xrightarrow{g^*} E^{n-1}(Y) \xrightarrow{f^*} E^{n-1}(X) \to E^n(Z) \xrightarrow{g^*} E^n(Y) \xrightarrow{f^*} E^n(X) \to E^{n+1}(Z) \xrightarrow{g^*} E^{n+1}(Y) \xrightarrow{f^*} \dots$$

for every cofiber sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$.

Exercise 2.4 (Eilenberg-MacLane spectra)**.** By Brown representability, reduced singular cohomology $\widetilde{H}^*(-;A)$ with coefficients in an abelian group A is represented by some spectrum E. Show that E is the Eilenberg-MacLane spectrum HA (up to homotopy equivalence).

Exercise 2.5 (Mayer-Vietoris sequence)**.** Consider a homotopy pushout square of pointed spaces

$$\begin{tikzcd} C \arrow[r, "k"] \arrow[d, "l"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] & X. \end{tikzcd}$$

[1]Strictly speaking it is the version of Exercise 2.2 for *reduced* homotopy pushouts, i.e. for homotopy pushouts of *pointed* spaces. The proof is the same as that for Exercise 2.2 and you may simply use it in Exercise 2.3.

[2]From now on, all cofibers of pointed maps are understood to be *reduced* cofibers.

(1) Show that the cofiber of the map $(i,j)\colon A\vee B\to X$ is homotopy equivalent to $\Sigma(C)$. *Hint: you may use the Fact below.*

(2) Show similarly that the cofiber of the resulting map $X\to\Sigma(C)$ is homotopy equivalent to $\Sigma A\vee\Sigma B$.

(3) Conclude that for a functor $F\colon \mathrm{h}\mathcal{S}_*^{\mathrm{op}}\to \mathrm{Ab}$ satisfying both the Wedge axiom and the Mayer-Vietoris property, we obtain a long exact sequence

$$\cdots\to F(\Sigma X)\xrightarrow{(\Sigma i^*,\Sigma j^*)}F(\Sigma A)\times F(\Sigma B)\xrightarrow{(\Sigma k)^*-(\Sigma l)^*}F(\Sigma C)\xrightarrow{\partial}F(X)\xrightarrow{(i^*,j^*)}F(A)\times F(B)\xrightarrow{k^*-l^*}F(C).$$

Fact. Consider a homotopy commutative diagram

$$\begin{tikzcd}
X_2 \arrow[rr, "f_2"] \arrow[dr, "k"] \arrow[dd, "l"'] & & Y_2 \arrow[dr, "k'"] \arrow[dd, "l'"' near start] & \\
& X_0 \arrow[rr, "f_0" near start] \arrow[dd, "i" near end] & & Y_0 \arrow[dd, "i'"] \\
X_1 \arrow[rr, "f_1" near start] \arrow[dr, "j"'] & & Y_1 \arrow[dr, "j'"] & \\
& X \arrow[rr, "f"'] & & Y,
\end{tikzcd}$$

meaning that all six faces commute up to given homotopy and that there is a homotopy of homotopies between the two induced homotopies $i'\circ f_0\circ k\sim j'\circ f_1\circ l$ of maps $X_2\to Y$.

Then passing to horizontal cofibers induces a homotopy commutative square

$$\begin{tikzcd}
C(f_2) \arrow[r] \arrow[d] & C(f_0) \arrow[d] \\
C(f_1) \arrow[r] & C(f).
\end{tikzcd}$$

If the left and right vertical face of the original cube are both homotopy pushout squares, then the induced square on homotopy cofibers is also a homotopy pushout square.

While one can prove this fact by hand, it will be an easy consequence of the general theory of colimits in ∞-categories, hence we will postpone its proof.

Exercise 2.6 (Bonus exercise)**.** Convince yourself that the Fact is true.

Exercise sheet 3

Exercise 3.1 (Yoneda lemma)**.** Let C be an ordinary category and consider an object X of C. Consider the functor

$$y_X\colon C^{\mathrm{op}} \to \mathrm{Set}, \qquad Y \mapsto \mathrm{Hom}_C(Y, X).$$

For any other functor $F\colon C^{\mathrm{op}} \to \mathrm{Set}$, show that evaluation at the identity $\mathrm{id}_X \in \mathrm{Hom}_C(X, X) = y_X(X)$ induces a bijection

$$\mathrm{Nat}(y_X, F) \xrightarrow{\cong} F(X),$$

where the left-hand side denotes the set of natural transformations $y_X \to F$.

Conclude from this that for two pointed spaces X and Y, any natural transformation $[-, X]_* \to [-, Y]_*$ of functors $\mathrm{h}\mathcal{S}_*^{\mathrm{op}} \to \mathrm{Set}$ is given by postcomposition with a pointed map $f\colon X \to Y$, and that this condition uniquely determines f up to homotopy.

Exercise 3.2. Let C and D be ∞-categories. Show that a functor $f\colon C \to D$ is an equivalence if and only if it admits both a section (i.e. a functor $g\colon D \to C$ such that $\mathrm{id}_D \cong fg$) and a retraction (i.e. a functor $h\colon D \to C$ such that $\mathrm{id}_C \cong hf$), and that in this case there is a natural isomorphism $g \cong h$.

Conclude that an inverse to f is unique up to isomorphism if it exists.

Exercise 3.3. Show that if $f\colon C \to D$ and $f'\colon D \to E$ are equivalences with inverses $g\colon D \to C$ and $g'\colon E \to D$, then also the composite $f'f\colon C \to E$ is an equivalence with inverse $gg'\colon E \to C$.

Exercise 3.4. Show that equivalences of ∞-categories satisfy the 2-out-of-6 property: given functors $f\colon C \to D$, $g\colon D \to E$ and $h\colon E \to F$ such that gf and hg are equivalences, also the functors f, g, h and hgf are equivalences.

Exercise 3.5. Construct for every morphism $f\colon x \to y$ commutative triangles in C of the form

$$\begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y. \end{tikzcd}$$

Exercise 3.6 (Bonus exercise)**.** Show that products of ∞-categories are associative: given ∞-categories C, D and E, produce an equivalence of ∞-categories

$$C \times (D \times E) \xrightarrow{\sim} (C \times D) \times E.$$

Exercise 3.7 (Bonus exercise)**.** Also the coproduct of ∞-categories is unital, associative and commutative. Given ∞-categories C, D and E, construct the necessary functors:

$$\begin{array}{llll} C \to C \sqcup \emptyset, & C \to \emptyset \sqcup C, & C \sqcup D \to D \sqcup C, & (C \sqcup D) \sqcup E \to C \sqcup (D \sqcup E), \\ C \sqcup \emptyset \to C, & C \to \emptyset \sqcup C, & D \sqcup C \to C \sqcup D, & C \sqcup (D \sqcup E) \to (C \sqcup D) \sqcup E. \end{array}$$

Verify for yourself that these functors are indeed inverse to one another. (You don't need to write out this verification.)

Exercise sheet 4

Exercise 4.1. Show that coproducts of ∞-categories are *disjoint*: given ∞-categories C and D, the commutative square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & C \arrow[d, "i_C"] \\ D \arrow[r, "i_D"] & C \sqcup D \end{tikzcd}$$

is a pullback square. In similar spirit, show that for every functor $f\colon C \to D$ the commutative square

$$\begin{tikzcd} \emptyset \arrow[r, "\mathrm{id}_\emptyset"] \arrow[d] & \emptyset \arrow[d] \\ C \arrow[r, "f"] & D \end{tikzcd}$$

is a pullback square. (In both cases also explain which square I mean when I say *the* commutative square.)

Exercise 4.2. Consider ∞-categories C and D, and let $p_C\colon C \to *$ and $p_D\colon D \to *$ be their functors to the terminal ∞-category. Construct a commutative square of the form

$$\begin{tikzcd} C \times D \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_C"'] & C \arrow[d, "p_C"] \\ D \arrow[r, "p_D"] & * \end{tikzcd}$$

and show it is a pullback square.

Exercise 4.3. Consider a commutative square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & D \arrow[d, "h"] \\ E \arrow[r, "k"] & F \end{tikzcd}$$

and assume that h is an equivalence. Show that g is an equivalence if and only if the square is a pullback square.

The goal of the following two exercises is to give an alternative description of the pullback $C \times_E D$ of two functors $f\colon C \to E$ and $g\colon D \to E$.

Exercise 4.4. Consider a functor $g\colon D \to E$ between ∞-categories.

(1) Construct for every ∞-category C a commutative diagram of the form

$$\begin{tikzcd} C \times D \arrow[r, "\mathrm{id}_C \times g"] \arrow[d, "\mathrm{pr}_D"'] & C \times E \arrow[r, "\mathrm{pr}_C"] \arrow[d, "\mathrm{pr}_E"] & C \arrow[d, "p_C"] \\ D \arrow[r, "g"] & E \arrow[r, "p_E"] & *. \end{tikzcd}$$

Combining the pasting lemma with Exercise 4.2, deduce that the left-hand square is a pullback square.

(2) Construct a commutative diagram of the following form:

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "(g{,}\mathrm{id}_D)"'] & E \arrow[d, "(\mathrm{id}_E{,}\mathrm{id}_E)"] \\ E \times D \arrow[r, "\mathrm{id}_E \times g"] \arrow[d, "\mathrm{pr}_D"'] & E \times E \arrow[d, "\mathrm{pr}_2"] \\ D \arrow[r, "g"] & E, \end{tikzcd}$$

where $\mathrm{pr}_2\colon E \times E \to E$ means the projection to the second factor of E. Show that the vertical composite of these squares

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "\mathrm{pr}_D \circ (g{,}\mathrm{id}_D)"'] & E \arrow[d, "\mathrm{pr}_1 \circ (\mathrm{id}_E{,}\mathrm{id}_E)"] \\ D \arrow[r, "g"] & E \end{tikzcd}$$

is a pullback square. *Hint:* use Exercise 4.3.

(3) Combine parts (1) and (2) with the pasting law of pullback squares to conclude that the commutative square

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[d, "(g{,}\mathrm{id}_D)"'] & E \arrow[d, "(\mathrm{id}_E{,}\mathrm{id}_E)"] \\ E \times D \arrow[r, "\mathrm{id}_E \times g"] & E \times E \end{tikzcd}$$

is a pullback square.

Exercise 4.5. Consider two functors $f\colon C \to E$ and $g\colon D \to E$ between ∞-categories.

(1) Construct a commutative diagram of the following form:

$$\begin{tikzcd}
C \times_E D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "(\mathrm{pr}_C, \mathrm{pr}_D)"'] & D \arrow[r, "g"] \arrow[d, "(g, \mathrm{id}_D)"] & E \arrow[d, "(\mathrm{id}_E, \mathrm{id}_E)"] \\
C \times D \arrow[r, "f \times \mathrm{id}_D"] \arrow[d, "\mathrm{pr}_C"'] & E \times D \arrow[r, "\mathrm{id}_E \times g"] \arrow[d, "\mathrm{pr}_E"] & E \times E \\
C \arrow[r, "f"] & E.
\end{tikzcd}$$

(In other words, produce for each of the three squares a natural isomorphism which makes the square commute.)

(2) Composing the two squares on the left, we obtain a commutative square of the form

$$\begin{tikzcd}
C \times_E D \arrow[r, "\mathrm{pr}_D"] \arrow[d, "\mathrm{pr}_C \circ (\mathrm{pr}_C, \mathrm{pr}_D)"'] & D \arrow[d, "\mathrm{pr}_E \circ (g, \mathrm{id}_D)"] \\
C \arrow[r, "f"] & E.
\end{tikzcd}$$

Show that this square is a pullback square (for example by proving this square is actually isomorphic to the canonical one defining $C \times_E D$.)

(3) Recall from Exercise 4.4 that the bottom-left and top-right squares in this diagram are also pullback squares. Deduce from the pasting law of pullback squares that the commutative square

$$\begin{tikzcd}
C \times_E D \arrow[r] \arrow[d, "(\mathrm{pr}_C, \mathrm{pr}_D)"'] & E \arrow[d, "(\mathrm{id}_E, \mathrm{id}_E)"] \\
C \times D \arrow[r, "f \times g"] & E \times E
\end{tikzcd}$$

is a pullback square.

Exercise 4.6 (Bonus exercise)**.** Explain heuristically why the pullback $(C \times D) \times_{E \times E} E$ appearing in part (3) of the previous exercise should be a fiber product of C and D over E: explain how we may think of a functor into it as specifying functors into C and D which agree after mapping further to E.

(Warning: this heuristic argument would not constitute a proof and is just meant to aid intuition.)

Exercise sheet 5

Exercise 5.1. Use the commutative square axiom to define functors

$$\min\colon [1] \times [1] \to [1], \qquad \text{and} \qquad \max\colon [1] \times [1] \to [1]$$

that behave like the 'maximum' and 'minimum' functors on the partial order $\{0 \le 1\}$, in the sense that they satisfy the equations

$$\max(x,0) = x = \max(0,x), \qquad \max(x,1) = 1 = \max(1,x),$$
$$\min(x,0) = 0 = \min(0,x), \qquad \min(x,1) = x = \min(1,x).$$

(Also show that the functors you define actually satisfy these equations.)

In the lecture we used the commutative square axiom to construct functors $p_0, p_2 \colon [1] \times [1] \to [2]$, pictorially represented by the following diagrams:

$$p_0 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d, equal] \arrow[dr] & 1 \arrow[d] \\ 0 \arrow[r] & 2, \end{tikzcd} \qquad p_2 \quad = \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 2 \arrow[d, equal] \\ 1 \arrow[r] & 2. \end{tikzcd}$$

As mentioned in the lecture, these functors provide a way of regarding commutative triangles in C as certain degenerate forms of commutative squares: a commutative triangle $\sigma \colon [2] \to C$ of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"] & & z \end{tikzcd}$$

may alternatively be encoded as a commutative square $[1] \times [1] \to C$ whose restriction $\{0\} \times [1]$ is the identity

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\mathrm{id}_x"'] & y \arrow[d, "g"] \\ x \arrow[r, "gf"] & z. \end{tikzcd}$$

Alternatively we may encode it as a commutative square $[1] \times [1] \to C$ whose restriction to $[1] \times \{1\}$ is the identity

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "gf"'] & y \arrow[d, "g"] \\ z \arrow[r, "\mathrm{id}_z"] & z. \end{tikzcd}$$

The following exercise makes this precise:

Exercise 5.2. Show that for every ∞-category C the following two commutative squares are pullback squares:

$$\begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_0^*"] \arrow[d, "\mathrm{ev}_0"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}(\{0\}\times[1],C) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Fun}([2],C) \arrow[r, "p_2^*"] \arrow[d, "\mathrm{ev}_2"'] & \mathrm{Fun}([1]\times[1],C) \arrow[d, "\mathrm{res}"] \\ C \arrow[r, "(p_{[1]})^*"] & \mathrm{Fun}([1]\times\{1\},C). \end{tikzcd}$$

(It suffices to prove it for one of them; the other case is symmetric.) The functors labeled ‘res’ are given by precomposition with the respective inclusions of $\{0\}\times[1]$ and $[1]\times\{1\}$ into $[1]\times[1]$.

Exercise 5.3. Let C_0 and C_1 be ∞-categories and consider morphisms $f_0\colon x_0 \to y_0$ and $g_0\colon y_0 \to z_0$ in C_0 and morphisms $f_1\colon x_1 \to y_1$ and $g_1\colon y_1 \to z_1$ in C_1. Show that the composite $(g_0,g_1)\circ(f_0,f_1)\colon (x_0,x_1) \to (z_0,z_1)$ in $C_0\times C_1$ is given by $(g_0\circ f_0, g_1\circ f_1)$.

Exercise 5.4. Let X be a topological space. Show that its singular complex $\mathrm{Sing}(X) \in \mathrm{sSet}$ is a Kan complex.

Exercise 5.5. Let C be a small category.

- Show that its nerve $N(C) \in \mathrm{sSet}$ is a quasicategory;
- Provide an example for which $N(C)$ is not a Kan complex;
- Give a complete characterization of those categories C for which $N(C)$ is a Kan complex.

Exercise 5.6 (Bonus exercise)**.** Let C be a Kan-enriched category. Show that its homotopy-coherent nerve $N^{\Delta}(C) \in \mathrm{sSet}$ is a quasicategory.

Exercise sheet 6

Exercise 6.1. Show that for every ∞-category C and every anima X the inclusion $\gamma_C\colon C^{\simeq} \to C$ induces an equivalence

$$(\gamma_C)_*\colon \mathrm{Map}(X, C^{\simeq}) \xrightarrow{\sim} \mathrm{Map}(X, C).$$

For the next exercise, we need the following fact:[3]

Fact. For every ∞-category C, the functor $p_{[1]}^*\colon C \to \mathrm{Fun}([1], C)$ is fully faithful.

Exercise 6.2. Let C be an ∞-category. Show that the following statements are equivalent:

(1) The ∞-category C is an anima;

(2) The functor $p_{[1]}^*\colon C \to \mathrm{Fun}([1], C)$ is an equivalence;

(3) The functor $\pi_{\mathrm{Iso}}\colon \mathrm{Iso}(C) \to \mathrm{Fun}([1], C)$ is an equivalence.

[3] This fact follows from Exercise 6.4. However, this reasoning is circular since the fact is indirectly used to prove Exercise 6.4. If we had taken characterization (2) in Exercise 6.2 as our definition of animae then the circularity can be avoided.

Exercise 6.3. Show that a morphism $f\colon X \to Y$ of animae is an equivalence iff the induced map $f^*\colon \operatorname{Map}(Y,Z) \to \operatorname{Map}(X,Z)$ is an equivalence for every anima Z.

Exercise 6.4. Show that the functor $p_{[1]}\colon [1] \to *$ exhibits the terminal ∞-category as a localization of $[1]$ at all its morphisms, i.e. show that $|[1]| \simeq *$.

Hint: use Exercise 6.2 and Exercise 6.3.

Exercise 6.5. Let C be a pointed ∞-category that admits fibers and cofibers. Show that the functor $\Sigma\colon C \to C$ is left adjoint to $\Omega\colon C \to C$: for all $X, Y \in C$ there is a natural equivalence

$$\operatorname{Hom}_C(\Sigma X, Y) \simeq \operatorname{Hom}_C(X, \Omega Y).$$

Exercise 6.6 (Bonus exercise)**.** Let C be an ∞-category and let x and y be objects of C. The goal of this exercise is to show that the Hom anima $\operatorname{Hom}_C(x,y)$ is indeed an anima.

(1) For an object z in an ∞-category D, we define the *slice category* $D_{/z}$ via the following pullback square:

$$\begin{tikzcd} D_{/z} \arrow[r] \arrow[d] & \operatorname{Fun}([1],D) \arrow[d, "\mathrm{ev}_1"] \\ * \arrow[r, "z"] & D. \end{tikzcd}$$

In other words, objects of $D_{/z}$ are morphisms $f\colon w \to z$ in D and morphisms are commutative squares in D of the form

$$\begin{tikzcd} w \arrow[r] \arrow[d] & w' \arrow[d] \\ z \arrow[r, equal] & z, \end{tikzcd}$$

or equivalently (by Exercise 5.2) commutative triangles in D of the form

$$\begin{tikzcd} w \arrow[rr] \arrow[dr] & & w' \arrow[dl] \\ & z. & \end{tikzcd}$$

Show that for every other ∞-category E there is an equivalence

$$\operatorname{Fun}(E, D_{/z}) \xrightarrow{\sim} \operatorname{Fun}(E,D)_{/\mathrm{const}_z}.$$

(2) Let $\pi\colon C_{/y} \to C$ denote the composite functor

$$C_{/y} \hookrightarrow \operatorname{Fun}([1],C) \xrightarrow{\mathrm{ev}_0} C.$$

Show that the following commutative square is a pullback square:

$$\begin{tikzcd}
\mathrm{Fun}([1], C_{/y}) \arrow[r, "\pi_*"] \arrow[d, "\mathrm{ev}_1"'] & \mathrm{Fun}([1], C) \arrow[d, "\mathrm{ev}_1"] \\
C_{/y} \arrow[r, "\pi"] & C.
\end{tikzcd}$$

Hint: Use the Segal axiom and Exercise 5.2.

(3) Show that there is a pullback square

$$\begin{tikzcd}
\mathrm{Hom}_C(x, y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C_{/y} \arrow[d, "\pi"] \\
* \arrow[r, "x"] & C.
\end{tikzcd}$$

(4) Show that $\mathrm{Hom}_C(x, y)$ is an anima.

Hint: Use Exercise 6.2.

Exercise sheet 7

Exercise 7.1. Let $f\colon X \to Y$ be a morphism in a stable ∞-category. Show that there is an isomorphism[4]

$$\mathrm{cofib}(f) \cong \mathrm{fib}(f)[1].$$

Exercise 7.2. Let $f\colon X \to Y$ be a morphism in a stable ∞-category C. Show that the following are equivalent:

(1) The morphism f is an isomorphism in C;

(2) The cofiber $\mathrm{cofib}(f)$ of f is a zero object in C;

(3) The fiber $\mathrm{fib}(f)$ of f is a zero object in C.

Recall that the *splitting lemma* provides equivalent conditions for a short exact sequence in an abelian category to be a *split short exact sequence*. In the following exercise we will prove a splitting lemma for stable ∞-categories:

Exercise 7.3. Let C be a stable ∞-category and let $X \xrightarrow{i} Y \xrightarrow{p} Z$ be an exact sequence in C. Show that the following are equivalent:

[4]Recall that the notation $X[1]$ is an alternative notation for $\Sigma(X)$ in a stable ∞-category.

(1) The associated morphism $\mathrm{fib}(i) \to X$ is null-homotopic;

(2) The associated morphism $Z \to \mathrm{cofib}(p)$ is null-homotopic;

(3) The map i admits a retract: there is a morphism $r\colon Y \to X$ satisfying $ri \simeq \mathrm{id}_X$;

(4) The map p admits a section: there is a morphism $s\colon Z \to Y$ satisfying $ps \simeq \mathrm{id}_Z$;

(5) There is an isomorphism $Y \cong X \oplus Z$ in C making the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "{(\mathrm{id}_X,0)}"'] & Y \arrow[r, "p"] \arrow[d, "\cong"] & Z \\ & X \oplus Z. \arrow[ur, "\mathrm{pr}_Z"'] & \end{tikzcd}$$

For the following exercise, you may use the fact (to be proved in two different ways in Exercise 7.6 and Exercise 7.7 below) that in a stable ∞-category C the set $[X,Y] := \pi_0 \mathrm{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y in C admits a canonical abelian group structure, and that the composition maps

$$- \circ f\colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ -\colon [X,Y] \to [X,Y']$$

are group homomorphisms for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C.

Exercise 7.4. Let C be a stable ∞-category and consider a commutative square in C of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "g"'] & W. \end{tikzcd}$$

Show that the following conditions are equivalent:

(1) The square is exact;

(2) The sequence

$$X \xrightarrow{(f,u)} Y \oplus Z \xrightarrow{(v,-g)} W$$

is an exact sequence.

Exercise 7.5. Let C and D be ∞-categories with finite limits. Show that there is an equivalence

$$\mathrm{Sp}(C \times D) \xrightarrow{\sim} \mathrm{Sp}(C) \times \mathrm{Sp}(D).$$

Exercise 7.6 (Bonus exercise)**.** Let C be a semiadditive ∞-category.

(1) Show that for all objects X and Y in C the set $[X,Y] := \pi_0 \operatorname{Hom}_C(X,Y)$ of homotopy classes of morphisms from X to Y admits a canonical structure of an abelian monoid, where addition of $f,g \in [X,Y]$ is given by the composite

$$f+g\colon \quad X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

(2) Show that for all morphisms $f\colon X' \to X$ and $g\colon Y \to Y'$ in C the composition maps

$$- \circ f\colon [X,Y] \to [X',Y] \qquad \text{and} \qquad g \circ -\colon [X,Y] \to [X,Y']$$

are homomorphisms of abelian monoids.

(3) Let M be an abelian monoid. Show that M is an abelian group if and only if the map

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\colon M \oplus M \to M \oplus M, \qquad (x,y) \mapsto (x, x+y)$$

is a bijection of sets.

(4) Conclude that the semiadditive ∞-category C is in fact *additive* if and only if the abelian monoid $[X,Y]$ is an abelian group for all X and Y.

Exercise 7.7 (Bonus exercise)**.** Let C be a pointed ∞-category with finite limits.

(1) Show that for every object X, the loop space ΩX admits the structure of a group object in the homotopy category $\operatorname{Ho}(C)$: there are maps

$$e\colon * \to \Omega X, \qquad m\colon \Omega X \times \Omega X \to \Omega X, \qquad \text{and} \qquad i\colon \Omega X \to \Omega X$$

that satisfy the usual group relations up to homotopy.

(2) Show that for objects X and Y in C, the set $[X, \Omega Y]$ is canonically a group. Show that $[X, \Omega^2 Y]$ is an abelian group.

(3) Deduce that when C is stable, the Hom-set $[X,Y]$ in $\operatorname{Ho}(C)$ is an abelian group for all X and Y, and that composition in C defines group homomorphisms.

(4) Show that the resulting abelian group structure on $[X,Y]$ agrees with the one defined in Exercise 7.6.

Remark 7.8. Applying the previous exercise to C^{op}, we deduce that for every X the suspension ΣX is a *cogroup object* in $\operatorname{Ho}(C)$, and hence that $[\Sigma X, Y]$ admits an abelian group structure for all Y. Applying this to $C = \mathrm{An}_*$, this provides a proof of the claim in Remark 1.3.9.

Exercise sheet 8

Exercise 8.1. Let C and D be ∞-categories. Assume that C is stable and that D has finite limits. Show that precomposition with the functor $\Omega^\infty\colon \mathrm{Sp}(D) \to D$ induces an equivalence

$$(\Omega^\infty)_*\colon \mathrm{Fun}^{\mathrm{lex}}(C, \mathrm{Sp}(D)) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(C, D),$$

where $\mathrm{Fun}^{\mathrm{lex}}(-,-)$ denotes the full subcategory of left-exact functors.

Exercise 8.2. Let C be a stable ∞-category. Use the previous exercise to show that the Hom functor

$$\mathrm{Hom}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathrm{An}$$

(essentially) uniquely lifts to a *mapping spectrum functor*

$$\mathrm{map}_C(-,-)\colon C^{\mathrm{op}} \times C \to \mathrm{Sp},$$

where 'lifts' means that $\Omega^\infty \mathrm{map}_C(X,Y) \cong \mathrm{Hom}_C(X,Y)$. Show that for $C = \mathrm{Sp}$ the resulting functor

$$\mathrm{map}_{\mathrm{Sp}}(-,-)\colon \mathrm{Sp}^{\mathrm{op}} \times \mathrm{Sp} \to \mathrm{Sp}$$

agrees with the mapping spectrum functor defined in the lecture.

Exercise 8.3. Let X be a pointed anima. Consider the prespectrum $\Sigma^{\infty,\mathrm{pre}}(X) := (\Sigma^n X, \sigma_n)$, where the structure maps are given by the unit maps

$$\Sigma^n X \to \Omega\Sigma(\Sigma^n X) = \Omega\Sigma^{n+1} X$$

of the adjunction $\Sigma \dashv \Omega$. Show that the spectrum associate to the prespectrum $\Sigma^{\infty,\mathrm{pre}}(X)$ is $\Sigma^\infty(X)$.

Hint: show that all the maps

$$(\Sigma^\infty\Sigma^n X)[-n] \to (\Sigma^\infty\Sigma^{n+1} X)[-(n+1)]$$

appearing in the definition of the associated spectrum are isomorphisms.

Remark. Since the prespectrum $\Sigma^{\infty,\mathrm{pre}}(X)$ consists of all the suspensions of X, we may call it the 'suspension prespectrum' of X. This is the historical reason for the name 'suspension spectrum'.

Exercise 8.4. Show that the functor $\Sigma^\infty\colon \mathrm{An}_* \to \mathrm{Sp}$ sends smash products of pointed animae to tensor products of spectra: produce a natural isomorphism

$$\Sigma^\infty(X) \otimes \Sigma^\infty(Y) \cong \Sigma^\infty(X \wedge Y)$$

for $X, Y \in \mathrm{An}_*$.

Exercise 8.5. Let C be an ordinary category. Show that the category $\mathrm{Mon}(C)$ is equivalent to the usual category of monoids in C:

(1) Given $X \in \mathrm{Mon}(C)$, show that the maps $e\colon * \to X_1$ and $m\colon X_1 \times X_1 \to X_1$ do indeed give X_1 the structure of a monoid;

(2) Given a monoid M in C, show that we obtain an object $X \in \mathrm{Mon}(C)$ by setting $X_n := M^n$, and by defining for $\varphi\colon [m] \to [n]$ in $\mathbf{\Delta}$ the map $\varphi^*\colon M^n \to M^m$ by

$$\varphi^*(x_1,\ldots,x_n) := (y_1,\ldots,y_m), \qquad y_i := \prod_{\varphi(i-1)<j\leq\varphi(i)} m_j.$$

(3) Show that these two constructions define the desired equivalence of categories.

Exercise sheet 9

Exercise 9.1. Let E be a spectrum and let X be a pointed anima. Show that our new definition of the E-cohomology of X,

$$E^k(X) := \pi_{-k}(\mathrm{map}(\Sigma^\infty(X), E)),$$

agrees with the definition from Construction 1.3.7 in Chapter 1.

Exercise 9.2. Consider the inclusion functor

$$j\colon [1] \hookrightarrow \mathbf{\Delta}_+, \qquad j(0) = [-1], \quad j(1) = [0].$$

Let C be an ∞-category with pullbacks. Using the pointwise formula for Kan extensions, show that the restriction functor

$$j^*\colon \mathrm{Fun}(\mathbf{\Delta}_+, C) \to \mathrm{Fun}([1], C)$$

admits a right adjoint

$$j_*\colon \mathrm{Fun}([1], C) \to \mathrm{Fun}(\mathbf{\Delta}_+, C).$$

Show that for a morphism $f\colon Y \to X$ in C, regarded as an object in $\mathrm{Fun}([1], C)$, its right Kan extension $j_*(f)$ along j is given by

$$(j_*(f))_n = Y^{\times_X^n} = Y \times_X Y \times_X \cdots \times_X Y.$$

Remark. The underlying simplicial object of this augmented simplicial object is called the *Čech nerve of f* and plays an important role in topos theory / ∞-categories of sheaves.

Exercise 9.3. Show that the inclusion functor

$$\mathrm{Grp}(\mathrm{An}) \hookrightarrow \mathrm{Mon}(\mathrm{An})$$

admits a left adjoint

$$(-)^{\mathrm{grp}} \colon \mathrm{Mon}(\mathrm{An}) \to \mathrm{Grp}(\mathrm{An})$$

which on objects is given by $M \mapsto \Omega \mathbf{B} M$. (This is called the *group completion functor*.)

Exercise 9.4. Show that the forgetful functor $\mathrm{Grp}(\mathrm{An}) \to \mathrm{An}_*$ admits a left adjoint

$$F^{\mathrm{Grp}}(-) \colon \mathrm{An}_* \to \mathrm{Grp}(\mathrm{An})$$

which is given on objects by $X \mapsto \Omega\Sigma X$. (This is called the *free group functor*.)

Exercise 9.5. Let x be an object of an ∞-category C. Denote by $\mathrm{Aut}_C(x)$ the connected component of $\mathrm{End}_C(x) := \mathrm{Hom}_C(x,x)$ spanned by those endomorphisms $x \to x$ that are invertible. Show that composition of morphisms in C can be enhanced to a group structure on $\mathrm{Aut}_C(x)$.

Remark. One can also show that $\mathrm{End}_C(x)$ itself has the structure of a monoid via composition of morphisms, but this requires some theory we haven't introduced.

Exercise 9.6 (Bonus exercise)**.** The goal of this exercise is to show that the forgetful functor $\mathrm{Mon}(\mathrm{An}) \to \mathrm{An}$ admits a left adjoint

$$F^{\mathrm{Mon}}(-) \colon \mathrm{An} \to \mathrm{Mon}(\mathrm{An})$$

given by the 'anima of words in X of arbitrary length':

$$F^{\mathrm{Mon}}(X)_1 \simeq \coprod_{n \geq 0} X^n.$$

(1) Let $\mathbf{\Delta}_{\mathrm{int}} \subseteq \mathbf{\Delta}$ denote the subcategory which has all objects, but which has only those morphisms $[n] \to [m]$ that are inclusions of an interval $\{i \leq i+1 \leq \cdots \leq j-1 \leq j\}$ into $[m]$ for some $0 \leq i \leq j \leq m$ (these are called the *inert* morphisms). Show that evaluation at $[1] \in \mathbf{\Delta}_{\mathrm{int}}$ defines an equivalence

$$\mathrm{Fun}^{\mathrm{Seg}}({\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}}, \mathrm{An}) \xrightarrow{\sim} \mathrm{An},$$

where the left-hand side denotes those functors $X \colon {\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}} \to \mathrm{An}$ satisfying the *Segal condition*, i.e. the map

$$(e_i^*)_{i=1}^n \colon X_n \to X_1^n$$

is an equivalence for all $n \geq 0$.

Hint: the inverse is given by right Kan extension.

(2) Show that the left Kan extension functor

$$i_!\colon \operatorname{Fun}({\mathbf{\Delta}_\infty}^{\mathrm{op}}, \mathrm{An}) \to \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An})$$

preserves Segal objects.

(3) Show that the resulting composite

$$F^{\mathrm{Mon}}(-)\colon \mathrm{An} \simeq \operatorname{Fun}^{\mathrm{Seg}}({\mathbf{\Delta}_{\mathrm{int}}}^{\mathrm{op}}, \mathrm{An}) \xrightarrow{i_!} \operatorname{Fun}^{\mathrm{Seg}}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{An}) = \mathrm{Mon}(\mathrm{An})$$

is left adjoint to the forgetful functor, and use the pointwise formula for Kan extensions to show that the formula

$$F^{\mathrm{Mon}}(X)_1 \simeq \bigsqcup_{n \geq 0} X^n$$

is satisfied.

Exercise sheet 10

Recall from Construction 4.3.9 the functor $\mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$.

Exercise 10.1. Let $M\colon \mathrm{Span}(\mathrm{Fin}) \to C$ be a commutative monoid. Show that $M \circ \mathrm{Cut}\colon \mathbf{\Delta}^{\mathrm{op}} \to C$ is a monoid. Show that $M \circ \mathrm{Cut}$ is a group if and only if M is a commutative group.

In particular, we obtain forgetful functors

$$U := \mathrm{Cut}^*\colon \mathrm{CMon}(C) \to \mathrm{Mon}(C) \qquad \text{and} \qquad U := \mathrm{Cut}^*\colon \mathrm{CGrp}(C) \to \mathrm{Grp}(C).$$

Exercise 10.2. Assume that C is an ordinary category with finite products.

(1) Show that the category $\mathrm{CMon}(C)$ is equivalent to the category of commutative monoids in C defined in the usual way: an object M of C equipped with maps $+\colon M \times M \to M$ and $0\colon * \to M$ such that the following three diagrams commute:

$$\begin{tikzcd} M \times * \arrow[r, "\mathrm{id}_M \times 0"] \arrow[dr, "\simeq"'] & M \times M \arrow[d, "+"] & M \arrow[l, "0 \times \mathrm{id}_M"'] \arrow[dl, "\simeq"] \\ & M, & \end{tikzcd} \qquad \begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id}_M \times +"] \arrow[d, "+ \times \mathrm{id}_M"] & M \times M \arrow[d, "+"] \\ M \times M \arrow[r, "+"] & M, \end{tikzcd} \qquad \begin{tikzcd} M \times M \arrow[rr, "\tau", "\simeq"'] \arrow[dr, "+"'] & & M \times M \arrow[dl, "+"] \\ & M. & \end{tikzcd}$$

(2) Combining part (1) with Exercise 8.5, deduce that in this case the forgetful functor $U\colon \mathrm{CMon}(C) \hookrightarrow \mathrm{Mon}(C)$ is fully faithful.

Remark. For a general ∞-category, it is *not* true that the forgetful functor $\mathrm{CMon}(C) \to \mathrm{Mon}(C)$ is fully faithful! In other words, in the world of ∞-categories, it is no longer true that commutativity is a *property* of a monoid: it is additional *structure* one needs to provide.

Exercise 10.3. For a finite set S, the inclusion $\mathrm{Fin} \hookrightarrow \mathrm{Span}(\mathrm{Fin})$ of the subcategory containing all objects but only the forward-pointing maps induces a functor on slice categories of the form

$$\mathrm{Fin}_{/S} \to \mathrm{Span}(\mathrm{Fin})_{/S}.$$

Show that this functor admits a left adjoint, which is given on objects by sending a span

$$T \xleftarrow{f} U \xrightarrow{g} S$$

to the map $g\colon U \to S$.

Exercise 10.4. Let $M \in \mathrm{Mon}(\mathrm{An})$ be a monoid in animae. Show that M is a group in An if and only if its monoid of path components $\pi_0(M)$ is a group in the ordinary sense.

Conclude that the commutative monoid $\mathbf{B}M$ is a commutative group for every commutative monoid M.

Exercise 10.5. Show that for two ∞-categories C and D, the projection maps $C \times D \to C$ and $C \times D \to D$ induce an equivalence

$$|C \times D| \xrightarrow{\sim} |C| \times |D|$$

on geometric realizations. (Recall that $|C|$ is obtained from C by inverting all morphisms: it is the localization $C[W^{-1}]$ with $W = \mathrm{Map}([1], C)$.)

Exercise sheet 11

Exercise 11.1. Let A be an abelian group, which under the inclusion $\mathrm{Ab} = \mathrm{CGrp}(\mathrm{Set}) \hookrightarrow \mathrm{CGrp}(\mathrm{An})$ we may see as a commutative group object in An.

(1) Show that $\mathbf{B}^n A$ is an Eilenberg-MacLane space $K(A, n)$ for all $n \geq 0$;

(2) Conclude that the spectrum $\mathbf{B}^\infty A$ is the Eilenberg-MacLane spectrum HA classifying singular cohomology with coefficients in A.

An important notion in the context of symmetric monoidal ∞-categories is that of a *dualizable object*, generalizing the concept of dual vector spaces.

Definition. Let C be a symmetric monoidal ∞-category. An object X of C is called *dualizable* if there exists another object $X^\vee$ together with morphisms $\mathrm{ev}_X : X^\vee \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^\vee$ that satisfy the triangle identities: the composites

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \qquad \text{and} \qquad X^\vee \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_X} X^\vee$$

are homotopic to the respective identities. The object $X^\vee$ is then determined uniquely up to isomorphism, and is called the *dual* of X.

Exercise 11.2. Let C be the ordinary category Vect_k of vector spaces over some field k. Show that every finite-dimensional vector space k is dualizable in Vect_k, with dual given by $V^\vee = V^* := \mathrm{Hom}_k(V, k)$.

Exercise 11.3. Let C be a symmetric monoidal ∞-category, and let X be an *invertible* object of C, in the sense that there exists an object Y and an isomorphism $X \otimes Y \cong \mathbb{1}_C$. Show that X is dualizable with $X^\vee = Y$.

Exercise 11.4. Let C be a symmetric monoidal ∞-category, and assume that C is stable and that the tensor product $X \otimes - : C \to C$ is exact for every object X of C.

(1) Show that the zero object is dualizable;

(2) Deduce that the subcategory of C consisting of the dualizable objects is closed under pushouts (*Hint:* show it is closed under direct sums and cofibers);

(3) Deduce that the (unreduced) suspension spectrum $\mathbb{S}[X]$ is dualizable in Sp for every finite anima $X \in \mathrm{An}^{\mathrm{fin}}$.

Exercise 11.5. Show that the map $p_{\mathrm{univ}} : E_{\mathrm{univ}} \to \mathrm{Gr}_k(\mathbb{C}^\infty)$ from Definition 5.2.2 is a rank k complex vector bundle. *Hint:* compare with the proof of Lemma 5.2.15.

Exercise sheet 12

Exercise 12.1. Let E_1 and E_2 be complex vector bundles over a topological space X. Show that their direct sum

$$E_1 \oplus E_2 := E_1 \times_X E_2$$

is again a complex vector bundle over X.

Here is a fact mentioned in the lecture that you may use in the next exercise:

Fact. Let X be a compact Hausdorff space. For every complex vector bundle $E \to X$ there exists another vector bundle $E' \to X$ such that the direct sum $E \oplus E'$ is a trivial vector bundle (i.e. isomorphic to $X \times \mathbb{C}^N$ for some $N \in \mathbb{N}$).

The Bonus Exercise 12.7 below provides hints on how to prove this fact.

Exercise 12.2. Let X be a compact Hausdorff space and consider the composite

$$\pi_0 \mathrm{Vect}(X) \to K^0(X) \to \widetilde{K}^0(X),$$

which is a morphism of abelian monoids. Show that this map is surjective, and that two vector bundles E and E' are sent to the same element of $\widetilde{K}^0(X)$ if and only if there exists an isomorphism $E \oplus \underline{\mathbb{C}}^n \cong E' \oplus \underline{\mathbb{C}}^m$ for some n and m.

Exercise 12.3. Using the commutative monoid structure on $\mathrm{Vect}(\mathrm{pt}) = \bigsqcup_{k=0}^{\infty} BU(k)$, show that $\mathrm{Vect}(X) := \mathrm{Map}(X, \mathrm{Vect}(\mathrm{pt}))$ admits a canonical commutative monoid structure as well. Show that the induced abelian monoid structure on $\pi_0 \mathrm{Vect}(X)$ is precisely the direct sum operation on vector bundles.

Exercise 12.4. Show that the description of the simplicial topological space $\mathrm{Gr}^{\mathbf{\Delta}} := \mathrm{Gr} \circ \mathrm{Cut} \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ given in the lecture is indeed correct: show that for a morphism $\varphi \colon [m] \to [n]$ in $\mathbf{\Delta}$, the map $\varphi^* \colon \mathrm{Gr}^{\mathbf{\Delta}}([n]) \to \mathrm{Gr}^{\mathbf{\Delta}}([m])$ is given by

$$\varphi^*(V_j) = (W_k), \qquad W_k := \bigoplus_{\varphi(k-1) < j \leq \varphi(k)} V_j.$$

Write down explicitly what this map does in the following four cases:

- $\varphi = d_0 \colon [n-1] \hookrightarrow [n]$;
- $\varphi = d_n \colon [n-1] \hookrightarrow [n]$;
- $\varphi = d_i \colon [n-1] \hookrightarrow [n]$ for $0 < i < n$;
- $\varphi = s_i \colon [n+1] \twoheadrightarrow [n]$ for $0 \leq i \leq n$.

Exercise 12.5. Let $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ be a simplicial topological space. The definition of its geometric realization $|X|$ uses the 'gluing relation' $(\varphi^*(x), y) \sim (x, |\Delta^{\varphi}|(y))$. Provide an explicit description of this relation in the following two special cases:

- When $\varphi = d_i \colon [n-1] \hookrightarrow [n]$;
- When $\varphi = s_i \colon [n+1] \twoheadrightarrow [n]$.

Describe in words why we may think of the relations for these two special cases as 'gluing faces' and 'collapsing degenerate simplices', respectively.

Exercise 12.6 (Bonus exercise)**.** Construct for two complex vector bundles E_1 and E_2 over X their *tensor product* $E_1 \otimes E_2$, and show that it is again a vector bundle. Show that your construction defines an operation

$$- \otimes -\colon \pi_0\mathrm{Vect}(X) \times \pi_0\mathrm{Vect}(X) \to \pi_0\mathrm{Vect}(X)$$

which is unital, commutative and associative. Show that it is distributive w.r.t. $- \oplus -$, making $\pi_0\mathrm{Vect}(X)$ into a *commutative semiring*.

Deduce from this that the complex K-theory group $K^0(X)$ of X is in fact a *commutative ring*.

Remark. The ring structure on $K^0(X)$ from the previous exercise can be lifted to the level of spectra: the connective K-theory spectrum ku can be given the structure of a commutative ring spectrum.

Exercise 12.7 (Bonus exercise)**.** Provide a proof of the above Fact. Use the following steps:

(1) Show that every complex vector bundle $p\colon E' \to X$ admits an *inner product*, i.e. a map $\langle -, - \rangle\colon E \oplus E \to \mathbb{C}$ such that the restriction $E_x \oplus E_x \to \mathbb{C}$ to each fiber $E_x := p^{-1}(x)$ is an (complex) inner product.

   Hint: use that for every open cover $\{U_\alpha\}$ of X there exists a partition of unity with respect to this open cover, i.e. collection of maps $\varphi_\beta\colon X \to [0,1]$ satisfying $\sum_\beta \varphi_\beta(x) = 1$ for all x, such that for each β the set $\varphi_\beta^{-1}((0,1])$ is contained in some U_α and such that for each x only finitely many $\varphi_\beta(x)$ are non-zero.

(2) Using this, show that every subbundle E of a vector bundle E' admits an orthogonal complement $E^\perp \subseteq E'$, which is itself a subbundle satisfying $E \oplus E^\perp \cong E'$.

(3) Show that for every vector bundle E there exist an embedding $E \hookrightarrow X \times \mathbb{C}^N$ which identifies E with a subbundle of the trivial bundle.

(4) Conclude.

Exercise sheet 13

Exercise 13.1. Compute all the homotopy groups of the spectra ku and KU.

Exercise 13.2. Let E be a spectrum. Show the following basic properties of E-local spectra:

(1) A spectrum L is E-local if and only if $\mathrm{map}(X, L) = 0$ for every E-acyclic spectrum X;

(2) Given an exact sequence $X \to Y \to Z$, if two of these three spectra are E-local then so is the third;

(3) In particular, if L is E-local then so are all its shifts $E[n]$;

(4) A limit $\lim_i L_i$ of E-local spectra is E-local;

(5) A retract of an E-local spectrum is E-local.

Exercise 13.3. Show that the mapping spectrum $\mathrm{map}(E,Y)$ is E-local for an arbitrary spectrum Y. Using this, complete the incomplete proof from the lecture: show that for a morphism $l\colon X \to X'$ of spectra the following are equivalent:

(1) The morphism $l\colon X \to X'$ is an E-equivalence;

(2) The induced map $-\circ l\colon \mathrm{Hom}_{\mathrm{Sp}}(X',Y) \to \mathrm{Hom}_{\mathrm{Sp}}(X,Y)$ is an equivalence for all E-local spectra Y.

Exercise 13.4. Let E and F be spectra. Show that the following two conditions are equivalent:

(1) Every E-acyclic spectrum X is also F-acyclic;

(2) Every F-local spectrum Y is also E-local.

Remark. Two spectra E and F are said to *have the same Bousfield class* if a spectrum X is E-acyclic if and only if it is F-acyclic. By the previous exercise, E and F have the same Bousfield class if and only if the subcategories $\mathrm{Sp}_E \subseteq \mathrm{Sp}$ and $\mathrm{Sp}_F \subseteq \mathrm{Sp}$ agree.

The following two exercises use material from the lecture on Monday February 3rd, and will therefore be marked as 'bonus'.

Exercise 13.5 (Bonus exercise)**.** Let $X \in \mathrm{Sp}_{\mathbb{Q}}$ be a rational spectrum (i.e. $\mathbb{Q}$-local). Assume that X is p-complete for some prime p. Show that X is the zero spectrum.

Exercise 13.6 (Arithmetic fracture square, Bonus exercise)**.** Show that for every spectrum X, the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_p X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

is a pullback square. Here the product is taken over all prime numbers p.

Exercise 13.7 (Bonus exercise)**.** Let E, F and X be spectra. Is it always true that there is an equivalence $L_E L_F X \simeq L_F L_E X$? If yes, provide a proof. If no, provide a counterexample.

Bibliography

[Ada74] J. F. Adams. *Stable homotopy and generalised homology*. Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press. X, 373 p. £ 3.00 (1974). 1974.

[And+14] Matthew Ando, Andrew J. Blumberg, David Gepner, Michael J. Hopkins, and Charles Rezk. "An ∞-categorical approach to R-line bundles, R-module Thom spectra, and twisted R-homology". *J. Topol.* 7.3 (2014), pp. 869–893.

[Ati61] Michael F. Atiyah. "Thom complexes". *Proc. Lond. Math. Soc. (3)* 11 (1961), pp. 291–310.

[Bar17] Clark Barwick. "Spectral Mackey functors and equivariant algebraic K-theory. I." *Adv. Math.* 304 (2017), pp. 646–727.

[BG99] James C. Becker and Daniel Henry Gottlieb. "A history of duality in algebraic topology". *History of topology*. Amsterdam: Elsevier, 1999, pp. 725–745.

[BH21] Tom Bachmann and Marc Hoyois. *Norms in motivic homotopy theory*. Vol. 425. Astérisque. Paris: Société Mathématique de France (SMF), 2021.

[Bot59] Raoul Bott. "The stable homotopy of the classical groups". *Ann. Math. (2)* 70 (1959), pp. 313–337.

[Bou79] A. K. Bousfield. "The localization of spectra with respect to homology". *Topology* 18 (1979), pp. 257–281.

[BR20] David Barnes and Constanze Roitzheim. *Foundations of stable homotopy theory*. English. Vol. 185. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2020.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Vol. 347. Lect. Notes Math. Springer, Cham, 1973.

[Cis+24] Denis-Charles Cisinski, Bastiaan Cnossen, Kim Nguyen, and Tashi Walde. "Formalization of higher categories". *Available at https://drive.google.com/file/d/1lKaq7watGGl3xvjqw9qHjm6SDPFJ2-0o/view* (2024).

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2019.

[CP86] Jean-Marc Cordier and Timothy Porter. "Vogt's theorem on categories of homotopy coherent diagrams". *Math. Proc. Camb. Philos. Soc.* 100 (1986), pp. 65–90.

[Die08] Tammo tom Dieck. *Algebraic topology*. EMS Textb. Math. Zürich: European Mathematical Society (EMS), 2008.

[DT24] Jack Davies and Liz Tatum. *Lecture notes for Algebraic Topology II (Stable and chromatic homotopy theory)*. 2024.

[ES52] Samuel Eilenberg and Norman Steenrod. *Foundations of algebraic topology*. Vol. 15. Princeton Math. Ser. Princeton University Press, Princeton, NJ, 1952.

[Fre37] H. Freudenthal. "Über die Klassen der Sphärenabbildungen. I. Große Dimensionen." *Compos. Math.* 5 (1937), pp. 299–314.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. "Universality of multiplicative infinite loop space machines". *Algebr. Geom. Topol.* 15.6 (2015), pp. 3107–3153.

[Gro16] Moritz Groth. "Characterizations of abstract stable homotopy theories". *arXiv preprint arXiv:1602.07632* (2016).

[Har80] Bruno Harris. "Bott periodicity via simplicial spaces". *J. Algebra* 62 (1980), pp. 450–454.

[Hat03] Allen Hatcher. "Vector bundles and K-theory". *Available online at http://www.math.cornell.edu/~{}hatcher* (2003).

[Hau+22] Rune Haugseng, Fabian Hebestreit, Sil Linskens, and Joost Nuiten. "Two-variable fibrations, factorisation systems and ∞-categories of spans". *arXiv preprint arXiv:2011.11042* (2022).

[Hei23] Hadrian Heine. "An equivalence between enriched ∞-categories and ∞-categories with weak action". *Adv. Math.* 417 (2023), p. 140.

[Hir15] Philip S. Hirschhorn. *Functorial CW-approximation*. Preprint, arXiv:1508.01944 [math.AT] (2015). 2015. URL: https://arxiv.org/abs/1508.01944.

[HM24] Claudius Heyer and Lucas Mann. *6-Functor Formalisms and Smooth Representations*. Preprint, arXiv:2410.13038 [math.CT] (2024). 2024. URL: https://arxiv.org/abs/2410.13038.

[Hov99] Mark Hovey. *Model categories*. Vol. 63. Math. Surv. Monogr. Providence, RI: American Mathematical Society, 1999. ISBN: 0-8218-1359-5.

[HS23] Fabian Hebestreit and Jan Steinebrunner. *A short proof that Rezk's nerve is fully faithful*. Preprint, arXiv:2312.09889 [math.CT] (2023). 2023. URL: https://arxiv.org/abs/2312.09889.

[HW21] Fabian Hebestreit and Ferdinand Wagner. *Algebraic and Hermitian K-Theory*. 2021.

[Joy08] André Joyal. "Notes on quasi-categories". *preprint* (2008).

[JT07] André Joyal and Myles Tierney. "Quasi-categories vs Segal spaces". *Categories in algebra, geometry and mathematical physics. Conference and workshop in honor of Ross Street's 60th birthday, Sydney and Canberra, Australia, July 11–16/July 18–21, 2005*. Providence, RI: American Mathematical Society (AMS), 2007, pp. 277–326.

[Lan21] Markus Land. *Introduction to infinity-categories*. Compact Textb. Math. Cham: Birkhäuser, 2021.

[LMS86] L. G. jun. Lewis, J. P. May, and M. Steinberger. *Equivariant stable homotopy theory. With contributions by J. E. McClure*. Vol. 1213. Lect. Notes Math. Springer, Cham, 1986.

[Lur09a] Jacob Lurie. *(Infinity,2)-Categories and the Goodwillie Calculus I*. Preprint, arXiv:0905.0462 [math.CT] (2009). 2009. URL: https://arxiv.org/abs/0905.0462.

[Lur09b] Jacob Lurie. *Higher topos theory*. Vol. 170. Ann. Math. Stud. Princeton, NJ: Princeton University Press, 2009.

[Lur17] Jacob Lurie. "Higher algebra". *https://www.math.ias.edu/~lurie/papers/HA.pdf* (2017).

[Lur24] Jacob Lurie. "Kerodon". *online textbook, available at https://kerodon.net* (2024).

[Mat76] Michael Mather. "Pull-backs in homotopy theory". *Can. J. Math.* 28 (1976), pp. 225–263.

[May72] J. P. May. *The geometry of iterated loop spaces*. Vol. 271. Lect. Notes Math. Springer, Cham, 1972.

[Mil59] John Milnor. "On spaces having the homotopy type of a CW-complex". *Transactions of the American Mathematical Society* 90.2 (1959), pp. 272–280.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Vol. 76. Ann. Math. Stud. Princeton University Press, Princeton, NJ, 1974.

[MS76] D. McDuff and G. Segal. "Homology fibrations and the 'group-completion' theorem". *Invent. Math.* 31 (1976), pp. 279–284.

[Nar21] Denis Nardin. *Introduction to stable homotopy theory*. 2021.

[Nik17] Thomas Nikolaus. “The group completion theorem via localizations of ring spectra”. *Available on author’s webpage* (2017).

[Qui67] D. G. Quillen. *Homotopical algebra*. Vol. 43. Lect. Notes Math. Springer, Cham, 1967.

[Rez01] Charles Rezk. “A model for the homotopy theory of homotopy theory”. *Trans. Am. Math. Soc.* 353.3 (2001), pp. 973–1007.

[Seg68] Graeme Segal. “Classifying spaces and spectral sequences”. *Publ. Math., Inst. Hautes Étud. Sci.* 34 (1968), pp. 105–112.

[Seg74] Graeme Segal. “Categories and cohomology theories”. English. *Topology* 13 (1974), pp. 293–312.

[Ser53] Jean-Pierre Serre. “Groupes d’homotopie et classes de groupes abéliens”. *Ann. Math. (2)* 58 (1953), pp. 258–294.

[Sta63] James Dillon Stasheff. “Homotopy associativity of H-spaces. II”. *Transactions of the American Mathematical Society* 108.2 (1963), pp. 293–312.

[SW53] Edwin H. Spanier and J. H. C. Whitehead. “A first approximation to homotopy theory”. *Proc. Natl. Acad. Sci. USA* 39 (1953), pp. 655–660.

[Swa62] R. G. Swan. “Vector bundles and projective modules”. *Trans. Am. Math. Soc.* 105 (1962), pp. 264–277.

[Swi75] Robert M. Switzer. *Algebraic topology - homotopy and homology*. Vol. 212. Grundlehren Math. Wiss. Springer, Cham, 1975.

[Tho54] René Thom. “Quelques propriétés globales des variétés différentiables”. *Comment. Math. Helv.* 28 (1954), pp. 17–86.