Seminar: Topological K-theory

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We will study topological K-theory, an invariant of topological spaces built out of vector bundles. Key properties are homotopy invariance, a form of excision, and Bott periodicity. We end the seminar with some classical applications: determining the precise number of linearly independent vector fields over spheres, and showing that the only real division algebras are the reals, the complex numbers, the quaternions and the octonions.

Date and time: Tuesday, 16:15-17:45.

Location: M009.

Organizational meeting: Wednesday, July 15th, 10:15, M102.

If you are interested in giving a talk in the seminar, please send me an email: bastiaan.cnossen@ur.de.

Seminar program

1. Vector bundles (October 13, 2026)

Introduce the notion of a vector bundle and of a morphism between vector bundles. Discuss examples of vector bundles: trivial bundles, tangent bundles (of spheres, and more generally, of manifolds), the tautological line bundle over projective spaces. Explain the pullback construction [Hat03, Proposition 1.5]. Construct the Whitney sum and tensor product of two vector bundles. Introduce the notion of a section and explain how choices of sections define trivial subbundles in a given vector bundle.

Literature: [Hat03, §1.1]

2. Inner products and complements (Leo Kreuzer, October 20, 2026)

Discuss the notion of paracompactness [Hat03, §1.2]. Use partitions of unity to construct inner products on vector bundles [Hat03, Proposition 1.2], and use this to show that every subbundle of a vector bundle over a paracompact base space has a complement [Hat03, Proposition 1.3]. Prove [Hat03, Proposition 1.4] and use this to establish the Serre-Swan theorem [Ati67, p. 30-31].

Literature: [Hat03, §1.1 & 1.2], [Ati67]

3. Classification of vector bundles (October 27, 2026)

Prove the homotopy invariance of the pullback construction [Hat03, Theorem 1.6]. Explain the classification of vector bundles over spheres in terms of clutching functions.

Literature: [Hat03, §1.2]

4. Classification of vector bundles II (November 3, 2026)

Introduce the tautological line bundle over the Grassmannian [Hat03, Lemma 1.15] and prove the classification theorem for vector bundles [Hat03, Theorem 1.16].

Explain how the Whitney sum operation is induced by maps between Grassiannians [Swi75, pp. 11.46–11.53].

Literature: [Hat03, §1.2], [Swi75]

5. K-theory I (Matthias Wagenhofer, November 10, 2026)

Introduce the group completion of a commutative monoid, and use this to define $K^0(X)$ for a compact Hausdorff space as the group completion of the commutative monoid of isomorphism classes of vector bundles over X, with respect to the Whitney sum. Explain how the tensor product of vector bundles induces a ring structure on $K^0(X)$, and how this induces an "exterior product". Define the reduced K-theory group $\widetilde{K}^0(X)$, and describe it alternatively via stable isomorphism classes of vector bundles.

Discuss the functoriality in X, and state [Hat03, Proposition 2.9] and [Hat03, Theorem 2.11]. Use these results to construct a 2-periodic cohomology theory [Hat03, p. 55].

Literature: [Hat03, §2.1]

6. K-theory II (Hongye Chang, November 17, 2026)

Prove that the K-theory ring is graded commutative [Hat03, Proposition 2.14] and prove [Hat03, Proposition 2.15].

Then start establishing the statements used as black obxes in the previous talk: prove [Hat03, Proposition 2.9, Lemma 2.10], state [Hat03, Theorem 2.2] as a black box, and use it to prove Bott periodicity [Hat03, Theorem 2.11].

Literature: [Hat03, §2]

7. Bott periodicity I (November 24, 2026)

Explain the general form of clutching functions for vector bundles over $X \times S^2$. Explain the overall strategy for proving the fundamental product theorem [Hat03, Theorem 2.2]. Prove [Hat03, Proposition 2.4] and [Hat03, Proposition 2.6].

Literature: [Hat03, §2]

8. Bott periodicity II (Ziang Li, December 1, 2026)

Finish the proof of [Hat03, Theorem 2.2], which includes proving [Hat03, Proposition2.7] and [Hat03, Lemma 2.8] as intermediate results.

Literature: [Hat03, §2]

9. The Adams operations (December 8, 2026)

Introduce the notion of a (special) λ-ring [Wei13, p. II.4.1], and discuss [Wei13, Example II.4.3]. Formulate the splitting principle for λ-rings with positive structure [Wei13].

Literature: [Wei13], [Hat03, §2.3]

10. The splitting principle (December 15, 2026) Determine $K^0(\mathbb{C}P^n)$ [Hat03, Proposition 2.24], prove the Leray–Hirsch theorem [Hat03, Theorem 2.25], and deduce the splitting principle.

Literature: [Hat03, §2.3]

11. Real division algebras and the parallelisability of spheres (December 22, 2026)

State [Hat03, Theorem 2.16], and reduce it to the Hopf invariant 1 problem [Hat03, Lemmas 2.17 & 2.18]. Prove [Hat03, Theorem 2.19].

Literature: [Hat03, §2.3]

12. Vector fields on spheres: lower bounds (January 12th, 2027)

Define vector fields of a vector bundle. Show that the tangent bundle of a sphere S^n admits a non-zero vector field if and only if n is odd [Hus93, Theorem 1.4]. Define orthogonal multiplications, and show they give rise to orthonormal vector fields on spheres [Hus93, Theorem 2.2]. Define quadratic forms (M, f) and their Clifford algebras [Hus93, §12.4 & 5], and spell out the example C_{k-1}. Show that there is a one-to-one correspondence between orthogonal multiplications $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ and C_{k-1}-module structures on $\mathbb{R}^n$ [Hus93, Theorem 2.4]. Explain the idea behind the proof of the claim that the maximum number k such that there exists an orthogonal multiplication $\mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is the Radon-Hurwitz number $k = \rho(n) = 8d(n) + 2^{c(n)}$, where $n = (2a+1)2^{4d(n)+c(n)}$ with $0 \leq c(n) \leq 3$ [Hus93, Theorem 8.2]. (In particular, there are at least $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.)

Literature: [Hus93, Chapter 12]

13. Vector fields on spheres: upper bounds I (January 19th, 2027)

Define the Stiefel manifold $V_{l,n} \subseteq (S^{n-1})^l$, and show that S^{n-1} admits k linearly independent vector fields if and only if the map $V_{k+1,n} \to S^{n-1}, (v_1, \ldots, v_{k+1}) \mapsto v_1$ admits a section. Use such a section to construct a map of sphere bundles $S(\underline{\mathbb{R}}^n) \to S(\gamma^{\oplus n})$ over $\mathbb{R}P^k$ between the sphere bundle of the trivial rank n bundle and the n-fold Whitney sum of the tautological line bundle.

Define fiber homotopy equivalences between sphere bundles, and state Dold's criterion for them. Use it to prove that the map $S(\underline{\mathbb{R}}^n) \to S(\gamma^{\oplus n})$ is a fiber homotopy equivalence [Rak15, Lemma 4.6]. Define the Grothendieck group $SF(X)$ of fiber homotopy equivalences of spherical fibrations, and define the J-homomorphism $J_{\mathbb{C}} \colon K(X) \to SF(X)$. Conclude that for even $n = 2m$ we have $J_{\mathbb{C}}(m(\gamma_{\mathbb{C}} - 1)) = 0 \in SF(\mathbb{R}P^k)$ whenever S^{n-1} admits k linearly independent vector fields.

Literature: [Rak15, §3-4]

14. Vector fields on spheres: upper bounds II (January 26th, 2027)

Define the Thom space $\mathrm{Th}(E)$ and Thom class u_E of a vector bundle, and explain the Thom isomorphism. Introduce the cannibalistic classes $\rho^l(E) = \psi^l(u_E)/u_E$. Show that if two vector bundles have fiber homotopy equivalent sphere bundles, then their ρ^l-classes agree up to a factor $\psi^l(1+y)/(1+y)$ [Rak15, Lemma 7.10]. State the calculation of $\widetilde{K}(\mathbb{R}P^k) \cong \mathbb{Z}/2^{\lfloor k/2 \rfloor}$. Define the subgroup $V(X) \subseteq K(X)$, with $J'(X) = K(X)/V(X)$, and show that the map $K(X) \to J'(X)$ factors through $J(X)$ [Rak15, Proposition 7.14]. Deduce that if $J(m(\gamma_{\mathbb{C}} - 1)) = 0$, then $2^{\lfloor k/2 \rfloor} \mid n$. Deduce the complex K-theoretic upper bound: if S^{n-1} admits k linearly independent vector fields, then $2^{\lfloor k/2 \rfloor} \mid n$. Compare this with the lower bound from Talk 12, and explain that the real KO-theoretic refinement gives Adams' sharp bound [Rak15, Remark 8.4].

Literature: [Rak15, §5-8]

References

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[Rak15] Arpon Raksit. *Vector fields and the J-homomorphism*. 2015. URL: %5Curl%7Bhttps://ncatlab.org/nlab/files/Raksit_JHomomorphism.pdf%7D.

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